1. a) $P(I_i=1 \mid \pi, \mu_1, \mu_2, \sigma_1, \sigma_2, x, I_{-i}) \propto p(x \mid I, \pi, \mu_1, \mu_2, \sigma_1, \sigma_2) \cdot p(I_i=1)$

$$= \prod_{i=1}^{n} p(x_i \mid I_i, \pi, \mu_1, \mu_2, \sigma_1, \sigma_2) \cdot \pi$$

$$\propto p(x_i \mid I_i=1, \pi, \mu_1, \mu_2, \sigma_1, \sigma_2) \cdot \pi$$

$$= \phi(x_i \mid \mu_1, \sigma_1^2) \cdot \pi$$

p.s.s.

$$P(I_i=2 \mid \pi, \mu_1, \mu_2, \sigma_1, \sigma_2, I_{-i}) \propto \phi(x_i \mid \mu_2, \sigma_2^2)(1-\pi)$$

So

$$P(I_i=1 \mid \cdot\,) = \frac{\pi \cdot \phi(x_i; \mu_1, \sigma_1^2)}{\pi\,\phi(x_i; \mu_1, \sigma_1^2) + (1-\pi)\,\phi(x_i; \mu_2, \sigma_2^2)}$$

1b) $P(\lambda \mid \beta, \sigma^2, y, x) = \dfrac{P(\beta, \sigma^2, \lambda \mid y, x)}{P(\beta, \sigma^2 \mid y, x)} \propto P(\beta, \sigma^2, \lambda \mid y, x)$

$$P(\beta, \sigma^2, \lambda \mid y) \propto P(y \mid \beta, \sigma^2, \lambda)\, P(\beta, \sigma^2, \lambda)$$

$$= P(y \mid \beta, \sigma^2, \lambda)\, P(\beta \mid \sigma^2, \lambda)\, P(\sigma^2 \mid \lambda)\, P(\lambda)$$

$$= \prod_{i=1}^{n} P(y_i \mid \beta, \sigma^2, \lambda) \cdot P(\beta \mid \sigma^2, \lambda)\, P(\sigma^2 \mid \lambda)\, P(\lambda)$$

$$= \underbrace{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y_i - x_i^T\beta)^T (y_i - x_i^T\beta)\right)}_{\text{constant with respect to } \lambda\,!} \cdot P(\beta \mid \sigma^2, \lambda)\, P(\sigma^2 \mid \lambda)\, P(\lambda)$$

So, $p(\lambda \mid \beta, \sigma^2, y, x) \propto p(\beta \mid \sigma^2, \lambda) \underbrace{p(\sigma^2 \mid \lambda)}_{\text{const.}} p(\lambda)$

Now $\beta \mid \sigma^2, \lambda \sim N(0, \lambda^{-1} I_p)$

$\sigma^2 \mid \lambda \sim \text{Inv-}\chi^2$ Does not depend on λ.

$\lambda \sim \text{Inv-}\chi^2(\eta_0, \lambda_0)$

$$p(\lambda \mid \beta, \sigma^2, y, x) \propto \prod_{i=1}^{p} \frac{1}{\sqrt{2\pi \frac{1}{\lambda}}} \exp\left(-\frac{1}{2} \cdot \frac{1}{\frac{1}{\lambda}} \beta_i^2\right) \cdot \frac{\exp\left(\frac{-\eta_0 \lambda_0}{2\lambda}\right)}{\lambda^{1+\eta_0/2}}$$

$$\propto \lambda^{p/2} \exp\left(-\frac{1}{2} \sum_{i=1}^{p} \beta_i^2\right) \exp\left(-\frac{\eta_0 \lambda_0}{2\lambda}\right) \cdot \lambda^{-(\eta_0/2+1)}$$

$$= \lambda^{\frac{p}{2} - \eta_0/2 - 1} \exp\left(-\frac{\lambda}{2} \sum_{i=1}^{p} \beta_i^2 - \frac{\eta_0 \lambda_0}{2\lambda}\right)$$

Not so good... Better if prior was of the form

$$\lambda^a \exp(-\lambda b).$$

This the Gamma distribution.

The way to go.

$$\frac{1}{\lambda} \sim \text{Inv-}\chi^2(\eta_0, \lambda_0) \Rightarrow$$

$$\frac{1}{\lambda} \sim \text{Inv Gamma}\left(\frac{\eta_0}{2}, \frac{\eta_0 \lambda_0}{2}\right) \Rightarrow$$

$$\lambda \sim \text{Gamma}\left(\alpha = \frac{\eta_0}{2}, \beta = \frac{\eta_0 \lambda_0}{2}\right)$$

$$\frac{1}{\lambda} \sim \text{Inv-}\chi^2(n_0, \lambda_0) \Rightarrow \frac{1}{\lambda} \sim \text{Inv-Gamma}\left(\frac{n_0}{2}, \frac{n_0\lambda_0}{2}\right)$$

$p(\lambda) \propto \exp\left(-\frac{\nu\sigma^2}{2\lambda}\right)\lambda^{-(1+\frac{\nu}{2})}$

$$\Rightarrow \lambda \sim \text{Gamma}\left(\frac{n_0}{2}, \frac{n_0\lambda_0}{2}\right)$$

$$p(\lambda \mid \beta, \sigma^2, y) \propto \lambda^{p/2} \exp\left(-\lambda \frac{\sum \beta_i^2}{2}\right) \lambda^{n_0/2 - 1} \exp\left(-\lambda \frac{n_0\lambda_0}{2}\right)$$

$$= \lambda^{(p+n_0)/2 - 1} \exp\left(-\lambda \left(\frac{\sum \beta_i^2 + n_0\lambda_0}{2}\right)\right)$$

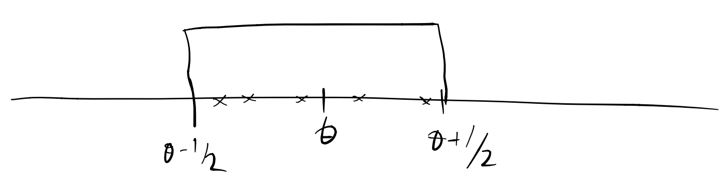
$$\lambda \mid \beta, \sigma^2, y \sim \text{Ga}\left(\frac{p+n_0}{2}, \frac{\sum \beta_i^2 + n_0\lambda_0}{2}\right)$$

$$\frac{1}{\lambda} \sim \text{Inv-}\chi^2\left(p+\nu, \frac{\sum \beta_i^2 + n_0\lambda_0}{p+n_0}\right)$$

$$E\left(\frac{1}{\lambda}\right) \approx \frac{\sum \beta_i^2 + n_0\lambda_0}{p+n_0} \approx \frac{\sum \beta_i^2}{p} = \text{Var}(\beta_i)$$

when $p >> n_0$

2a)



$\text{Var}(\bar{x}) = \frac{\sigma^2}{n}$ $\quad \sigma^2 = \text{Var}(x_i) \quad x_i \sim U(\theta - 1/2, \theta + 1/2)$

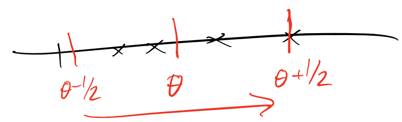
$x \sim U(0,1) \quad \text{Var}(x) = \frac{1}{12}$

So $\text{Var}(\bar{x}) = \frac{1}{12n}$

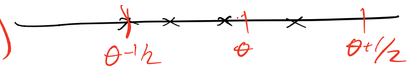
2b) $P(\theta | x_1, \ldots, x_n) \propto P(\cancel{x}_1 \cdots x_n | \theta) P(\theta)$

$$= \prod_{i=1}^{n} P(x_i | \theta) P(\theta)$$

$$= \prod_{i=1}^{n} I\left(\theta - \tfrac{1}{2} \le x_i \le \theta + \tfrac{1}{2}\right) \cdot 1$$



$\theta + 1/2 \ge x_{max}$
$\theta - 1/2 \le x_{min}$ $\Rightarrow \theta \in [x_{max} - 1/2, x_{min} + 1/2)$



$P(\theta | x_1 \ldots x_n) \propto 1$ for $\theta \in [x_{max} - 1/2, x_{min} + 1/2]$

$= 0$ therwise.

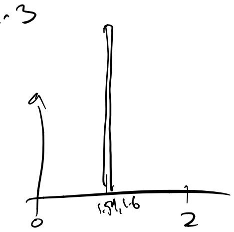
$\theta | x_1 \ldots x_n \sim U(x_{max} - 1/2, x_{min} + 1/2)$

2c) Frequentist : $\hat{\theta} = \bar{x} = 1.53$

$Var(\hat{\theta}) = \frac{1}{12n} = \frac{1}{12 \cdot 3} = 0.027777$

$SD(\hat{\theta}) = 0.1666$

Bayesian : $\theta | x_1 x_2 x_3 \sim U(1.59, 1.6)$



3 a) $\theta | x_1 \dots x_n \sim Beta(\alpha+s, \beta+f)$

$$p(\theta|x) \propto \theta^{\alpha+s-1}(1-\theta)^{\beta+f-1} \Rightarrow \ln p(\theta|x) \propto (\alpha+s-1)\ln\theta + (\beta+f-1)\ln(1-\theta)$$

$$\frac{\partial \ln p(\theta|x)}{\partial\theta} = \frac{\alpha+s-1}{\theta} + \frac{\beta+f-1}{1-\theta}(-1)$$

$$\frac{\partial \ln p(\theta|x)}{\partial\theta} = 0 \Rightarrow \frac{\alpha+s-1}{\theta} = \frac{\beta+f-1}{1-\theta}$$

$$\Rightarrow \hat{\theta} = \frac{\alpha+s-1}{\alpha+\beta+n-2}$$

3b) $\theta|x_1 \dots x_n \overset{approx}{\sim} N(\hat{\theta}, -J^{-1}_{\hat{\theta}|x})$ $\quad 1-\hat{\theta} = \frac{\beta+f-1}{\alpha+\beta+n-2}$

$$\frac{\partial^2 \ln p(x|\theta)}{\partial\theta^2} = -\frac{\alpha+s-1}{\theta^2} + \frac{\beta+f-1}{(1-\theta)^2}(-1)$$

$$\left.\frac{\partial^2 \ln p(x|\theta)}{\partial\theta^2}\right|_{\theta=\hat{\theta}} = -\left(\frac{\alpha+s-1}{\left(\frac{\alpha+s-1}{\alpha+\beta+n-2}\right)^2} + \frac{\beta+f-1}{\left(\frac{\beta+f-1}{\alpha+\beta+n-2}\right)^2}\right)$$

$$= -(\alpha+\beta+n-2)^2\left(\frac{1}{\alpha+s-1} + \frac{1}{\beta+f-1}\right)$$

$$= -(\alpha+\beta+n-2)^2\left(\frac{\alpha+\beta+n-2}{(\alpha+s-1)(\beta+f-1)}\right)$$

$$= -\frac{(\alpha+\beta+n-2)^3}{(\alpha+s-1)(\beta+f-1)}$$

$$\theta \mid x_1 \ldots x_n \overset{\text{approx}}{\sim} N\left(\hat{\theta} = \frac{\alpha+s-1}{\alpha+\beta+n-2}, \; -J^{-1}_{\hat{\theta},x} = \frac{(\alpha+s-1)(\beta+f-1)}{(\alpha+\beta+n-2)^3} \right)$$

$\theta \sim \text{Beta}(\alpha+s, \beta+f)$

Check: $$\text{Var}(\theta) = \frac{(\alpha+s)(\beta+f)}{(\alpha+s+\beta+f)^2(\alpha+s+\beta+f+1)}$$

$$= \frac{(\alpha+s)(\beta+f)}{(\alpha+\beta+n)^2(\alpha+\beta+n+1)}$$

3 c) Se R-code

3 d) —— " ——

4 a) $n_A = 5 \quad n_B = 5 \quad n_C = 10$

$$P(A \mid T_1, T_2) \propto P(T_1, T_2 \mid A) \, P(A)$$

$$= P(T_1 \mid A) \cdot P(T_2 \mid A) \, P(A)$$

$P(A) \quad P(B) \quad P(C)$ From Dirichlet

$\theta_1 \quad \theta_2 \quad \theta_3$

$$(\theta_1 \, \theta_2 \, \theta_3) \mid n_A, n_B, n_C \sim \text{Dirichlet}(5+1, 5+1, 10+1)$$

$$\text{Dirichlet}(6, 6, 11)$$

$$E(\theta_1 \mid n_A, n_B, n_C) = \frac{6}{23} \approx 0.26$$

$$E(\theta_2 \mid \cdot) = \frac{6}{23} = 0.26$$

$$E(\theta_3 \mid \cdot) = \frac{11}{23} \approx 0.48$$

$P(T_1 | A)$ $\qquad N(\mu_{1A}, 1)$

$$\boxed{\mu \sim N(\bar{x}, \tfrac{\sigma^2}{n})}$$

$\mu_{1A} | \bar{x}_1 = 1.2 \sim N(1.2, \frac{1}{5})$

$\mu_{1B} | \bar{x}_1 = 1.4 \sim N(1.4, \frac{1}{5})$

$\mu_{1C} | \bar{x}_1 = 0.7 \sim N(0.7, \frac{1}{10})$

$\mu_{2A} | \bar{x}_2 = 2.1 \sim N(2.1, \frac{1}{5})$

$\mu_{2B} | \bar{x}_2 = 3.5 \sim N(3.5, \frac{1}{5})$

$\mu_{2C} | \bar{x}_2 = 4.7 \sim N(4.7, 1/10)$

$$P(A | T_1 = 1.3, T_2 = 4.2) \propto P(T_1 = 1.3 | A) \cdot P(T_2 = 4.2 | A) \cdot P(A)$$

$$= \phi(1.3, \mu = 1.2, \sigma^2 = 1 + \tfrac{1}{5}) \cdot \phi(4.2, \mu = 2.1, \sigma^2 = 1 + \tfrac{1}{5}) \cdot \tfrac{6}{23} \quad =0.006$$

$$P(B | T_1 = 1.3, T_2 = 4.2) \propto \phi(1.3 | 1.4, 1 + \tfrac{1}{5}) \phi(4.2, 3.5, 1 + \tfrac{1}{5}) \cdot \tfrac{6}{23} \quad =0.028$$

$$P(C | T_1 = 1.3, T_2 = 4.2) \propto \phi(1.3 | 0.7, 1 + \tfrac{1}{10}) \cdot \phi(4.2 | 4.7, 1 + \tfrac{1}{10}) \cdot \tfrac{11}{23} \quad =0.052$$

Normalized posterior probabilities: (0.06, 0.33, 0.61)