Deligne-Mumford compactification of the moduli space of Painlevé V connections

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Meromorphic Connections

Meromorphic Connections over $\mathbb{P}^1$

Definition

A meromorphic connection (∇, E, D) over $\mathbb{P}^1$ is the data of:

- a holomorphic vector bundle $E \to \mathbb{P}^1$,
- a morphism $\nabla \colon \mathcal{E} \to \mathcal{E} \otimes \Omega^1(D)$

where D is a effective divisor of $\mathbb{P}^1$ called the polar divisor of ∇.

Leibnitz Rule

For the $\mathscr{O}$-module structure of $\mathcal{E}$.

$$\nabla(f\sigma) = df \cdot \sigma + f\nabla\sigma$$

Warning

We will work only with rank 2 connections, that are connections (∇, E, D) such that $\mathrm{rk}(E) = 2$.

Example : Connections on the trivial bundle

Fact

Any connection on the trivial bundle $\mathscr{O} \oplus \mathscr{O} \cong \mathbb{C}^2 \times \mathbb{P}^1$ express as

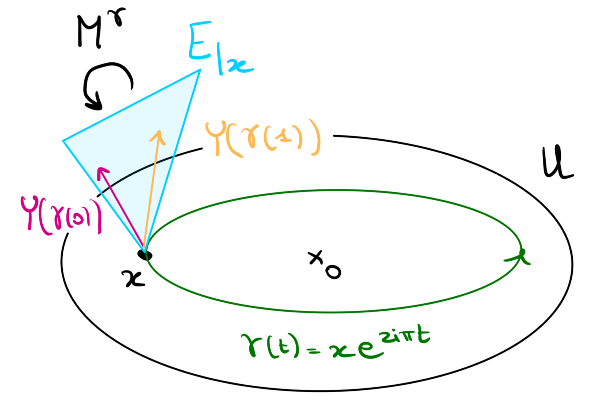
$$\nabla = d + \Omega \quad \text{for} \quad \Omega = \begin{pmatrix} \omega_{1,1} & \omega_{1,2} \\ \omega_{2,1} & \omega_{2,2} \end{pmatrix} \in \mathfrak{gl}\left(\Omega^1(D)\right)$$

Local Form

If we suppose that $0 \in \mathbb{P}^1$ is a singularity and we consider U such that $U \cap |D| = \{0\}$, then we have a local expression of Ω.

$$\Omega_{|U} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \frac{dx}{x^n} + \cdots + \underbrace{\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \frac{dx}{x}}_{\text{Residual matrix at 0}} + \text{holomorphic terms}$$

Local Solutions and their Monodromy



- Defined by the process of analytic continuation of $Y(x)$
- Only depends on the homotopy class of γ in $U \setminus \{0\}$.

Connections on a non trivial Bundle

Local Form

Let $E \to \mathbb{P}^1$ a vector bundle and U be a simply connected open set trivialising E.

$$\nabla_U = d + \Omega_U \quad \text{with} \quad \Omega_U = \begin{pmatrix} \omega_{1,1} & \omega_{1,2} \\ \omega_{2,1} & \omega_{2,2} \end{pmatrix} \in \mathfrak{gl}\Big(\Omega^1(D \cap U)\Big)$$

Gluing Conditions

On the overlaps $U_i \cap U_j$ it holds that

$$\Omega_i = g_{i,j}^{-1} \cdot \Omega_j \cdot g_{i,j} + g_{i,j}^{-1} \cdot dg_{i,j},$$

where $\{g_{i,j}\}$ is the cocycle of the bundle E.

Horizontal Sections

Horizontal sections

A local section $Y: U \to U \times \mathbb{C}^2$ is horizontal if :

$$\nabla Y = 0 \quad \Longleftrightarrow \quad \begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix} + \begin{pmatrix} \omega_{1,1} & \omega_{1,2} \\ \omega_{2,1} & \omega_{2,2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

Theorem

Let (∇, E) be a connection on $\mathbb{P}^1$ and U a simply connected open set of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Then:

- There exists a fundamental matrix of solution $Y(x)$ defined everywhere over U.
- Any other matrix of solutions $Y'(x)$ differs from $Y(x)$ by a constant matrix factor $Y'(x) = Y(x) \cdot C$.

Equivalence of Connections

Gauge equivalence

We say that $(\nabla, E) \sim (\nabla', E')$ if there exists a meromorphic (i.e. rational) morphism $\Phi\colon E \to E'$ sending ∇-horizontal sections in ∇'-horizontal sections.

The connections matrices are then related by these local equalities:

$$\Omega_U = \Phi^{-1}\Omega'_U\Phi + \Phi^{-1}d\Phi \qquad \text{for each trivialising } \; U \subseteq \mathbb{P}^1$$

Fact

Equivalent connections have conjugated monodromy:

$$(\nabla, E) \sim (\nabla', E) \quad \Longrightarrow \quad M_\nabla \sim M_{\nabla'}$$

Meromorphic Gauge Transformations

Example

$$\Psi := \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{x-q} \end{pmatrix} \quad : \quad (\nabla, \mathscr{O} \oplus \mathscr{O}, D) \;\mapsto\; (\tilde{\nabla}, \mathscr{O} \oplus \mathscr{O}(1), D + [q])$$

acts on a connection in this way:

$$\begin{pmatrix} \omega_{1,1} & \omega_{1,2} \\ \omega_{2,1} & \omega_{2,2} \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} \omega_{1,1} & \frac{\omega_{1,2}}{x-q} \\ (x-q) \cdot \omega_{2,1} & \omega_{2,2} - \frac{dx}{x-q} \end{pmatrix}$$

Conclusion

Since we are interested in isomonodromic deformations, we have to study connections up to meromorphic gauge transformations.

Painlevé V Connections

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Painlevé V Connections

Meromorphic connections (∇, E, D) such that

- $X = \mathbb{P}^1$
- $D = [0] + 2[1] + [\infty]$

Remark

Technical convention: the polar divisor D is minimal w.r.t. meromorphic gauge transformations.
Consequence: in D appear only poles with non trivial local monodromy.

Normal Form

Normal Form on $\mathscr{O} \oplus \mathscr{O}(2)$ (Diarra, Loray 2019)

$$\nabla_{|0} = d + \Omega_0 =$$

$$d + \begin{pmatrix} 0 & 1 \\ 0 & t \end{pmatrix} \frac{dx}{(x-1)^2} + \begin{pmatrix} 0 & -1 \\ 0 & \kappa_1 \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} 0 & 1 \\ 0 & -\kappa_0 \end{pmatrix} \frac{dx}{x}$$

$$+ \begin{pmatrix} 0 & 0 \\ \kappa_\infty & 0 \end{pmatrix} x dx + \begin{pmatrix} 0 & 0 \\ p & -1 \end{pmatrix} \frac{dx}{x-q} + \begin{pmatrix} 0 & 0 \\ \hat{K} & 0 \end{pmatrix} dx,$$

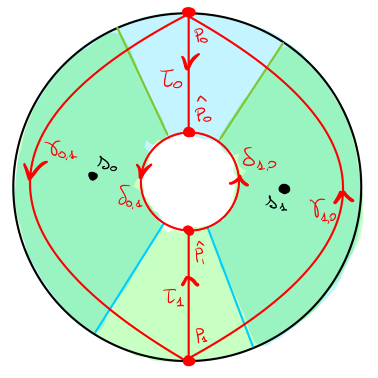
Fixed Parameters (local monodromy)

$$\Theta := \{\kappa_0, \kappa_1, \kappa_\infty\} \subseteq \mathbb{C}$$

Monodromy

Stokes Phenomena

The local monodromy around $x = 1$ is more complicated: solutions are defined only on sectors around the singularity and the change of sector give rise to a monodromy.



Monodromy

Problem

Isomonodromic deformations

Tools

- Moduli space of connections
- Deligne-Mumford compactification via stable curves
- Painlevé V differential equation

Painlevé Equations (Jimbo, Miwa, Ueno 1981)

Are second order differential equations whose solutions parametrise isomonodromic deformations of rank 2 meromorphic connections with $\deg D = 4$.

Moduli Space and Compactification

Non-Compact Moduli Space

What to do?

We have to find a suitable algebraic variety of dimension 3 in which live our parameters

$$(t, q, p).$$

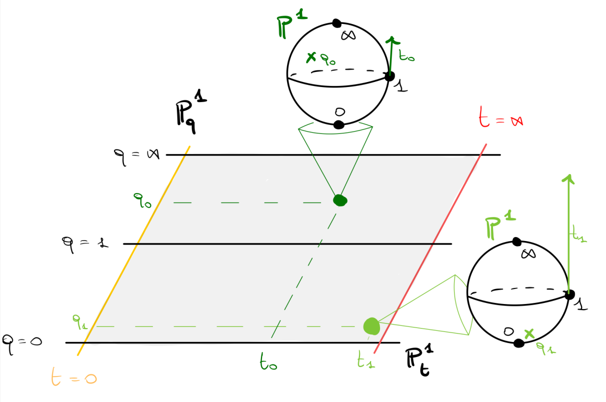
Recall that

- $t \in \mathbb{P}^1 \setminus \{0, \infty\} \cong T_1\mathbb{P}^1 \setminus \{0\}$,
- $q \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$,
- $p \in \mathbb{C}$.

PARAMETERS (t, q)

Moduli space of stable curves $\mathcal{M}$

$$\mathcal{M} := \left\{ q \in \mathbb{P}^1 \setminus \{0, 1, \infty\};\ \ t \in T_1\mathbb{P}^1 \setminus \{0\} \right\}.$$



Parameter p

Line Bundle

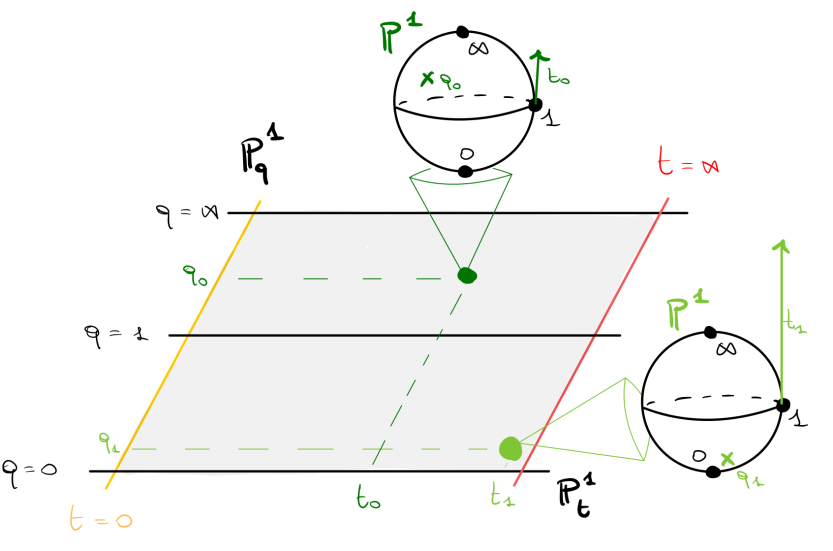
We can add to the picture the parameter $p \in \mathbb{C}$ as the coordinate on the fiber of a trivial line bundle.

Moduli space of Connections $\mathfrak{Con}_V^{\Theta}$

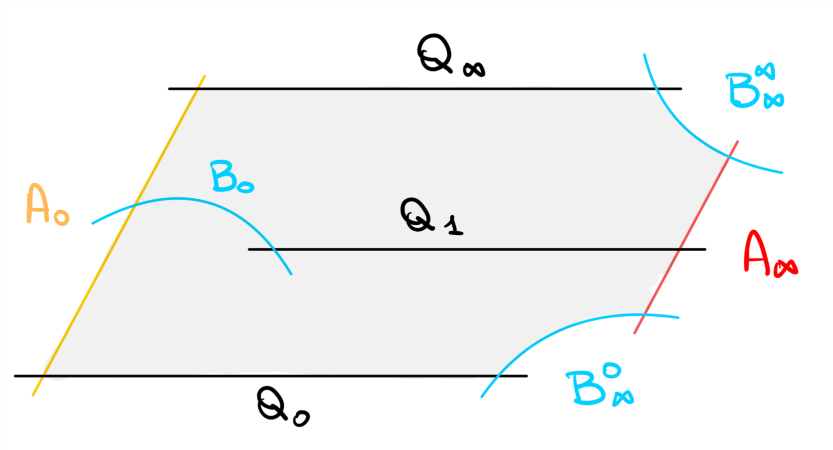
$$\mathfrak{Con}_V^{\Theta} \supseteq \mathcal{M} \times \mathbb{C} \ni (t, q, p)$$

Our moduli space contains as a Zarisky open set the trivial line bundle $\mathcal{M} \times \mathbb{C}$.

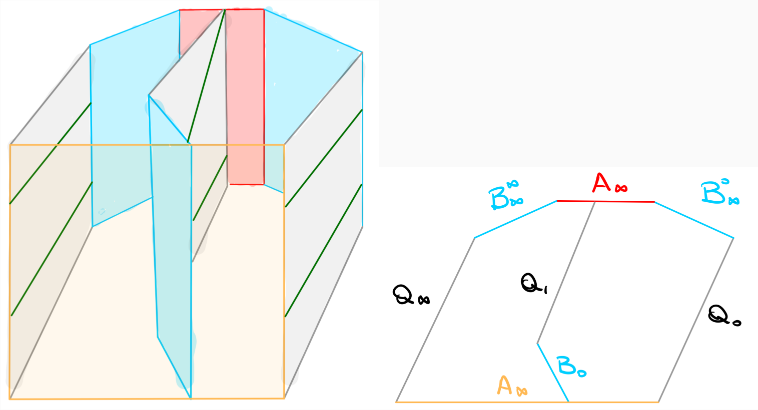
Compactification $\overline{\mathcal{M}}$



COMPACTIFICATION $\overline{\mathcal{M}}$



COMPACTIFICATION $\overline{\mathfrak{Con}_V^\Theta}$



Compactification (M., 2025)

$$\overline{\mathfrak{Con}_V^\Theta} \to \mathscr{O}_{\overline{\mathcal{M}}}\left(2Q_0 + B_\infty^0 - B_\infty^\infty + A_0\right) \cup s_\infty$$

Painlevé V Foliation

Painlevé V Equation

The Equation (PV)

$$q''(t) = \left(\frac{1}{2q(t)} + \frac{1}{q(t)-1}\right) q'(t)^2 - \frac{1}{t} q(t)'$$

$$+ \frac{(q(t)-1)^2}{t^2}\left(\alpha q(t) + \frac{\beta}{q(t)}\right) + \gamma \frac{q(t)}{t} + \delta \frac{q(t)(q(t)+1)}{q(t)-1}$$

Where $\alpha, \beta, \gamma, \delta$ are parameters depending on Θ.

Painlevé V Equation

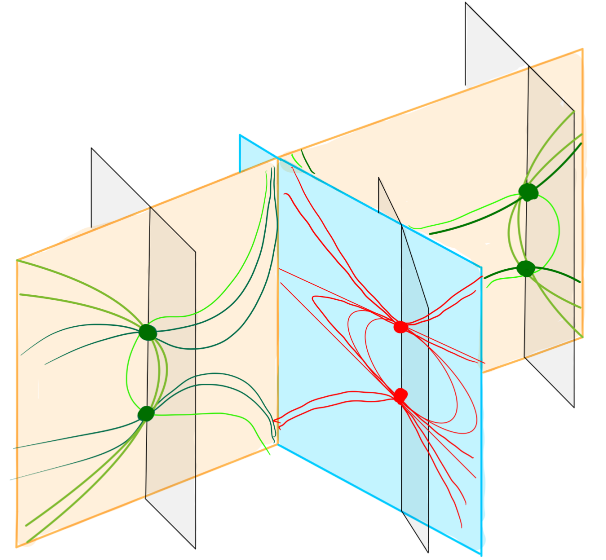
The Hamiltonian System

$$\begin{cases} \dfrac{\partial H^V}{\partial p} = \dfrac{dq}{dt} \\[2em] \dfrac{\partial H^V}{\partial q} = -\dfrac{dp}{dt} \end{cases}$$

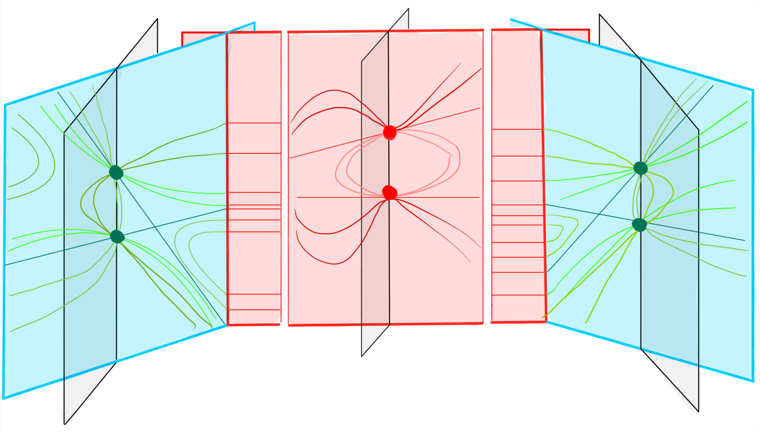
The Hamiltonian Function (Ohyama, 2006)

$$H^V := \frac{q\,(q-1)^2\,p^2 - \Big(\kappa_0\,(q-1)^2 + \kappa_1 q\,(q-1) - tq\Big)\,p + \kappa_\infty\,(q-1)}{t}$$

First Integrals around $t = 0$



First Integrals around $t = \infty$



Goals

Short terms goals

- Compute first integrals in a neighbourhood of the hypersurfaces $t = 0, \infty$.
- Understand the singularities of the foliation.
- Eventually apply MMP to have a "good" model.

Long terms goals

- Study the dynamics of the foliation in relation to the dynamics of the wild character variety associated (moduli space of monodromies).
- Apply the same study to other Painlevé equations

THANKS :)

Obrigado pela sua atenção !!