CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
COMPUTATIONAL BIOLOGY A

COURSE CODES: **FFR 110, FIM740GU, PhD**

Time:	June 9, 2017, at $08^{30} - 12^{30}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once at 10^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 18 points (need 7 points to pass).
CTH $\geq$17 grade 3; $\geq$22 grade 4; $\geq$26 grade 5,
GU $\geq$17 grade G; $\geq$ 24 grade VG.

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1. Effect of fishing on a predator-prey model [2 points] Consider a simple predator-prey model (Lotka-Volterra)

$$\begin{aligned} \dot{N} &= N(a - bP) \\ \dot{P} &= P(cN - d)\,. \end{aligned} \tag{1}$$

Assume N denotes the size of a prey population of fish that is eaten by predator fish of population size P. Assume a, b, c, d are positive constants.

a) Explain the forms of the terms on the right-hand side of Eq. (1).

b) Modify the model in Eq. (1) to include the effect of fishing by humans. Assume that the fishing tools do not distinguish between predator and prey fish, and assume that the total number of fish caught is proportional to the total fish population with proportionality coefficient f.

c) Find all the steady states of the resulting model. How does the number of biologically relevant steady states depend on f?

d) Most solutions of Eq. (1) show oscillations with some period T. It is possible to show that the averages of the populations over one period (denoted $\overline{N}$ and $\overline{P}$) are equal to the values of the non-trivial steady state (denoted N^* and P^*), i.e.

$$\overline{N} \equiv \frac{1}{T}\int_0^T \mathrm{d}t' N(t') = N^* = \frac{d}{c}\,, \quad \overline{P} \equiv \frac{1}{T}\int_0^T \mathrm{d}t' P(t') = P^* = \frac{a}{b}\,.$$

By using this result, explain what effect fishing has on the average populations of prey and predators.

2. Discrete growth models [2.5 points] Consider the following discrete growth model for a population of size N:

$$N_{t+1} = N_t \exp\left[r\left(1 - \frac{N_t}{K}\right)\right], \tag{2}$$

where r and K are positive parameters.

a) Determine the non-negative steady states of the model (2) and give an interpretation of the parameter K.

b) Find a limit where the model (2) shows discrete Malthus growth:

$$N_{t+1} = N_t(1 + r).$$

Give an interpretation of r in this limit (note that r must be dimensionless in Eq. (2)).

c) Determine the stability of the fixed points found in subtask a).

d) Show that Eq. (2) has a period-doubling bifurcation at $r = 2$.

3. Reaction-diffusion and travelling waves [2.5 points] Consider a reaction-diffusion equation in one spatial dimension

$$\frac{\partial n}{\partial t}(x,t) = rn(x,t)\left(1 - \frac{n(x,t)}{K}\right)\left(\frac{n(x,t)}{A} - 1\right) + D\frac{\partial^2 n}{\partial x^2}(x,t). \tag{3}$$

Here $n(x,t)$ denotes a population density of some species at position x at time t. Moreover r, K, A and D are non-negative parameters. Assume that $A < K$.

a) First consider the case where $D = 0$ in Eq. (3) and consider a homogeneous initial condition (i.e. you can neglect the spatial coordinate). By for example sketching the resulting flow, explain and give possible interpretations of the remaining parameters r, K, and A.

b) To simplify the analysis, let $r = 1$, $A = 1/2$, $K = 1$ and $D = 1$. Assume that $n(x,t) = u(z)$ only depends on the combination $z = x - ct$. Starting from Eq. (3) derive an ordinary differential equation for $u(z)$.

c) Does the resulting equation in subtask b) allow travelling wave solutions? If so, for which values of c?

4. Diffusion-driven instability and pattern formation [2 points]
Turing showed that a two-dimensional reaction-diffusion system

$$\begin{aligned} \frac{\partial u}{\partial t} &= \gamma f(u,v) + \nabla^2 u \\ \frac{\partial v}{\partial t} &= \gamma g(u,v) + d\nabla^2 v \end{aligned}$$

is unstable to spatial wave-like perturbations in a range of wave numbers if the following conditions hold

$$\begin{aligned} \mathrm{tr}\mathbb{J} &< 0 \\ \det J &> 0 \\ dJ_{11} + J_{22} &> 0 \\ \frac{(dJ_{11} + J_{22})^2}{4d} &> \det \mathbb{J}\,. \end{aligned}$$

Here J_{ij} are elements of the Jacobian matrix of the homogeneous steady state

$$\mathbb{J} = \gamma \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}\,.$$

a) Consider the following reaction-diffusion system in one spatial dimension written in dimensionless units:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{u^2}{v} - bu + \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial v}{\partial t} &= u^2 - v + d\frac{\partial^2 v}{\partial x^2}\,. \end{aligned} \tag{4}$$

Here b and d are positive constants.

Determine the positive spatially homogeneous steady states of the system (4). Is this state stable?

b) Determine the conditions for the steady state in subtask a) to be driven unstable by diffusion. Sketch the b-d parameter space in which diffusion-driven instability occurs.

c) Explain without calculations what it means that a system has a diffusion-driven instability, for example by sketching the response of the system to a small, suitable perturbation.

d) An observation in nature is that there is (almost) no animal with striped body and spotted tail, but there is animal with spotted body and striped tail. Give one possible explanation (without calculations) for this observation.

5. Disease spreading in large but finite populations [2 points] Assume that a population consists of N (constant in time) individuals. Each individual is either infected by, or susceptible to a disease. Assume that recovered individuals once again become susceptible (SIS model).

a) Denote the number of susceptible and infected individuals by S and I respectively. Assume that the rate at which susceptibles S turn into infectives I is $\beta SI/N$ and that the rate at which infectives turn into susceptibles is γI, where β and γ are positive constants.

   Explain why the forms of these rates are reasonable. In particular, give an interpretation of β considering that the first rate is divided by the total population size N.

b) Assume that in a short time interval the number of infected individuals changes by $+1$ or -1 because of infections or due to recovery.

   Using the rates introduced in subtask a), write down a Master equation for the probability $\rho_n(t)$ to observe n infected individuals at time t in a finite population consisting of N individuals.

c) Contrast stochastic to deterministic models of disease spreading. Discuss under what conditions one should use either and discuss typical differences between the dynamics of the models.

6. Phase resetting of oscillators [1 point] Note: The theory for this problem is not covered in the course this year

a) Explain what is meant by phase resetting of an oscillating system.

b) Give two examples of applications of phase resetting.