Locally Stable Time-Varying Multi-Seasonal AR Models

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Co-authors, slides and paper

- Joint work with
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  - **Robert Kohn**, University of New South Wales

- **Slides**: http://mattiasvillani.com/news.
- **Paper**: arXiv 2409.18640

Motivation

- **Box-Jenkins ARIMA methodology**
  - **Global stationarity**. Restrictive.
  - **Differencing**. Overdifferencing. Levels are non-stationary.
  - **Data in most recent regime**. Wasteful. No probability for regime change.
- **Local stationarity** [1]
- **Time-varying parameters**
  - long history [2], [3]
  - work on more realistic parameter evolutions [4], [5], [6], [7]
  - parameters can visit **explosive region**
- **Seasonality**
  - is predictable. Model it! Time-varying?
  - time series with **multiple seasonal periods** is now common.

Multi-seasonal AR models

- **Seasonal AR(p, P)** with season s

$$\phi_p(L)\Phi_P(L^s)(x_t - \mu) = \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} N(0, \sigma^2)$$

- Regular **lag polynomial**

$$\phi_p(L) = 1 - \phi_1 L - \ldots - \phi_p L^p$$

- Seasonal lag polynomial

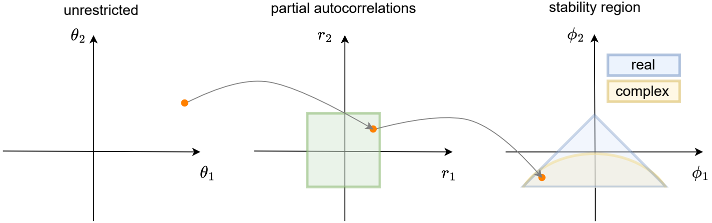
$$\Phi_p(L^s) = 1 - \Phi_1 L^s - \ldots - \Phi_P L^{Ps}$$

- **Multiple seasonal periods**, e.g. hourly data with daily, weekly and yearly cycles.

- **Multi-seasonal AR** models with M polynomials

$$\prod_{j=1}^{M} \phi_j(L^{s_j})(x_t - \mu) = \varepsilon_t$$

Enforce stability in AR models



- Composite map to **stability**

$$\boldsymbol{\theta} \to \boldsymbol{r} \to \boldsymbol{\phi}$$

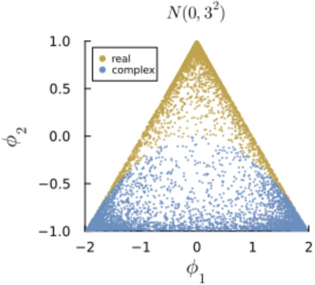
- Unrestricted $\to$ partial autocorrelations [8]

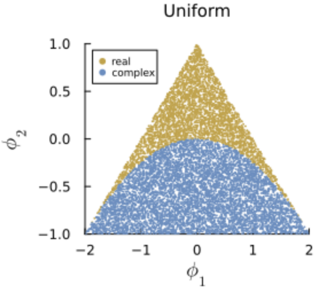
$$r_k = \frac{\theta_k}{(1 + \theta_k^2)^{1/2}}$$

- Partial autocorrs $\boldsymbol{r} \to$ **stable AR coeffs** ϕ by a recursion [9]
- Same parameterization for all polynomial factors.

Uniform distribution over stability region

- Unrestricted parameters θ_k have no interpretation. Priors?
- **Uniform distribution over stability region** $\mathbb{S}_p$ for ϕ.





- Lemma: If, independently,

$$\theta_k \sim \begin{cases} t\left(k+1, 0, \frac{1}{\sqrt{k+1}}\right) & \text{if } k \text{ is odd} \\ t_{\text{skew}}\left(\frac{k}{2}, \frac{k+2}{2}, 0, \frac{1}{\sqrt{k+1}}\right) & \text{if } k \text{ is even,} \end{cases}$$

then $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)^\top$ is uniformly distributed over $\mathbb{S}_p$.
- The skew-t distribution is the one in [10].

TVSAR with dynamic shrinkage process priors

- Time-varying multi-seasonal AR

$$\prod_{j=1}^{M} \phi_{jt}(L^{s_j})(x_t - \mu_t) = \varepsilon_t \qquad \varepsilon_t \overset{\text{iid}}{\sim} N(0, \sigma_t^2)$$

- **Dynamic shrinkage process** prior [6] for TVSAR

$$\boldsymbol{\phi}_t = \boldsymbol{g}(\boldsymbol{\theta}_t)$$

$$\theta_{kt} = \theta_{k,t-1} + \nu_{kt}, \qquad \nu_{kt} \overset{\text{indep}}{\sim} N(0, \exp(h_{kt}))$$

$$h_{kt} = \mu_k + \kappa_k(h_{k,t-1} - \mu_k) + \eta_{kt}, \qquad \eta_{kt} \overset{\text{iid}}{\sim} Z(1/2, 1/2, 0, 1)$$

- Time-series extension of **horseshoe prior** ($\kappa = 0$) [11]
- **Global volatility** μ_k
- Persistent **local volatility** η_{kt} from heavy-tailed Z-distr.
- Constant periods, periods of rapid change and jumps.

Bayesian inference for TVSAR

- Multi-seasonal TVSAR: **regression** with **nonlinear restrictions** by multiplying out polynomials in

$$\prod_{j=1}^{M} \phi_{jt}(L^{s_j}) x_t = \varepsilon_t$$

- TVSAR with conditional likelihood:

$$y_t = \boldsymbol{z}_t^\top \tilde{\boldsymbol{g}}(\boldsymbol{\theta}_t) + \varepsilon_t \qquad \varepsilon_t \overset{\text{indep}}{\sim} N(0, \sigma_t^2)$$
$$\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} + \boldsymbol{\nu}_t, \qquad \boldsymbol{\nu}_t \overset{\text{indep}}{\sim} N\left(0, \mathrm{Diag}(\exp(\boldsymbol{h}_t))\right)$$
$$\boldsymbol{h}_t = \boldsymbol{\mu} + \boldsymbol{\kappa}(\boldsymbol{h}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{\eta}_t, \qquad \eta_{kt} \overset{\text{iid}}{\sim} Z(1/2, 1/2, 0, 1)$$

- Aim: posterior conditional on time series $y_{1:T}$

$$p(\boldsymbol{\phi}_{0:T}, \boldsymbol{h}_{0:T}, \boldsymbol{\mu}, \boldsymbol{\kappa}, \sigma_{1:T} | y_{1:T})$$

- Gibbs sampling by data augmentation: [12] and [6]

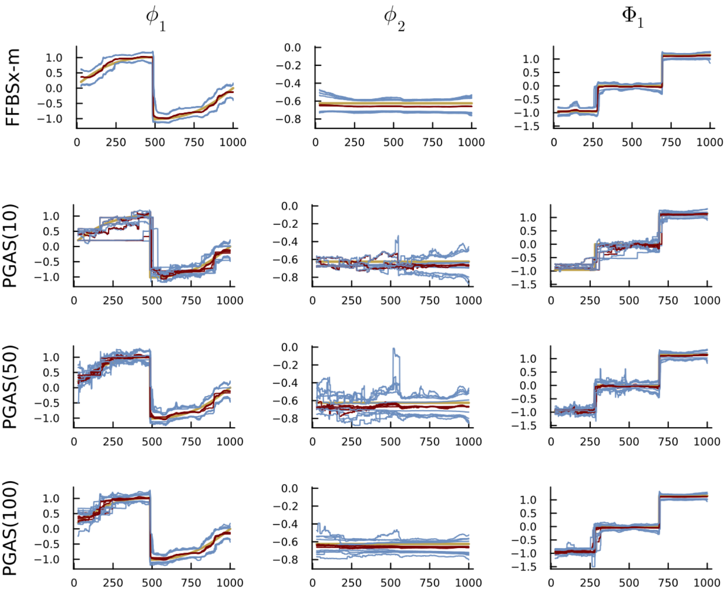
Sampling from conditional posterior for $\theta_{0:T}$

- Full conditional posterior for $\boldsymbol{\theta}_{0:T}$:
  - state-space model with state vector $\boldsymbol{\theta}_t$.
  - **nonlinear Gaussian observation model**.
  - linear (heteroscedastic) Gaussian state transition.

- **Particle Gibbs with Ancestor Sampling** (**PGAS**) [13]
  - simulation consistent
  - slow
  - particle degeneracy when parameters are approx constant

- **FFBSx - FFBS with extended Kalman filter**
  - fast
  - robust to near-degeneracy
  - approximate, but shown to be very accurate for TVSAR
  - **automatic differentiation** $\tilde{\phi}_t = \tilde{\boldsymbol{g}}(\boldsymbol{\theta}_t)$ makes it all beautiful.

Simulation TVSAR with $p = (2, 2)$ and $s = (1, 12)$

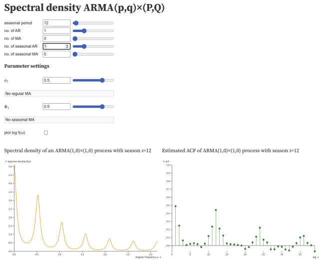


Spectral density

- Spectral density for Seasonal AR

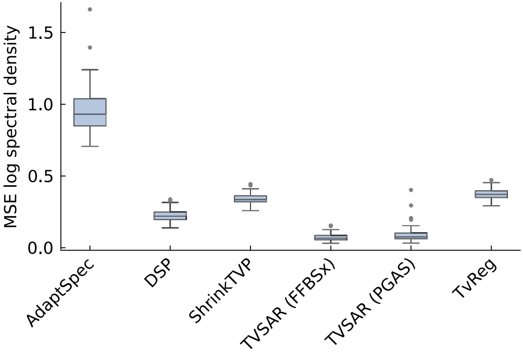
$$f(\omega) = \frac{\sigma^2}{\pi} \frac{1}{|\phi_p(e^{-i\omega})|^2} \frac{1}{|\Phi_P(e^{-is\omega})|^2} \text{ for } \omega \in (0, \pi)$$

- **Time-varying spectral density** $f(\omega, t)$.

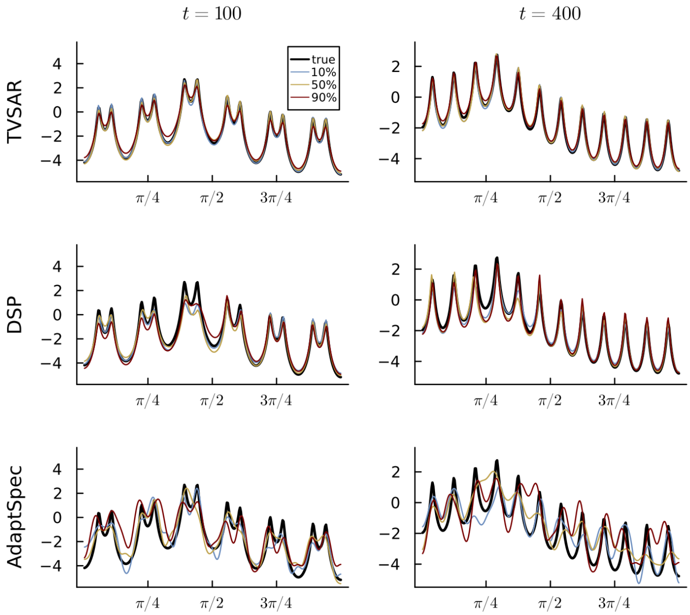


Fitting the time-varying spectral density

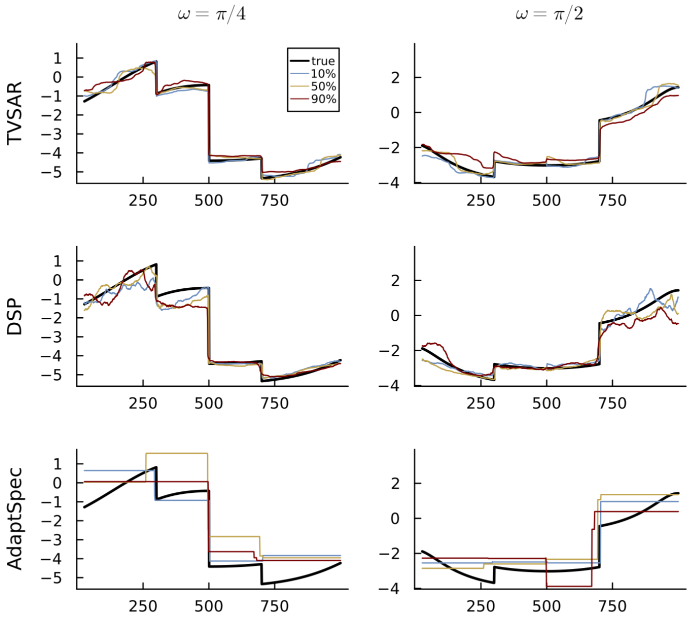
- MSE of log spectrogram estimate: $\widehat{\log f(\omega, t)}$ integrated over frequency $\omega \in (0, \pi]$ and time $t = 1, \ldots, 1000$.



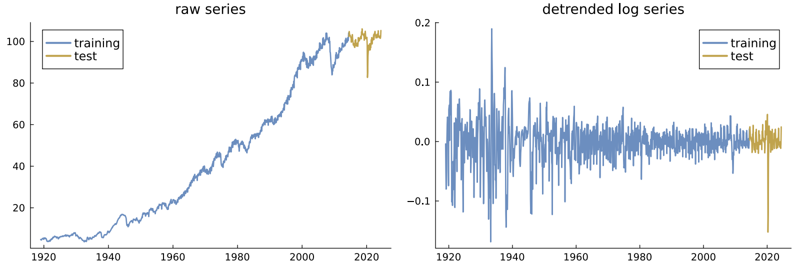
Log spectral density snapshots



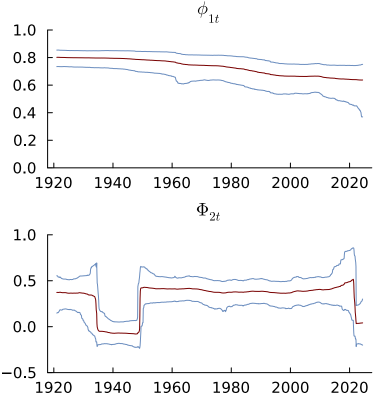
Log spectral density over time



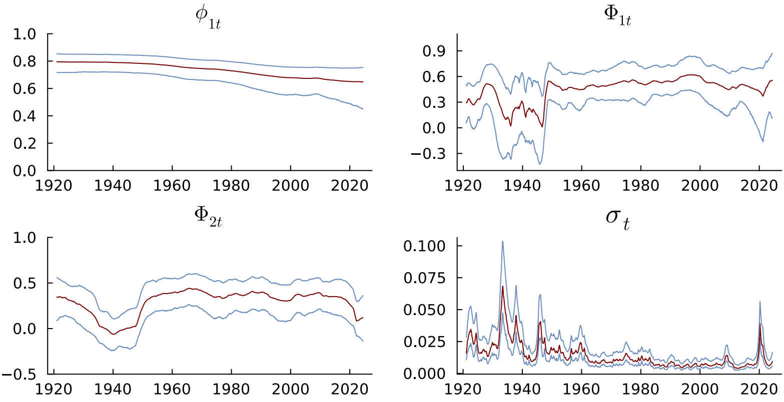
US industrial production 1919-2024



TVSAR(1,2) - dynamic shinkage prior



TVSAR(1,2) - homoscedastic Gaussian evolution



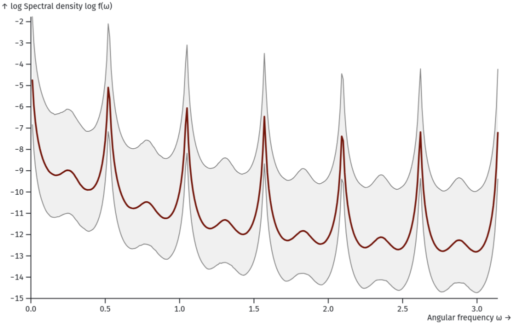
TVSAR(1,2) spectral density snapshots

year 1975

month 1

plot: ☑ 95 % intervals ☐ median static

Log spectral density in Jan, 1975



Conclusions

- **Multi-seasonal AR** model with **time-varying** parameters.
- Stability restrictions at every time point. **Locally stable**.
- Parameter evolution by **dynamic shrinkage processes**.
- Parameters can stand still, move rapidly or jump.
- Challenging inference problem:
  - **non-linear** - stability restrictions and multiplicative seasonality
  - dynamic shrinkage prior tends to give **near-degeneracy**.
- Gibbs sampler with **particle/extended Kalman update** step.
- In progress:
  - Extension to ARMA and exact likelihood.
  - Seasonal VARMA with Ansley-Kohn stability restrictions.
  - Models directly in the spectral domain.
- julia package for arbitrary number of polynomials is coming.

R. Dahlhaus, "Locally stationary processes," in *Handbook of statistics*, vol. 30, pp. 351–413, Elsevier, 2012.

T. Doan, R. Litterman, and C. Sims, "Forecasting and conditional projection using realistic prior distributions," *Econometric reviews*, vol. 3, no. 1, pp. 1–100, 1984.

G. E. Primiceri, "Time varying structural vector autoregressions and monetary policy," *The Review of Economic Studies*, vol. 72, no. 3, pp. 821–852, 2005.

R. Gerlach, C. Carter, and R. Kohn, "Efficient bayesian inference for dynamic mixture models," *Journal of the American Statistical Association*, vol. 95, no. 451, pp. 819–828, 2000.

M. Kalli and J. E. Griffin, "Time-varying sparsity in dynamic regression models," *Journal of Econometrics*, vol. 178, no. 2, pp. 779–793, 2014.

D. R. Kowal, D. S. Matteson, and D. Ruppert, "Dynamic shrinkage processes," *Journal of the Royal Statistical Society:*

Series B (Statistical Methodology), vol. 81, no. 4, pp. 781–804, 2019.

P. Knaus, A. Bitto-Nemling, A. Cadonna, and S. Frühwirth-Schnatter, "Shrinkage in the time-varying parameter model framework using the R package shrinkTVP," *Journal of Statistical Software*, vol. 100, no. 13, 2021.

J. F. Monahan, "A note on enforcing stationarity in autoregressive-moving average models," *Biometrika*, vol. 71, no. 2, pp. 403–404, 1984.

O. Barndorff-Nielsen and G. Schou, "On the parametrization of autoregressive models by partial autocorrelations," *Journal of Multivariate Analysis*, vol. 3, no. 4, 1973.

M. C. Jones and M. Faddy, "A skew extension of the t-distribution, with applications," *Journal of the Royal Statistical Society Series B: Statistical Methodology*, vol. 65, no. 1, pp. 159–174, 2003.

- C. M. Carvalho, N. G. Polson, and J. G. Scott, “The horseshoe estimator for sparse signals,” *Biometrika*, vol. 97, no. 2, pp. 465–480, 2010.
- S. Kim, N. Shephard, and S. Chib, “Stochastic volatility: likelihood inference and comparison with arch models,” *The review of economic studies*, vol. 65, no. 3, pp. 361–393, 1998.
- F. Lindsten, M. Jordan, and T. B. Schön, “Particle gibbs with ancestor sampling,” *Journal of Machine Learning Research*, vol. 15, p. 2145—2184, 06 2014.

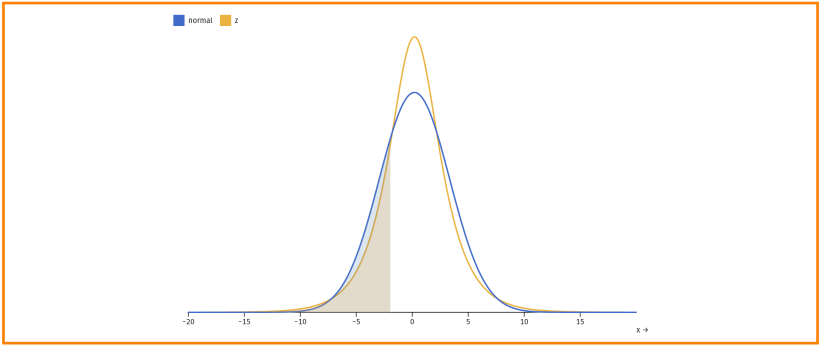
Z-distribution

- Also called **Logistic-Beta distribution** since

$$X \sim \mathrm{Beta}(\alpha, \beta) \implies \log\left(\frac{X}{1-X}\right) \sim Z(\alpha, \beta, 0, 1)$$

- Four-parameter version by location-scale

$$X \sim Z(\alpha, \beta, 0, 1) \implies \mu + \sigma X \sim Z(\alpha, \beta, \mu, \sigma)$$



Extended Kalman filter

- State-space model

$$\boldsymbol{\theta}_t = \boldsymbol{A}\boldsymbol{\theta}_{t-1} + \boldsymbol{B}\boldsymbol{u}_t + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$
$$y_t = C(\boldsymbol{\theta}_t) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \boldsymbol{\Sigma}_\varepsilon)$$

- Yesterday's posterior: $\boldsymbol{\theta}_{t-1}|y_{1:t-1} \sim N(\boldsymbol{\mu}_{t-1|t-1}, \boldsymbol{\Omega}_{t-1|t-1})$
- Today's prior: $\boldsymbol{\theta}_t|y_{1:t-1} \sim N(\boldsymbol{\mu}_{t|t-1}, \boldsymbol{\Omega}_{t|t-1})$
- Today's posterior: $\boldsymbol{\theta}_t|y_{1:t} \sim N(\boldsymbol{\mu}_{t|t}, \boldsymbol{\Omega}_{t|t})$

```
# Prior propagation step
μ̄ = A*μ .+ B*u
Ω̄ = A*Ω*A' + Σₙ

# Measurement update
K = Ω̄*C' / (C*Ω̄*C' .+ Σₑ)
μ = μ̄ + K*(y .- C*μ̄)
Ω = (I - K*C)*Ω̄
```

```
# Prior propagation step
μ̄ = A*μ + B*u
Ω̄ = A*Ω*A' + Σₙ
C̄ = ∂C(μ̄, Cargs)

# Measurement update
K = Ω̄*C̄' / (C̄*Ω̄*C̄' .+ Σₑ)
μ = μ̄ + K*(y .- C(μ̄, Cargs))
Ω = (I - K*C̄)*Ω̄
```

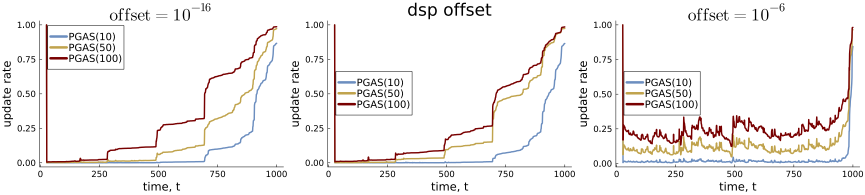
PGAS - larger offset improves convergence

- Volatility models

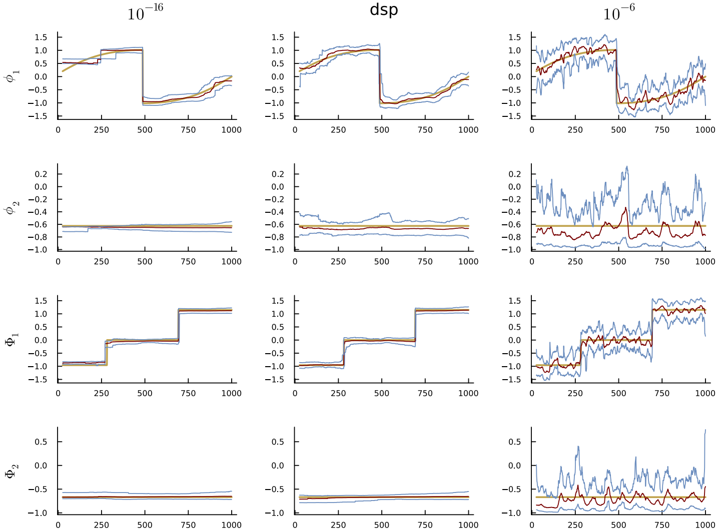
$$\nu_t = \exp(h_t/2)\epsilon_t, \qquad \epsilon_t \sim N(0,1)$$

- The usual trick

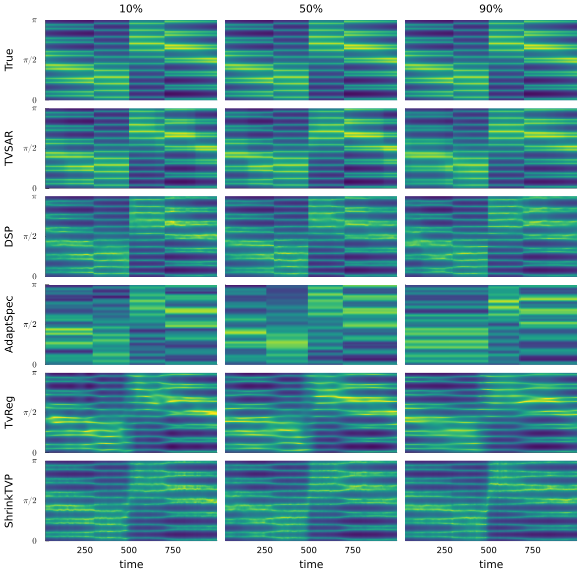
$$\log(\nu_t^2 + \text{offset}) = h_t + \log \epsilon_t^2$$



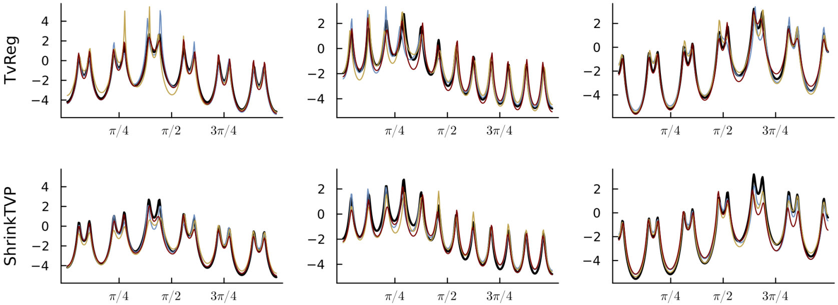
PGAS - but makes parameters more wiggly



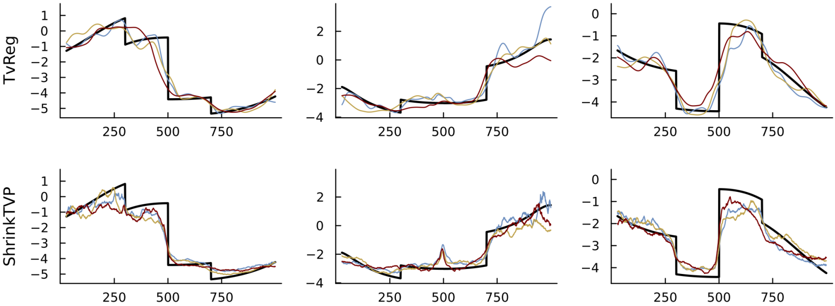
Fitted log time-varying spectral densities



Fitted log spectrogram snapshots



Fitted log spectral density over time



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