Locally Stable Time-Varying Multi-Seasonal ARMA Models

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Collaborators

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Structure of the talk

- **Time-varying multi-seasonal AR models**
- **Parameter evolution process**
- **Bayesian inference**
- **Simulation experiments**
- **Application**

- **Slides**: http://mattiasvillani.com/news
- **Paper**: arXiv: 2409.18640 (ARMA extension soon on arXiv)

Motivation

- **Box-Jenkins. Global stationarity** is restrictive.
- **Differencing**. Information loss. Overdifferencing. Levels are non-stationary. Poor short-run forecasts.
- **Data in most recent regime**. Wasteful. No probability for regime change.

- **Time-varying parameters** has a long history. [1], [2], [3]
- Recently: more realistic parameter evolutions. [4], [5], [6], [7]
- Parameters can enter **explosive region** for some periods.

- **Seasonality** is predictable. Model it! Time-varying?
- Time series with **multiple seasonal periods** is now common.

Multi-seasonal AR models

- **Seasonal AR(p, P)** with season s

$$\phi_p(L)\Phi_P(L^s)(y_t - \mu) = \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} N(0, \sigma^2)$$

- Lag operator: $L^j y_t = y_{t-j}$
- Regular **lag polynomial**

$$\phi_p(L) = 1 - \phi_1 L - \ldots - \phi_p L^p$$

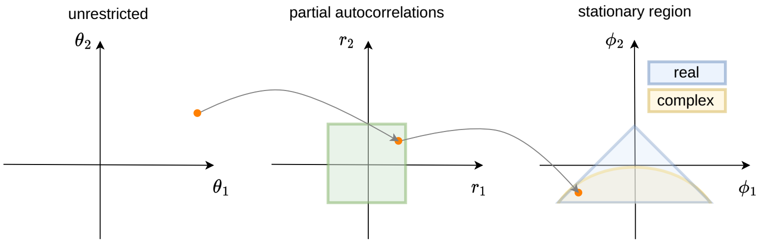
- Seasonal lag polynomial

$$\Phi_p(L^s) = 1 - \Phi_1 L^s - \ldots - \Phi_P L^{Ps}$$

- **Multiple seasonal periods**, e.g. hourly data with daily, weekly and yearly cycles.
- **Multi-seasonal AR** models with M polynomials

$$\prod_{j=1}^{M} \phi_j(L^{s_j})(y_t - \mu) = \varepsilon_t$$

Enforce stability in AR models



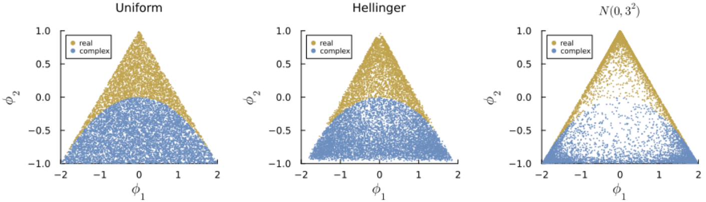
- **Unrestr. params** $\theta_k \in \mathbb{R} \Rightarrow$ **Partial autocorr** $r_k \in (-1, 1)^p$

$$r_k = \frac{\theta_k}{(1 + \theta_k^2)^{1/2}}$$

- Partial autocorrs $\boldsymbol{r} \Rightarrow$ **stable AR coeffs** ϕ by a recursion [8]
- Same parameterization for all polynomial factors.
- **Invertibility in MA** by the same parameterization.

Uniform distribution over stability region

- Unrestricted parameters θ_k have no interpretation. Priors?
- **Uniform distribution over stability region** $\mathbb{S}_p$ for $\boldsymbol{\phi}$.



- Lemma: If, independently,

$$\theta_k \sim \begin{cases} t\left(k+1, 0, \frac{1}{\sqrt{k+1}}\right) & \text{if } k \text{ is odd} \\ t_{\text{skew}}\left(\frac{k}{2}, \frac{k+2}{2}, 0, \frac{1}{\sqrt{k+1}}\right) & \text{if } k \text{ is even,} \end{cases}$$

then $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)^\top$ is uniformly distributed over $\mathbb{S}_p$.

Time-varying multi-seasonal AR

- Time-varying multi-seasonal AR

$$\prod_{j=1}^{M} \phi_{jt}(L^{s_j})(y_t - \mu_t) = \varepsilon_t \qquad \varepsilon_t \overset{\text{iid}}{\sim} N(0, \sigma_t^2)$$

  ϕ_{jkt} is kth AR parameter in jth polynomial at time t.

- $\boldsymbol{\phi}_t = (\phi_{1t}, \dots, \phi_{p_j t})^\top$ parameters in AR polynomial at time t.

- $\boldsymbol{\phi}_t = \boldsymbol{g}(\boldsymbol{\theta}_t)$ map from unrestricted $\boldsymbol{\theta}_t$ to stable $\boldsymbol{\phi}_t$.

Dynamic shrinkage process priors for TVSAR

- **Dynamic shrinkage process** prior [6] for TVSAR

$$\boldsymbol{\phi}_t = \boldsymbol{g}(\boldsymbol{\theta}_t)$$

$$\theta_{kt} = \theta_{k,t-1} + \nu_{kt}, \qquad \nu_{kt} \overset{\text{indep}}{\sim} N(0, \exp(h_{kt}))$$

$$h_{kt} = \mu_k + \kappa_k(h_{k,t-1} - \mu_k) + \eta_{kt}, \qquad \eta_{kt} \overset{\text{iid}}{\sim} Z(1/2, 1/2, 0, 1)$$

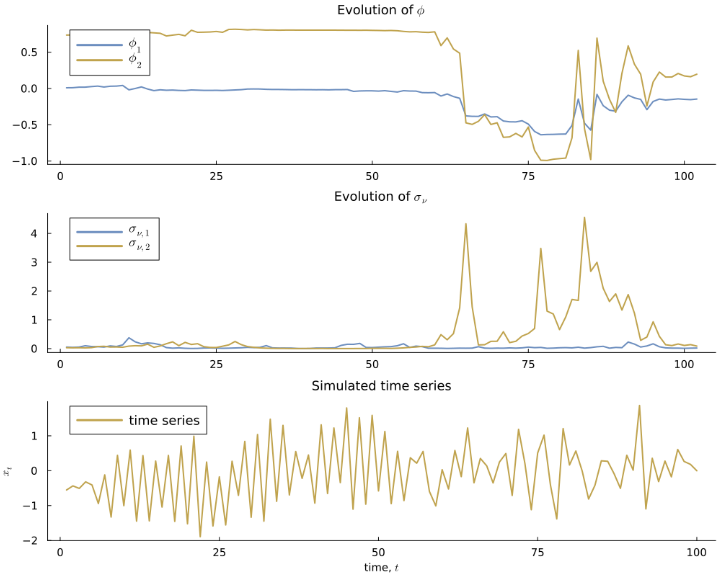
- **Global log-volatility** μ_k
- **Local log-volatility** η_{kt} from heavy-tailed Z-distr.
- Constant periods, periods of rapid change and jumps.
- Time-series extension of **horseshoe prior** [9], since for $\kappa = 0$

$$\mathrm{Var}(\nu_t) = \exp(h_t) = \exp(\mu)\exp(\eta_t) = \tau^2\lambda_t^2$$

and

$$\eta_t \sim Z(1/2, 1/2, 0, 1) \implies \lambda_t = \exp(\eta_t/2) \sim C^+(0, 1)$$

Dynamic shrinkage process priors for TVSAR



Conditional likelihood function

- Multi-seasonal TVSAR can be expressed as **regression** with **nonlinear restrictions** by multiplying out polynomials in

$$\prod_{j=1}^{M} \phi_{jt}(L^{s_j}) y_t = \varepsilon_t$$

- Example (static case):

$$(1 - \phi_1 L)(1 - \Phi_1 L^s) y_t = (1 - \phi_1 L - \Phi_1 L^s + \phi_1 \Phi_1 L^{1+s}) y_t$$

$$y_t = \boldsymbol{x}_t^\top \tilde{\boldsymbol{\phi}} + \varepsilon_t$$

  - $\boldsymbol{x}_t = (y_{t-1}, y_{t-s}, y_{t-(1+s)})^\top$
  - $\tilde{\boldsymbol{\phi}} = (\phi_1, \Phi_1, \phi_1 \Phi_1)^\top$.
  - Lags $2, \ldots, s-1$ have zero coefficients.

Bayesian inference TVSAR

- **TVSAR** with conditional likelihood:

$$
\begin{aligned}
y_t &= \boldsymbol{x}_t^\top \tilde{\phi}_t + \varepsilon_t & \varepsilon_t &\overset{\text{indep}}{\sim} N(0, \sigma_t^2) \\
\tilde{\phi}_t &= \tilde{\boldsymbol{g}}(\boldsymbol{\theta}_t) & & \\
\boldsymbol{\theta}_t &= \boldsymbol{\theta}_{t-1} + \boldsymbol{\nu}_t, & \boldsymbol{\nu}_t &\overset{\text{indep}}{\sim} N\left(0, \mathrm{Diag}(\exp(\boldsymbol{h}_t))\right) \\
\boldsymbol{h}_t &= \boldsymbol{\mu} + \boldsymbol{\kappa}(\boldsymbol{h}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{\eta}_t, & \eta_{kt} &\overset{\text{iid}}{\sim} Z(1/2, 1/2, 0, 1)
\end{aligned}
$$

- Aim: posterior conditional on time series $y_{1:T}$

$$
p(\boldsymbol{\theta}_{0:T}, \boldsymbol{h}_{1:T}, \boldsymbol{\mu}, \boldsymbol{\kappa}, \sigma_{1:T} | y_{1:T})
$$

- Gibbs sampling by data augmentation:
  - Mixture of normal for $\log \chi_1^2$ for volatility $\boldsymbol{h}_{0:T}$ [10]
  - Polya-Gamma augmentation for Z-distribution. [6]
- After´ augmentation: **transition model is linear Gaussian**.

Bayesian inference TVSARMA

- **TVSARMA** with exact likelihood:

$$
\begin{aligned}
y_t &= \boldsymbol{x}_t^\top \tilde{\boldsymbol{\phi}}_t + \boldsymbol{z}_t^\top \tilde{\boldsymbol{\psi}}_t + \varepsilon_t, && \varepsilon_t \overset{\text{indep}}{\sim} N(0, \sigma_t^2) \\
(\tilde{\boldsymbol{\phi}}_t, \tilde{\boldsymbol{\psi}}_t) &= \tilde{\boldsymbol{g}}(\boldsymbol{\theta}_t) \\
\boldsymbol{\theta}_t &= \boldsymbol{\theta}_{t-1} + \boldsymbol{\nu}_t, && \boldsymbol{\nu}_t \overset{\text{indep}}{\sim} N\left(0, \mathrm{Diag}(\exp(\boldsymbol{h}_t))\right) \\
\boldsymbol{h}_t &= \boldsymbol{\mu} + \boldsymbol{\kappa}(\boldsymbol{h}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{\eta}_t, && \eta_{kt} \overset{\text{iid}}{\sim} Z(1/2, 1/2, 0, 1)
\end{aligned}
$$

- Add Gibbs update steps for
  - past disturbances: $\boldsymbol{z}_t = \left(\varepsilon_{t-1}, \varepsilon_{t-s}, \varepsilon_{t-(1+s)}\right)^\top$
  - pre-sample lags: $y_0, y_{-1}, y_{-2}, \ldots$

Sampling from conditional posterior for $\theta_{0:T}$

- Full conditional posterior for $\boldsymbol{\theta}_{0:T}$:
  - state-space model with state vector $\boldsymbol{\theta}_t$.
  - **nonlinear Gaussian observation model**.
  - Linear (heteroscedastic) Gaussian state transition.

- **Particle Gibbs with Ancestor Sampling** (**PGAS**) [11]
  - simulation consistent
  - slow
  - **particle degeneracy when model is near-degenerate** (parameters standing still for extended periods).

- **FFBSx - FFBS with extended Kalman filter**
  - fast
  - robust to near-degeneracy
  - approximate, but shown to be very accurate for TVSAR
  - **automatic differentiation** $\tilde{\phi}_t = \tilde{\boldsymbol{g}}(\boldsymbol{\theta}_t)$ makes it all beautiful.

Extended Kalman filter

- State-space model

$$\boldsymbol{\theta}_t = \boldsymbol{A}\boldsymbol{\theta}_{t-1} + \boldsymbol{B}\boldsymbol{u}_t + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$
$$y_t = \boldsymbol{C}(\boldsymbol{\theta}_t) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \boldsymbol{\Sigma}_\varepsilon)$$

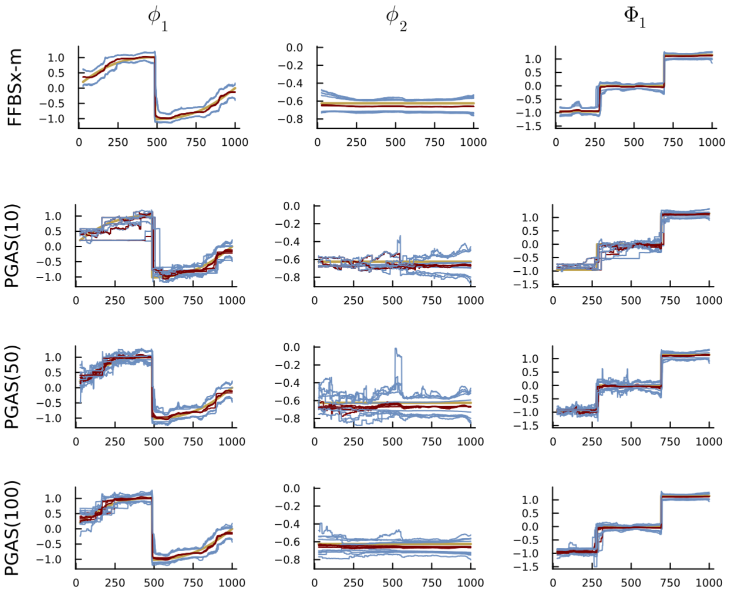
Standard Kalman filter

```
# Prior propagation step
μ̄ = A*μ .+ B*u
Ω̄ = A*Ω*A' + Σn

# Measurement update
K = Ω̄*C' / (C*Ω̄*C' .+ Σe)
μ = μ̄ + K*(y .- C*μ̄)
Ω = (I - K*C)*Ω̄
```

Extended Kalman filter

```
# Prior propagation step
μ̄ = A*μ + B*u
Ω̄ = A*Ω*A' + Σn
C̄ = ∂C(μ̄, Cargs)

# Measurement update
K = Ω̄*C̄' / (C̄*Ω̄*C̄' .+ Σe)
μ = μ̄ + K*(y .- C(μ̄, Cargs))
Ω = (I - K*C̄)*Ω̄
```

Simulation TVSAR with $p = (2, 2)$ and $s = (1, 12)$



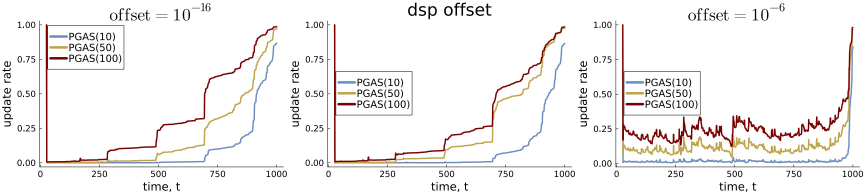
PGAS - larger offset improves convergence

- Volatility models

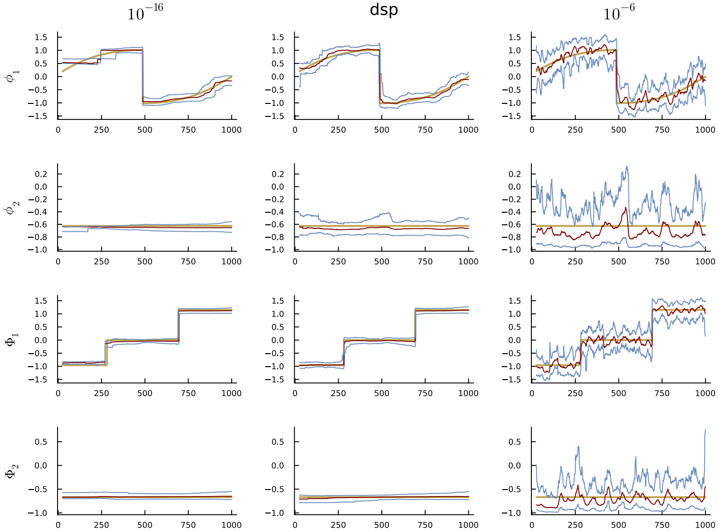
$$\nu_t = \exp(h_t/2)\epsilon_t, \qquad \epsilon_t \sim N(0,1)$$

- The usual trick

$$\log(\nu_t^2 + \text{offset}) = h_t + \log \epsilon_t^2$$



PGAS - but makes parameters more wiggly



Spectral density ARMA

Spectral density ARMA(p,q)×(P,Q)

seasonal period	12
no. of AR	1
no. of MA	0
no. of seasonal AR	1
no. of seasonal MA	0

Parameter settings

ϕ_1 0.5

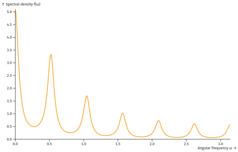
No regular MA

Φ_1 0.5

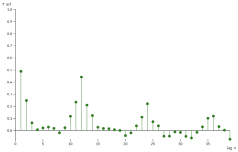
No seasonal MA

plot log f(ω)

Spectral density of an ARMA(1,0)×(1,0) process with season s=12

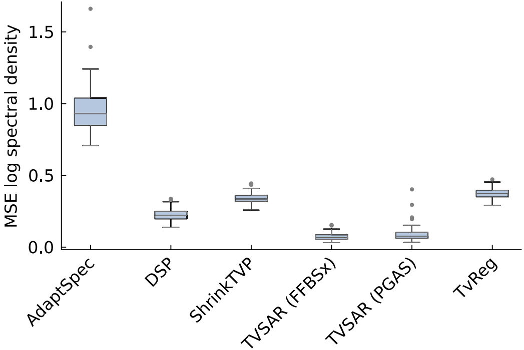


Estimated ACF of ARMA(1,0)×(1,0) process with season s=12

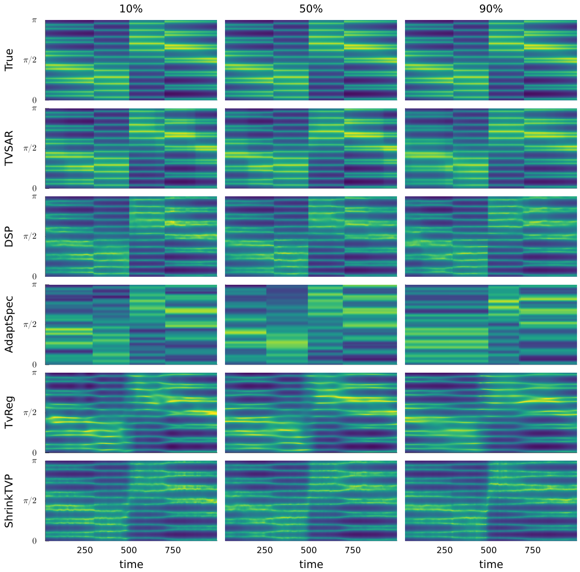


Fitting the time-varying spectral density

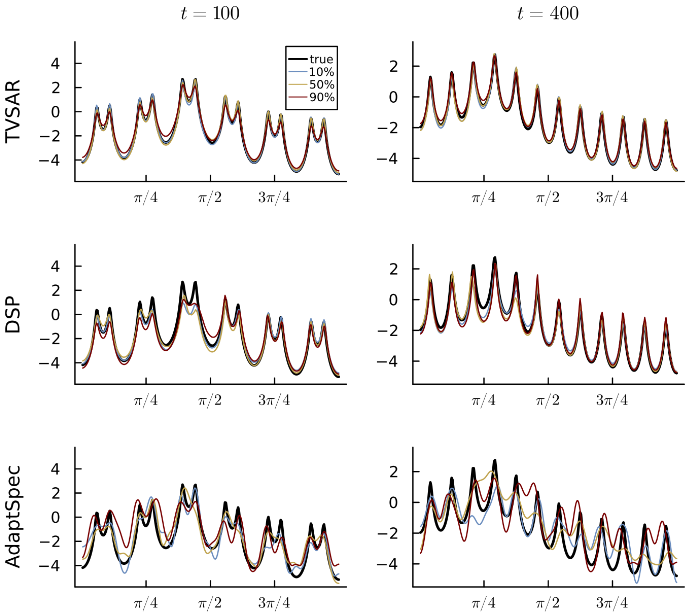
- Benchmark models are linear:
  - Fit $y_t = \boldsymbol{z}_t^\top \tilde{\boldsymbol{\phi}}_t + \varepsilon_t$ with zero restrictions
  - Coeff on lag $x_{t-(1+s)}$ fitted without multiplicative restriction.
- **Time-varying spectral density** $f(\omega, t)$. Spectrogram.
- MSE of log spectrogram estimate: $\widehat{\log f(\omega, t)}$ integrated over frequency $\omega \in (0, \pi]$ and time $t = 1, \dots, 1000$.



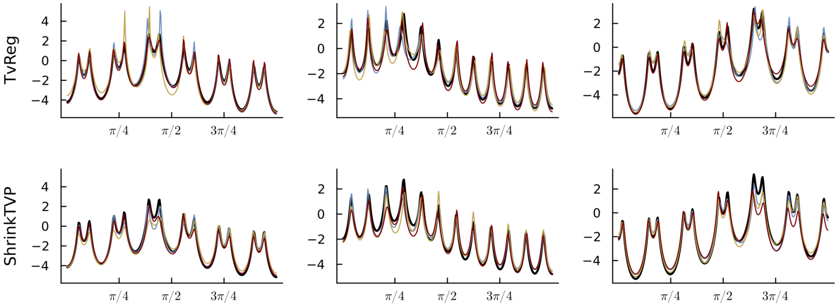
Fitted log spectrogram



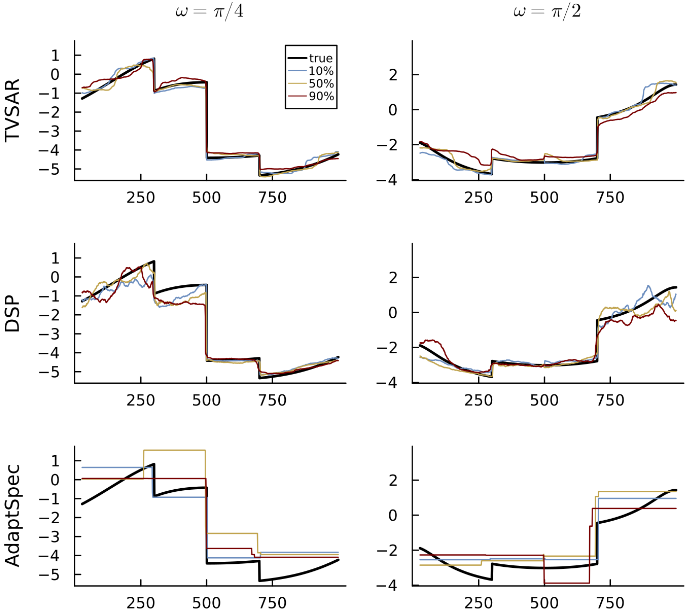
Fitted log spectrogram snapshots



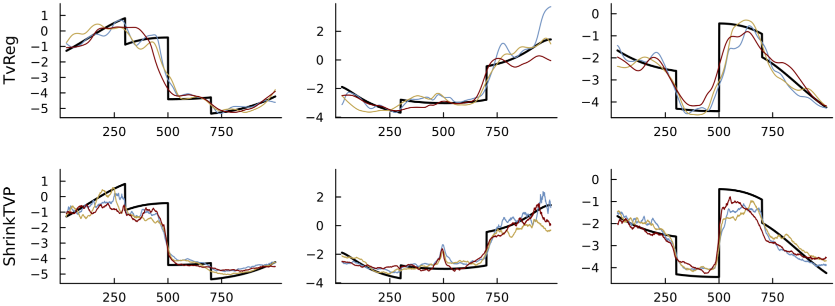
Fitted log spectrogram snapshots



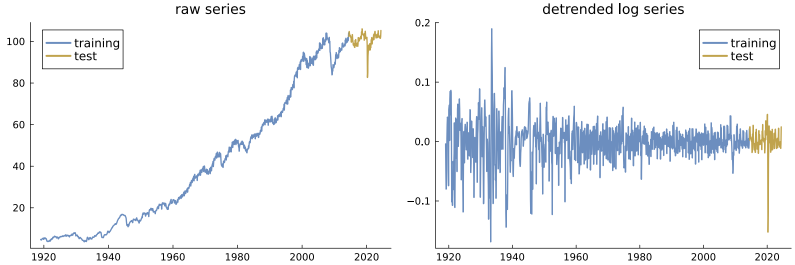
Fitted log spectral density over time



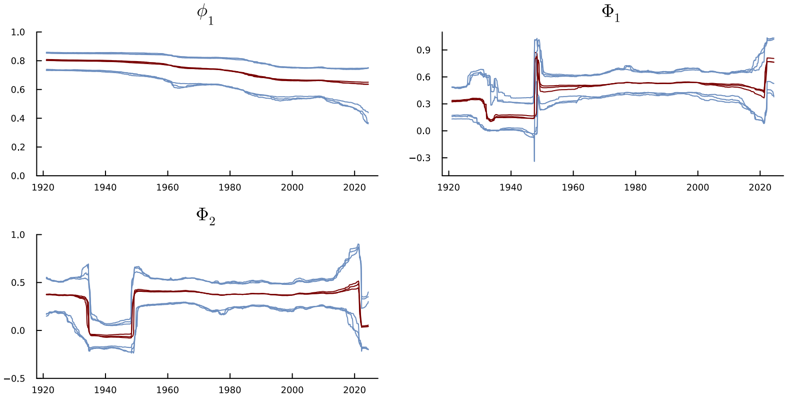
Fitted log spectral density over time



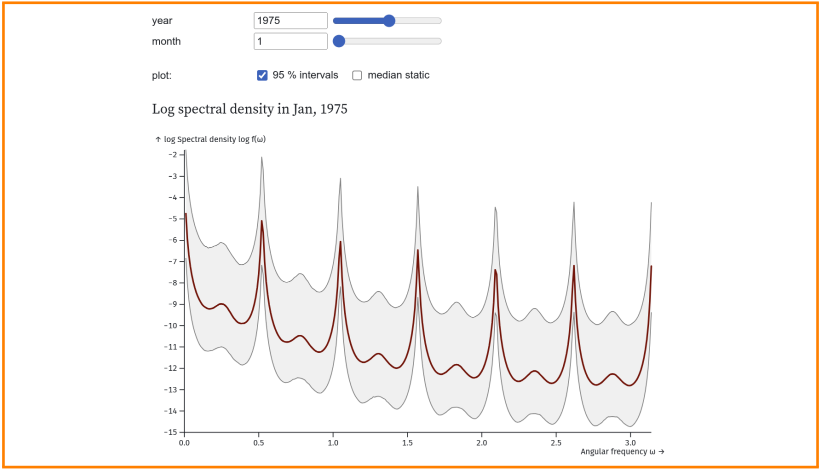
US industrial production 1919-2024



SAR(1,2) - FFBSx three random initial values



SAR(1,2) spectral density snapshots



Conclusions

- **Multi-seasonal AR** model with **time-varying** parameters.
- Stability restrictions at every time point. **Locally stable**.
- Parameter evolution by **dynamic shrinkage processes**.
- Challenging inference problem:
  - **non-linear** - stability restrictions and multiplicative seasonality
  - dynamic shrinkage prior tends to give **near-degeneracy**.
- Gibbs sampler with **particle/extended Kalman update** step.
- In progress:
  - Beyond Extended Kalman.
  - Better prior eliciatation and posterior sampling for DSP prior.
  - julia package for arbitrary number of polynomials.

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