

Inference in non-linear and non-Gaussian models

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Taxonomy of state-space models

■ Linear Gaussian models

Measurement model: $\mathbf{y}_t = \mathbf{C}\mathbf{z}_t + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_{\varepsilon})$

Transition model: $\mathbf{z}_t = \mathbf{A}\mathbf{z}_{t-1} + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_{\eta})$

■ Non-linear (additive) Gaussian models

Measurement model: $\mathbf{y}_t = \mathbf{C}(\mathbf{z}_t) + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_{\varepsilon})$

Transition model: $\mathbf{z}_t = \mathbf{A}(\mathbf{z}_{t-1}) + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \stackrel{\text{iid}}{\sim} N(\mathbf{0}, \Sigma_{\eta})$

■ General distribution (non-linear and non-Gaussian) models

Measurement model: $\mathbf{y}_t \sim p(\mathbf{y}_t | \mathbf{z}_t)$

Transition model: $\mathbf{z}_t \sim p(\mathbf{z}_t | \mathbf{z}_{t-1})$

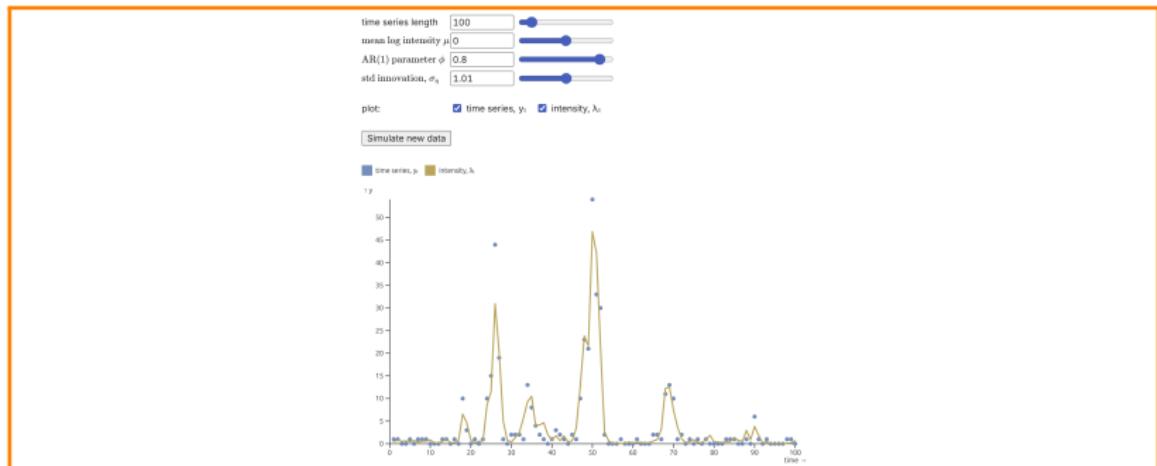
■ Hybrids: nonlinear measurement + linear Gaussian transition.

Poisson time series for counts

■ Poisson model with time-varying intensity

$$y_t | z_t \sim \text{Poisson}(\exp(z_t))$$

$$z_t = \mu + \phi(z_{t-1} - \mu) + \eta_t, \quad \eta_t \stackrel{\text{iid}}{\sim} N(0, \sigma_\eta^2)$$



Common Bayesian computational approaches

- Kalman filtering and **FFBS sampling** for linear Gaussian models. [1, 2]
- **Particle MCMC** and **SMC** [3]
- **Hamiltonian Monte Carlo** (Stan/Turing.jl) [4]
- **Variational approximations** [5]
- **INLA** [6]
- **Gaussian approximations** [7]

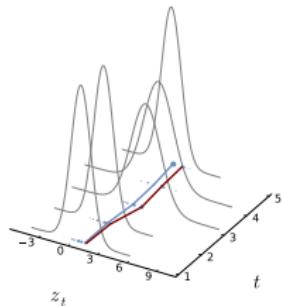
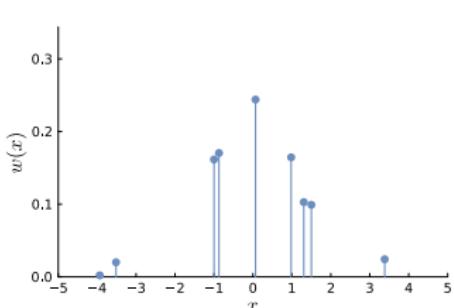
Particle filters

- Approximate filtering posteriors by **weighted particle system**

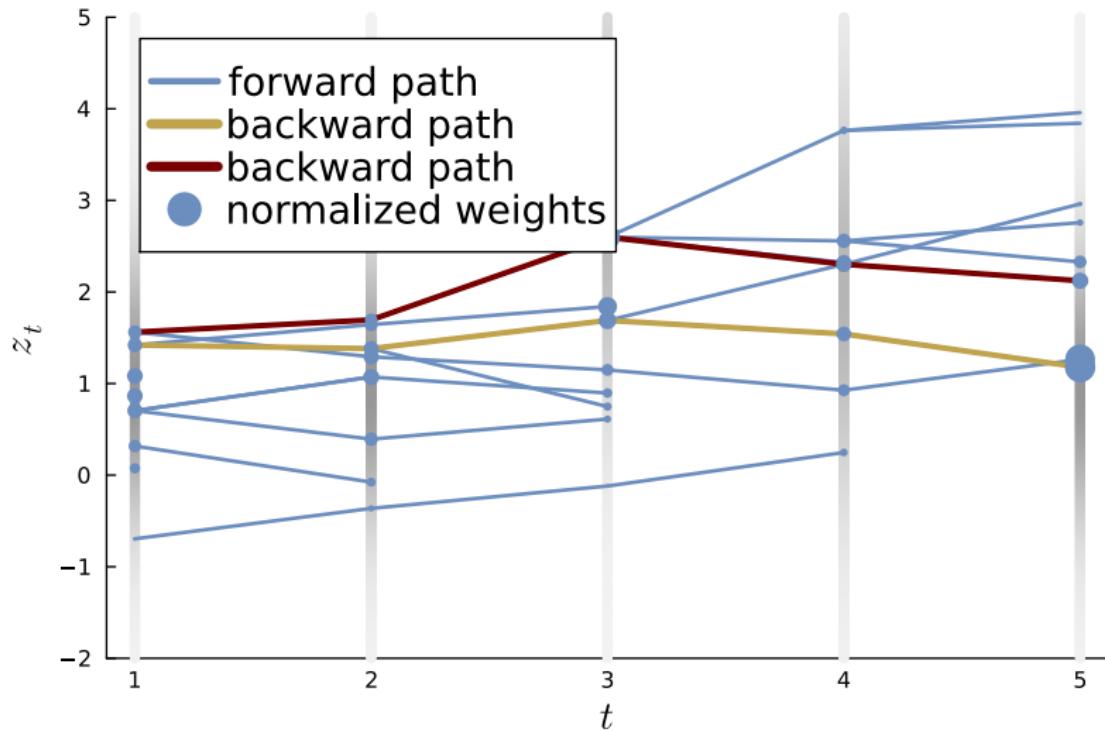
$$\hat{p}(\mathbf{z}_t | \mathbf{y}_{1:t}) = \sum_{j=1}^M \omega_t^{(j)} \delta_{\mathbf{z}_t^{(j)}}(\mathbf{z}_t)$$

- **Sequential importance sampling** moves particles over time.
- **Bootstrap filter**: importance density is the prior at time t :

$$q(\mathbf{z}_t | \mathbf{y}_{1:t}) = p(\mathbf{z}_t | \mathbf{y}_{1:t-1})$$



Particle filters



The Bootstrap filter

Bootstrap filter update for general state-space model

$$p(\mathbf{z}_{t-1} | \mathbf{y}_{1:t-1}) \Rightarrow p(\mathbf{z}_t | \mathbf{y}_{1:t-1}) \Rightarrow p(\mathbf{z}_t | \mathbf{y}_{1:t})$$

Input: particles from previous step $\mathbf{z}_{t-1}^{(1)}, \dots, \mathbf{z}_{t-1}^{(M)}$
measurement \mathbf{y}_t
control signal \mathbf{u}_t

for j in $1:M$ **do**

Prior propagation

draw state particle $\mathbf{z}_t^{(j)} \sim g(\mathbf{z}_t | \mathbf{z}_{t-1}^{(j)})$

Measurement update

compute unnormalized weight $\tilde{w}_t^{(j)} = p(\mathbf{y}_t | \mathbf{z}_t^{(j)}, \theta)$

end

normalize weights

$w_t^{(j)} = \tilde{w}_t^{(j)} / \sum_{k=1}^M \tilde{w}_t^{(k)}$ for $j = 1, \dots, M$

Resampling

for j in $1:M$ **do**

$k_j \sim \text{Cat}(w_t^{(1)}, \dots, w_t^{(M)})$

$\mathbf{z}_t^{(j)} \leftarrow \mathbf{z}_t^{(k_j)}$

end

Output: M draws $\mathbf{z}_t^{(1)}, \dots, \mathbf{z}_t^{(M)}$ from $p(\mathbf{z}_t | \mathbf{y}_{1:t})$.

Particle Gibbs with Ancestor Sampling (PGAS)

- Particle Gibbs samples from the joint smoothing posterior $p(\mathbf{z}_{1:T} | \mathbf{y}_{1:T}, \theta)$ via a particle filter inside a Gibbs sampler. [8]
- **Conditional particle filter** that conditions on a **reference particle trajectory** from the previous Gibbs iteration.
- Markov kernel that leaves the target posterior invariant.
- Extended target with state + SMC randomness.
- Partially collapsed (SMC randomness integrated out).
- **Particle Gibbs with Ancestor Sampling (PGAS)** [9]
 - ▶ Resamples the reference particle ancestor at every time step.
 - ▶ Mitigates degeneracy and gives better mixing.
 - ▶ Smaller number of particles needed, but more costly.

Extended Kalman filter (EKF)

■ State-space model

$$\theta_t = \mathbf{a}(\theta_{t-1}) + \eta_t, \quad \eta_t \sim N(\mathbf{0}, \Sigma_\eta)$$

$$y_t = \mathbf{c}(\theta_t) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \Sigma_\varepsilon)$$

■ Linearized model

$$\theta_t = \mathbf{a}'(\theta_{t-1}) + \eta_t, \quad \eta_t \sim N(\mathbf{0}, \Sigma_\eta)$$

$$y_t = \mathbf{c}'(\theta_t) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \Sigma_\varepsilon)$$

Standard Kalman filter

```
# Prior propagation step
```

$$\bar{\mu} = A * \mu .+ B * u$$

$$\bar{\Omega} = A * \Omega * A' + \Sigma_n$$

Extended Kalman filter

```
# Prior propagation step
```

$$\bar{\mu} = A * \mu + B * u$$

$$\bar{\Omega} = A * \Omega * A' + \Sigma_n$$

$$\bar{C} = \partial C(\bar{\mu}, Cargs)$$

```
# Measurement update
```

$$K = \bar{\Omega} * C' / (C * \bar{\Omega} * C' .+ \Sigma_e)$$

$$\mu = \bar{\mu} + K * (y .- C * \bar{\mu})$$

$$\Omega = (I - K * C) * \bar{\Omega}$$

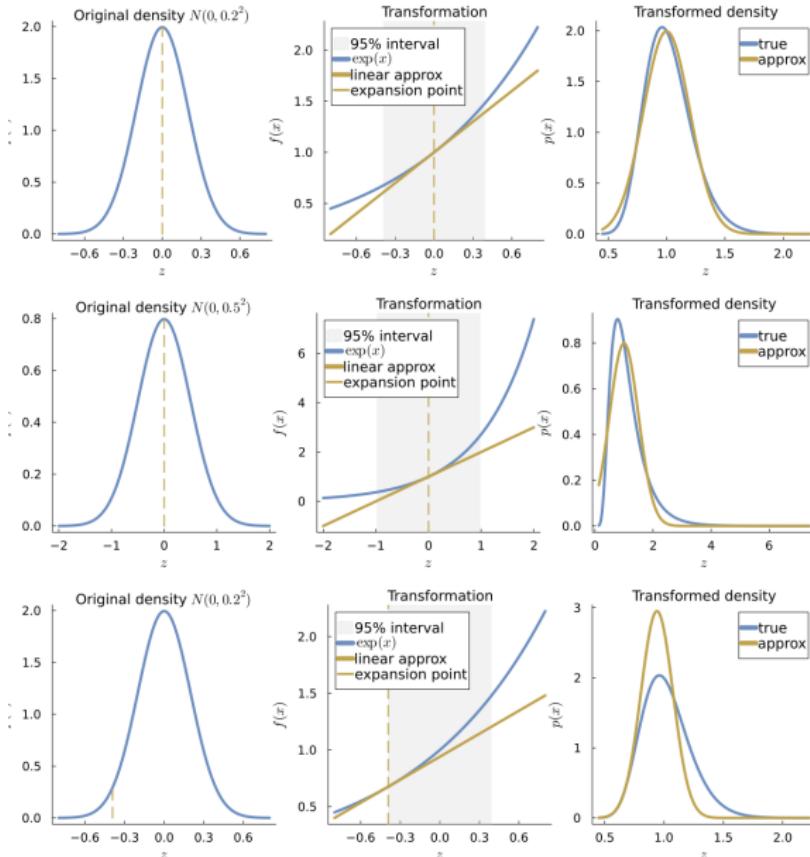
```
# Measurement update
```

$$K = \bar{\Omega} * \bar{C}' / (\bar{C} * \bar{\Omega} * \bar{C}' .+ \Sigma_e)$$

$$\mu = \bar{\mu} + K * (y .- C(\bar{\mu}, Cargs))$$

$$\Omega = (I - K * \bar{C}) * \bar{\Omega}$$

Extended Kalman filter



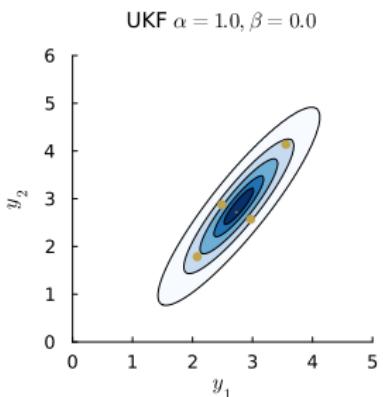
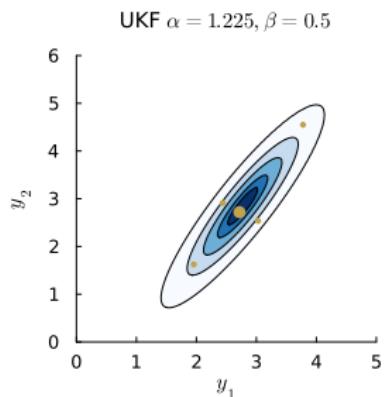
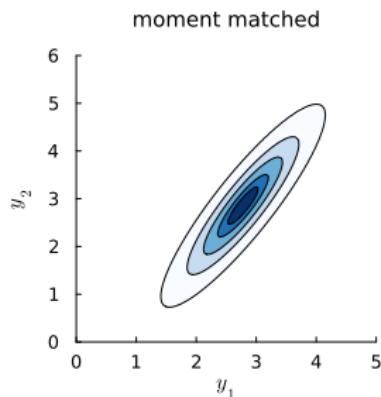
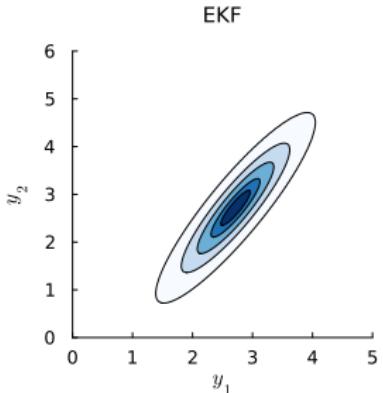
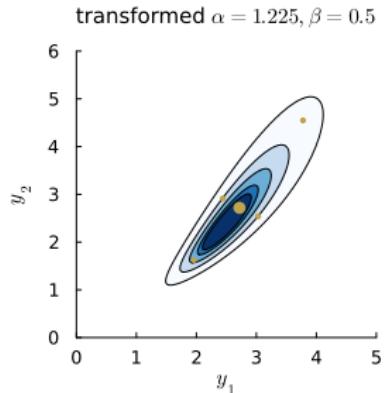
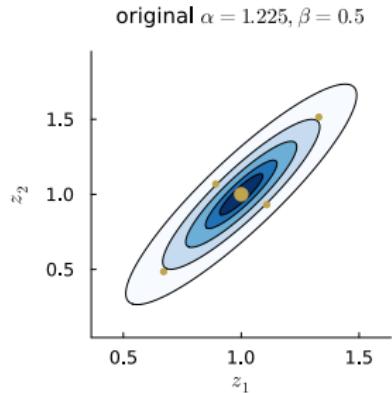
FFBS-EKF

- EKF: **automatic differentiation** makes it all easy.
- Forward filtering with EKF gives $\mu_{t|t}, \Omega_{t|t}, \mu_{t|t-1}, \Omega_{t|t-1}$.
- **Sampling from joint smoothing density** by backward sampling using same multivariate normal distributions as in FFBS.
- Much faster than PGAS.
- **Robust to near-degeneracy** in the transition model.
Important when global-local shrinkage process priors is used for the parameter evolution. See Lecture 4.

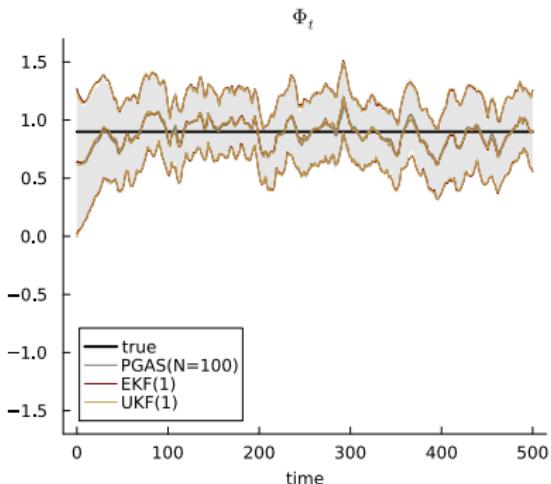
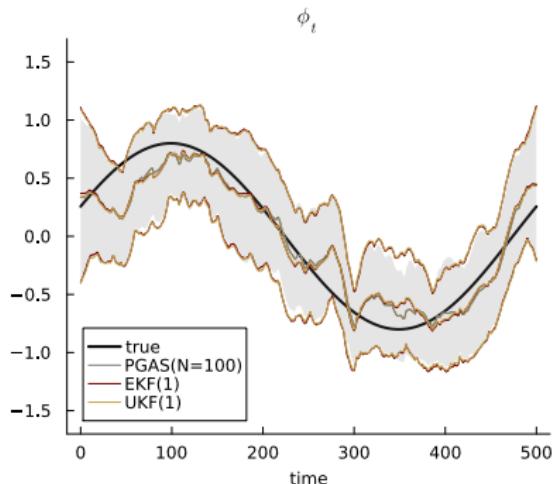
Alternative approximate Gaussian filtering

- **Moment matching** by numerical integration. [7]
- **Unscented Kalman Filter (UKF)**: map sigma points through nonlinear transition and measurement functions. [10]
- **Iterated EKF** and **Iterated UKF**: expands around posterior mean instead of prior mean. [11, 12]
- Iterated EKF/UKF with **line search**. [11, 12]
- **Prior Linearization filter**. Linearizes $\mathbf{y} = \mathbf{c}(\mathbf{x})$ by minimizing $\mathbb{E}_{\text{prior}} \left(\|\mathbf{c}(\mathbf{x}) - (\mathbf{c}_0 + \mathbf{C}\mathbf{x})\|^2 \right)$. [13]
- **Posterior linearization filter** minimizes $\mathbb{E}_{\text{post}} \left(\|\mathbf{c}(\mathbf{x}) - (\mathbf{c}_0 + \mathbf{C}\mathbf{x})\|^2 \right)$ by iterating. [13]
- **Laplace approximation**: approximates by posterior mode and inverse negative Hessian. [14]

Gaussian approximations to bivariate log-normal

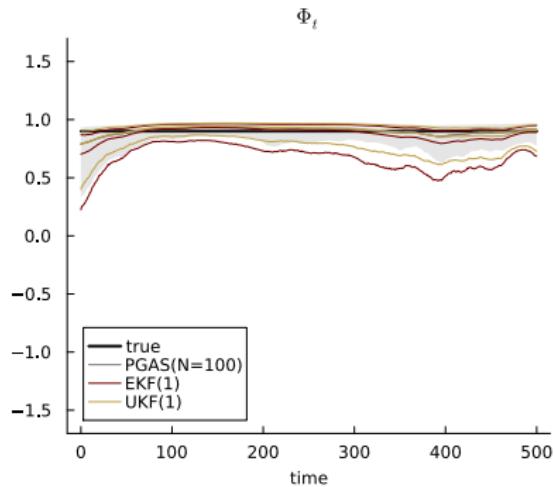
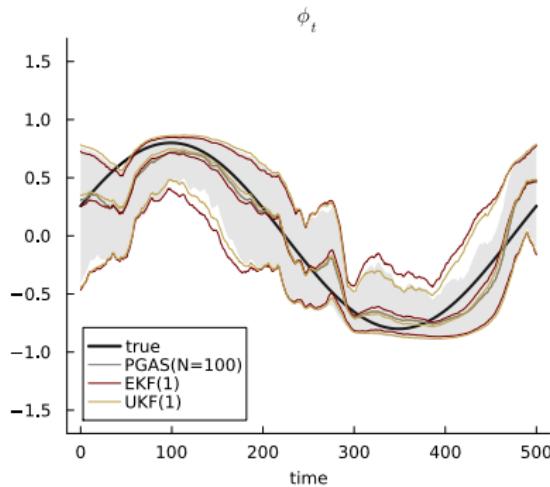


Seasonal AR - without stability restrictions



- Compute time 10000 draws:
 - ▶ PGAS(100): 211.71 sec
 - ▶ FFBS-EKF: 11.92 sec
 - ▶ FFBS-UKF: 14.72 sec

Seasonal AR - with stability restrictions



- This can be improved by alternative stability parameterizations.

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