Chapter 3 - Multi-parameter models: Exercise solutions

Mattias Villani

Click on the arrow to see a solution.

Exercise 3.1

A dataset contains observations on measurements of pesticide for $n = 24$ pike fish in a lake in southern Italy. The sample mean pesticide level is $\bar{x} = 12.5$ with a sample standard deviation of $s = 3.1$. Assume that the pesticide levels $X_1, \dots, X_n | \theta, \sigma^2 \sim \mathrm{N}(\theta, \sigma^2)$ are independent and identically distributed normal random variables with unknown mean θ and unknown variance σ^2.

Compute the joint posterior distribution for θ and σ^2 using a conjugate prior distribution. Use a prior mean for θ of 8 and an (imaginary) prior sample size of 5. Use $\sigma_0^2 = 1^2$ and $\nu_0 = 3$ degrees of freedom in your prior for σ^2. Plot the marginal posterior distribution of θ.

Solution

The posterior distribution is given by

$$\begin{aligned}
\theta | \sigma^2, x &\sim N\Big(\mu_n, \frac{\sigma^2}{\kappa_n}\Big) \\
\sigma^2 | x &\sim \mathrm{Inv-}\chi^2(\nu_n, \sigma_n^2)
\end{aligned}$$

where

$$\begin{aligned}
\mu_n &= w\bar{x} + (1-w)\mu_0 \\
w &= \frac{n}{\kappa_0 + n} \\
\kappa_n &= \kappa_0 + n \\
\nu_n &= \nu_0 + n \\
\nu_n \sigma_n^2 &= \nu_0 \sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n}(\bar{x} - \mu_0)^2
\end{aligned}$$

We have

```
# data
n = 24
xBar = 12.5
s2 = 3.1^2

# prior
mu0 = 8
kappa0 = 5
nu0 = 3
sigma02 = 1^2

# posterior
w = n/(kappa0 + n)
mun = w*xBar + (1-w)*mu0
kappan = kappa0 + n
nun = nu0 + n
sigman2 = (nu0*sigma02 + (n-1)*s2  +
             (kappa0*n/(kappa0 + n))*(xBar - mu0)^2 )/nun
```

So the joint posterior is

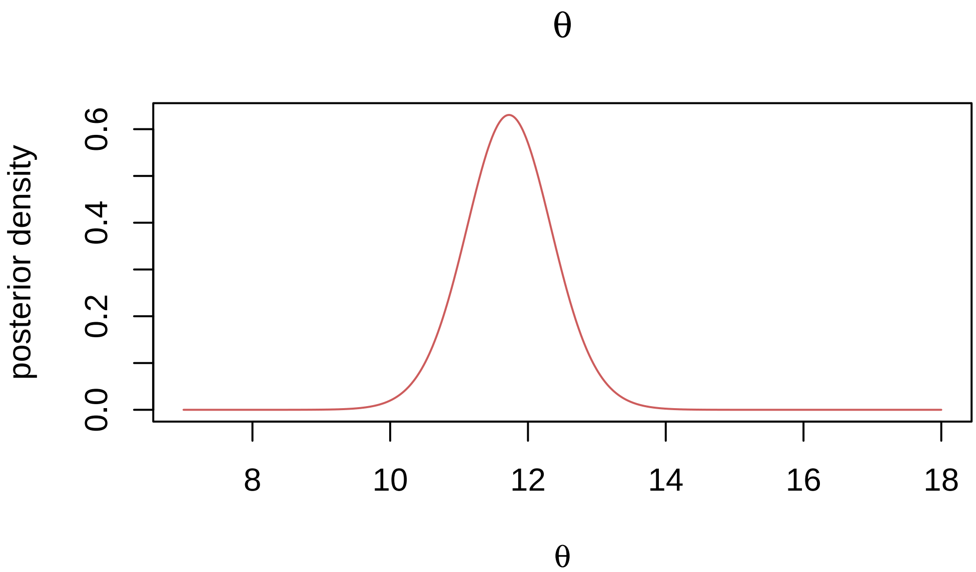
$$\theta|\sigma^2, x \sim N\Big(11.724, \frac{\sigma^2}{29}\Big)$$
$$\sigma^2|x \sim \text{Inv}{-}\chi^2(27, 11.401)$$

The marginal posterior for θ is

$$\theta|x \sim t\Big(\mu_n, \frac{\sigma_n^2}{\kappa_n}, \nu_n\Big) = t\Big(27, \frac{11.401}{29}, 27\Big)$$

Plotting the marginal posterior of θ:

```
thetaGrid = seq(7, 18, length = 1000)
sigma2Grid = seq(0.001, 30, length = 1000)
dtdist <- function(x, mu, sigma2, nu){
  return((1/sqrt(sigma2))*dt(x = (x - mu)/sqrt(sigma2), df = nu))
}
plot(thetaGrid, dtdist(thetaGrid, mun, sigman2/kappan, nun), type = "l",
     xlab = expression(theta), ylab = "posterior density",
     col = "indianred", main = expression(theta))
```



Exercise 3.3

Let $X_1, \dots, X_n \mid \theta, \sigma^2 \overset{\text{iid}}{\sim} \mathrm{N}(\theta, \sigma^2)$, where θ is assumed known. Show that the Inv$-\chi^2$ distribution is a conjugate prior for σ^2.

Solution

The normal density function for a single observation x_i is

$$p(x_i|\theta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(x_i - \theta)^2\Big) \propto \frac{1}{(\sigma^2)^{1/2}} \exp\Big(-\frac{1}{2\sigma^2}(x_i - \theta)^2\Big)$$

The likelihood for the iid normal model with known mean θ is therefore the product of n such density functions:

$$\begin{aligned} p(x_1, \dots, x_n|\theta, \sigma^2) = \prod_{i=1}^{n} p(x_i|\theta, \sigma^2) &\propto \frac{1}{(\sigma^2)^{n/2}} \exp\Big(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \theta)^2\Big) \\ &\propto \frac{1}{(\sigma^2)^{n/2}} \exp\Big(-\frac{ns^2}{2\sigma^2}\Big) \end{aligned}$$

where we have defined $s^2 = \frac{\sum_{i=1}^{n}(x_i-\theta)^2}{n}$ as the sample standard deviation (dividing by n instead of $n-1$ since the mean θ is assumed known).

The density of the $\sigma^2 \sim \text{Scaled}-\text{Inv}-\chi^2(\nu_0, \sigma_0^2)$ prior is of the form

$$p(\sigma^2) \propto \frac{1}{(\sigma^2)^{1+\nu_0/2}} \exp\Big(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\Big)$$

The posterior distribution from using the $\sigma^2 \sim \text{Scaled}-\text{Inv}-\chi^2(\nu_0, \sigma_0^2)$ prior is given by Bayes' theorem as (to avoid cluttering the notation, we do not write out the conditioning on θ since it is known)

$$\begin{aligned}
p(\sigma^2|x_1,\ldots,x_n) &\propto p(x_1,\ldots,x_n|\sigma^2)p(\sigma^2) \\
&\propto \frac{1}{(\sigma^2)^{n/2}} \exp\Big(-\frac{ns^2}{2\sigma^2}\Big)\frac{1}{(\sigma^2)^{1+\nu_0/2}} \exp\Big(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\Big) \\
&\propto \frac{1}{(\sigma^2)^{1+(\nu_0+n)/2}} \exp\Big(-\frac{1}{2\sigma^2}\big(\nu_0\sigma_0^2 + ns^2\big)\Big) \\
&= \frac{1}{(\sigma^2)^{1+(\nu_0+n)/2}} \exp\Big(-\frac{\nu_0+n}{2\sigma^2}\frac{\nu_0\sigma_0^2 + ns^2}{\nu_0+n}\Big)
\end{aligned}$$

which is proportional to a $\sigma^2 \sim \chi^2(\nu_0 + n, \frac{\nu_0\sigma_0^2+ns^2}{\nu_0+n})$ density. Note how the location parameter in the posterior

$$\frac{\nu_0\sigma_0^2 + ns^2}{\nu_0+n} = \frac{\nu_0}{\nu_0+n}\sigma_0^2 + \frac{n}{\nu_0+n}s^2$$

is a weighted average of prior location σ_0^2 and the data estimate s^2, with more weight placed on the strongest information source (prior with ν_0 imaginary sample data points versus the data with n actual data points).

Exercise 3.6

A Swedish poll in 2024 asked 2311 persons the question: *Which political party would you vote for if there was an election today?* The table below gives the poll results across the eight parties in parliament and the nineth option *Other*.

	M	L	C	KD	S	V	MP	SD	Other
Votes	410	88	83	81	721	196	238	434	60
Percent	17.7	3.8	3.6	3.5	31.2	8.5	10.3	18.8	2.6