Problem set 2
for February 28, 23.59, 2022

Problem 1, Travelling waves [6p] In this task you will analyse the spatial spread of a prey population with growth rate obtained by the more realistic prey model introduced in the lectures. Let n denote the concentration of prey and assume that the predator population is constant during the time scale at which the prey population spreads. The time evolution of n in a given habitat is then described as

$$\frac{\partial n}{\partial t} = rn\left(1 - \frac{n}{K}\right) - \frac{An}{1 + n/B} + D\frac{\partial^2 n}{\partial x^2}\,. \tag{1}$$

Here, r, K, A, B, and D stand for the per-individual growth rate, carrying capacity, predator-prey interaction, predator appetite saturation, and diffusion constant, respectively. These parameters are assumed to be positive constants. The habitat is assumed to be one-dimensional with position x.

a) [**1.5p**] Introduce dimensionless time $\tau = At$, position $\xi = x\sqrt{A/D}$ and population size $u(\xi, \tau) = n(x, t)/B$ and rewrite the dynamics (1) in term of these coordinates and the dimensionless parameters $\rho = r/A$ and $q = K/B$. Neglect diffusion and determine the spatially homogeneous steady states of the dimensionless system in terms of ρ and q.

In what follows, set $\rho = 0.5$ and $q = 8$ (if you did not solve subtask a), you can set $r = 0.5$, $K = 8$ and $B = A = D = 1$). Assume that the habitat size (number of discrete patches) L is $L = 100$, i.e. $\xi = 1, 2, \ldots, 100$. Simulate the dimensionless dynamics in this habitat assuming zero-flux boundary conditions:

$$\frac{\partial u}{\partial \xi}(0, \tau) = \frac{\partial u}{\partial \xi}(L, \tau) = 0\,.$$

Hint: Use Euler's method to integrate time and discretise the Laplacian using the second order symmetric derivative in one dimension:

$$\frac{d^2 f}{dx^2} = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}\,.$$

b) [**3p**] To model domain invasion of the prey population from one direction, assume an initial population size given by a smoothed ramp function:

$$u(\xi, 0) = \frac{u_0}{1 + e^{\xi - \xi_0}}\,,$$

where ξ_0 is the ramp location and u_0 is an upper bound on the ramp. For each of the three following parameter sets for this ramp, simulate the dynamics for a suitable time range

(i) $\xi_0 = 20$ and $u_0 = u_1^*$, where u_1^* is the largest homogeneous steady state.

(ii) $\xi_0 = 50$ and $u_0 = u_2^*$, where u_2^* is the second largest homogeneous steady state.

(iii) $\xi_0 = 50$ and $u_0 = 1.1 \cdot u_2^*$.

For each case do the following

- Describe how the total population expands over the habitat. Do you find a travelling wave?
- If you find a travelling wave, numerically estimate its velocity c.
- Make a plot with two panels. The left panel should show the wave profile $u(\xi, \tau)$ against ξ at a fixed time τ where the wave is well developed. The right panel should show the corresponding trajectory in the phase plane spanned by u and $v = \frac{\mathrm{d}u}{\mathrm{d}\xi}$ at the fixed value of τ.
- Does the wave connect between two fixed points in the phase plane? If so, classify these fixed points using your numerically estimated velocity c.

c) [**1.5p**] To model the spread of a local outbreak, assume an initial population size given by a smoothed peak function:

$$u(\xi, 0) = u_0 e^{-(\xi - \xi_0)^2} \, .$$

Assume $\xi_0 = 50$. For the two cases $u_0 = u_1^*$ (largest homogeneous steady state) and $u_0 = 3 \cdot u_1^*$, describe how the total population expand over the habitat. Do you find a travelling wave?

Problem 2, Diffusion driven instability [6p] In this task you will analyse the effect of diffusion on the stability of the steady states in the following model for two interacting chemicals undergoing a Belousov-Zhabotinsky reaction

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= a - (b+1)u + u^2 v + D_u \boldsymbol{\nabla}^2 u \,, \\
\frac{\partial v}{\partial t} &= bu - u^2 v + D_v \boldsymbol{\nabla}^2 v \,.
\end{aligned}
\tag{2}
$$

Here u and v denote concentrations of the two chemicals, $\boldsymbol{\nabla}^2$ is the Laplace operator in a two-dimensional space, a and b are positive constants, and D_u and D_v denote diffusion coefficients.

a) [**1.5p**] Neglect diffusion and determine the spatially homogeneous steady states of the system (2) and their stability in terms of a and b.

In what follows, assume $a = 3$, $b = 8$, $D_u = 1$ and $D_v > 1$.

b) [**1.5p**] For which values of D_v does the homogeneous stable steady state(s) of the system (2) exhibit a diffusion-driven instability?

c) [**3p**] Simulate the dynamics of the system (2) on a square grid of size $L \times L$ with $L = 128$. Initially each grid point takes the value of the spatially homogeneous stable steady state plus a small local random perturbation (of order 10%). Discretize the Laplacian in Eq. (2) and implement periodic boundary conditions. Describe your implementations. For a reliable integration, set the time step to a small value (say, 0.01). Consider four values of D_v: $D_v = \{2.3, 3, 5, 9\}$. For each value of D_v show a snapshot in the form of a heat map of the spatial distribution of u during a transient state (say, after 1000 iterations), as well as when the system has reached a spatially inhomogeneous steady state. Make sure to use the same color range in all heat maps. Describe the patterns observed. What is the effect of increasing D_v?

Problem 3, Synchronisation [5p] The Kuramoto model (see lectures notes) describes the dynamics of the phases θ_i of N coupled oscillators ($i = 1, ..., N$):

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\,. \tag{3}$$

The frequencies ω_i are random, drawn from a symmetric distribution $g(\omega)$ with one maximum. Assume that $g(\omega) = (\gamma/\pi)[\omega^2 + \gamma^2]^{-1}$.

a) [**1p**] In the lecture notes it was shown that the degree of synchronisation in this model is, within a mean-field approximation, described by the order parameter r which satisfies the self-consistent equation

$$1 = K\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta\cos^2(\theta)g(Kr\sin\theta)\,.$$

By expanding this equation in the vicinity of the bifurcation (onset of synchronisation), the bifurcation point K_c was found. It was further shown that, in the vicinity of the bifurcation, the order parameter is given by $r = C\sqrt{\mu}$ where $0 < \mu = (K - K_\mathrm{c})/K_\mathrm{c} \ll 1$. Determine the coefficient C.

b) [**4p**] Perform numerical simulations of the Kuramoto model (3) for a value of K below K_c, and for two values of K above K_c. Of the latter two, let one value be very close to K_c. For each simulation, initialise the phases of the oscillators to random numbers between $-\pi/2$ and $\pi/2$. Plot how the order parameter depends on time. Compare the results of your simulations to the predictions from subtask a). In the lecture notes it is argued that the meanfield theory is expected to work well when the number of oscillators N is sufficiently large. Check this by simulating the Kuramoto model for several different values of N (for example, choose $N = 20, 100, 300$). Discuss: how well does the mean-field theory work? How do your results depend on the choice of N?