Multigrid preconditioner for saddle point problems with highly variable coefficients

Mattia Penati and Edie Miglio

Politecnico di Milano - Department of Mathematics - MOX

Objectives

The aim of this work is the developement of a robust preconditioning technique for the saddle point problem with highly variable coefficients, especially the Darcy problem. The idea is based on the augmented Lagrangian formulation of saddle point problem and the multigrid preconditioning technique in $H(\mathrm{div})$ developed by Arnold *et al.*. This kind of preconditioner avoid the construction of Schur complement and all the operators are approxymated by the means of conforming finite element spaces.

Introduction

The Darcy's law describes the flow of fluids through porous media. This equation is used to model various type of phenomena such as groundwater flows in aquifers and expulsion of hydrocarbons from the source rock and subsequent seeping into reservoirs.

Even though the model might look simple, its numerical solution is not trivial, in particular due of the high variability of the equation's coefficients. Numerical iterative methods need proper preconditioning in order to find a solution of the problem in a reasonable time, and research on the best preconditioning techniques is still ongoing, especially in the context of parallel programming. This work aims to provide an implementation of an optimal preconditioner, *i.e.* a preconitioner whose performance does not depend on the mesh size and on the variations of the equation coefficients. The advocated method is based on the work of Arnold, Falk and Winther in [1].

A block-diagonal preconditioner based on the augmented Lagrangian formulation of the saddle point problem will be presented. As pointed out by other authors the use of this formulation instead of the original one allows us to avoid the necessity to construct a good approximation of the Schur complement and its inverse. This approach seems to be more natural for the mixed problems embedded in $H(\mathrm{div}) \times L^2$, like the Darcy problem.

Mathematical introduction

Let V be an Hilbert space and denote its inner product with $(\cdot,\cdot)_V$. A linear continuous operator $A : V \to V^*$ will be considered. Let Q be another Hilbert space and $B : V \to Q^*$ a linear continuous operator and $B^T : Q \to V^*$ its own adjoint. Given $\boldsymbol{f} \in V^*$ and $g \in Q^*$, the aim is to find $\boldsymbol{u} \in V$ and $p \in Q$ solutions of the problem

$$\begin{cases} A\boldsymbol{u} + B^T p = \boldsymbol{f} & \text{in } V^*, \\ B\boldsymbol{u} = g & \text{in } Q^*, \end{cases}$$

If the operator A is invertible on $\mathrm{Ker}\, B$ and the operator B is closed, that is, there exists $\beta > 0$ such that the inf-sup condition is satisfied

$$\sup_{\boldsymbol{v}\in V} \frac{\langle B\boldsymbol{u}, q\rangle_{Q^*\times Q}}{\|\boldsymbol{v}\|_V} \geq \beta \|q\|_Q, \qquad \forall q \in Q.$$

For all $\boldsymbol{f} \in V^*$ and $g \in \mathrm{Im}\, B \subseteq Q^*$ there exists a unique solution $\boldsymbol{u} \in V$ and $p \in Q/\mathrm{Ker}\, B^T$ of the problem.

Augmented Lagrangian-type preconditioners

If the hypotheses of the existence and uniqueness of solution theorem are fulfilled and both the operators

$$\hat{A} := A + B^T M^{-1} B : V \to V^* \qquad \text{and} \qquad M : Q \to Q^*,$$

define an equivalent inner product on V and Q, then all the eigenvalues of the generalized problem

$$B\hat{A}^{-1}B^T q = \lambda M q, \qquad \text{in } Q^*,$$

belong to the interval $[\beta^2, 1]$.

This result can be used to prove that the following block operators

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} \hat{A} & 0 \\ 0 & M \end{bmatrix},$$

are spectrally equivalent, the spectrum lies in $[-1, -\beta^2] \cup \{1\}$. For the Darcy problem the operator M is defined as the identity operator on Q, leading to the following $\hat{A}$ operator

$$\hat{A} = \frac{1}{\mu} I_V - \nabla \,\mathrm{div}\,.$$

Multigrid preconditioning

Solving the preconditioner with a direct method can have serious drawbacks when the dimension of the problem is large. The two biggest concerns regard poor scaling and excessive memory consumption. A geometric multigrid preconditioner has been implemented for our problem, reproducing the work by Arnold, Falk and Winther in [1].

This method is based on a standard V-cycle multigrid, but a proper smoother operator R_j has to be defined on each level $j = 1, \dots, J$. The authors proposed two different smoothers, which are based respectively on an additive and a multiplicative Schwarz operator.

The additive smoother $R_j : V_j \to V_j$ is defined as a multiple of the additive Schwarz operator applied to the following decomposition of V_j: let $\mathcal{N}_j$ be the set of vertices in the triangulation T_j; for every $\nu \in \mathcal{N}_j$, $T_{j,\nu}$ is defined as the patch made of mesh elements meeting at the vertex ν. Then we can define our smoother as

$$R_j = \eta \sum_{\nu\in\mathcal{N}_j} P_{j,\nu}\, \Lambda_j^{-1}.$$

Implementation details

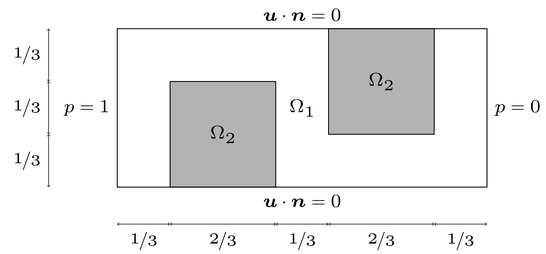
The code is based on the finite element library deal.II [2], in particular version 8.1.0, which provides us with all the building blocks we needed.

Data: $\boldsymbol{x}_j^0$, f_j, j
for $i < m$ **do**
  Compute Residual $\boldsymbol{r}_j^i$;
  for *Every vertex* **do**
    Solve $\hat{A}_v \tilde{\boldsymbol{x}}_v = \boldsymbol{r}_v$;
    Distribute $\tilde{\boldsymbol{x}}_v$ into $\tilde{\boldsymbol{x}}$;
  end
  $\boldsymbol{x}_j^{i+1} = \boldsymbol{x}_j^i + \eta\tilde{\boldsymbol{x}}$;
end
return $\boldsymbol{x}_j^m$

Results

The first problem is defined in $\Omega = (0, 7/3) \times (0, 1)$ and it is partitioned into two subdomains Ω_1 and Ω_2. In the first subdomain μ is taken constant and equal to $1, 10^1, 10^2, 10^4, 10^8$, in the second one $\mu = 1$. The forcing term g is zero and the boundary conditions are chosen in order to have a flux from left to right:

- on the top and the bottom of the domain the normal component of $\boldsymbol{u}$ is taken equal to zero,
- on the right boundary $p = 1$ and on the left one $p = 0$.



The number of outer iterations of GMRES method for each numerical experiment of this case is constant and it does not depend on h and μ.

h	$\mu_1 = 1$	$\mu_1 = 10^1$	$\mu_1 = 10^2$	$\mu_1 = 10^4$	$\mu_1 = 10^8$
0.1	2	2	2	2	2
0.05	2	2	2	2	2
0.025	2	2	2	2	2
0.0125	2	2	2	2	2
0.00625	2	2	2	2	2

Conclusion

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Future developments

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References

[1] Douglas N. Arnold, Richard S. Falk, and R. Winther. Preconditioning in $H(div)$ and applications. *Math. Comp.*, 66(219):957–984, 1997.

[2] W. Bangerth, R. Hartmann, and G. Kanschat. deal.II – a general purpose object oriented finite element library. *ACM Trans. Math. Softw.*, 33(4):24/1–24/27, 2007.

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Contact Information

- Web: http://www.university.edu/smithlab
- Email: john@smith.com
- Phone: +1 (000) 111 1111

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