

# Multigrid preconditioner for saddle point problems with highly variable coefficients

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## Objectives

The aim of this work is the development of a robust preconditioning technique for saddle point problems with highly variable coefficients, especially the Darcy problem. The idea is based on the augmented Lagrangian formulation of saddle point problem and the multigrid preconditioning technique in  $H(\text{div})$  developed by Arnold *et al.* This kind of preconditioner avoids the construction of Schur complement and all the operators are approximated by the means of conforming finite element spaces.

## Introduction

The Darcy's law describes the flow of fluids through porous media. This equation is used to model various type of phenomena such as groundwater flows in aquifers and expulsion of hydrocarbons from the source rock and subsequent seeping into reservoirs.

Even though the model might look simple, its numerical solution is not trivial, in particular due of the high variability of the equation's coefficients. Numerical iterative methods need proper preconditioning in order to find a solution of the problem in a reasonable time, and research on the best preconditioning techniques is still ongoing, especially in the context of parallel programming. This work aims to provide an implementation of an optimal preconditioner, *i.e.* a preconditioner whose performance does not depend on the mesh size and on the variations of the equation coefficients. The advocated method is based on the work of Arnold, Falk and Winther in [1].

A block-diagonal preconditioner based on the augmented Lagrangian formulation of the saddle point problem will be presented. As pointed out by other authors the use of this formulation instead of the original one allows us to avoid the necessity to construct a good approximation of the Schur complement and its inverse. This approach seems to be more natural for the mixed problems embedded in  $H(\text{div}) \times L^2$ , like the Darcy problem.

## Mathematical introduction

Let  $V$  be an Hilbert space and denote its inner product with  $(\cdot, \cdot)_V$ . A linear continuous operator  $A : V \rightarrow V^*$  will be considered. Let  $Q$  be another Hilbert space and  $B : V \rightarrow Q^*$  a linear continuous operator and  $B^T : Q \rightarrow V^*$  its own adjoint. Given  $\mathbf{f} \in V^*$  and  $g \in Q^*$ , the aim is to find  $\mathbf{u} \in V$  and  $p \in Q$  solutions of the problem

$$\begin{cases} A\mathbf{u} + B^T p = \mathbf{f} & \text{in } V^*, \\ B\mathbf{u} = g & \text{in } Q^*, \end{cases}$$

If the operator  $A$  is invertible on  $\text{Ker } B$  and the operator  $B$  is closed, that is, there exists  $\beta > 0$  such that the inf-sup condition is satisfied

$$\sup_{\mathbf{v} \in V} \frac{(B\mathbf{u}, q)_{Q^* \times Q}}{\|\mathbf{v}\|_V} \geq \beta \|q\|_Q, \quad \forall q \in Q.$$

For all  $\mathbf{f} \in V^*$  and  $g \in \text{Im } B \subseteq Q^*$  there exists a unique solution  $\mathbf{u} \in V$  and  $p \in Q / \text{Ker } B^T$  of the problem.

## Augmented Lagrangian-type preconditioners

If the hypotheses of the existence and uniqueness of solution theorem are fulfilled and both the operators

$$\hat{A} := A + B^T M^{-1} B : V \rightarrow V^* \quad \text{and} \quad M : Q \rightarrow Q^*,$$

define an equivalent inner product on  $V$  and  $Q$ , then all the eigenvalues of the generalized problem

$$B\hat{A}^{-1}B^T q = \lambda Mq, \quad \text{in } Q^*,$$

belong to the interval  $[\beta^2, 1]$ .

This result can be used to prove that the following block operators

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \hat{A} & 0 \\ 0 & M \end{bmatrix},$$

are spectrally equivalent; the spectrum lies in  $[-1, -\beta^2] \cup \{1\}$ . For the Darcy problem the operator  $M$  is defined as the identity operator on  $Q$ , leading to the following  $\hat{A}$  operator

$$\hat{A} = \frac{1}{\mu} I_V - \nabla \text{div}.$$

## Multigrid preconditioning

Solving the preconditioner with a direct method can have serious drawbacks when the dimension of the problem is large. The two biggest concerns regard poor scaling and excessive memory consumption. A geometric multigrid preconditioner has been implemented for our problem, reproducing the work by Arnold, Falk and Winther in [1].

This method is based on a standard V-cycle multigrid, but a proper smoother operator  $R_j$  has to be defined on each level  $j = 1, \dots, J$ . The authors proposed two different smoothers, which are based respectively on an additive and a multiplicative Schwarz operator.

The additive smoother  $R_j : V_j \rightarrow V_j$  is defined as a multiple of the additive Schwarz operator applied to the following decomposition of  $V_j$ : let  $\mathcal{N}_j$  be the set of vertices in the triangulation  $T_j$ ; for every  $\nu \in \mathcal{N}_j$ ,  $T_{j,\nu}$  is defined as the patch made of mesh elements meeting at the vertex  $\nu$ . Then we can define our smoother as

$$R_j = \eta \sum_{\nu \in \mathcal{N}_j} P_{j,\nu} \Lambda_j^{-1}.$$

## Implementation details

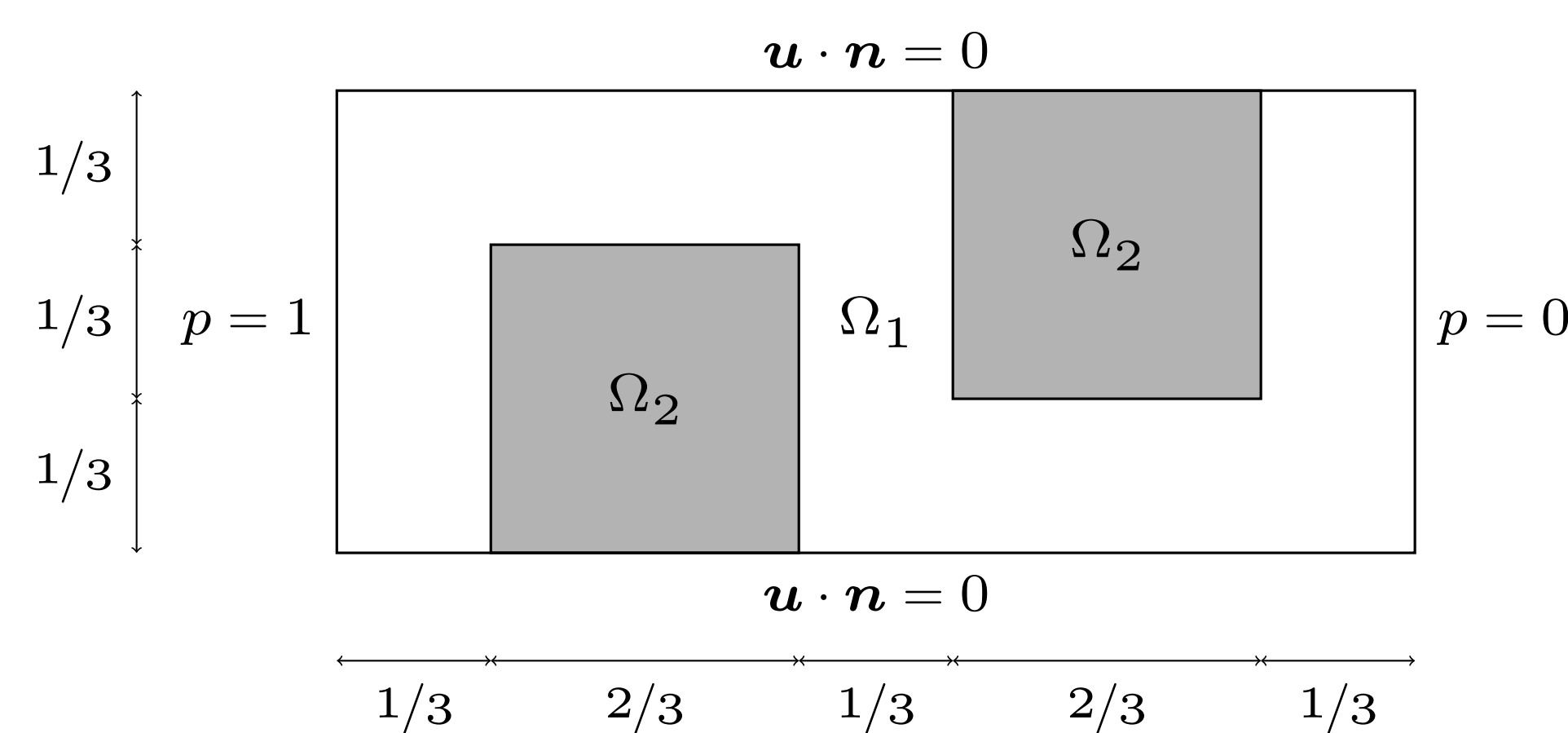
The code is based on the finite element library deal.II [2], version 8.1.0, which provided us with all the building blocks we needed. In particular, a module in the library implements the geometric multigrid method on hierarchical meshes, along with various helper functions.

Thanks to these features we were able to quickly implement the preconditioner and plug it in the deal.II solving procedures. We aimed at hybrid parallelization using intel TBB library and MPI-enabled algebraic backends. MPI support for distributed meshes is still experimental, but we were able to run scaling tests on the preconditioner initialization procedures. On the other hand, support for shared memory parallelism is complete.

## Results

The first problem is defined in  $\Omega = (0, 7/3) \times (0, 1)$  and it is partitioned into two subdomains  $\Omega_1$  and  $\Omega_2$ . In the first subdomain  $\mu$  is taken constant and equal to 1,  $10^1$ ,  $10^2$ ,  $10^4$ ,  $10^8$ , in the second one  $\mu = 1$ . The forcing term  $g$  is zero and the boundary conditions are chosen in order to have a flux from left to right:

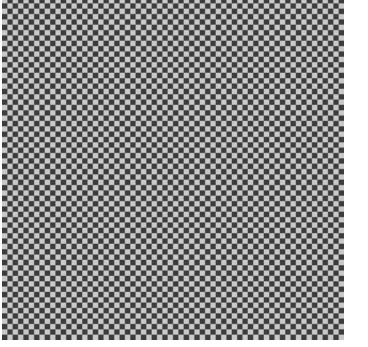
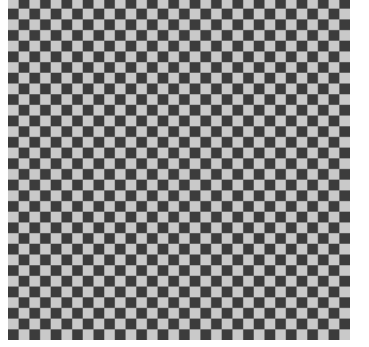
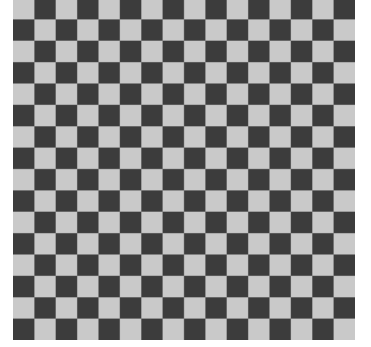
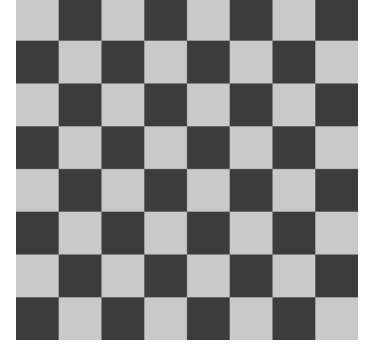
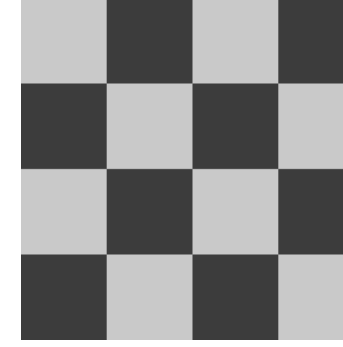
- on the top and the bottom of the domain the normal component of  $\mathbf{u}$  is taken equal to zero,
- on the right boundary  $p = 1$  and on the left one  $p = 0$ .



The number of outer iterations of GMRES method for each numerical experiment of this case is constant and it does not depend on  $h$  and  $\mu$ .

$h$	$\mu_1 = 1$	$\mu_1 = 10^1$	$\mu_1 = 10^2$	$\mu_1 = 10^4$	$\mu_1 = 10^8$
0.1	2	2	2	2	2
0.05	2	2	2	2	2
0.025	2	2	2	2	2
0.0125	2	2	2	2	2
0.00625	2	2	2	2	2

The next test case is inspired by the one proposed by Sundar *et al.* in [3]. The problem is defined in the  $\Omega = (-1, 1) \times (-1, 1)$  square, with homogeneous Dirichlet BCs and  $f = 1$  on the whole  $\Omega$ . The grid step  $h$  is a fixed  $h = 2^{-7}$ . The  $\mu$  function is discontinuous, and has the checkerboard pattern shown below; on white tiles  $\mu = 1$  while on black tiles  $\mu = 10^7$ . The main focus of this test is to make sure that the solver maintains a good behaviour when the frequency  $K$  of the tiles becomes higher, *i.e.* the number of discontinuities in the permeability grows. In the table we compare the iterations needed by the GMRES when solving the preconditioner with a direct method (It<sub>d</sub>), and when applying the multigrid method (It<sub>mg</sub>).

$\mu$					
$K$	6	5	4	3	2
It <sub>d</sub>	8	8	8	8	8
It <sub>mg</sub>	14	16	18	19	15

As one might expected, the multigrid method performs generally worse than the preconditioner solved with a direct method. However, neither method is affected by the change in frequency of the checkerboard pattern.

A similar test has been run by stretching the mesh along one spatial dimension to simulate anisotropic coefficients. The preconditioner still showed an almost-optimal performance.

## Conclusions

The numerical performance of the ideal preconditioner with internal direct solver is as good as we expected: in all our test cases it proved to be robust to mesh refinement, highly variable coefficients, non-squared domains with local refinement and anisotropic meshes.

The numerical performance of the ideal preconditioner with internal multigrid solver was satisfactory as well: even though the number of the outer GMRES iterations is always higher than its direct counterpart, the number of iterations stays constant regardless of the level of mesh refinement, permeability coefficient and grid anisotropy.

## Future developments

- The first important enhancement planned for the code is having the multigrid-based preconditioner running on the MPI version of the code.
- Using a mixed-hybrid discretization technique to reduce the number of the unknowns and obtaining a more efficient solver.

## References

- [1] Douglas N. Arnold, Richard S. Falk, and R. Winther. Preconditioning in  $H(\text{div})$  and applications. *Math. Comp.*, 66(219):957–984, 1997.
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- [3] Hari Sundar, George Biros, Carsten Burstedde, Johann Rudi, Omar Ghattas, and Georg Stadler. Parallel geometric-algebraic multigrid on unstructured forests of octrees. In *Proceedings of the International Conference on High Performance Computing, Networking, Storage and Analysis, SC '12*, pages 43:1–43:11, Los Alamitos, CA, USA, 2012. IEEE Computer Society Press.

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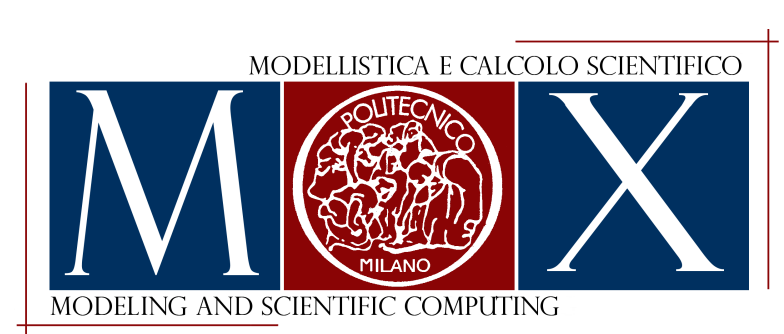
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The poster can be downloaded from  
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