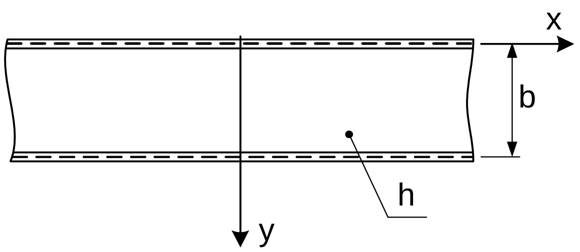
III.4. Kuvan mukaisen laattakaistan pituus on hyvin suuri sen leveyteen b verrattuna. x-suuntaisilla reunoilla $y=0$ ja $y=b$ on niveltuennat. Määritä laatan taipuman w ja



jännityskomponenttien σ_x, σ_y, σ_z, τ_{xy}, τ_{xz} ja τ_{xz} lausekkeet sekä niiden maksimiarvot, kun laatalla on a) kuormitus $p(y)=p_0 \sin\frac{\pi y}{b}$ ja b) tasainen kuormitus p_0. $\nu = 0{,}3$.

Ratkaisu:
Laatta taipuu sylinteripinnaksi, jonka emäviivat ovat x-akselin suuntaisia, jos kaista on äärettömän pitkä. Äärellisen ja pitkän kaistan tapauksessa tämä on voimassa laatan päissä olevia pieniä häiriöalueita lukuun ottamatta. Tällöin on voimassa

$$w=w(y) \quad \Rightarrow \quad w,_x=0 \quad w,_{xx}=0 \quad w,_{xy}=0$$

Laatan perusdifferentiaaliyhtälö menee näin ollen muotoon $w,_{yyyy}=\frac{p}{D}$.

a) $p(y)=p_0\sin\frac{\pi y}{b} \quad \Rightarrow \quad D\,w,_{yyyy}=p_0\sin\frac{\pi y}{b} \quad \Rightarrow \quad D\,w,_{yyy}=-\frac{b}{\pi}p_0\cos\frac{\pi y}{b}+C_1 \quad \Rightarrow$

$$D\,w,_{yy}=-\left(\frac{b}{\pi}\right)^2 p_0\sin\frac{\pi y}{b}+C_1 y+C_2 \quad \Rightarrow \quad D\,w,_y=\left(\frac{b}{\pi}\right)^3 p_0\cos\frac{\pi y}{b}+\frac{C_1}{2}y^2+C_2 y+C_3$$

$$\Rightarrow \quad D\,w=\left(\frac{b}{\pi}\right)^4 p_0\sin\frac{\pi y}{b}+\frac{C_1}{6}y^3+\frac{C_2}{2}y^2+C_3 y+C_4$$

Reunaehdot: $w(x{,}0)=0 \Rightarrow C_4=0 \qquad M_y(x{,}0)=-D\,w,_{yy}(x{,}0)=0 \Rightarrow C_2=0$

$$M_y(x{,}b)=-D\,w,_{yy}(x{,}b)=0 \Rightarrow C_1=0 \qquad w(x{,}b)=0 \Rightarrow C_3=0$$

$$\Rightarrow \quad w(x,y)=w(y)=\left(\frac{b}{\pi}\right)^4\frac{p_0}{D}\sin\frac{\pi y}{b} \quad \Rightarrow \quad w_{max}=\frac{p_0 b^4}{\pi^4 D} \quad \text{kun } y=\frac{b}{2}$$

Laattamomentit:

$$M_x=-D(w,_{xx}+\nu\,w,_{yy}) \quad \Rightarrow \quad M_x=\nu\left(\frac{b}{\pi}\right)^2 p_0\sin\frac{\pi y}{b}$$

$$M_y=-D(w,_{yy}+\nu\,w,_{xx}) \quad \Rightarrow \quad M_y=\left(\frac{b}{\pi}\right)^2 p_0\sin\frac{\pi y}{b}$$

$$M_{xy}=-D(1-\nu)w,_{xy} \quad \Rightarrow \quad M_{xy}=0$$

Laatan leikkausvoimat:

$$Q_x = -D(w_{,xx} + w_{,yy})_{,x} \quad \Rightarrow \quad Q_x = 0$$

$$Q_y = -D(w_{,xx} + w_{,yy})_{,y} \quad \Rightarrow \quad Q_y = \frac{b}{\pi} p_0 \cos\frac{\pi y}{b}$$

Laatan jännitykset:

$$\sigma_x = \frac{M_x}{I} z \quad \Rightarrow \quad \sigma_x = \frac{\nu}{I}\left(\frac{b}{\pi}\right)^2 p_0 \sin\frac{\pi y}{b} \cdot z$$

$$\sigma_y = \frac{M_y}{I} z \quad \Rightarrow \quad \sigma_y = \frac{1}{I}\left(\frac{b}{\pi}\right)^2 p_0 \sin\frac{\pi y}{b} \cdot z$$

$$\tau_{xy} = \frac{M_{xy}}{I} z \quad \Rightarrow \quad \tau_{xy} = 0 \qquad \tau_{xz} = \frac{3Q_x}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \quad \Rightarrow \quad \tau_{xz} = 0$$

$$\tau_{yz} = \frac{3Q_y}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \quad \Rightarrow \quad \tau_{yz} = \frac{3}{2}\frac{b}{\pi h} p_0 \cos\frac{\pi y}{b} \cdot \left[1 - \left(\frac{z}{h/2}\right)^2\right]$$

$$\sigma_z = -\frac{p(x,y)}{4}\left[2 - 3\left(\frac{z}{h/2}\right) + \left(\frac{z}{h/2}\right)^3\right] \quad \Rightarrow \quad \sigma_z = -\frac{p_0}{4}\sin\frac{\pi y}{b}\left[2 - 3\left(\frac{z}{h/2}\right) + \left(\frac{z}{h/2}\right)^3\right]$$

Jännitysten maksimiarvot:

$$\max\sigma_x = \frac{0{,}3}{h^3/12} \cdot \frac{b^2}{\pi^2} \cdot p_0 \cdot 1 \cdot \frac{h}{2} \quad \text{kun} \quad y = \frac{b}{2} \ \text{ja} \ z = \frac{h}{2} \quad \Rightarrow \quad \max\sigma_x = 0{,}182 \cdot p_0 \left(\frac{b}{h}\right)^2$$

$$\max\sigma_y = \frac{1}{h^3/12} \cdot \frac{b^2}{\pi^2} \cdot p_0 \cdot 1 \cdot \frac{h}{2} \quad \text{kun} \quad y = \frac{b}{2} \ \text{ja} \ z = \frac{h}{2} \quad \Rightarrow \quad \max\sigma_y = 0{,}608 \cdot p_0 \left(\frac{b}{h}\right)^2$$

$$\max\tau_{yz} = \frac{3}{2} \cdot \frac{b}{\pi h} \cdot p_0 \cdot 1 \cdot 1 \quad \text{kun} \quad y = 0 \ \text{ja} \ z = 0 \quad \Rightarrow \quad \max\tau_{yz} = 0{,}477 \cdot p_0 \left(\frac{b}{h}\right)$$

$$\max|\sigma_z| = p_0 \cdot 1 \cdot 1 \quad \text{kun} \quad y = \frac{b}{2} \ \text{ja} \ z = -\frac{h}{2} \quad \Rightarrow \quad \max|\sigma_z| = p_0$$

Jos esimerkiksi $\frac{b}{h} = 20 \quad \Rightarrow$

$$\frac{\max|\sigma_z|}{\max\sigma_x} = \frac{p_0}{0{,}182\, p_0 \cdot 20^2} = 0{,}0137 \qquad \frac{\max\tau_{yz}}{\max\sigma_x} = \frac{0{,}477\, p_0 \cdot 20}{0{,}182\, p_0 \cdot 20^2} = 0{,}131$$

b) $p(y) = p_0 \quad \Rightarrow \quad D w,_{yyyy} = p_0 \quad \Rightarrow \quad D w,_{yyy} = p_0\, y + C_1 \quad \Rightarrow$

$$D w,_{yy} = \frac{p_0}{2} y^2 + C_1 y + C_2 \quad \Rightarrow \quad D w,_y = \frac{p_0}{6} y^3 + \frac{C_1}{2} y^2 + C_2\, y + C_3 \quad \Rightarrow$$

$$D w = \frac{p_0}{24} y^4 + \frac{C_1}{6} y^3 + \frac{C_2}{2} y^2 + C_3\, y + C_4$$

Reunaehdot: $w(x{,}0) = 0 \quad \Rightarrow \quad C_4 = 0 \qquad M_y(x{,}0) = -D w,_{yy}(x{,}0) = 0 \quad \Rightarrow \quad C_2 = 0$

$$M_y(x{,}b) = -D w,_{yy}(x{,}b) = 0 \quad \Rightarrow \quad \frac{p_0}{2} b^2 + C_1 b = 0 \quad \Rightarrow \quad C_1 = -\frac{1}{2} p_0\, b$$

$$w(x{,}b) = 0 \quad \Rightarrow \quad \frac{p_0}{24} b^4 - \frac{p_0\, b}{2 \cdot 6} b^3 + C_3\, b = 0 \quad \Rightarrow \quad C_3 = \frac{1}{24} p_0\, b^3$$

$$\Rightarrow \qquad w(x, y) = w(y) = \frac{p_0\, b^4}{24D}\left(\frac{y^4}{b^4} - 2\frac{y^3}{b^3} + \frac{y}{b}\right)$$

$$\Rightarrow \quad w_{max} = \frac{p_0\, b^4}{24D}\left(\frac{b^4/16}{b^4} - 2\frac{b^3/8}{b^3} + \frac{b/2}{b}\right) \quad \Rightarrow \quad w_{max} = \frac{5 p_0\, b^4}{384D} \quad \text{kun } y = \frac{b}{2}$$

Laattamomentit:

$$M_x = -D(w,_{xx} + \nu\, w,_{yy}) \quad \Rightarrow \quad M_x = -\frac{\nu p_0\, b}{2} y\left(\frac{y}{b} - 1\right)$$

$$M_y = -D(w,_{yy} + \nu\, w,_{xx}) \quad \Rightarrow \quad M_y = -\frac{p_0\, b}{2} y\left(\frac{y}{b} - 1\right)$$

$$M_{xy} = -D(1-\nu)\, w,_{xy} \quad \Rightarrow \quad M_{xy} = 0$$

Laatan leikkausvoimat:

$$Q_x = -D(w,_{xx} + w,_{yy}),_x \quad \Rightarrow \quad Q_x = 0$$

$$Q_y = -D(w,_{xx} + w,_{yy}),_y \quad \Rightarrow \quad Q_y = -\frac{p_0\, b}{2}\left(2\frac{y}{b} - 1\right)$$

Laatan jännitykset:

$$\sigma_x = \frac{M_x}{I} z \quad \Rightarrow \quad \sigma_x = -\frac{\nu p_0 b}{2I} y\left(\frac{y}{b} - 1\right) \cdot z$$

$$\sigma_y = \frac{M_y}{I} z \quad \Rightarrow \quad \sigma_y = -\frac{p_0 b}{2I} y\left(\frac{y}{b} - 1\right) \cdot z$$

$$\tau_{xy} = \frac{M_{xy}}{I} z \quad \Rightarrow \quad \tau_{xy} = 0 \qquad \tau_{xz} = \frac{3Q_x}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \quad \Rightarrow \quad \tau_{xz} = 0$$

$$\tau_{yz} = \frac{3Q_y}{2h}\left[1 - \left(\frac{z}{h/2}\right)^2\right] \quad \Rightarrow \quad \tau_{yz} = -\frac{3}{2}\frac{p_0 b}{2h}\left(2\frac{y}{b} - 1\right) \cdot \left[1 - \left(\frac{z}{h/2}\right)^2\right]$$

$$\sigma_z = -\frac{p(x,y)}{4}\left[2 - 3\left(\frac{z}{h/2}\right) + \left(\frac{z}{h/2}\right)^3\right] \quad \Rightarrow \quad \sigma_z = -\frac{p_0}{4}\left[2 - 3\left(\frac{z}{h/2}\right) + \left(\frac{z}{h/2}\right)^3\right]$$

Jännitysten maksimiarvot:

$$\max \sigma_x = -\frac{\nu p_0 b}{2h^3/12} \cdot \frac{b}{2} \cdot \left(\frac{b/2}{b} - 1\right) \cdot \frac{h}{2} = \frac{3 \cdot 0{,}3 \cdot p_0 b^2}{4h^2} \quad \text{kun} \quad y = \frac{b}{2} \ \text{ ja } \ z = \frac{h}{2} \quad \Rightarrow$$

$$\max \sigma_x = 0{,}225 \cdot p_0 \left(\frac{b}{h}\right)^2 \quad \Rightarrow \quad \max \sigma_y = 0{,}750 \cdot p_0 \left(\frac{b}{h}\right)^2$$

$$\max \tau_{yz} = -\frac{3}{2} \cdot \frac{p_0 b}{2h} \cdot (-1) \cdot 1 \quad \text{kun} \quad y = 0 \ \text{ ja } \ z = 0 \quad \Rightarrow \quad \max \tau_{yz} = 0{,}750 \cdot p_0 \left(\frac{b}{h}\right)$$

$$\max |\sigma_z| = p_0 \cdot 1 \cdot 1 \quad \text{kun} \quad y = \frac{b}{2} \ \text{ ja } \ z = -\frac{h}{2} \quad \Rightarrow \quad \max |\sigma_z| = p_0$$

Jos esimerkiksi $\frac{b}{h} = 20 \quad \Rightarrow$

$$\frac{\max |\sigma_z|}{\max \sigma_x} = \frac{p_0}{0{,}225 p_0 \cdot 20^2} = 0{,}0111 \qquad \frac{\max \tau_{yz}}{\max \sigma_x} = \frac{0{,}750 p_0 \cdot 20}{0{,}225 p_0 \cdot 20^2} = 0{,}167$$