LINMA2710 : Scientific computing

Project 2 : Finite volumes

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April 2023

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1 Methodology and results

Considering the following diffusion problem with a Dirichlet boundary condition :

$$-\nabla \cdot (a(\mathbf{x})\nabla u) = f \qquad \text{where} \qquad u = g \text{ on } \partial\Omega \tag{1}$$

More specifically, the scalar functions from 1 are :

$$a(x) = \sin(0.1x) + \cos(0.1y) \qquad f(x,y) = x + 0.3y \qquad g(x,y) = \sqrt{x^2 + 0.1y^2} + x \tag{2}$$

The domain Ω considered is a circle centered in $(0,0)$ with a radius of $r = 10$.

The diffusion problem is solved using the finite volume (FV) method. Given a set of points, the Voronoi cells are computed for these points. Then for each control volume, the PDE is integrated. Using the divergence theorem, the volume integral is converted into a surface integral. Finally, the solution is approximated with a finite difference on each side of the control volume [1]. The final scheme reads :

$$-\int_{\partial V_i} a(\mathbf{x})\nabla u \cdot \mathbf{n}\, ds \approx \sum_{j\sim i} a_{ij} \frac{u_j - u_i}{h_{ij}} l_{ij} = \mathrm{Vol}(V_i) f_i \tag{3}$$

For all the grid points, the following linear system is obtained

$$A\mathbf{u} = \mathbf{f} \tag{4}$$

For a given point (x_i, y_i) and its corresponding volume Vol_i and neighbouring volume Vol_j, the coefficient $A_{ij} = \frac{a_{ij} l_{ij}}{h_{ij}}$ is first computed and then stored in the matrix at the line i and the column corresponding to the custom index of the volume j. For the volumes on the boundary of the domain, A_{ij} is set to 1 in order to respect the boundary condition. Regarding the right-hand side of the linear system, the area of the Voronoi cell is multiplied by the source term function evaluated at the point (x_i, y_i). This procedure is applied to each control volume and to each one of its neighbouring volumes. Finally, the system is solved using the `Armadillo` library. The result is then plotted as shown on figure 1.

2 Convergence analysis

In order to undertake the convergence analysis, various mesh resolution where used. The mesh resolution corresponds to the variable `delta` in the furnished `main.cpp` code. At each refinement, the mesh resolution is doubled. The various resolutions used where : 4, 2, 1, 0.5 and 0.25. The corresponding meshes have 1022, 1070, 1306, 2246 and 6006 points, respectively. All the solution plots relative to each mesh are presented in the appendix.

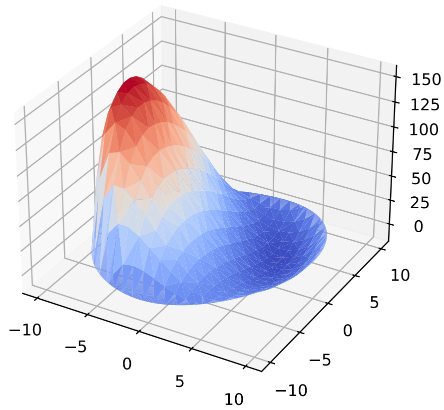


Figure 1: Solution to the diffusion problem with a mesh resolution of 1

The root-mean-square error (RMSE) for the solution $u(x, y)$ at a set of given points for two meshes of different sizes is computed. This set of points, refereed as the reference points, corresponds to all the points of the mesh of a resolution of 4.

Figure 2 shows that the RMSE relative to the solution u computed on the reference points decreases as the mesh resolution is increased. This was expected as the solution becomes more and more accurate, as it can be visually observed on figure 4 presented in the appendix. Moreover, it seems that increasing the resolution up to 0.25 is not relevant as the RMSE does not improve much compared to a mesh of resolution of 0.5. Considering the additional computation resources to solve this problem on such a fine mesh, this choice is irrelevant.

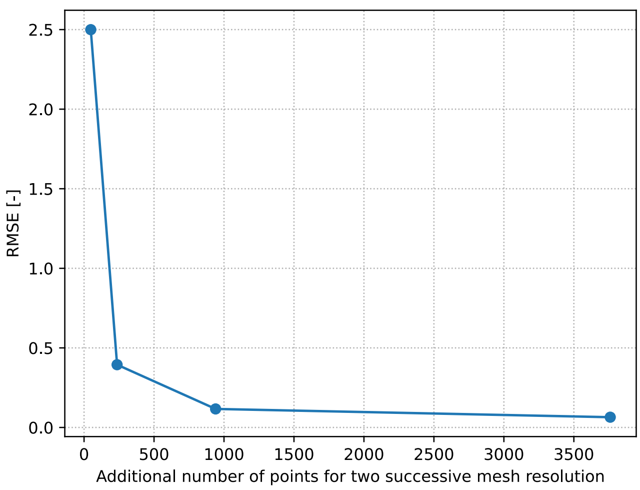


Figure 2: Evolution of RMSE of the solution $u(x, y)$ for various mesh resolutions

Finally, the order of convergence is studied. This is done using a log-log plot and then observing the slope on the graph. Indeed if $y = x^n$ then $\log y = n \log x$ and the relationship between y and x is linear and the slope is n. In this case, as there seems to be an inverse relationship on figure 2, the function $y = 1/x$ is applied to the data points for the sake of easier visual interpretation.

It is observed on figure 3 that the two lines seem to have very similar slopes. Since the orange line corresponds to the function $y = 1/x$, the slope is -1 for both lines. Thus it can be concluded that the convergence order of the scheme is 1.

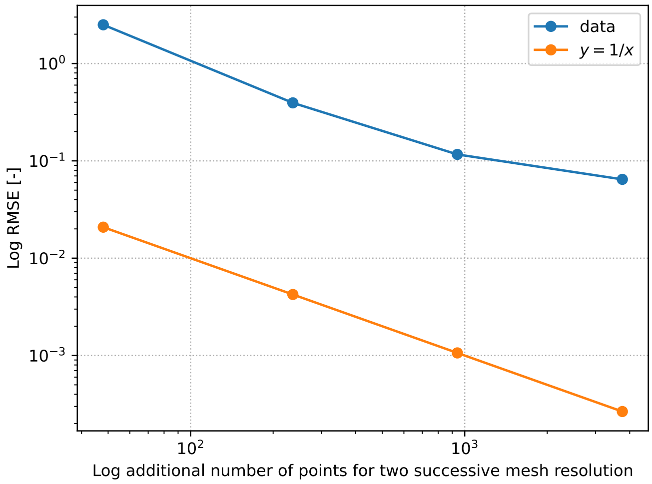


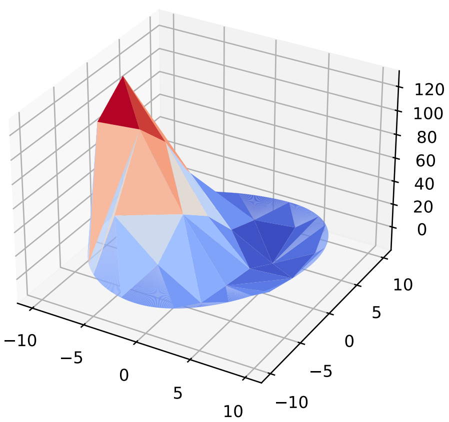
Figure 3: Log-log plot

3 References

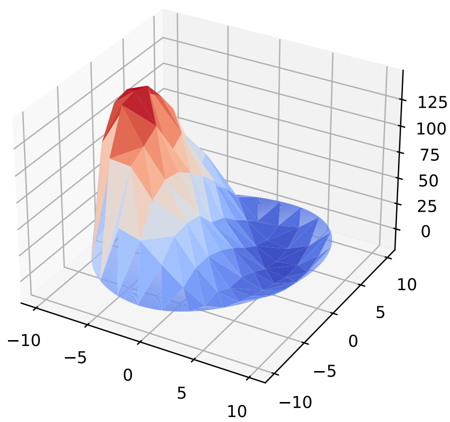
[1] Martin J Gander and Felix Kwok. *Numerical analysis of partial differential equations using maple and MATLAB*. SIAM, 2018.

A Appendix

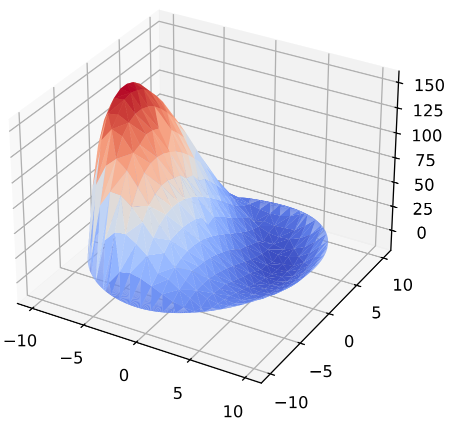
A.1 Solution plot for convergence analysis



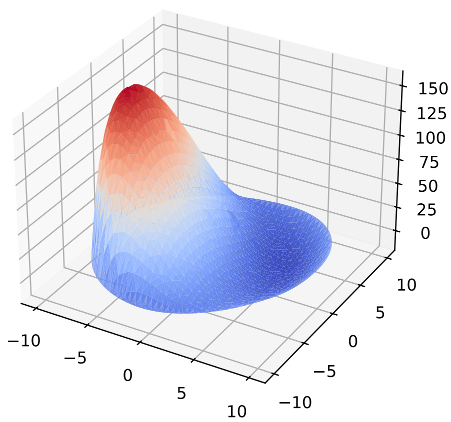
(a) resolution = 4



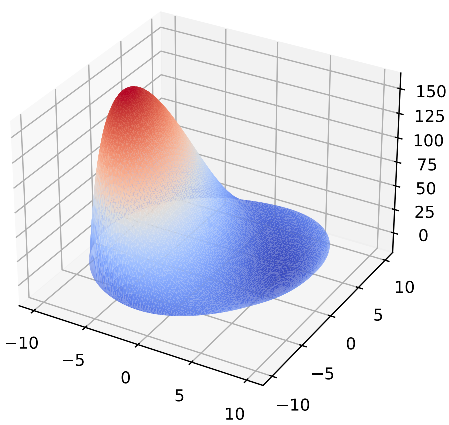
(b) resolution = 2



(c) resolution = 1



(d) resolution = 0.5



(e) resolution = 0.25

Figure 4: Evolution of solution for various mesh resolutions