Bayesian inference in time-varying parameter models with global-local shrinkage process priors

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Collaborators

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Plan for the talk

- **Setting the stage: Bayesian inference in state-space models**

- **Motivation: Time-varying multi-seasonal AR models** with **dynamic shrinkage process prior**

- **Challenge: Algorithms for Bayesian inference based on Gaussian approximations**

- **Slides**: http://mattiasvillani.com/news
- **Paper**: arXiv: 2409.18640 (ARMA extension soon on arXiv)

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Time-varying parameter models

- **Regression** with **time-varying parameters**

$$\text{Measurement model:} \quad y_t = \boldsymbol{x}_t^\top \boldsymbol{\beta}_t + \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} N(0, \sigma_\varepsilon^2)$$

$$\text{Transition model:} \quad \boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \boldsymbol{\nu}_t, \qquad \boldsymbol{\nu}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\nu}})$$

- Linear Gaussian **state-space model** with state: $\boldsymbol{z}_t = \boldsymbol{\beta}_t$

- **Filtering posterior**: $p(\boldsymbol{z}_t | y_{1:t})$. Instantaneous posterior.

- **Smoothing posterior**: $p(\boldsymbol{z}_t | y_{1:T})$. Retrospective posterior.

- **Joint smoothing posterior**: $p(\boldsymbol{z}_{1:T} | y_{1:T})$.

Parameter evolutions matter

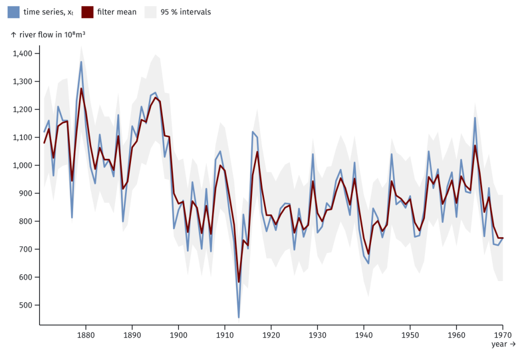
- **Regression** with **time-varying parameters**

$$\text{Measurement model:} \quad y_t = \boldsymbol{x}_t^\top \boldsymbol{\beta}_t + \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} N(0, \sigma_\varepsilon^2)$$

$$\text{Transition model:} \quad \boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \boldsymbol{\nu}_t, \qquad \boldsymbol{\nu}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\nu}})$$

- Homoscedastic Gaussian parameter innovations $\boldsymbol{\nu}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\nu}})$ often a problem. Over- and under-smoothing.
- Ideal: parameters can stand still, move rapidly and jump. [1]
- **Dynamic global-local priors** is a great step forward. [2, 3, 4]
- ... but can be tricky computationally.

Local level model for the Nile data



Taxonomy of state-space models

- **Linear Gaussian** models

$$\text{Measurement model:} \quad \boldsymbol{y}_t = \boldsymbol{C}\boldsymbol{z}_t + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$$

$$\text{Transition model:} \quad \boldsymbol{z}_t = \boldsymbol{A}\boldsymbol{z}_{t-1} + \boldsymbol{\nu}_t, \qquad \boldsymbol{\nu}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\nu}})$$

- **Non-linear** (additive) **Gaussian** models

$$\text{Measurement model:} \quad \boldsymbol{y}_t = \boldsymbol{c}(\boldsymbol{z}_t) + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$$

$$\text{Transition model:} \quad \boldsymbol{z}_t = \boldsymbol{a}(\boldsymbol{z}_{t-1}) + \boldsymbol{\nu}_t, \qquad \boldsymbol{\nu}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\nu}})$$

- **General distribution** (non-linear and non-Gaussian) models

$$\text{Measurement model:} \quad \boldsymbol{y}_t \sim p(\boldsymbol{y}_t | \boldsymbol{z}_t)$$

$$\text{Transition model:} \quad \boldsymbol{z}_t \sim p(\boldsymbol{z}_t | \boldsymbol{z}_{t-1})$$

- Hybrids: nonlinear measurement + linear Gaussian transition.

Common Bayesian computational approaches

- **Kalman** filtering and **FFBS sampling** for linear Gaussian models. [5, 6]
- **Particle MCMC** and **SMC** [7]
- **Hamiltonian Monte Carlo** (Stan/Turing.jl)
- **Variational approximations** [8]
- **INLA** [9]
- **Gaussian approximations** [10]

Multi-seasonal AR models

- **Seasonal AR(p, P)** with season s

$$\phi_p(L)\Phi_P(L^s)y_t = \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} N(0, \sigma^2)$$

- Seasonal AR can be written as a **non-linear regression**:

$$(1-\phi_1 L)(1-\Phi_1 L^s)y_t = (1-\phi_1 L - \Phi_1 L^s + \phi_1\Phi_1 L^{1+s})y_t$$

$$y_t = \boldsymbol{x}_t^\top \tilde{\boldsymbol{\phi}} + \varepsilon_t$$

  - covariates $\boldsymbol{x}_t = (y_{t-1}, y_{t-s}, y_{t-(1+s)})^\top$ and
  - regression coefficients $\tilde{\boldsymbol{\phi}} = (\phi_1, \Phi_1, \phi_1\Phi_1)^\top$.

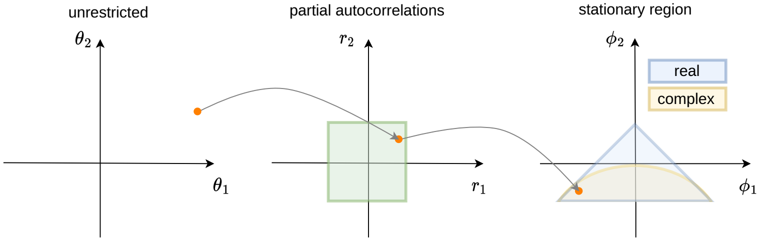
- **Multiple seasonal periods**, e.g. hourly data with daily, weekly and yearly cycles.

- **Multi-seasonal AR** models with M polynomials

$$\prod_{j=1}^{M} \phi_j(L^{s_j})(y_t - \mu) = \varepsilon_t$$

- Direct extension to **ARMA**. Need to infer the ε_t.

Enforce stability/invertibility in ARMA models



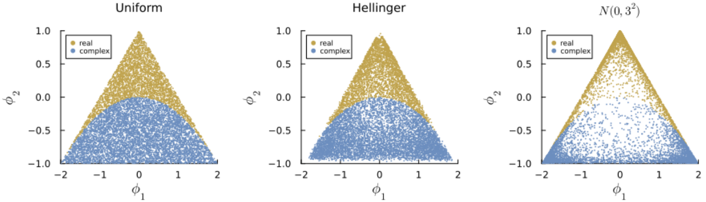
- $\phi = \boldsymbol{g}(\boldsymbol{\theta})$ composite map: [11, 12]

  unrestricted $\boldsymbol{\theta} \rightarrow$ partial autocorr $\boldsymbol{r} \rightarrow$ stable AR ϕ

- Same parameterization for all polynomial factors.
- For seasonal models $\tilde{\phi}_t = \tilde{\boldsymbol{g}}(\boldsymbol{\theta}_t)$ is the nonlinear mapping from
  1. stability restrictions
  2. multiplication of polynomials
- **Invertibility in MA** by the same parameterization.

Uniform distribution over stability region

- Unrestricted parameters θ_k have no interpretation. Priors?
- **Uniform distribution over stability region** $\mathbb{S}_p$ for $\boldsymbol{\phi}$.



- Lemma: If, independently,

$$\theta_k \sim \begin{cases} t\left(k+1, 0, \frac{1}{\sqrt{k+1}}\right) & \text{if } k \text{ is odd} \\ t_{\text{skew}}\left(\frac{k}{2}, \frac{k+2}{2}, 0, \frac{1}{\sqrt{k+1}}\right) & \text{if } k \text{ is even,} \end{cases}$$

then $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)^\top$ is uniformly distributed over $\mathbb{S}_p$.

Time-varying multi-seasonal AR with dynamic shrinkage process prior

- **Time-varying multi-seasonal AR**

$$\prod_{j=1}^{M} \phi_{jt}(L^{s_j})(y_t - \mu_t) = \varepsilon_t \qquad \varepsilon_t \overset{\text{iid}}{\sim} N(0, \sigma_t^2)$$

- **Dynamic shrinkage process** prior [3] for TVSAR

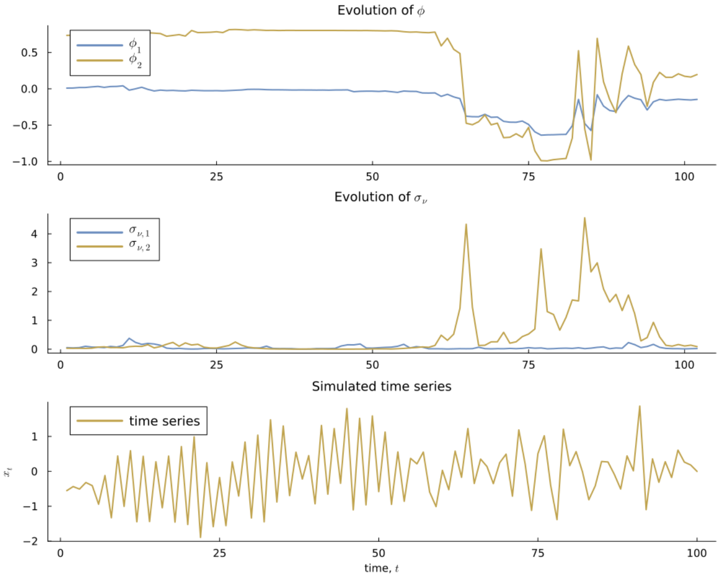
$$\tilde{\phi}_t = \tilde{\boldsymbol{g}}(\boldsymbol{\theta}_t)$$

$$\theta_{kt} = \theta_{k,t-1} + \nu_{kt}, \qquad \nu_{kt} \overset{\text{indep}}{\sim} N(0, \exp(h_{kt}))$$

$$h_{kt} = \mu_k + \kappa_k(h_{k,t-1} - \mu_k) + \eta_{kt}, \qquad \eta_{kt} \overset{\text{iid}}{\sim} Z(1/2, 1/2, 0, 1)$$

- **Global log-variance** μ_k and **local log-variance** η_{kt}.
- Constant periods, periods of rapid change and jumps.
- Time-series extension of the **horseshoe prior** [13].

Dynamic shrinkage process priors for TVAR(2)



Bayesian inference for TVSAR

- Aim: posterior conditional on time series $y_{1:T}$

$$p(\boldsymbol{\theta}_{0:T}, \boldsymbol{h}_{1:T}, \boldsymbol{\mu}, \boldsymbol{\kappa}, \sigma_{1:T} | y_{1:T})$$

- Gibbs sampling by data augmentation:
  - Mixture of normal for $\log \chi_1^2$ for volatility $\boldsymbol{h}_{0:T}$ [14]
  - Polya-Gamma augmentation for Z-distribution. [3]
- Conditional on $h_{1:T}$, the TVSAR is a state-space model with:
  - **nonlinear additive Gaussian measurement** model
    - Nonlinearity 1: multiplicative seasonality
    - Nonlinearily 2: stability restrictions
  - **linear** (heteroscedastic) **Gaussian transition** model

Sampling from conditional posterior for $\theta_{0:T}$

- **Particle Gibbs with Ancestor Sampling (PGAS)** [15]
  - simulation consistent
  - slow
  - **particle degeneracy when model is near-degenerate** (parameters standing still for extended periods).
  - Adding an offset fixes particle degeneracy, but parameters can then no longer stand still. [16]

- **FFBSx - FFBS with extended Kalman filter**
  - **Local linearization** of transition and measurement functions.
  - fast
  - robust to near-degeneracy
  - approximate, but generally accurate for TVSAR
  - **automatic differentiation** of $\tilde{\phi}_t = \tilde{\mathbf{g}}(\boldsymbol{\theta}_t)$ makes it beautiful.

Extended Kalman filter

- State-space model

$$\boldsymbol{\theta}_t = \boldsymbol{A}\boldsymbol{\theta}_{t-1} + \boldsymbol{B}\boldsymbol{u}_t + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$
$$y_t = \boldsymbol{C}(\boldsymbol{\theta}_t) + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \boldsymbol{\Sigma}_\varepsilon)$$

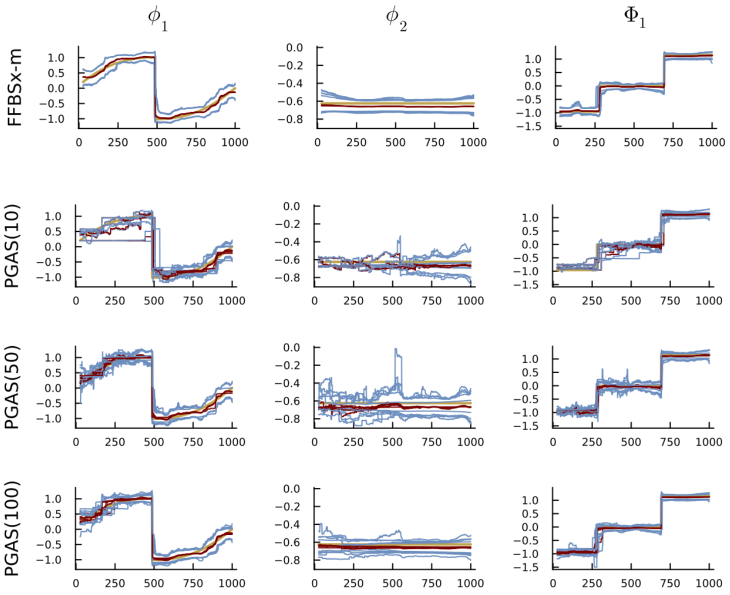
Standard Kalman filter

```
# Prior propagation step
μ̄ = A*μ .+ B*u
Ω̄ = A*Ω*A' + Σₙ

# Measurement update
K = Ω̄*C' / (C*Ω̄*C' .+ Σₑ)
μ = μ̄ + K*(y .- C*μ̄)
Ω = (I - K*C)*Ω̄
```

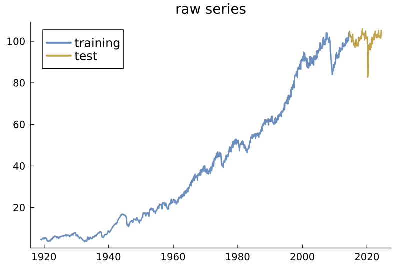
Extended Kalman filter

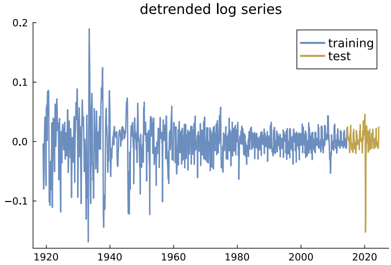
```
# Prior propagation step
μ̄ = A*μ + B*u
Ω̄ = A*Ω*A' + Σₙ
C̄ = ∂C(μ̄, Cargs)

# Measurement update
K = Ω̄*C̄' / (C̄*Ω̄*C̄' .+ Σₑ)
μ = μ̄ + K*(y .- C(μ̄, Cargs))
Ω = (I - K*C̄)*Ω̄
```

Simulation TVSAR with $p = (2, 2)$ and $s = (1, 12)$

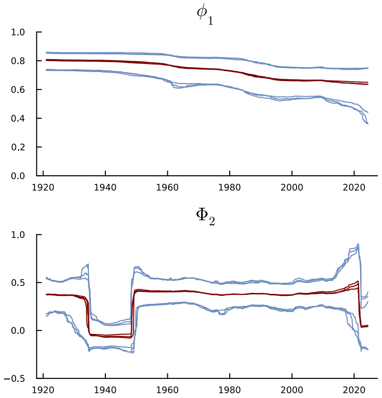


US industrial production 1919-2024

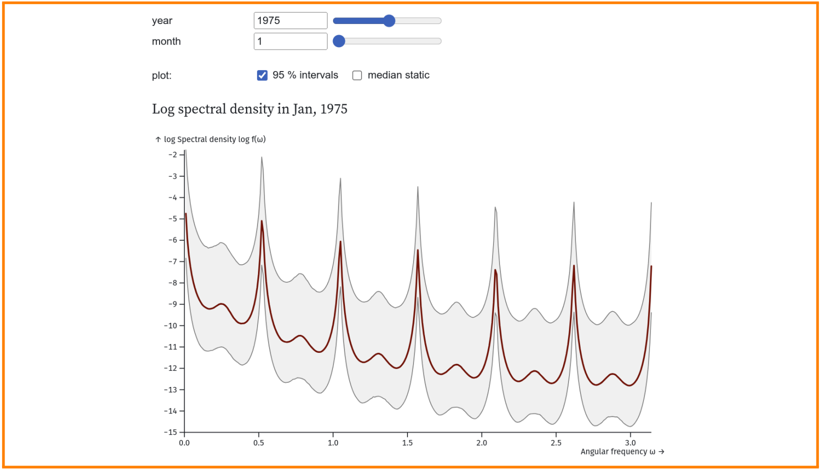




SAR(1,2) - FFBSx three random initial values



SAR(1,2) spectral density snapshots



Approximate Gaussian Kalman-based samplers

- Key for **prior propagation update**: approximate the joint distribution $p(\boldsymbol{z}_t, \boldsymbol{z}_{t-1} | \boldsymbol{y}_{1:t-1})$

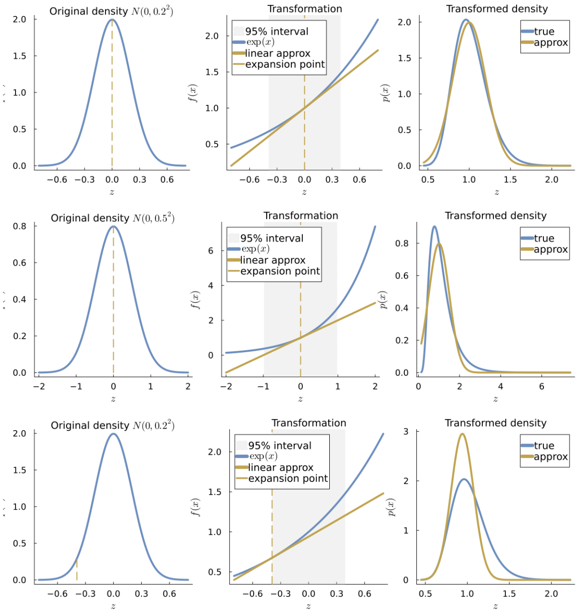
$$\begin{pmatrix} \boldsymbol{z}_t \\ \boldsymbol{z}_{t-1} \end{pmatrix} \mid \boldsymbol{y}_{1:t-1} \overset{\text{approx}}{\sim} N\left[\begin{matrix} \mathbb{E}(\boldsymbol{z}_t) \\ \mathbb{E}(\boldsymbol{z}_{t-1}) \end{matrix}, \begin{pmatrix} \mathbb{V}(\boldsymbol{z}_t) & \mathbb{C}(\boldsymbol{z}_t, \boldsymbol{z}_{t-1}) \\ \mathbb{C}(\boldsymbol{z}_t, \boldsymbol{z}_{t-1})^\top & \mathbb{V}(\boldsymbol{z}_{t-1}) \end{pmatrix} \right]$$

- Key for **measurement update**: approximate the joint distribution $p(\boldsymbol{y}_t, \boldsymbol{z}_t | \boldsymbol{y}_{1:t})$

$$\begin{pmatrix} \boldsymbol{y}_t \\ \boldsymbol{z}_t \end{pmatrix} \mid \boldsymbol{y}_{1:t} \overset{\text{approx}}{\sim} N\left[\begin{matrix} \mathbb{E}(\boldsymbol{y}_t) \\ \mathbb{E}(\boldsymbol{z}_t) \end{matrix}, \begin{pmatrix} \mathbb{V}(\boldsymbol{y}_t) & \mathbb{C}(\boldsymbol{y}_t, \boldsymbol{z}_t) \\ \mathbb{C}(\boldsymbol{y}_t, \boldsymbol{z}_t)^\top & \mathbb{V}(\boldsymbol{z}_t) \end{pmatrix} \right]$$

- **Extended Kalman filter** (**EKF**)
  - linearizes transition model around posterior mean at $t-1$.
  - linearizes measurement model around prior mean at t.

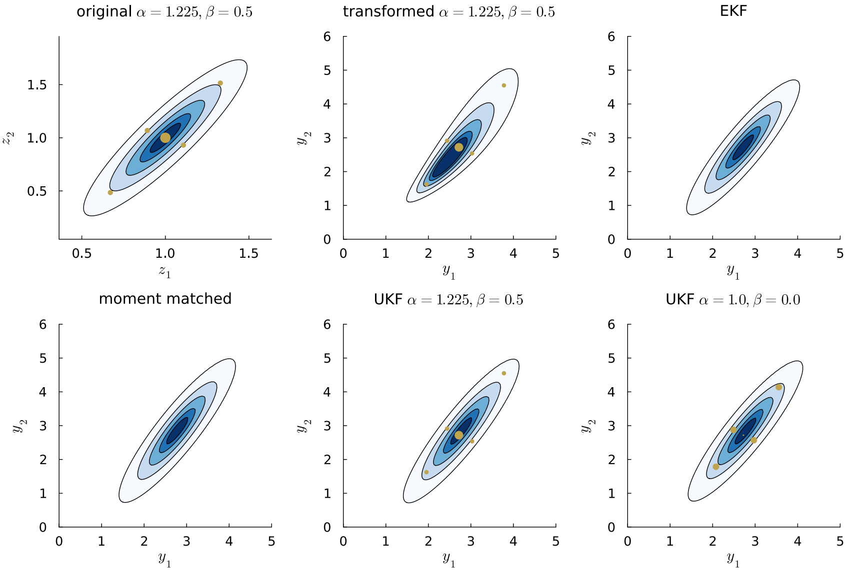
Extended Kalman filter



Alternative Gaussian Kalman-based samplers

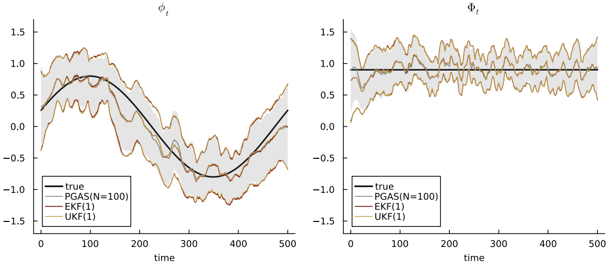
- **Moment matching** by numerical integration. [10]
- **Unscented Kalman Filter** (**UKF**): map sigma points through nonlinear transition and measurement functions. [17]
- **Iterated EKF** and **Iterated UKF**: expands around posterior mean instead of prior mean. [18, 19]
- Iterated EKF/UKF with **line search**. [18, 19]
- **Prior Linearization filter**. Linearizes $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x})$ by minimizing $\mathbb{E}_{\text{prior}} \left(\| \boldsymbol{g}(\boldsymbol{x}) - (\boldsymbol{a} + \boldsymbol{C}\boldsymbol{x}) \|^2 \right)$. [20]
- **Posterior linearization filter** minimizes $\mathbb{E}_{\text{post}} \left(\| \boldsymbol{g}(\boldsymbol{x}) - (\boldsymbol{a} + \boldsymbol{C}\boldsymbol{x}) \|^2 \right)$ by iterating. [20]
- **Laplace approximation**: approximates by posterior mode and inverse negative Hessian. [21]

Gaussian approximations to bivariate log-normal

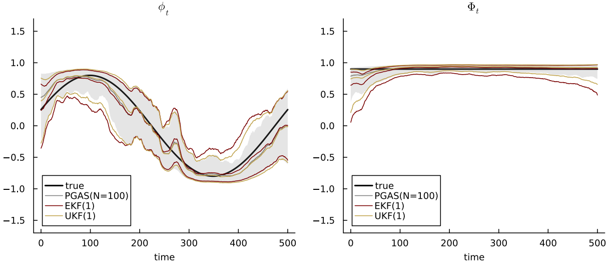


UKF improves on EKF

Near non-stable DGP - fit without stability restrictions



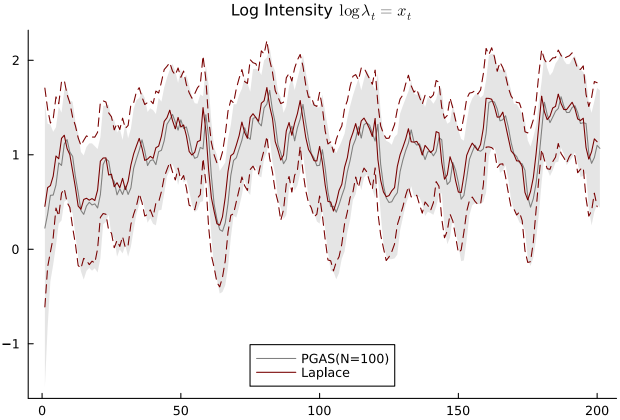
Near non-stable DGP - fit with stability restrictions



Laplace approximation illustration

$$y_t | x_t \sim \text{Poisson}(\exp(x_t))$$

$$x_t = \mu + a(x_{t-1} - \mu) + \nu_t, \qquad \nu_t \overset{\text{iid}}{\sim} N(0, \sigma_\nu^2)$$



Summary

- **Dynamic global-local shrinkage priors** have interesting parameter evolutions.
- The **near-constancy of parameters** makes particle methods problematic.
- **Gaussian approximations work well** in many settings.
- **Multi-seasonal ARMA** models with **time-varying** parameters following a **dynamic shrinkage process**
  - can be made stable at every time period
  - the extended Kalman filter works well in many settings
  - large changes in seasonality in US industrial production data
  - Julia package for arbitrary number of polynomials is coming

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Bayesian inference TVSARMA

- **TVSARMA** with exact likelihood:

$$
\begin{aligned}
y_t &= \boldsymbol{x}_t^\top \tilde{\boldsymbol{\phi}}_t + \boldsymbol{z}_t^\top \tilde{\boldsymbol{\psi}}_t + \varepsilon_t, & \varepsilon_t &\overset{\text{indep}}{\sim} N(0, \sigma_t^2) \\
(\tilde{\boldsymbol{\phi}}_t, \tilde{\boldsymbol{\psi}}_t) &= \tilde{\boldsymbol{g}}(\boldsymbol{\theta}_t) \\
\boldsymbol{\theta}_t &= \boldsymbol{\theta}_{t-1} + \boldsymbol{\nu}_t, & \boldsymbol{\nu}_t &\overset{\text{indep}}{\sim} N(0, \mathrm{Diag}(\exp(\boldsymbol{h}_t))) \\
\boldsymbol{h}_t &= \boldsymbol{\mu} + \boldsymbol{\kappa}(\boldsymbol{h}_{t-1} - \boldsymbol{\mu}) + \boldsymbol{\eta}_t, & \eta_{kt} &\overset{\text{iid}}{\sim} Z(1/2, 1/2, 0, 1)
\end{aligned}
$$

- Add Gibbs update steps for
  - past disturbances: $\boldsymbol{z}_t = \left(\varepsilon_{t-1}, \varepsilon_{t-s}, \varepsilon_{t-(1+s)}\right)^\top$
  - pre-sample lags: $y_0, y_{-1}, y_{-2}, \ldots$

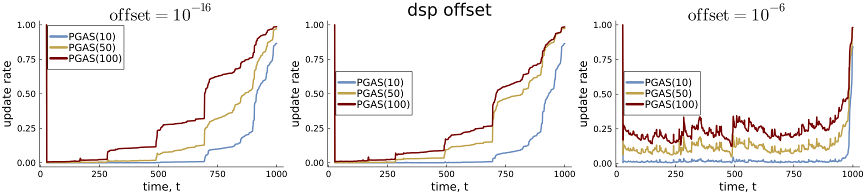
PGAS - larger offset improves convergence

- Volatility models

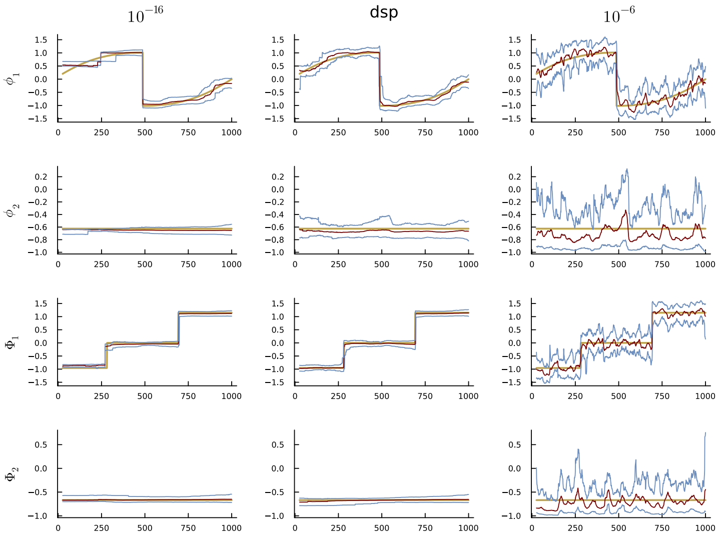
$$\nu_t = \exp(h_t/2)\epsilon_t, \qquad \epsilon_t \sim N(0,1)$$

- The usual **square-and-log trick**

$$\log(\nu_t^2 + \text{offset}) = h_t + \log \epsilon_t^2$$



… but makes parameters more "wiggly"



Spectral density Seasonal ARMA

Spectral density ARMA(p,q)×(P,Q)

seasonal period 12
no. of AR 1
no. of MA 0
no. of seasonal AR 1
no. of seasonal MA 0

Parameter settings

ϕ_1 0.5

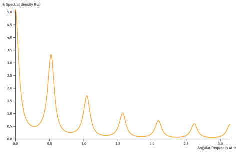
No regular MA

Φ_1 0.5

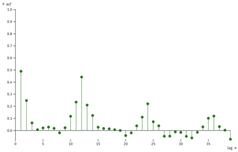
No seasonal MA

plot log f(ω)

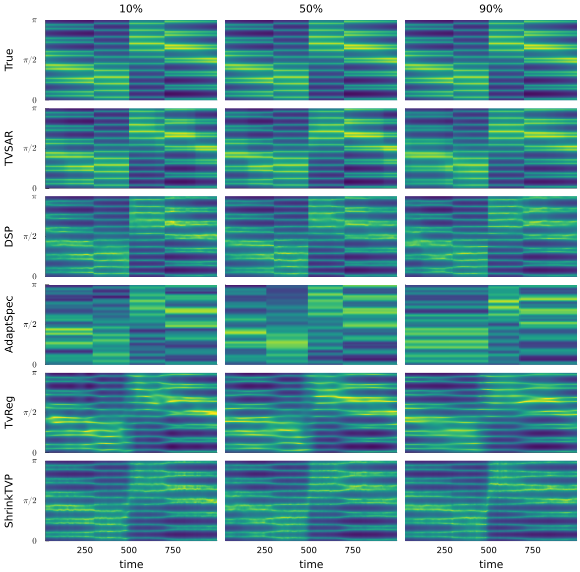
Spectral density of an ARMA(1,0)×(1,0) process with season s=12



Estimated ACF of ARMA(1,0)×(1,0) process with season s=12



Fitted log spectrogram



Quantifying the spectral density fit