Machine Learning

Lecture 2 - Bonus on Entropy

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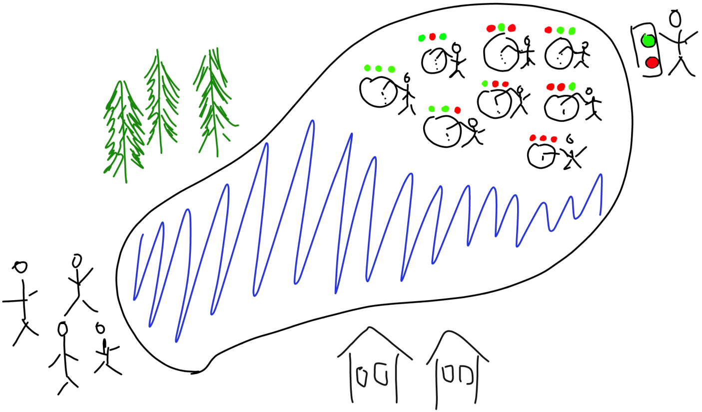
Binary representation

- **Bit** = 0-1, True-False, On-Off (binary digit).
- Representing four different outcomes in two bits:
  - Option A: 00
  - Option B: 01
  - Option C: 10
  - Option D: 11
- General: n bits can encode 2^n different outcomes.

"Entropy by the lake"



Entropy

- **Entropy** = The **smallest number of bits** needed to encode a message using an **optimal coding scheme**.
- **Measure of information**. **Measure of unorder**.
- Entropy of a random variable X with discrete support $\mathcal{X}$:

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \cdot \log_2 p(x)$$

- If all 8 fishermen are equally skilled: $p(x) = \frac{1}{8}$ and

$$H(X) = -\left(\frac{1}{8}\log_2\frac{1}{8} + \ldots + \frac{1}{8}\log_2\frac{1}{8}\right) = -(\log_2 1 - \log_2 8) = 3 \text{ bits}$$

- Uniform distribution has largest entropy. Least informative.

Entropy and Huffman coding

- Entropy of a random variable:

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \cdot \log_2 p(x)$$

- If the fishermen are not equally skilled and

$$\begin{array}{rcccccccc} x: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ p(x): & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} & \frac{1}{64} \end{array}$$

- Entropy:

$$H(X) = -\left(\frac{1}{2}\log_2\frac{1}{2} + \ldots + \frac{1}{64}\log_2\frac{1}{64}\right) = 2 \text{ bits}$$

- The optimal scheme sends only two bits *on average* (**Huffman coding**).

$$\begin{array}{rcccccccc} x: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \text{Code}: & 0 & 10 & 110 & 1110 & 111100 & 111101 & 111110 & 111111 \end{array}$$

Entropy as expected surprise

- The entropy can be written

$$H(X) = \sum p(x) \cdot \log_2 \frac{1}{p(x)} = \mathrm{E}\left(\log_2 \frac{1}{p(x)}\right)$$

- $\frac{1}{p(x)}$ is a measure how *surprising* the outcome x is.
- Entropy is the **expected surprise** when values are drawn from $p(x)$.
- Entropy is a **measure of uncertainty** in a distribution.
- Entropy of a continuous variable

$$H(X) = -\int p(x) \cdot \log_2 p(x) dx$$

- $X \sim N(\mu, \sigma^2) \rightarrow H(X) = \frac{1}{2} \ln\left(2\pi e\sigma^2\right)$ [Entropy defined using natural logs].

Joint and conditional entropy

- **Joint entropy**

$$H(X,Y) = -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\cdot\log_2 p(x,y)$$

- Conditional entropy of Y given $X = x$

$$H(Y|X=x) = -\sum_{y\in\mathcal{Y}} p(y|x)\cdot\log_2 p(y|x)$$

- **Conditional entropy** of Y

$$H(Y|X) = \sum_{x\in\mathcal{X}} p(x)\cdot H(Y|X=x)$$

- Chain rule for entropy [corresponds to $p(X,Y) = p(X)\cdot p(Y|X)$]

$$H(X,Y) = H(X) + H(Y|X)$$

Mutual information

- **Mutual information** (reduction in entropy of X from knowing Y)

$$I(X;Y) = H(X) - H(X|Y)$$

- Kullback-Leibler divergence between distributions (**relative entropy**)

$$D(p||q) = \sum_{x \in \mathcal{X}} p(x) \cdot \log \frac{p(x)}{q(x)}$$

- Alternative formulation of mutual information:

$$I(X;Y) = \sum_{x,y} p(x,y) \cdot \log \frac{p(x,y)}{p(x) \cdot p(y)}$$

- $I(X;Y)$ measures how far a joint distribution is from independence:

$$I(X;Y) = D\left[p(x,y) || p(x) \cdot p(y)\right]$$

Evaluating models using entropy

- Cross-entropy of a distribution $q(x)$ wrt distribution $p(x)$

$$H(p,q) = -\sum_{x \in \mathcal{X}} p(x) \cdot \log q(x) = \mathrm{E}_p \left[\log \frac{1}{q(x)} \right]$$

- Expected surprise from $q(x)$ when data comes from $p(x)$.´
- Low $H(p,q)$ means good q.
- The cross-entropy $\geq$ entropy since

$$H(p,q) = H(p) + D\left(p || q\right).$$

- Maximum likelihood estimator minimizes cross entropy:
  - $q(x)$ is the probability model with parameters $\boldsymbol{\theta}$
  - $p(x)$ is the empirical distribution of the sample $x_1, \ldots, x_n$

$$p(x) = \sum_{i=1}^{n} \delta_{x_i}(x)$$

  Dirac's point mass: $\delta_{x_i}(x) = 1$ if $x = x_i$ and $\delta_{x_i}(x) = 0$ otherwise.