Inference in non-linear and non-Gaussian models

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Taxonomy of state-space models

- **Linear Gaussian** models

$$\text{Measurement model:} \quad \boldsymbol{y}_t = \boldsymbol{C}\boldsymbol{z}_t + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$$

$$\text{Transition model:} \quad \boldsymbol{z}_t = \boldsymbol{A}\boldsymbol{z}_{t-1} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$

- **Non-linear** (additive) **Gaussian** models

$$\text{Measurement model:} \quad \boldsymbol{y}_t = \boldsymbol{C}(\boldsymbol{z}_t) + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$$

$$\text{Transition model:} \quad \boldsymbol{z}_t = \boldsymbol{A}(\boldsymbol{z}_{t-1}) + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$

- **General distribution** (non-linear and non-Gaussian) models

$$\text{Measurement model:} \quad \boldsymbol{y}_t \sim p(\boldsymbol{y}_t|\boldsymbol{z}_t)$$

$$\text{Transition model:} \quad \boldsymbol{z}_t \sim p(\boldsymbol{z}_t|\boldsymbol{z}_{t-1})$$

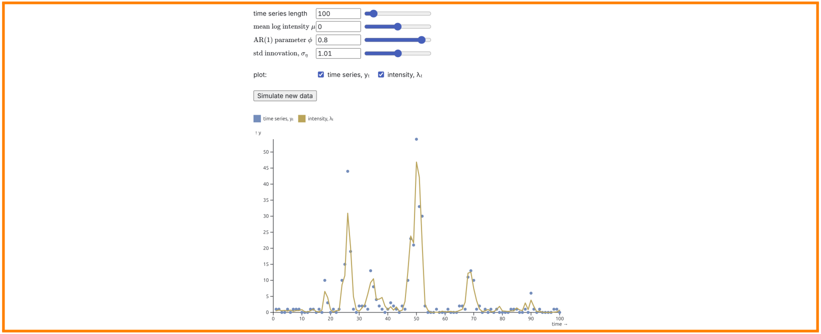
- Hybrids: nonlinear measurement + linear Gaussian transition.

Poisson time series for counts

Poisson model with time-varying intensity

$$y_t | z_t \sim \text{Poisson}\big(\exp(z_t)\big)$$

$$z_t = \mu + \phi(z_{t-1} - \mu) + \eta_t, \qquad \eta_t \overset{\text{iid}}{\sim} N(0, \sigma_\eta^2)$$



Common Bayesian computational approaches

- **Kalman** filtering and **FFBS sampling** for linear Gaussian models. [1, 2]
- **Particle MCMC** and **SMC** [3]
- **Hamiltonian Monte Carlo** (Stan/Turing.jl) [4]
- **Variational approximations** [5]
- **INLA** [6]
- **Gaussian approximations** [7]

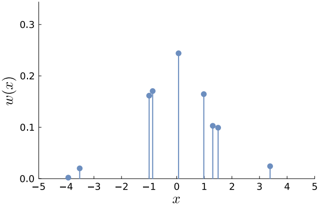
Particle filters

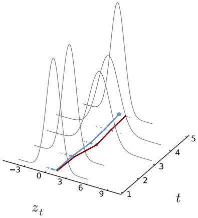
- Approximate filtering posteriors by **weighted particle system**

$$\hat{p}(\boldsymbol{z}_t|\boldsymbol{y}_{1:t}) = \sum_{j=1}^{M} \omega_t^{(j)} \delta_{\boldsymbol{z}_t^{(j)}}(\boldsymbol{z}_t)$$

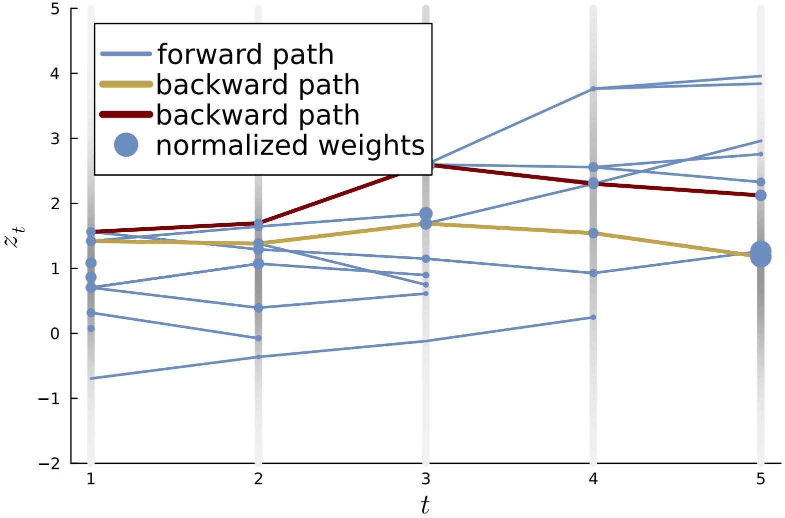
- **Sequential importance sampling** moves particles over time.
- **Bootstrap filter**: importance density is the prior at time t:

$$q(\boldsymbol{z}_t|\boldsymbol{y}_{1:t}) = p(\boldsymbol{z}_t|\boldsymbol{y}_{1:t-1})$$





Particle filters



The Bootstrap filter

Bootstrap filter update for general state-space model

$$p(\mathbf{z}_{t-1}|\mathbf{y}_{1:t-1}) \Rightarrow p(\mathbf{z}_t|\mathbf{y}_{1:t-1}) \Rightarrow p(\mathbf{z}_t|\mathbf{y}_{1:t})$$

Input: particles from previous step $\mathbf{z}_{t-1}^{(1)}, \dots, \mathbf{z}_{t-1}^{(M)}$
measurement $\mathbf{y}_t$
control signal $\mathbf{u}_t$

for j in 1:M **do**

Prior propagation

draw state particle $\mathbf{z}_t^{(j)} \sim g(\mathbf{z}_t|\mathbf{z}_{t-1}^{(j)})$

Measurement update

compute unnormalized weight $\tilde{w}_t^{(j)} = p(\mathbf{y}_t|\mathbf{z}_t^{(j)}, \boldsymbol{\theta})$

end

normalize weights
$w_t^{(j)} = \tilde{w}_t^{(j)} / \sum_{k=1}^{M} \tilde{w}_t^{(k)}$ for $j = 1, \dots, M$

Resampling

for j in 1:M **do**

$k_j \sim \mathrm{Cat}(w_t^{(1)}, \dots, w_t^{(M)})$

$\mathbf{z}_t^{(j)} \leftarrow \mathbf{z}_t^{(k_j)}$

end

Output: M draws $\mathbf{z}_t^{(1)}, \dots, \mathbf{z}_t^{(M)}$ from $p(\mathbf{z}_t|\mathbf{y}_{1:t})$.

Particle Gibbs with Ancestor Sampling (PGAS)

- Particle Gibbs samples from the joint smoothing posterior $p(\boldsymbol{z}_{1:T}|\boldsymbol{y}_{1:T}, \boldsymbol{\theta})$ via a particle filter inside a Gibbs sampler. [8]
- **Conditional particle filter** that conditions on a **reference particle trajectory** from the previous Gibbs iteration.
- Markov kernel that leaves the target posterior invariant.
- Extended target with state + SMC randomness.
- Partially collapsed (SMC randomness integrated out).
- **Particle Gibbs with Ancestor Sampling** (**PGAS**) [9]
  - Resamples the reference particle ancestor at every time step.
  - Mitigates degeneracy and gives better mixing.
  - Smaller number of particles needed, but more costly.

Extended Kalman filter (EKF)

- State-space model

$$\boldsymbol{\theta}_t = \boldsymbol{a}(\boldsymbol{\theta}_{t-1}) + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$
$$y_t = \boldsymbol{c}(\boldsymbol{\theta}_t) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \boldsymbol{\Sigma}_\varepsilon)$$

- Linearized model

$$\boldsymbol{\theta}_t = \boldsymbol{a}'(\boldsymbol{\theta}_{t-1}) + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$
$$y_t = \boldsymbol{c}'(\boldsymbol{\theta}_t) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \boldsymbol{\Sigma}_\varepsilon)$$

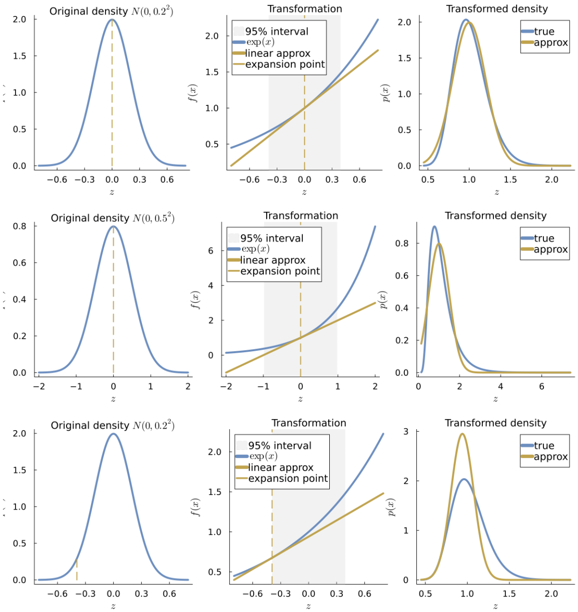
Standard Kalman filter

```
# Prior propagation step
μ̄ = A*μ .+ B*u
Ω̄ = A*Ω*A' + Σₙ

# Measurement update
K = Ω̄*C' / (C*Ω̄*C' .+ Σₑ)
μ = μ̄ + K*(y .- C*μ̄)
Ω = (I - K*C)*Ω̄
```

Extended Kalman filter

```
# Prior propagation step
μ̄ = A*μ + B*u
Ω̄ = A*Ω*A' + Σₙ
C̄ = ∂C(μ̄, Cargs)

# Measurement update
K = Ω̄*C̄' / (C̄*Ω̄*C̄' .+ Σₑ)
μ = μ̄ + K*(y .- C(μ̄, Cargs))
Ω = (I - K*C̄)*Ω̄
```

Extended Kalman filter



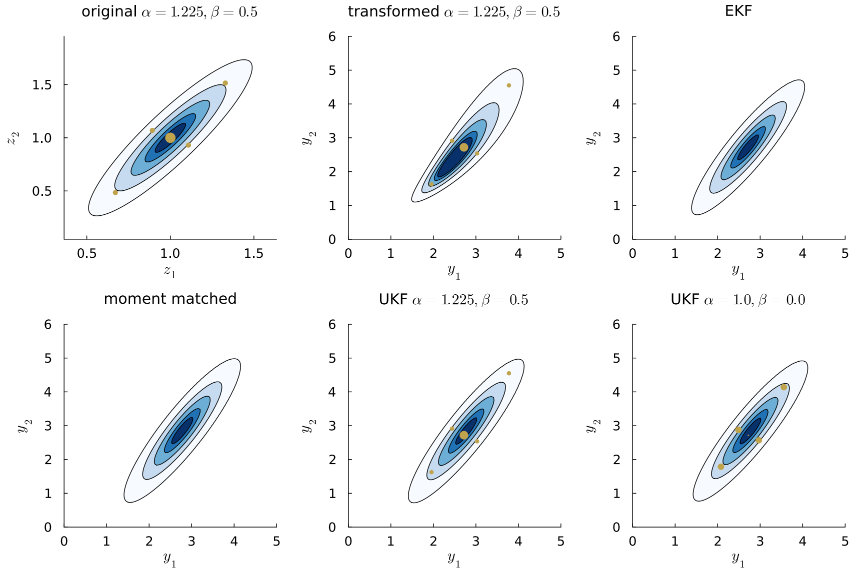
FFBS-EKF

- EKF: **automatic differentiation** makes it all easy.
- Forward filtering with EKF gives $\boldsymbol{\mu}_{t|t}, \boldsymbol{\Omega}_{t|t}, \boldsymbol{\mu}_{t|t-1}, \boldsymbol{\Omega}_{t|t-1}$.
- **Sampling from joint smoothing density** by backward sampling using same multivariate normal distributions as in FFBS.
- Much faster than PGAS.
- **Robust to near-degeneracy** in the transition model. Important when global-local shrinkage process priors is used for the parameter evolution. See Lecture 4.

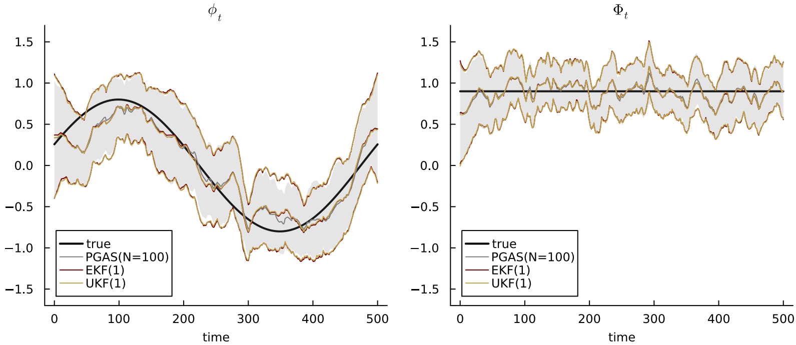
Alternative approximate Gaussian filtering

- **Moment matching** by numerical integration. [7]
- **Unscented Kalman Filter** (**UKF**): map sigma points through nonlinear transition and measurement functions. [10]
- **Iterated EKF** and **Iterated UKF**: expands around posterior mean instead of prior mean. [11, 12]
- Iterated EKF/UKF with **line search**. [11, 12]
- **Prior Linearization filter**. Linearizes $\boldsymbol{y} = \boldsymbol{c}(\boldsymbol{x})$ by minimizing $\mathbb{E}_{\text{prior}}\left(\|\boldsymbol{c}(\boldsymbol{x}) - (\boldsymbol{c}_0 + \boldsymbol{C}\boldsymbol{x})\|^2\right)$. [13]
- **Posterior linearization filter** minimizes $\mathbb{E}_{\text{post}}\left(\|\boldsymbol{c}(\boldsymbol{x}) - (\boldsymbol{c}_0 + \boldsymbol{C}\boldsymbol{x})\|^2\right)$ by iterating. [13]
- **Laplace approximation**: approximates by posterior mode and inverse negative Hessian. [14]

Gaussian approximations to bivariate log-normal

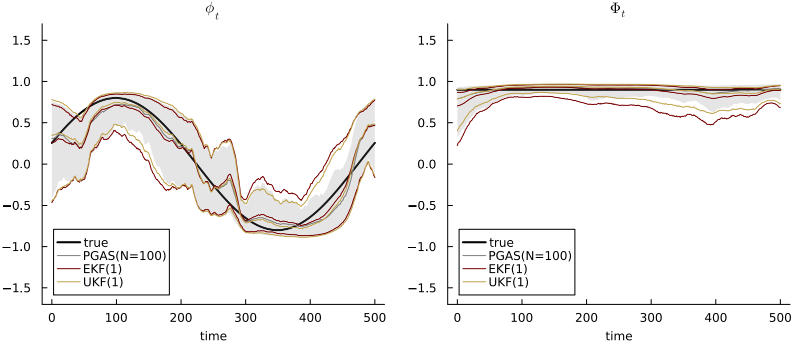


Seasonal AR - without stability restrictions



- Compute time 10000 draws:
  - PGAS(100): 211.7 sec
  - FFBS-EKF: 11.9 sec
  - FFBS-UKF: 14.7 sec

Seasonal AR - with stability restrictions



- This can be improved by alternative stability parameterizations.

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