

Global-local shrinkage processes for more realistic parameter evolution

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Global-local shrinkage priors in linear regression

- **Ridge regression** - one global shrinkage parameter λ

$$\beta_j | \sigma_\varepsilon^2 \stackrel{\text{iid}}{\sim} N\left(0, \frac{\sigma_\varepsilon^2}{\lambda}\right)$$

- **Lasso regression**

$$\beta_j | \sigma_\varepsilon^2 \stackrel{\text{iid}}{\sim} \text{Laplace}\left(0, \frac{\sigma_\varepsilon^2}{\lambda}\right)$$

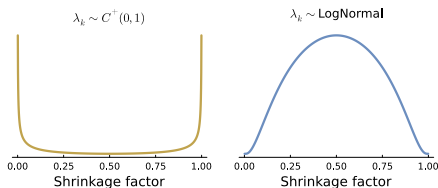
- **Global-Local shrinkage**: global + local shrinkage for each β_j .

- **Horseshoe prior**:

$$\beta_j | \lambda_j^2, \tau^2, \sigma_\varepsilon^2 \sim N\left(0, \tau^2 \lambda_j^2 \sigma_\varepsilon^2\right)$$

$$\lambda_j \stackrel{\text{iid}}{\sim} C^+(0, 1) \quad \text{and} \quad \tau \sim C^+(0, 1)$$

- **Shrinkage factor** c_j wrt OLS: $\tilde{\beta}_j = (1 - c_j)\hat{\beta}_j$



Dynamic horseshoe in state-space models

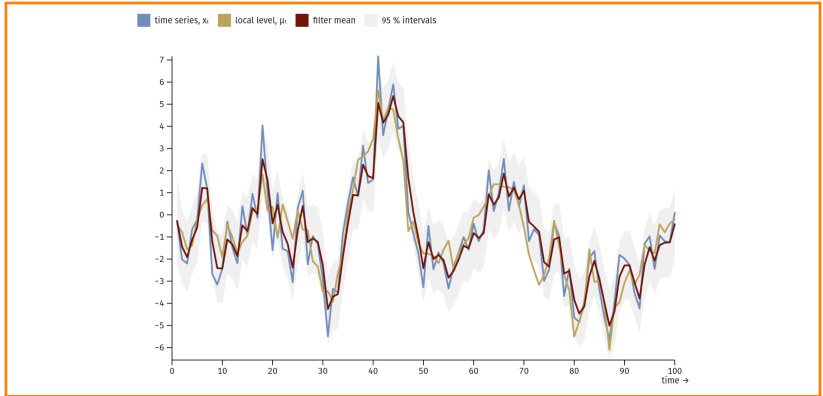
- **Local level model** (state-space) for time series

$$x_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$$

$$\mu_t = \mu_{t-1} + \nu_t, \quad \nu_t \sim N(0, \sigma_\nu^2)$$

- **Innovation variance** $\sigma_\nu^2 \Rightarrow$ how fast the mean evolves.
- Same normal $N(0, \sigma_\nu^2)$ for all ν_t . Compare Ridge regression.
- Restrictive parameter evolution. Can't get all of this:
 - 1 $\nu_t \approx 0$ for a stretch of t (parameters stand still)
 - 2 large ν_t for some t (jumps)
 - 3 persistent periods of rapid changes

Local level model with Gaussian innovations



Dynamic horseshoe process prior

- **Horseshoe prior** for time series

$$\mu_t = \mu_{t-1} + \nu_t, \quad \nu_t \sim N(0, \tau^2 \lambda_t^2)$$

$$\lambda_t \stackrel{\text{iid}}{\sim} C^+(0, 1)$$

$$\tau \sim C^+(0, 1)$$

- This gives us Property 1 and 2 above.
- Local variances λ_t^2 are independent. No Property 3.
- **Dynamic horseshoe process** [1]

$$\mu_t = \mu_{t-1} + \nu_t, \quad \nu_t \sim N(0, \tau^2 \exp(h_t))$$

$$h_t = \phi h_{t-1} + \eta_t, \quad \eta_t \sim Z(1/2, 1/2, 0, 1)$$

$$\tau \sim C^+(0, 1)$$

- The horseshoe prior is the special case with $\phi = 0$

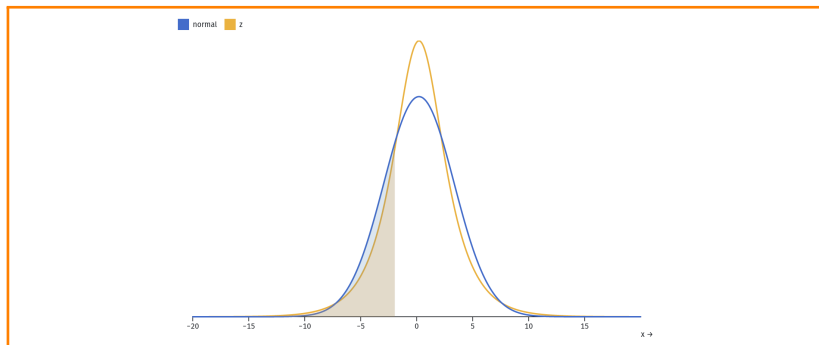
$$\eta_t \sim Z(1/2, 1/2, 0, 1) \iff \lambda_t = \exp(\eta_t/2) \sim C^+(0, 1)$$

Z-distribution

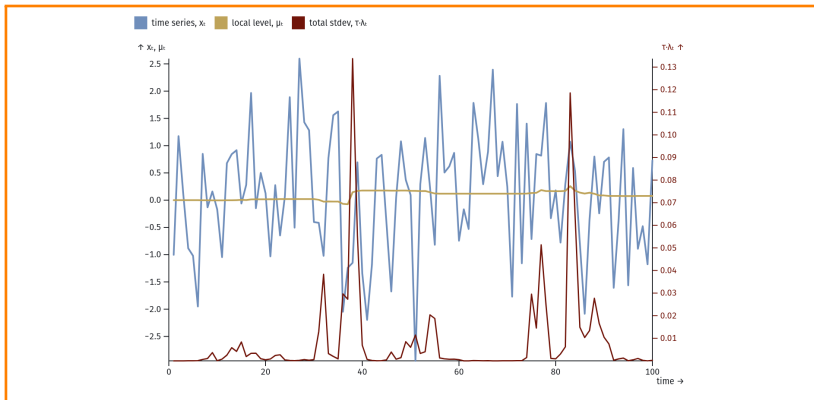
- Also called **Logistic-Beta distribution** since

$$X \sim \text{Beta}(\alpha, \beta) \implies \log \left(\frac{X}{1-X} \right) \sim Z(\alpha, \beta, 0, 1)$$

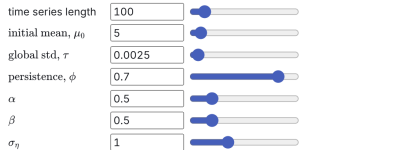
- The $Z(\alpha, \beta, 0, 1)$ distribution is heavy tailed.
- Linearly decaying log density.



Local level with dynamic shrinkage process



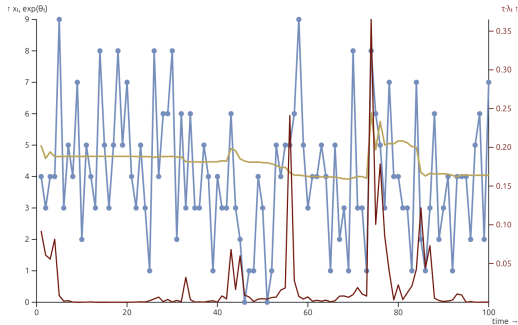
Poisson time series with dynamic shrinkage process



plot: ☒ time series, x_t ☒ data mean, μ_t ☒ total stdev, $\tau \lambda_t$

Simulate new data

☒ time series, x_t ☒ data mean, μ_t ☒ total standard deviation, $\tau \lambda_t$



Bayesian inference for models with DSP priors

- Aim: posterior conditional on time series $\mathbf{y}_{1:T}$

$$p(\mathbf{z}_{0:T}, \mathbf{h}_{1:T}, \boldsymbol{\tau}, \boldsymbol{\phi} | \mathbf{y}_{1:T})$$

- Gibbs sampling by data augmentation:
 - ▶ Mixture of normal for $\log \chi_1^2$ for volatility $\mathbf{h}_{1:T}$ [2]
 - ▶ Polya-Gamma augmentation for Z-distribution. [1]
- Draw from full conditional of $\mathbf{h}_{1:T}$ by tri-diagonal multivariate Gaussian.
- Conditional on $\mathbf{h}_{1:T}$ and Polya-Gamma draws, draw the state $\mathbf{z}_{0:T}$ by sampling from posterior in a state-space model with:
 - ▶ **nonlinear/non-Gaussian measurement** model
 - ▶ **linear** (heteroscedastic) **Gaussian transition** model.

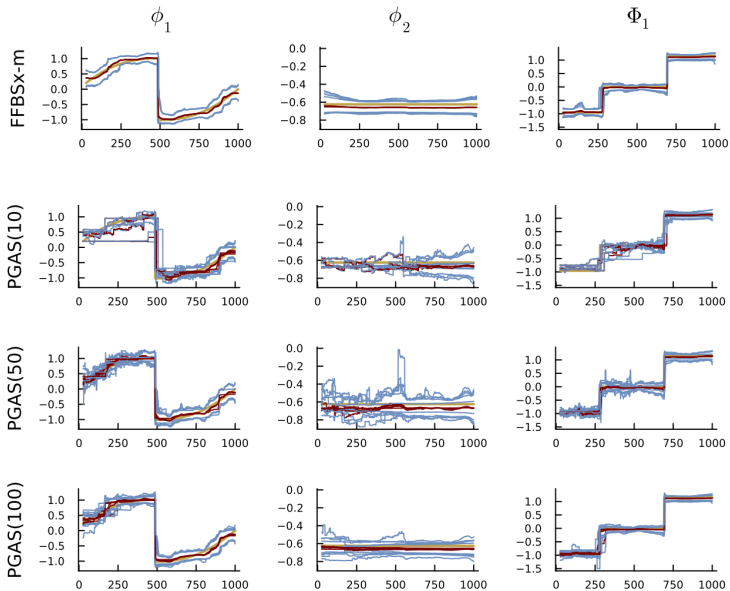
Stable seasonal AR with DSP prior

- **Time-varying Seasonal AR**(p, P) with season s [3]

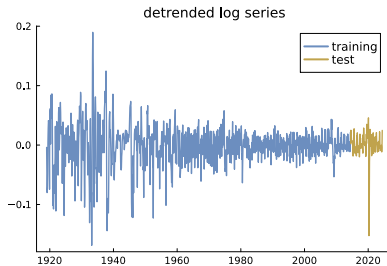
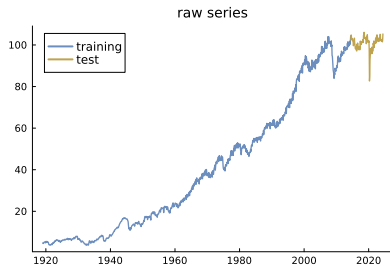
$$\phi_{p,t}(L)\Phi_{P,t}(L^s)y_t = \varepsilon_t, \quad \varepsilon_t \stackrel{\text{iid}}{\sim} N(0, \sigma_t^2)$$

- Parameters $\phi_{1,t}, \dots, \phi_{p,t}$ and $\Phi_{1,t}, \dots, \Phi_{P,t}$ follow DSP priors.
- Can be directly extended to multi-seasonal AR.
- Two non-linearities:
 - ▶ **Multiplicative structure** of regular and seasonal polynomials.
 - ▶ **Stability restrictions** at every t (non-linear recursive).

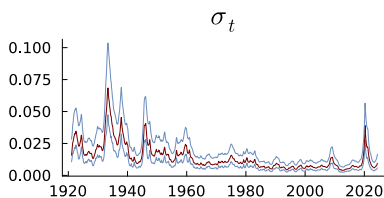
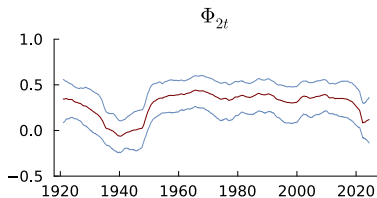
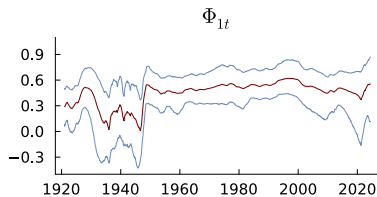
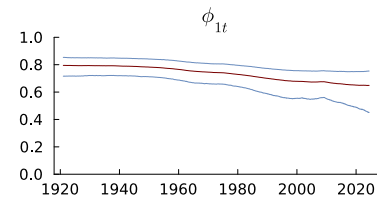
PGAS performs poorly due to near-degeneracy



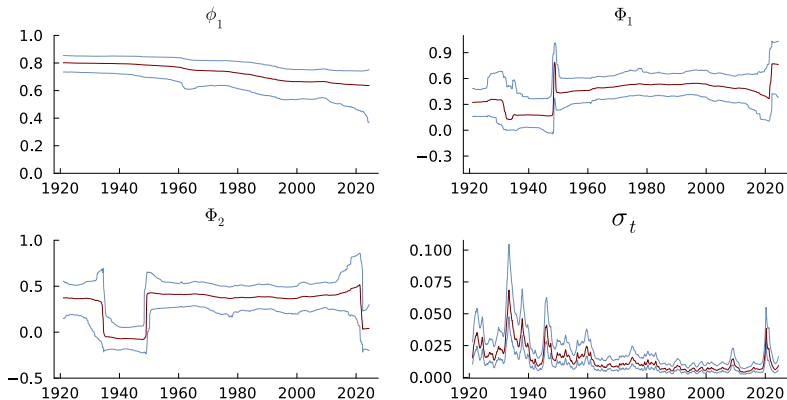
US industrial production 1919-2024



Posterior with homoscedastic Gaussian evolution



Posterior with DSP evolution





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