CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
COMPUTATIONAL BIOLOGY A

COURSE CODES: **FFR 110, FIM740GU, PhD**

Time:	August 28, 2019, at $08^{30} - 12^{30}$
Place:	Johanneberg
Teachers:	Anshuman Dubey, 072-190 6469 (mobile), visits once around 10^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 18 points (need 7 points to pass).
CTH $\geq$15 grade 3; $\geq$20 grade 4; $\geq$25 grade 5,
GU $\geq$15 grade G; $\geq$ 23 grade VG.

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1. Short questions [3 points] For each of the following questions give a concise answer within a few lines per question.

a) Discuss why we study mathematical models of biological systems. What are the advantages and disadvantages compared to other approaches such as computer models or experimental models?

b) Explain what a cobweb plot is and how it is generated. Illustrate using an explicit example.

c) Show that the Ricker map

$$N_{\tau+1} = \rho N_\tau e^{-N_\tau}$$

with $\rho > 0$ has a period-doubling bifurcation at a critical value ρ_c.

d) What does the law of mass action state? Explain the form of the law of mass action.

e) The following is a stochastic model for Malthus growth:

$$Q_N(t) = Q_N(t-\delta t) + r\delta t(N-1)Q_{N-1}(t-\delta t) - r\delta t N Q_N(t-\delta t)\,.$$

Here $Q_N(t)$ is the probability to have a population of size N at discrete time steps separated by δt. Explain the meaning of r and the form of the different terms in this equation.

f) Discuss similarities and differences between molecular diffusion and population diffusion.

g) Consider a generic reaction diffusion equation in one spatial dimension

$$\frac{\partial}{\partial t} n(x,t) = f(n(x,t)) + D\frac{\partial^2}{\partial x^2} n(x,t)\,,$$

where f only implicitly depends on t and x through $n(x,t)$. Explain why the solutions to $f(n) = 0$ are important for travelling wave solutions of the reaction diffusion equation.

h) Write down the SIR model for spreading of diseases and explain all involved variables and parameters.

2. Delay model for white blood cells [2.5 points] White blood cells are produced in the bone marrow to be released in the body. Since it takes several days to produce white blood cells in response to a deficit, the number of white blood cells N in the blood stream at time t can be modeled using

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\gamma N(t) + \frac{\beta N(t-T)\theta^m}{\theta^m + N(t-T)^m}\,. \tag{1}$$

Here $0 < \gamma < \beta$, $\theta > 0$ and $T > 0$. m is a positive integer.

a) Assuming N has dimension 'size' and t has dimension 'time', what are the dimensions of the parameters γ, β, θ and T? Explain the roles of T, γ, β and θ in the model.

b) Introduce dimensionless units and write Eq. (1) in terms of m, a dimensionless delay time, and one additional dimensionless parameter.

c) Find all steady states ($N(t) = \text{const.}$) of the dimensionless system in subtask b) [if you failed subtask b) you can use Eq. (1) in what follows]. Verify that the steady states are biologically relevant for the parameter constraints given below Eq. (1).

d) To simplify, consider the case $m = 1$ in this subtask. Derive the dynamics of a small time-dependent perturbation $\eta(t)$ close to the most positive steady state.

e) It can be shown (you do not need to show this) that the ansatz $\eta(t) = e^{\lambda t}$ has solutions with positive real part of λ for certain parameter values, while for other parameters all solutions have negative real part of λ. Discuss possible long-term behaviours of the system (1).

3. Plants in dry environments [2.5 points] In dry environments plant growth is mainly limited by the access to water. Assuming a small constant supply of water, the population size N of plants and amount W of accessible water can be modeled by the following system

$$\begin{aligned} \dot{N} &= aNW - bN \\ \dot{W} &= S - cW - dNW \end{aligned} \tag{2}$$

where a, b, c, d and S are positive parameters.

a) Give plausible interpretations of the different terms in Eq. (2).

b) Introduce dimensionless units in Eq. (2). Choose units such that the dimensionless growth rate of W does not depend on any parameter. Which parameter combinations govern the dimensionless growth of N?

c) Find all steady states of the dimensionless system [or (2) if you failed subtask b)] and determine conditions for which they are biologically relevant. Discuss the biological meaning of the different steady states.

d) Determine the stability of the fixed points as a function of the system parameters. Discuss the possible long-term states of the system for different parameters.

4. A linear reaction-diffusion equation [2 points] Consider the reaction-diffusion equation in two spatial dimensions

$$\begin{aligned} \frac{\partial u}{\partial t} &= 2 - u + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial v}{\partial t} &= u - 2v + 4 + 3\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{aligned} \tag{3}$$

a) Find the homogeneous steady state (u^*, v^*) of the system (3) and determine its stability.

b) By making an ansatz $(u, v) = (u^*, v^*) + e^{\lambda t + \mathrm{i}(k_x x + k_y y)}(\delta u_0, \delta v_0)$ where $(\delta u_0, \delta v_0)$ are (possibly complex) constants, find all solutions λ as functions of $k = \sqrt{k_x^2 + k_y^2}$.

c) An observation in nature is that there is (almost) no animal with striped body and spotted tail, but there is animal with spotted body and striped tail. Give one possible explanation (without calculations) for this observation.

5. The Kuramoto model [2 points] Consider a number N of coupled oscillators with phases $\theta_1, \theta_2, \ldots \theta_N$ with the following time evolution

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \,. \tag{4}$$

a) Introduce the order parameters $r(t)$ and $\psi(t)$

$$r e^{\mathrm{i}\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{\mathrm{i}\theta_j} \tag{5}$$

and show that Eq. (4) can be written on the following form

$$\dot{\theta}_i = \omega_i + K r \sin(\psi - \theta_i) \,. \tag{6}$$

b) What is the significance of r and ψ? Explain the significance of Eq. (6), what is gained compared to Eq. (4)?

c) In the limit of $N \to \infty$ we may, for each ω, interpret the oscillators as a continuum on the interval $-\pi < \theta < \pi$. Denote by $n(\theta, t)$ the concentration of oscillators with angle θ at time t. Use the dynamics of individual oscillators (6) to derive a continuity equation for $n(\theta, t)$.

d) Find a steady state solution (by setting $\frac{\partial n}{\partial t} = 0$) to the continuity equation you derived in subtask c) for the case of $r = 0$. Does your result correspond to your expectations?