CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
COMPUTATIONAL BIOLOGY A

COURSE CODES: **FFR 110, FIM740GU, PhD**

Time:	March 16, 2017, at $14^{00} - 18^{00}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once at 15^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 18 points (need 7 points to pass).
CTH $\geq$17 grade 3; $\geq$22 grade 4; $\geq$26 grade 5,
GU $\geq$17 grade G; $\geq$ 24 grade VG.

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1. Pest control [2 points] Assume that a population of size $N(t)$ of fertile insects shows the following time evolution:

$$\dot{N} = bN - dN - cN^2\,, \tag{1}$$

where b, c and d are positive parameters and $b > d$.

a) Explain the form of Eq. (1). Give plausible interpretations for the parameters b, d, and c. How large is the carrying capacity of the environment?

b) One method for pest control of insects is the release of sterile insects. Assume that a population of sterile insects is introduced into a population of N fertile insects. Assume that the number s of sterile insects is kept constant by a steady supply of new sterile insects balancing deaths. Assume further that the two populations (N and s) are at all times well mixed.

   Assume that the sterile insects show identical behaviour as fertile insects (equal mating rate and equal competition for resources), with the only exception that mating involving sterile insects results in failed births. Finally, assume that the sterile insects are male or female with equal probability, i.e. you do not need to take the sex of the insects into consideration.

   Modify Eq. (1) to model how a number s of sterile insects affect the time evolution of N.

c) Show, using the model you derived in subtask b), that the ratio

$$\rho = \frac{N}{N+s}$$

satisfies the following dynamics

$$\dot{\rho} = \rho((b\rho - d)(1-\rho) - cs)\,.$$

d) Assume that $d = 0$, $b = 1$, and $c = 1$. Under this assumption, determine the smallest number of sterile insects, s_{c}, needed to make the insect population go extinct for any allowed initial value of ρ.

2. SIRS model [2 points] A simple model for the spreading of influenza is the SIRS model (in contrast to the SIR model discussed in the lectures). The SIRS model has the following dynamics:

$$\begin{aligned}
\dot{S} &= -rSI + \gamma R \\
\dot{I} &= rSI - \alpha I \\
\dot{R} &= \alpha I - \gamma R\,.
\end{aligned} \tag{2}$$

The different population sizes correspond to susceptibles $S(t)$, infectives $I(t)$, and removed $R(t)$. Assume that the initial population size is $N = S(0)+I(0)$ and that $R(0) = 0$.

a) Give brief, plausible explanations of the different parameters r, α, and γ (all assumed to be positive).

b) What are the differences between Eq. (2) and the SIR model discussed in the lectures?

c) Show that Eq. (2) can be written as a two-dimensional system:

$$\begin{aligned}
\dot{S} &= f(S,I) \\
\dot{I} &= g(S,I)\,.
\end{aligned}$$

Explicitly write down $f(S,I)$ and $g(S,I)$.

d) The system in subtask c) has two possible fixed points, determine these. Give a condition on the parameters for which the system has two biologically relevant (non-negative I and S) fixed points.

e) Determine the stability of the fixed points for the case where the system has one biologically relevant fixed point, and for the case where the system has two biologically relevant fixed points. Discuss the long-term fate of the system for these two cases.

3. Reaction kinetics [2 points] Assume that a chemical reaction proceeds as follows

$$\mathrm{S} + \mathrm{E} \underset{\mathrm{k}_{-1}}{\overset{\mathrm{k}_1}{\rightleftharpoons}} \mathrm{SE} \xrightarrow{\mathrm{k}_2} \mathrm{P} + \mathrm{E}\,, \tag{3}$$

where S denotes a substrate, E an enzyme, and P a product. The parameters k_1, k_{-1} and k_2 are rate constants.

a) What is the law of mass action? What does it assume about the underlying chemical reaction?

b) Use the law of mass action together with the reaction in Eq. (3) to set up a dynamical system for the concentrations $s = [S]$, $e = [E]$, $c = [SE]$ and $p = [P]$ of the reactants. Assume appropriate initial conditions, that are determined just before the reaction starts.

c) What is the role of the enzyme E? How is this role reflected in the dynamics you found in subtask b)?

d) Consider the following reaction between two reactants A and X

$$\mathrm{A} + \mathrm{X} \underset{\mathrm{k}_{-1}}{\overset{\mathrm{k}_1}{\rightleftharpoons}} 2\,\mathrm{X}\,.$$

Assume that A is maintained at constant concentration $a = [A] =$ const. Using the law-of mass action, set up a dynamical system for the concentrations $a = [A]$ and $x = [X]$.

e) How would the dynamics change in subtask d) if you remove the assumption that a is maintained at constant concentration, i.e. if the concentration a is only influenced by the reaction and not from any external sources?

4. Macroscopic diffusion [2 points] The diffusion equation in one spatial dimension can be written on the form

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}\,, \tag{4}$$

where n is a concentration, time t has unit T and position x has unit L.

a) What are the dimensional units of the diffusion coefficient

   i) for the diffusion equation in one spatial dimension?

   ii) for the diffusion equation in two spatial dimensions?

   iii) if the concentration n (unit L^{-d}, where d is the spatial dimension) in Eq. (4) was replaced by a probability density Q?

b) Derive the one-dimensional diffusion equation (4) starting from Fick's law.

c) Explain how the derivation in subtask b) must be modified to result in Fisher's equation instead of the diffusion equation. Write down Fisher's equation.

d) In one of the problem sets you simulated a one-dimensional version of Fisher's equation. The length unit in your simulations was (at least supposed to be) unity. Under what conditions do you expect this to be a good choice for your simulations?

5. Coupled oscillators [2 points] Consider N coupled oscillators with phases $\theta_1, \theta_2, \ldots \theta_N$ and with the following time evolution

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\,. \tag{5}$$

a) When $N = 2$, show that the two oscillators approaches a phase-locked dynamics. What is the relative phase between the oscillators?

b) What is the angular velocity of the phase-locked dynamics you found in subtask a)?

c) Introduce the order parameter

$$re^{\mathrm{i}\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}\theta_j} \tag{6}$$

Show how to rewrite Eq. (5) on the following form

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i)\,.$$

d) Briefly explain what the order parameter quantifies and briefly explain what the purpose is of introducing the order parameter in the Kuramoto model.

6. Difference filters [2 points] A linear filter denotes the procedure of convoluting a time series x_n with a discrete weight function a_n to form a filtered time series y_n:

$$y_n \equiv \sum_{m=-\infty}^{\infty} a_m x_{n-m} \,. \tag{7}$$

a) A first-order difference filter has weights $a_0 = 1$, $a_1 = -1$ and all other $a_n = 0$. Evaluate Eq. (7) using these weights.

b) A second-order difference filter is obtained by applying a first-order difference filter two times on a time series. Write down the form of Eq. (7) for a second-order filter and read off the non-zero weights a_n of the second-order difference filter.

c) Show that the first-order difference filter removes linear trends in a time series up to a constant, by applying it to a time series with a linear trend: $x_n = An + \eta_n$, where η_n denotes fluctuations around the linear trend and A is a constant.

d) Show that the second-order difference filter may be used to remove a quadratic trend in a time series.

e) Show that a p:th order difference filter, obtained by applying a first-order difference filter p times, can be used to eliminate a time series that is on the form of a general polynomial in n of degree $p-1$.