Development of a Numerical Framework for the Study of Solid Earth Tides

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Objectives

The aim of this work is the development of some numerical tools for the investigation of the effects of the Moon and Sun attraction on the Earth. The problem shows an intrinsic time dependency, given by both the dynamics in time of the celestial bodies and by the Earth deformation model.

Introduction

The effects of the gravitational field of other celestial bodies are evident on the Earth's seas levels varying daily because of the Moon tide. Even if not immediatly visible, tidal effects have been measured on continents; this phenomenon is known as solid Earth tide, that is the deformation of the Earth's crust due to the Moon and Sun tidal forces.

The solid Earth tides have been already studied from an analytical and theoretical point of view [1]. A numerical finite element approximation of the Moon tidal force has been proposed in [2], based on a very simple model of the Earth. Aim of this work is to construct a scalable software suite implementing a numerical solver for the solid Earth tide with larger complexity.

This problem shows an intrinsic time dependency, given by the dynamics of the celestial bodies and the Earth's deformation model. First the relative position of the Earth, Moon and Sun in time are determined, then the Earth's deformation is modeled.

The modeling of the Earth deformation can approached assuming the planet to be an isotropic viscoelastic medium. The choice of the viscoelastic rheological model leads us to the solution of a linear viscoelasticity problem. A solver for this problem has been developed and validated. A massive MPI parallelization has been implemented to speed up the computations.

Viscoelastic model with tidal forces

The Earth has been modeled as a continuum body, with an isotropic viscoelastic rheology. The conservation of linear momentum and the definition of the pressure are given by the equations

$$\operatorname{div}\sigma + \mathbf{f} = 0 \quad \text{and} \quad \operatorname{div}\mathbf{u} + \frac{d}{2\mu}\frac{1-2\nu}{1+\nu}p = 0;$$

with the stress tensor given by the constitutive equation

$$\sigma = -pI + \int_0^t e^{-\frac{t-s}{\tau}} 2\mu \operatorname{dev}\nabla_s \dot{\mathbf{u}}(s)\, ds,$$

where μ, ν and τ are the shear modulus, the Poisson ratio and relaxation time, respectively. These quantities are not constrained to a particular model of the Earth and this gives us the ability to approach different scenarios with little effort.

The term $\mathbf{f}$ is the tidal force acting on the Earth exerted by the other celestial bodies:

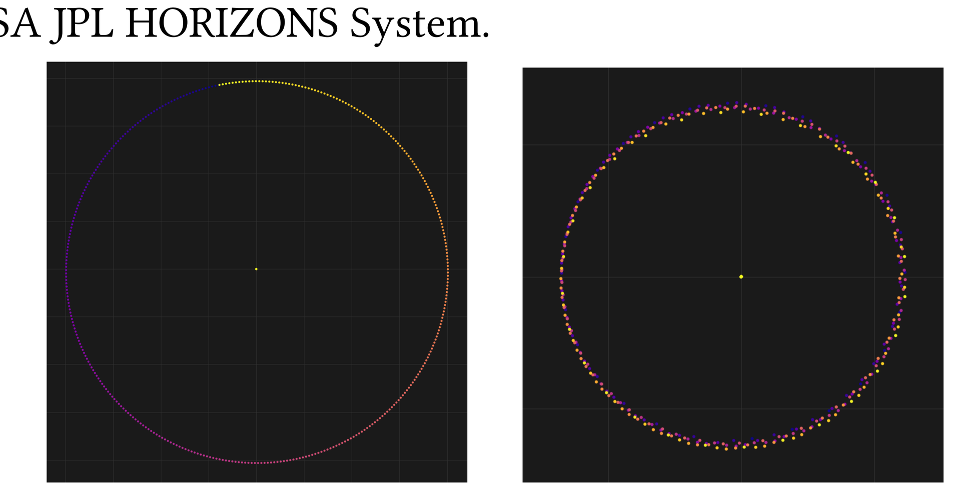
$$\mathbf{f}(\mathbf{x}, t) = -\rho G \left[\sum_i \frac{M_i}{\|\mathbf{r}_i\|^3(t)} \left(I - 3\frac{\mathbf{r}_i(t) \otimes \mathbf{r}_i(t)}{\|\mathbf{r}_i(t)\|^2} \right) \right] \mathbf{x}.$$

Where ρ and G are the Earth density and the Cavendish gravitation constant, respectively. The sum ranges over the considered celestial bodies with mass M_i and located at position $\mathbf{r}_i(t)$.

Simulation of the celestial mechanics

This task requires the numerical simulation of the gravitational influence of each body of the solar system to determine the configuration of Sun, Earth and Moon. Since we are dealing with a mechanical problem, it is natural to choose a geometric integrator, then *Galerkin variational integrators* [3] will be used.

These schemes has been implemented in a C suite, the validation of the solver has been done comparing the results generated by the implemented software with the ones provided by NASA JPL HORIZONS System.



Simulated trajectories over 1 year long time range. On the left Sun-Earth trajectory and on the right Earth-Moon one.

Viscoelastic numerical solver

A standard one-step, unconditionally stable and second-order accurate formula based on mid-point quadrature is used to discretize the constitutive equation with respect to time:

$$\begin{cases} \sigma_n = -p_n I + 2\mu e^{-\frac{\Delta t}{2\tau}} \operatorname{dev}\nabla_s(\mathbf{u}_n - \mathbf{u}_{n-1}) + e^{-\frac{\Delta t}{\tau}} h_{n-1}, \\ h_n = e^{-\frac{\Delta t}{\tau}} h_{n-1} + 2\mu e^{-\frac{\Delta t}{2\tau}} \operatorname{dev}\nabla_s(\mathbf{u}_n - \mathbf{u}_{n-1}). \end{cases}$$

A Discontinuous Galerkin scheme with Simmetric Weighted Interior Penalty has been used to obtain a fully discretized problem, with the saddle point structure

$$\begin{bmatrix} 2\hat{\mu}A & B^T \\ B & -\frac{d}{2\mu}\frac{1-2\nu}{1+\nu}M \end{bmatrix} \begin{bmatrix} \mathbf{u}_n \\ p_n \end{bmatrix} = \begin{bmatrix} \mathbf{f}_n \\ 0 \end{bmatrix} \quad \text{with } \hat{\mu} = \mu e^{-\frac{\Delta t}{2\tau}}.$$

If both the operators $\hat{A} := A + B^T M^{-1} B : V \to V^*$ and $M : Q \to Q^*$ define an equivalent inner product on V and Q, then all the eigenvalues of the generalized problem

$$B\hat{A}^{-1}B^T q = \lambda M q, \qquad \text{in } Q^*,$$

belong to the interval $[\beta^2, 1]$. This result can be used to prove that the following block operators

$$\begin{bmatrix} 2\hat{\mu}A & B^T \\ B & -\frac{d}{2\mu}\frac{1-2\nu}{1+\nu}M \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2\hat{\mu}\hat{A} & -B^T \\ 0 & \frac{d}{2\mu}M \end{bmatrix},$$

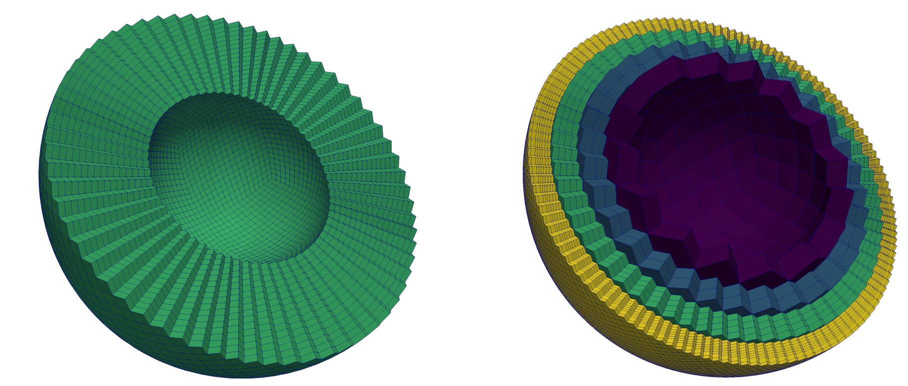
are spectrally equivalent; the spectrum lies in $[-1, -\beta^2] \cup \{1\}$. This defines an optimal block preconditioner for the problem.

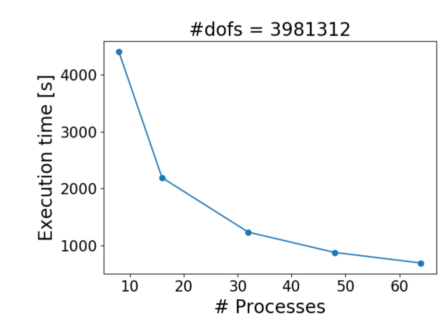
Implementation details

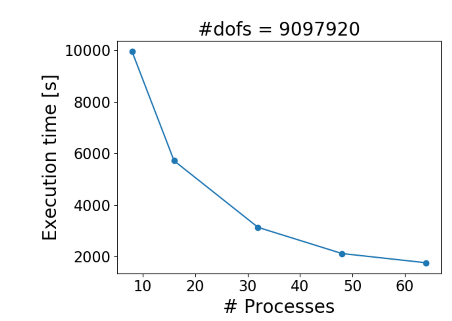
The code is based on the finite element library deal.II [4], version 9.0.0, which provided us with all the building blocks we needed. In particular, a module in the library implements the geometric multigrid method on hierarchical meshes with MPI support. This method has been used to implement a preconditioner for the $\hat{A}$ block; the matrix M is diagonal if a proper pair of finite element space and quadrature rule is choosen.

Results

Two different problems have been built to measure the scaling performance of the solver: one with a uniformly refined three dimensional hyper-shell and one with a non-uniform grid.



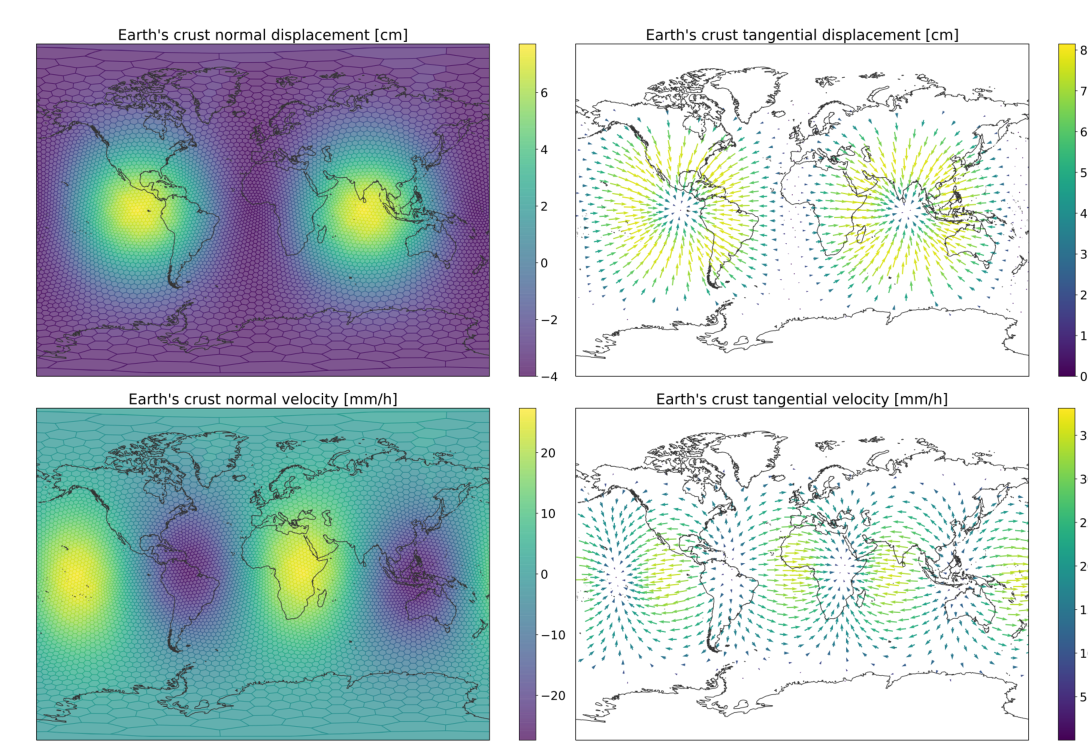


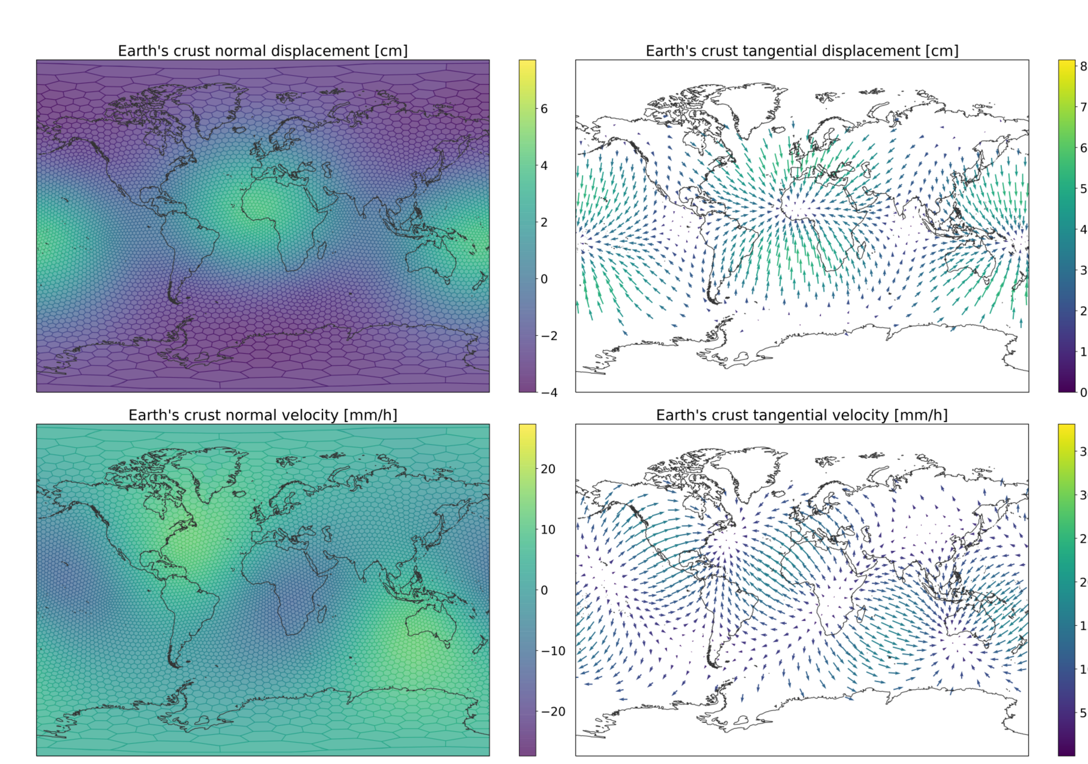


# procs	Elapsed time	Speed-up
8	73m 23s	–
16	36m 28s	2.01
32	20m 31s	3.58
48	14m 33s	5.04
64	11m 28s	6.40

# procs	Elapsed time	Speed-up
8	165m 51s	–
16	95m 13s	1.74
32	52m 07s	3.18
48	35m 11s	4.71
64	29m 12s	5.68

We evaluate the solid tide effects on a time span of 28 days with constant $\Delta t = 1$ hour. Two different configurations are shown: the first where the Moon and Sun are almost aligned with the Earth; the second where the Moon and the Sun form an angle of $\simeq 90°$ with respect to the Earth. The obtained results are of the same order of magnitude of the available measurements.





Conclusions

Solvers for these generic problems have been constructed, coded in a parallel, scalable framework and strongly tested and validated.

The correctness of the convergence rate both in space and time of the adopted numerical methods has been verified; moreover the optimality of the Geometric Multigrid procedure adopted to precondition the algebraic version of the problem has been confirmed.

The obtained results for the solid tides are exactly in the range of the available measurements on the phenomenon and the ratio between the computed lunar and solar tides is the expected one.

The developed tool results to be robust and is able to provide quantitative results in the constructed scenarios.

Future developments

The first important enhancement required for the code is having a multiscale time advancing method. This improvement will allow us to approach simulation on wide time ranges, on which we expect viscous effects to be more relevant.

At the current stage the effects on the crust are visibile only if a proper level of mesh refinement is reached. In order to reduce the number of the unknowns a shell model can be used to obtain a more accurate description of the Earth.

References

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`https://bitbucket.org/mattiapenati/siamgs19-poster`