CORRIGENDUM TO 'TIME-INCONSISTENT MEAN-FIELD OPTIMAL STOPPING: A LIMIT APPROACH'

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A CORRECTION OF THE PROOF OF THEOREM 3.3

We provide a correction of the proof of Theorem 3.3 in [DM23]. This result plays a key role in the proof of the main result of [DM23], Theorem 3.4. In the proof of Theorem 3.3, we used a well known law of large numbers whose statement in Eq. (3.24) is unfortunately wrong.

First, we recall that the family of interacting Snell envelopes $\{Y^{i,n}\}_{i=1}^n$ and the family of finite horizon stopping problems $\{Y^i\}_{i\geq 1}$ are defined by (2.4) and (2.5) in [DM23]. The function h and sequence $\{\xi^i\}_{i\geq 1}$ satisfy Assumption 2.1 in [DM23]. Moreover, we assume that, $\mathbb{P}$-a.s.,

$$\xi^i \geq h(\xi^i, \frac{1}{n}\sum_{j=1}^n \xi^j), \qquad i = 1, \ldots, n. \tag{1.1}$$

We refer to this new set of hypotheses as Assumption 2.1 -1.1, and in what follows we suppose that it holds. For the new proof, we need to introduce a (fixed) sequence of random variables $\{\alpha_j\}_{j\geq 1}$ which are all independent of $\{\mathbb{F}^i\}_{i\geq 1}$ and for which

$$\mathbb{E}[\alpha_j] = 1 - 2^{-j},\ \ \mathrm{Var}(\alpha_j) \leq a^j,\ \ j \geq 1, \quad |\mathbb{E}[\alpha_j \alpha_k]| \leq a^{|j-k|}, \quad j,k \geq 1, \tag{1.2}$$

for a given $a \in (0, 1)$. We note that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n (\mathbb{E}[\alpha_j] - 1)^2, \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n \mathrm{Var}(\alpha_j) = 0, \qquad \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n a^j = 0. \tag{1.3}$$

The next lemma is the main ingredient in the new proof of Theorem 3.3.

Lemma 1.1. *The following law of large numbers (LLN) holds*

$$\lim_{n\to\infty} \sup_{1\leq i\leq n} \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i} \left(\frac{1}{n}\sum_{j=1}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j])\right)^2\right] = 0. \tag{1.4}$$

Moreover,

$$\lim_{n\to\infty} \sup_{1\leq i\leq n} \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i} \left(\frac{1}{n}\sum_{j=1}^n (1-\alpha_j) Y_\tau^j\right)^2\right] = 0, \tag{1.5}$$

and

$$\lim_{n\to\infty} \sup_{1\leq i\leq n} \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i} \left(\frac{1}{n}\sum_{j=1}^n \alpha_j (Y_\tau^j - Y_\tau^i)\right)^2\right] = 0. \tag{1.6}$$

Proof. To show (1.4), we note that since

$$\frac{1}{n}\sum_{j=1}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]) = \frac{1}{n}(\alpha_i Y_\tau^i - \mathbb{E}[\alpha_i Y_\tau^i]) + \frac{n-1}{n}\frac{1}{n-1}\sum_{j=1, j\neq i}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j]),$$

by Dominated Convergence it suffices to show

$$\lim_{n\to\infty} \sup_{1\leq i\leq n} \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i} \left(\frac{1}{n-1}\sum_{j=1, j\neq i}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j])\right)^2\right] = 0. \tag{1.7}$$

By the properties of the essential supremum, for each $n \geq 2$, there exists a sequence $\{\tau_m^n\}_{m\geq 1}$ from $\mathcal{T}_0^i$ such that

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i}\left(\frac{1}{n-1}\sum_{j=1,j\neq i}^{n}(\alpha_j Y_\tau^j-\mathbb{E}[\alpha_j Y_\tau^j])\right)^2=\lim_{m\to\infty}\left(\frac{1}{n-1}\sum_{j=1,j\neq i}^{n}(\alpha_j Y_{\tau_m^n}^j-\mathbb{E}[\alpha_j Y_{\tau_m^n}^j])\right)^2 \quad \text{a.s.}$$

and by Dominated Convergence, we have

$$\begin{aligned}\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i}\left(\tfrac{1}{n-1}\textstyle\sum_{j\neq i}^n(\alpha_j Y_\tau^j-\mathbb{E}[\alpha_j Y_\tau^j])\right)^2\right] &= \lim_{m\to\infty}\mathbb{E}\left[\left(\tfrac{1}{n-1}\textstyle\sum_{j\neq i}^n(\alpha_j Y_{\tau_m^n}^j-\mathbb{E}[\alpha_j Y_{\tau_m^n}^j])\right)^2\right]\\ &\leq \sup_{\tau\in\mathcal{T}_0^i}\mathbb{E}\left[\left(\tfrac{1}{n-1}\textstyle\sum_{j\neq i}^n(\alpha_j Y_\tau^j-\mathbb{E}[\alpha_j Y_\tau^j])\right)^2\right].\end{aligned}$$

Noting that, since for any $j\neq i$, Y^j is independent of τ, we have

$$\begin{aligned}\mathbb{E}[Y_\tau^j]=\textstyle\int\mathbb{E}[Y_\tau^j\,|\,\tau=t]F_\tau(dt) &= \textstyle\int\mathbb{E}[Y_t^j\,|\,\tau=t]F_\tau(dt)=\int\mathbb{E}[Y_t^j]F_\tau(dt)\\ &= \textstyle\int\mathbb{E}[Y_t^1]F_\tau(dt):=\mathbb{E}[\mathbb{E}[Y_s^1]\big|_{s=\tau}],\end{aligned}$$

since the Y^j's are i.i.d., where $F_\tau(dt)$ denotes the probability distribution of τ. Similar calculations yield, for every $\tau\in\mathcal{T}_0^i$ and every $\ell,j\neq i$,

$$\mathbb{E}[Y_\tau^\ell Y_\tau^j]=\mathbb{E}[(\mathbb{E}[Y_s^1])^2\big|_{s=\tau}]\geq 0.$$

Moreover, the independence of (α_j,α_k) from (Y_τ^j,Y_τ^k) entails

$$\operatorname{cov}(\alpha_j Y_\tau^j,\alpha_k Y_\tau^k)=\operatorname{cov}(\alpha_j,\alpha_k)\mathbb{E}[Y_\tau^j Y_\tau^k]+\mathbb{E}[\alpha_j]\mathbb{E}[\alpha_k]\operatorname{cov}(Y_\tau^j,Y_\tau^k).$$

Thus,

$$\begin{aligned}\mathbb{E}\left[\left(\tfrac{1}{n-1}\textstyle\sum_{j=1,j\neq i}^n(\alpha_j Y_\tau^j-\mathbb{E}[\alpha_j Y_\tau^j])\right)^2\right] &= \tfrac{1}{(n-1)^2}\textstyle\sum_{j,k,j,k\neq i}^n\operatorname{cov}(\alpha_j Y_\tau^j,\alpha_k Y_\tau^k)\\ &\leq\left(\tfrac{1}{n-1}+\tfrac{n^2}{(n-1)^2}\tfrac{2}{n}\textstyle\sum_{m=1}^n a^m\right)\mathbb{E}[\sup_{t\in[0,T]}|Y_t^1|^2].\end{aligned}$$

This bound, being uniform in τ and in $i, 1\leq i\leq n$, yields (1.4) due to (1.3).

The limit (1.5) is derived as follows. Due to (1.2), the Cauchy-Schwarz inequality and the independence of α_j from Y_τ^j, we have

$$\begin{aligned}\mathbb{E}\big[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i}\left(\tfrac{1}{n}\textstyle\sum_{j=1}^n(1-\alpha_j)Y_\tau^j\right)^2\big] &\leq\mathbb{E}\big[\tfrac{1}{n}\textstyle\sum_{j=1}^n(1-\alpha_j)^2\big]\mathbb{E}[\sup_{t\in[0,T]}|Y_t^1|^2]\\ &\leq 2\mathbb{E}[\sup_{t\in[0,T]}|Y_t^1|^2]\left(\tfrac{1}{n}\textstyle\sum_{j=1}^n\operatorname{Var}(\alpha_j)+\tfrac{1}{n}\sum_{j=1}^n(\mathbb{E}[\alpha_j]-1)^2\right)\end{aligned}$$

which tends to 0, by (1.3).

We now show (1.6). We have $\mathbb{P}$-a.s.,

$$\begin{aligned}&\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i}\left|\tfrac{1}{n}\textstyle\sum_{j=1}^n a_j(Y_\tau^j-Y_\tau^i)\right|=\sup_{t\in[0,T]}\left|\tfrac{1}{n}\textstyle\sum_{j=1}^n a_j(Y_t^j-Y_t^i)\right|\\ &\leq\sup_{t\in[0,T]}\left|\tfrac{1}{n}\textstyle\sum_{j=1}^n\alpha_j Y_t^j-\mathbb{E}[\alpha_j]\mathbb{E}[Y_t^j]\right|+\sup_{t\in[0,T]}\left|\tfrac{1}{n}\textstyle\sum_{j=1}^n\alpha_j Y_t^i-\mathbb{E}[\alpha_j]\mathbb{E}[Y_t^i]\right|,\\ &\leq\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i}\left|\tfrac{1}{n}\textstyle\sum_{j=1}^n\alpha_j Y_\tau^j-\mathbb{E}[\alpha_j]\mathbb{E}[Y_\tau^j]\right|+\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i}\left|\tfrac{1}{n}\textstyle\sum_{j=1}^n\alpha_j Y_\tau^i-\mathbb{E}[\alpha_j]\mathbb{E}[Y_\tau^i]\right|,\end{aligned}$$

where we have used the fact that $\mathbb{E}[Y_t^i]=\mathbb{E}[Y_t^j]$ for each $t\in[0,T]$, since the Y^i's are i.i.d. Therefore,

$$\begin{aligned}\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i}\left(\tfrac{1}{n}\textstyle\sum_{j=1}^n\alpha_j(Y_\tau^j-Y_\tau^i)\right)^2\right] &\leq 2\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i}\left(\tfrac{1}{n}\textstyle\sum_{j=1}^n\alpha_j Y_\tau^j-\mathbb{E}[\alpha_j]\mathbb{E}[Y_\tau^j]\right)^2\right]\\ &\quad+2\mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i}\left(\tfrac{1}{n}\textstyle\sum_{j=1}^n\alpha_j Y_\tau^i-\mathbb{E}[\alpha_j]\mathbb{E}[Y_\tau^i]\right)^2\right].\end{aligned}\tag{1.8}$$

Following the same steps as in the proof of (1.4) (in this case the calculations are simplified), we obtain

$$\lim_{n\to\infty} \sup_{1\le i\le n} \mathbb{E}\left[\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i} \left(\frac{1}{n}\sum_{j=1}^n \alpha_j Y_\tau^i - \mathbb{E}[\alpha_j]\mathbb{E}[Y_\tau^i]\right)^2\right] = 0. \tag{1.9}$$

The limit (1.6) follows by further applying (1.4) to the first term of the r.h.s. of (1.8). □

We will now state a new and correct version of [DM23, Theorem 3.3] and sketch its proof. Notice that we also substitute the condition $\gamma_1^2 + \gamma_2^2 < \frac{1}{16}$ with a new (and more efficient) one.

Theorem 1.2. *Let Assumption 2.1-1.1 holds and let us assume that γ_1 and γ_2 satisfy the new condition*

$$\gamma_1^2 + \gamma_2^2 < \frac{1}{12}. \tag{1.10}$$

Then, it holds that

$$\lim_{n\to\infty} \sup_{1\le i\le n} \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^{i,n} - Y_t^i|^2\right] = 0. \tag{1.11}$$

Proof. As in the proof of [DM23, Theorem 3.3], we have that for any $t \le T$,

$$|Y_t^{i,n} - Y_t^i| \le \mathbb{E}\left[\gamma_1 \sup_{s\in[0,T]} |Y_s^{i,n} - Y_s^i| + \tfrac{\gamma_2}{n}\textstyle\sum_{j=1}^n \sup_{s\in[0,T]} |Y_s^{j,n} - Y_s^j| + \gamma_2 G^{i,n} \mid \mathcal{F}_t^i\right], \tag{1.12}$$

where

$$\begin{aligned} G^{i,n} := {} & \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i} |\tfrac{1}{n}\textstyle\sum_{j=1}^n (\alpha_j Y_\tau^j - \mathbb{E}[\alpha_j Y_\tau^j])| + \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i} |\tfrac{1}{n}\textstyle\sum_{j=1}^n \alpha_j (Y_\tau^j - Y_\tau^i)| \\ & + \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_0^i} |\tfrac{1}{n}\textstyle\sum_{j=1}^n (1-\alpha_j) Y_\tau^j| + |\tfrac{1}{n}\textstyle\sum_{j=1}^n (\mathbb{E}[\alpha_j] - 1)|\mathbb{E}[\sup_{s\in[0,T]} |Y_t^1|]. \end{aligned}$$

In view of the exchangeability of $\{Y^{j,n}, Y^j\}_{j=1}^n$, the Cauchy-Schwarz inequality and Doob's inequality, if we set $C := 12\gamma_2^2(1 - 12(\gamma_1^2+\gamma_2^2))^{-1}$, by (1.2) and Lemma 1.1 we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \mathbb{E}[\sup_{t\in[0,T]} |Y_t^{i,n} - Y_t^i|^2] \le C \lim_{n\to\infty} \sup_{1\le i\le n} \mathbb{E}[(G^{i,n})^2] = 0. \tag{1.13}$$

Now, from (1.12), we get

$$\begin{aligned} (1 - 12\gamma_1^2) \sup_{1\le i\le n} \mathbb{E}[\sup_{t\in[0,T]} |Y_t^{i,n} - Y_t^i|^2] &\le 12\gamma_2^2 \frac{1}{n}\sum_{j=1}^n \mathbb{E}[\sup_{t\in[0,T]} |Y_t^{j,n} - Y_t^j|^2] \\ &\quad + 12\gamma_2^2 \sup_{1\le i\le n} \mathbb{E}[(G^{i,n})^2]. \end{aligned}$$

Finally, since (1.10) entails $12\gamma_1^2 < 1$ and in view of Lemma 1.1 and (1.13), we obtain

$$\lim_{n\to\infty} \sup_{1\le i\le n} \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^{i,n} - Y_t^i|^2\right] = 0.$$

□

References

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