Time-varying parameter models as state-space models

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Workshop overview

- **Time-varying parameter models and state-space models**
- **Inference in linear Gaussian models**
- **Inference in non-linear and non-Gaussian models**
- **Global-local shrinkage processes for more realistic parameter evolution**

- Light **demos**, using packages in R.

Local level model

- **Local level model** has a time-varying mean:

$$\text{Measurement model:} \quad y_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \overset{\text{iid}}{\sim} N(0, \sigma_\varepsilon^2)$$

$$\text{Transition model:} \quad \mu_t = \mu_{t-1} + \eta_t, \quad \eta_t \overset{\text{iid}}{\sim} N(0, \sigma_\eta^2)$$

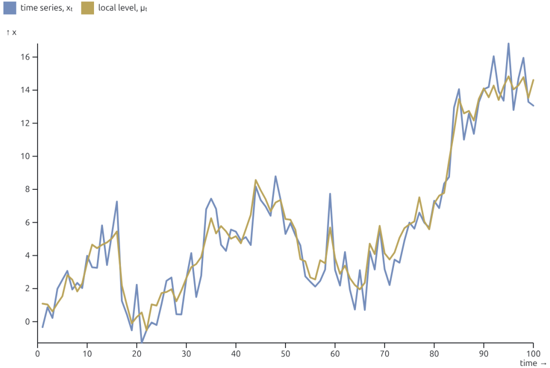
- Static parameters:
  - **Measurement variance** σ_ε^2
  - Parameter **innovation variance** σ_η^2
- Speed of parameter change is determined by σ_η^2

Interactive: local level model

time series length	100	
std error, σ_ε	1	
std innovation, σ_η	1	

plot: ☑ time series, x_t ☑ local level, μ_t

Simulate new data

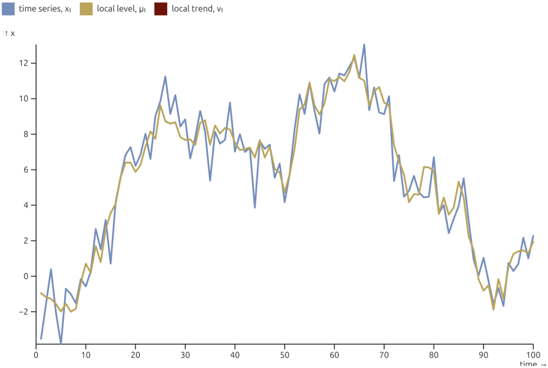


Interactive: local trend model

time series length	100	
std error, σ_ε	1	
std innovation, σ_η	1	
std innovation, σ_ω	0.01000...	

plot: ☑ time series, x_t ☑ local level, μ_t ☐ local trend, v_t

Simulate new data



Time-varying regression

- **Regression** with **time-varying parameters**

$$\text{Measurement model:} \quad y_t = \boldsymbol{x}_t^\top \boldsymbol{\beta}_t + \varepsilon_t, \qquad \varepsilon_t \stackrel{\text{iid}}{\sim} N(0, \sigma_\varepsilon^2)$$

$$\text{Transition model:} \quad \boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$

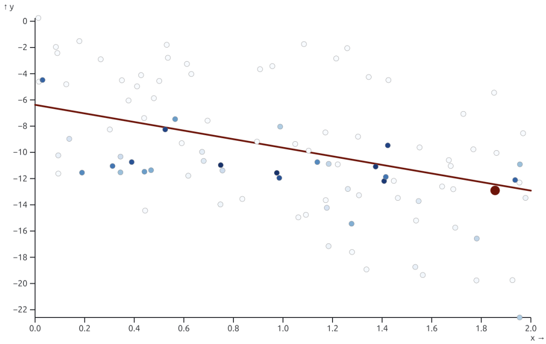
- Often $\boldsymbol{\Sigma}_\eta = \mathrm{Diag}\big(\sigma_{\eta_1}^2, \ldots, \sigma_{\eta_s}^2\big)$.
- Local level model is a special case with a intercept-only model.
- Time-varying autoregressive (AR) models. Covariates $\boldsymbol{x}_t = (y_{t-1}, \ldots, y_{t-p})^\top$.
- Time-varying vector autoregressive (VAR) models. Data: $\boldsymbol{y}_t$.

Interactive: time-varying regression

time series length	100	
σ_ε	1	
σ_α	1	
σ_β	1	
regr line at time	51	

Simulate new data

Darker points are closer to chosen time period $t = 51$. Red dot is data at $t = 51$.



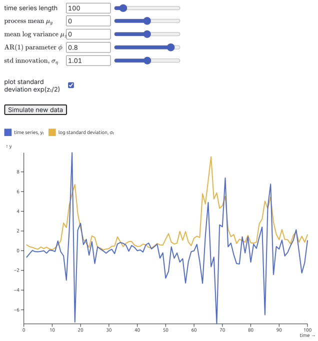
Stochastic volatility

- **Stochastic volatility** model with a time-varying variance

$$y_t = \mu_y + \varepsilon_t, \qquad \varepsilon_t \sim N(0, \exp(z_t))$$

$$z_t = \mu_z + \phi(z_{t-1} - \mu_z) + \eta_t, \qquad \eta_t \overset{\text{iid}}{\sim} N(0, \sigma_\eta^2)$$

- This is a **non-linear model**.



Linear Gaussian state-space models

- **Linear Gaussian State-Space** (LGSS) model

$$\text{Measurement model:} \quad \boldsymbol{y}_t = \boldsymbol{C}\boldsymbol{z}_t + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$$

$$\text{Transition model:} \quad \boldsymbol{z}_t = \boldsymbol{A}\boldsymbol{z}_{t-1} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$

- **Measurement** vector $\boldsymbol{y}_t$ and **State** vector $\boldsymbol{z}_t$
- Local level model:

$$y_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \overset{\text{iid}}{\sim} N(0, \sigma_\varepsilon^2)$$

$$\mu_t = \mu_{t-1} + \eta_t, \quad \eta_t \overset{\text{iid}}{\sim} N(0, \sigma_\eta^2)$$

  - $\boldsymbol{z}_t = \mu_t$
  - $\boldsymbol{A} = \boldsymbol{C} = 1$
  - $\boldsymbol{\Sigma}_\varepsilon = \sigma_\varepsilon^2$ and $\boldsymbol{\Sigma}_\eta = \sigma_\eta^2$

Linear Gaussian state-space models

- **Linear Gaussian State-Space** (LGSS) model

$$\text{Measurement model:} \quad \boldsymbol{y}_t = \boldsymbol{C}\boldsymbol{z}_t + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$$

$$\text{Transition model:} \quad \boldsymbol{z}_t = \boldsymbol{A}\boldsymbol{z}_{t-1} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$

- Time-varying regression:

$$y_t = \boldsymbol{x}_t^\top \boldsymbol{\beta}_t + \varepsilon_t, \qquad \varepsilon_t \overset{\text{iid}}{\sim} N(0, \sigma_\varepsilon^2)$$

$$\boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$

  - $\boldsymbol{z}_t = \boldsymbol{\beta}_t$
  - $\boldsymbol{A} = \boldsymbol{I}_s$
  - $\boldsymbol{C}_t = \boldsymbol{x}_t$ (depends on t)

State space models are very general

- State-space models are much more general than time-varying parameter models.

- The state can be some interpretable latent process:
  - potential output and other *stars* in economics ($\boldsymbol{y}_t =$ macro data)
  - coordinates and velocity of a robot ($\boldsymbol{y}_t =$ sensor data)
  - continuous intensity for modeling count data
  - latent surface in 2D/3D for modeling spatio-temporal data
  - etc etc

- Great for handling **missing data** or **mixed frequency** data.

LGSS model - assumptions

- LGSS model:

$$\text{Measurement model:} \quad \boldsymbol{y}_t = \boldsymbol{C}\boldsymbol{z}_t + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$$

$$\text{Transition model:} \quad \boldsymbol{z}_t = \boldsymbol{A}\boldsymbol{z}_{t-1} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$

- **State is Markov** (only depends on most recent past $\boldsymbol{z}_{t-1}$)

$$p(\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_T) = p(\boldsymbol{z}_0) \prod_{t=1}^{T} p(\boldsymbol{z}_t | \boldsymbol{z}_{t-1})$$

- **Measurements are conditionally independent** given state

$$p(\boldsymbol{y}_1, \ldots, \boldsymbol{y}_T | \boldsymbol{z}_1, \ldots, \boldsymbol{z}_T) = \prod_{t=1}^{T} p(\boldsymbol{y}_t | \boldsymbol{z}_t)$$

- Both measurement and transition models are **linear**
- Both measurement and transition models are **Gaussian**

The linear Gaussian state-space model

- Models can include exogenous variables/controls

$$\boldsymbol{y}_t = \boldsymbol{C}\boldsymbol{z}_t + \boldsymbol{D}\boldsymbol{x}_t + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$$

$$\boldsymbol{z}_t = \boldsymbol{A}\boldsymbol{z}_{t-1} + \boldsymbol{B}\boldsymbol{x}_t + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$

- Parameters can be time-varying (deterministically)

$$\boldsymbol{y}_t = \boldsymbol{C}_t\boldsymbol{z}_t + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_{\varepsilon,t})$$

$$\boldsymbol{z}_t = \boldsymbol{A}_t\boldsymbol{z}_{t-1} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_{\eta,t})$$

- Needed for example in time-varying regression: $\boldsymbol{C}_t = \boldsymbol{x}_t$.
- The $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\eta}_t$ are typically independent, but can be relaxed.

Taxonomy of state-space models

- **Linear Gaussian** models

$$\text{Measurement model:} \quad \boldsymbol{y}_t = \boldsymbol{C}\boldsymbol{z}_t + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$$

$$\text{Transition model:} \quad \boldsymbol{z}_t = \boldsymbol{A}\boldsymbol{z}_{t-1} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$

- **Non-linear** (additive) **Gaussian** models

$$\text{Measurement model:} \quad \boldsymbol{y}_t = \boldsymbol{c}(\boldsymbol{z}_t) + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$$

$$\text{Transition model:} \quad \boldsymbol{z}_t = \boldsymbol{a}(\boldsymbol{z}_{t-1}) + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \overset{\text{iid}}{\sim} N(\mathbf{0}, \boldsymbol{\Sigma}_\eta)$$

- **General distribution** (non-linear and non-Gaussian) models

$$\text{Measurement model:} \quad \boldsymbol{y}_t \sim p(\boldsymbol{y}_t | \boldsymbol{z}_t)$$

$$\text{Transition model:} \quad \boldsymbol{z}_t \sim q(\boldsymbol{z}_t | \boldsymbol{z}_{t-1})$$

- Hybrids: nonlinear measurement + linear Gaussian transition.