CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
COMPUTATIONAL BIOLOGY A

COURSE CODES: **FFR 110, FIM740GU, PhD**

Time:	Test exam
Place:	
Teachers:	
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 18 points (need 7 points to pass).
CTH $\geq$15 grade 3; $\geq$20 grade 4; $\geq$25 grade 5,
GU $\geq$15 grade G; $\geq$ 23 grade VG.

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1. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) Consider the two growth equations below

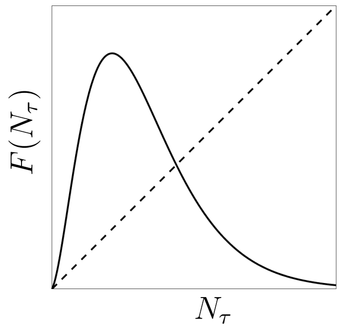
$$\frac{\mathrm{d}N}{\mathrm{d}t}(t) = rN(t)$$
$$\frac{\mathrm{d}N}{\mathrm{d}t}(t) = rN(t-T)\,.$$

with $r > 0$ and $T > 0$. Both these equations can be solved using the following ansatz $N(t) = \sum_i A_i e^{\lambda_i t}$. Without doing any calculations, explain the difference in the spectrum of allowed λ_i for the two equations and what this difference implies in terms of oscillations of the solutions.

b) Consider a discrete growth model for a single species of population size N

$$N_{\tau+1} = F(N_\tau)\,.$$

The figure below shows a particular choice of the map F (solid line) and the curve $N_{\tau+1} = N_\tau$ (dashed line). The scales of the axes are equal. On your answer sheet, sketch this map and mark a minimal upper bound $N_{\max}$ and a maximal lower bound $N_{\min}$ for the long-term dynamics. Explain your answer.



c) What does the law of mass action state? Explain the form of the law of mass action.

d) What is the dimensionality of the diffusion coefficient? The characteristic law of diffusion describes how the mean squared displacement of diffusing particles grows with time for large times in an infinite domain. How does this growth depend on the diffusion coefficient and on time?

e) What is meant by the phase of oscillation? Why is it useful?

f) Explain what the difference is between measurement noise and dynamical noise in a time series obtained from experimental measurements of a biological process.

2. Competition and hunting [2.5 points] Rabbits, being very rapidly reproducing and invasive herbivores, are introduced into a land patch where it comes to compete with a species of native herbivores. One possible model for this situation is:

$$\begin{aligned}\dot{N} &= r_1 N\left(1-\frac{N+b_{12}R}{K_1}\right)\\ \dot{R} &= r_2 R\left(1-\frac{b_{21}}{K_2}N\right),\end{aligned} \tag{1}$$

where N is the population size of the native species of herbivores and R is the population size of rabbits. Assume that all parameters r_1, r_2, b_{12}, b_{21}, K_1, and K_2 are positive.

a) Explain the different parameters and the assumptions used to derive Eq. (1). Explain why Eq. (1) is a suitable model for the situation described above.

b) Change to suitable dimensionless parameters and population sizes such that Eq. (1) takes the form:

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}\tau} &= u\left(1-u-v\right)\\ \frac{\mathrm{d}v}{\mathrm{d}\tau} &= \rho v\left(1-a_{21}u\right)\end{aligned}. \tag{2}$$

What expressions do you obtain for ρ and a_{21}?

c) In what follows, assume that $a_{21} < 1$. Sketch the phase-plane dynamics (you can use the null-clines as guide lines) and explain the long-term fates of the two populations.

d) After some time it is decided that the rabbit population must decrease and rabbits are hunted with a population-dependent hunting rate $h(v)$. Modify the equations to take this into account. Find a hunting strategy that result in the rabbit population going extinct within the model.

3. Stochastic models [1.5 points]

a) Write down a gain-loss equation (Master equation) for the probability $Q_N(t)$ to have a population of size N at time t. Assume stochastic dynamics where the population can change in a small time interval δt either due to a birth with probability $b\delta t$, or due to a death with probability $d\delta t$.

b) Show that in the limit of large N the average population size approaches a deterministic dynamics. Which growth model does the deterministic dynamics correspond to?

c) What are the fundamental differences between the stochastic and deterministic models if the initial population size N_0 is small? What are the differences when N_0 is large?

4. Travelling waves [2 points] Consider a reaction-diffusion equation in one spatial dimension

$$\frac{\partial n}{\partial t}(x,t) = f(n(x,t)) + D\frac{\partial^2 n}{\partial x^2}n(x,t)\,. \qquad (3)$$

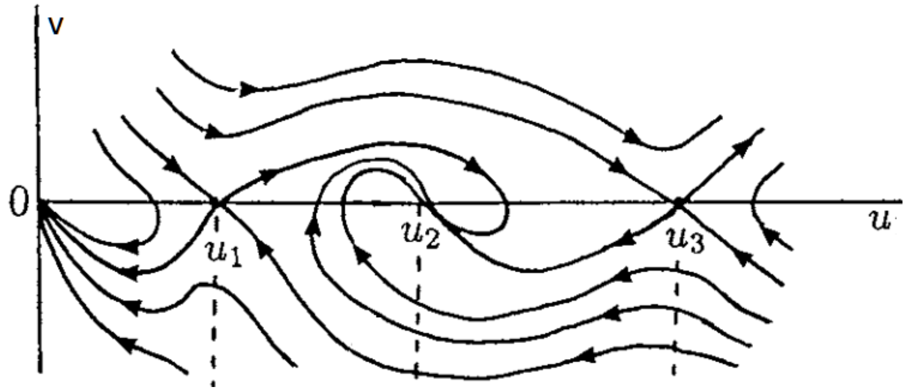
a) In a few lines give interpretations of the two terms on the right-hand side of Eq. (3).

b) Assume that $n(x,t) = u(z)$ only depends on the combination $z = x - ct$. Starting from Eq. (3) derive an ordinary differential equation for $u(z)$.

c) Give an interpretation of c. What changes if the value of c is changed?

d) Rewrite the ordinary differential equation for $u(z)$ obtained in subtask b) in terms of a first order system for $u(z)$ and $v(z) = u'(z)$.

e) Assume that a particular combination of $f(u)$ and c gives the following phase-plane trajectories (phase portrait) for the dynamics of u and v:



Here $(0,0)$, $(u_1,0)$, $(u_2,0)$, and $(u_3,0)$ are the fixed points (u^*,v^*) of the system. Sketch the allowed travelling wave solution(s) $u(z)$ of Eq. (3) as function(s) of the z coordinate.

5. Disease spreading [2 points] A simple model for disease spreading is the SIR model with the following dynamics

$$\begin{aligned} \dot{S} &= -rSI \\ \dot{I} &= rSI - \alpha I \\ \dot{R} &= \alpha I \,. \end{aligned} \tag{4}$$

a) Give brief explanations of the different population sizes S, I and R, and the parameters r and α.

b) By solving the system (4) find the maximal value $I_{\max}$ of I and the final value S_∞ of S as $t \to \infty$.

c) Discuss why the values $I_{\max}$ and S_∞ are important in an epidemic.

6. The Kuramoto model [2 points] Consider a number N of coupled oscillators with phases $\theta_1, \theta_2, \ldots \theta_N$ with the following time evolution

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\,. \tag{5}$$

a) In the Kuramoto model the frequencies ω_i are drawn from a symmetric distribution $g(\omega)$ with one maximum. Explain the reason $g(\omega)$ is symmetric. What would be the difference if it was not?

b) The order parameter r satisfies

$$1 = K\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \cos^2(\theta) g(Kr\sin\theta)\,.$$

Determine for which value of $K = K_\mathrm{c}$ the system shows a phase transition if

$$g(\omega) = \frac{\gamma}{\pi}\frac{1}{\omega^2+\gamma^2}\,.$$

c) What would happen if the distribution g was replaced by a Dirac delta function $g(\omega) = \delta(\omega)$ in subtask b)?

d) Explain how the result in subtask c) can be obtained from subtask b).