

Time-varying parameter models as state-space models

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Workshop overview

- Time-varying parameter models and state-space models
- Inference in linear Gaussian models
- Inference in non-linear and non-Gaussian models
- Global-local shrinkage processes for more realistic parameter evolution
- Light **demos**, using packages in R.

Local level model

- **Local level model** has a time-varying mean:

Measurement model: $y_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} N(0, \sigma_\varepsilon^2)$

Transition model: $\mu_t = \mu_{t-1} + \eta_t, \quad \eta_t \stackrel{\text{iid}}{\sim} N(0, \sigma_\eta^2)$

- Static parameters:

- ▶ **Measurement variance** σ_ε^2

- ▶ Parameter **innovation variance** σ_η^2

- Speed of parameter change is determined by σ_η^2

Interactive: local level model

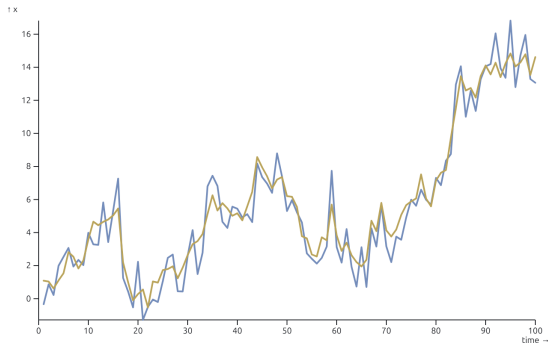
time series length

std error, σ_ε

std innovation, σ_η

plot: ☒ time series, x_t ☒ local level, μ_t

time series, x_t local level, μ_t



Interactive: local trend model

time series length

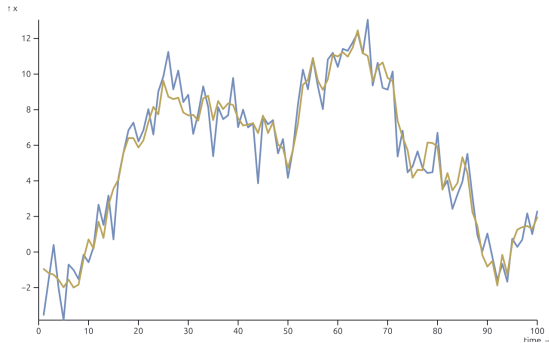
std error, σ_ε

std innovation, σ_η

std innovation, σ_ω

plot: ☒ time series, x_t ☒ local level, μ_t ☐ local trend, v_t

time series, x_t local level, μ_t local trend, v_t



Time-varying regression


■ Regression with time-varying parameters


Measurement model: $y_t = \mathbf{x}_t^\top \boldsymbol{\beta}_t + \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} N(0, \sigma_\varepsilon^2)$


Transition model: $\boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$


- Often $\boldsymbol{\Sigma}_\eta = \text{Diag}(\sigma_{\eta_1}^2, \dots, \sigma_{\eta_s}^2)$.
- Local level model is a special case with intercept-only model.
- Time-varying autoregressive (AR) models. Covariates $\mathbf{x}_t = (y_{t-1}, \dots, y_{t-p})^\top$.
- Time-varying vector autoregressive (VAR) models. Data: \mathbf{y}_t .


Interactive: time-varying regression

time series length 

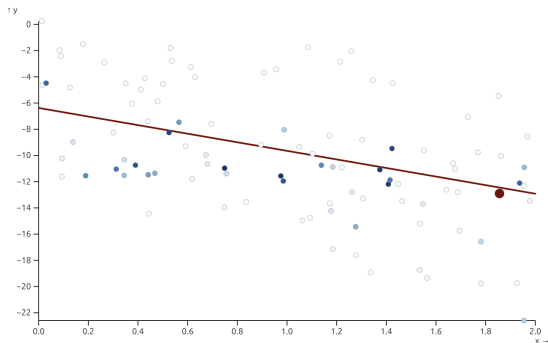
σ_ε 

σ_α 

σ_β 

regr line at time 

Darker points are closer to chosen time period $t = 51$. Red dot is data at $t = 51$.



Stochastic volatility

- **Stochastic volatility** model with a time-varying variance

$$y_t = \mu_y + \varepsilon_t, \quad \varepsilon_t \sim N(0, \exp(z_t))$$

$$z_t = \mu_z + \phi(z_{t-1} - \mu_z) + \eta_t, \quad \eta_t \stackrel{\text{iid}}{\sim} N(0, \sigma_\eta^2)$$

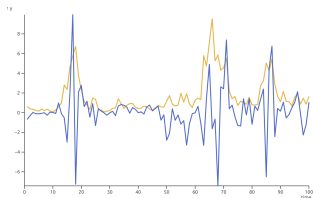
- This is a **non-linear model**.

time series length 100
process mean μ_y 0
mean log variance μ_z 0
AR(1) parameter ϕ 0.8
std innovation, σ_η 1.01

plot standard deviation $\exp(z_t/2)$ ☒

Simulate new data

■ time series, y_t ■ log standard deviation, z_t



Linear Gaussian state-space models

■ Linear Gaussian State-Space (LGSS) model

Measurement model: $\mathbf{y}_t = \mathbf{C}\mathbf{z}_t + \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_\varepsilon)$

Transition model: $\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \eta_t, \quad \eta_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_\eta)$

■ **Measurement** vector \mathbf{y}_t and **State** vector \mathbf{z}_t

■ Local level model:

$$y_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} N(0, \sigma_\varepsilon^2)$$

$$\mu_t = \mu_{t-1} + \eta_t, \quad \eta_t \stackrel{\text{iid}}{\sim} N(0, \sigma_\eta^2)$$

▶ $\mathbf{z}_t = \mu_t$

▶ $\mathbf{A} = \mathbf{C} = 1$

▶ $\Sigma_\varepsilon = \sigma_\varepsilon^2$ and $\Sigma_\eta = \sigma_\eta^2$

Linear Gaussian state-space models

■ Linear Gaussian State-Space (LGSS) model

Measurement model: $\mathbf{y}_t = \mathbf{C}\mathbf{z}_t + \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_\varepsilon)$

Transition model: $\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_\eta)$

■ Time-varying regression:

$$y_t = \mathbf{x}_t^\top \boldsymbol{\beta}_t + \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} N(0, \sigma_\varepsilon^2)$$

$$\boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_\eta)$$

▶ $\mathbf{z}_t = \boldsymbol{\beta}_t$

▶ $\mathbf{A} = \mathbf{I}_s$

▶ $\mathbf{C}_t = \mathbf{x}_t$ (depends on t)

State space models are very general

- State-space models are much more general than time-varying parameter models.
- The state can be some interpretable latent process:
 - ▶ potential output and other *stars* in economics (\mathbf{y}_t = macro data)
 - ▶ coordinates and velocity of a robot (\mathbf{y}_t = sensor data)
 - ▶ continuous intensity for modeling count data
 - ▶ latent surface in 2D/3D for modeling spatio-temporal data
 - ▶ etc etc
- Great for handling **missing data** or **mixed frequency** data.

LGSS model - assumptions

■ LGSS model:

Measurement model: $\mathbf{y}_t = \mathbf{C}\mathbf{z}_t + \varepsilon_t$, $\varepsilon_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_\varepsilon)$

Transition model: $\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \eta_t$, $\eta_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_\eta)$

■ **State is Markov** (only depends on most recent past \mathbf{z}_{t-1})

$$p(\mathbf{z}_0, \mathbf{z}_1, \dots, \mathbf{z}_T) = p(\mathbf{z}_0) \prod_{t=1}^T p(\mathbf{z}_t | \mathbf{z}_{t-1})$$

■ **Measurements are conditionally independent** given state

$$p(\mathbf{y}_1, \dots, \mathbf{y}_T | \mathbf{z}_1, \dots, \mathbf{z}_T) = \prod_{t=1}^T p(\mathbf{y}_t | \mathbf{z}_t)$$

■ Both measurement and transition models are **linear**

■ Both measurement and transition models are **Gaussian**

The linear Gaussian state-space model

- Models can include exogenous variables/controls

$$\begin{aligned} \mathbf{y}_t &= \mathbf{C}\mathbf{z}_t + \mathbf{D}\mathbf{x}_t + \varepsilon_t, & \varepsilon_t &\stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_\varepsilon) \\ \mathbf{z}_t &= \mathbf{A}\mathbf{z}_{t-1} + \mathbf{B}\mathbf{x}_t + \eta_t, & \eta_t &\stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_\eta) \end{aligned}$$

- Parameters can be time-varying (deterministically)

$$\begin{aligned} \mathbf{y}_t &= \mathbf{C}_t\mathbf{z}_t + \varepsilon_t, & \varepsilon_t &\stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_{\varepsilon,t}) \\ \mathbf{z}_t &= \mathbf{A}_t\mathbf{z}_{t-1} + \eta_t, & \eta_t &\stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_{\eta,t}) \end{aligned}$$

- Needed for example in time-varying regression: $\mathbf{C}_t = \mathbf{x}_t$.
- The ε_t and η_t are typically independent, but can be relaxed.

Taxonomy of state-space models

■ Linear Gaussian models

Measurement model: $\mathbf{y}_t = \mathbf{C}\mathbf{z}_t + \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_\varepsilon)$

Transition model: $\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \eta_t, \quad \eta_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_\eta)$

■ Non-linear (additive) Gaussian models

Measurement model: $\mathbf{y}_t = \mathbf{C}(\mathbf{z}_t) + \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_\varepsilon)$

Transition model: $\mathbf{z}_t = \mathbf{A}(\mathbf{z}_{t-1}) + \eta_t, \quad \eta_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_\eta)$

■ General distribution (non-linear and non-Gaussian) models

Measurement model: $\mathbf{y}_t \sim p(\mathbf{y}_t | \mathbf{z}_t)$

Transition model: $\mathbf{z}_t \sim p(\mathbf{z}_t | \mathbf{z}_{t-1})$

■ Hybrids: nonlinear measurement + linear Gaussian transition.