Proof for the statement:

$$\{x_{0:k}^j, w^j\} \sim p(x_{0:k} | y_{1:k})$$

Then: $\{x_k^j, w^j\} \sim p(x_k | y_{1:k})$

---

(1) $E[G(x_{0:k})] \approx \frac{1}{Ne} \sum_{j=1}^{Ne} G(x_{0:k}^j)\, w^j$

pick: $G(x_k) = g(x_k)$

$$E[g(x_k)] = \frac{1}{Ne} \sum_{j=1}^{Ne} g(x_k^j)\, w^j$$

(2) We need to show that

$$\int \cdots \int g(x_k)\, p(x_{0:k} | y_{1:k})\, dx_0 \cdots dx_k$$

$$= \int g(x_k)\, p(x_k | y_{1:k})\, dx_k .$$

$\hookrightarrow \int \cdots \int g(x_k)\, p(x_{0:k} | y_{1:k})\, dx_0 \cdots dx_k$

$$= \int g(x_k) \left[\underbrace{\int \cdots \int p(x_k | x_{0:k-1}, y_{1:k})\, p(x_{0:k-1} | y_{1:k})\, dx_0 \cdots dx_{k-1}}_{\overset{*}{=} p(x_k | y_{1:k})} \right] dx_k$$

$= \int g(x_k)\, p(x_k | y_{1:k})\, dx_k$ ✓

Helper proof:

$$\int p(x|y,z)\, p(y|z)\, dy$$

$$= \int p(y|x,z) \frac{p(x|z)}{\cancel{p(y|z)}} \cancel{p(y|z)}\, dy$$

$$= p(x|z) \underbrace{\int p(y|x,z)\, dy}_{=1} = p(x|z)$$