Making EnKF practical → localization & inflation.

Recall: Issues with EnKF & KF: (i) linear problems
relative
↳ fake
(ii) we need to be able to work with small N_e.

Can be addressed by localization & inflation.

(1) Inflation

If N_e is small, sampling error is large (recall Cholesky)

Underestimating variances is particularly dangerous.

↳ To prevent the covariances are too small, we make them artificially bigger

For EnKF: $\{x_j^i\}$ is forecast ensemble
$\bar{x}_j = \frac{1}{N_e} \sum x_j^i$ forecast mean
$P_j = \frac{1}{N_e - 1} \sum (x_j^i - \bar{x}_j)(x_j^i - \bar{x}_j)$ forecast cov.

Inflation: Replace P_j by $(1+\alpha) P_j$ for $\alpha > 0$, "small".
(tuning, see later)

→ we need to inflate the ensemble to have the inflated covariance:

define: $\tilde{x}_j^i = \bar{x}_j + \sqrt{1+\alpha}\,(x_j^i - \bar{x}_j)$

note $E[\tilde{x}_j^i] = \bar{x}_j$ ✓ (as before)

$\mathrm{cov}(\tilde{x}_j^i) = (1+\alpha) P_j$ ✓ inflated covariance!

Localization:

Example: $H = I$ $\quad R = I$ $\quad n = 10^3$

$P_k = I$ $\quad \mu_k = 0$

ensemble $\{x_k^j\} \sim N(0, I)$ $\quad j = 1 \dots N_e$, $N_e \ll n$

forecast: $\{x_f^j\}$ $\quad P_f$ (Perhaps inflated)

Note: P_f should be diagonal

but due to sampling error we see off-diagonal elements.

(Computer demonstration)

↳ Suppose we know that off-diagonal elements of P_f should be small/zero

↳ Then we can enforce this by setting small off-diagonal elements to zero or make large off-diagonal elements small.

↳ This is the idea of localization.

Replace P_f by P_f^e ← localized, with properly adjusted off-diagonal elements.

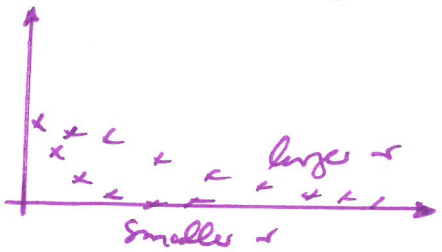
How to do this:

Define a localization matrix:

$$L_{ij} = \exp\left(-\frac{i-j}{r}\right) \quad \text{or} \quad L_{ij} = \exp\left(-\left(\frac{i-j}{r}\right)^2\right)$$

where r is localization radius.

Each row of L looks like



Define: $P_f^L = L \circ P_f$

↑ Schur / Hadamard / element-wise matrix product (careful in Python)

$$[P_f^L]_{ij} = [L]_{ij}\,[P_f]_{ij}$$

↳ Localized P_f has small off-diagonal elements

↳ We can control & tune localization radius r.

↳ Localized EnKF for large r has the "good" asymptotic properties of EnKF (= right estimate for $N_e \to \infty$).

↳ Localization introduces bias, but bias is controllable.

↳ Localization is "bad" if you are dealing with a problem for which P_f is not "local".

↝ This does not cover all problems but many, because many problems have this structure

Why? · If M comes from PDE or ODE, M is local, and so is P_f because many PDE/ODE are local

· local correlation - far away variables decorrelate

Example: Weather in Tucson uncorrelated from weather in Australia.

EnKF algorithm

$\{x_k^j\}$ $\quad j = 1 \dots N_e$

$x_f^j = H x_k^j$

$\bar{x}_f = \frac{1}{N_e} \sum x_f^j$

$x_f^j = \bar{x}_f + \sqrt{1+\alpha}\,(x_f^j - \bar{x}_f)$ // inflation

$P_f = L \circ \mathrm{Cov}(x_f^j)$ // localization

$K = P_f H^T (H P_f H^T + R)^{-1}$

$\mu_a = \mu_f + K(y - H\mu_f)$ // analysis mean

perturbed obs ↙ $\qquad$ ↘ square root

$x_a^j = x_f^j + K(\tilde{y}^j - H x_f^j)$

$x_a^j = \mu_a + \sqrt{N_e - 1}\,\tilde{x}_a$

$(ZZ^T = I - V(R + V^T V)V^T)$

$V = X^T H^T$

$X = \frac{1}{\sqrt{N_e - 1}}(x_f - x_f^j)$

$\tilde{x}_a = XZ$)

Last task: tuning → how to pick loc rad & inflation.

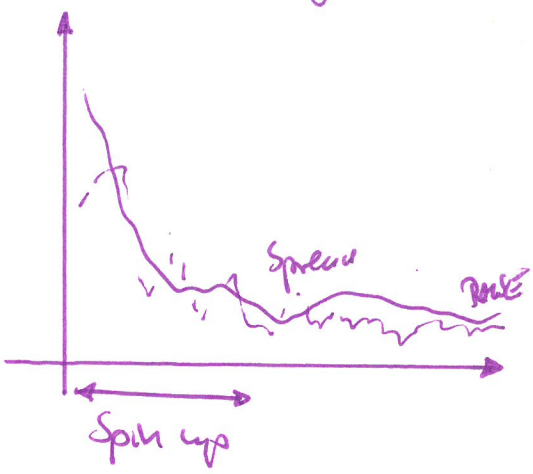
pick α, r.

Do a synthetic data experiment.

Compute RMSE & Spread

$$RMSE = \left(\frac{1}{n} \sum_{i=1}^{n} (x^t_i - [\mu_a]_i) \right)^{0.5}$$

$$Spread = \left(\frac{1}{n} \text{trace}(P_a) \right)^{0.5}$$



Spin-up: artefacts of initial ensemble, which may not be so good (see below)

→ Disregard (make sure it is large enough)

Compute RMSE average over time

Repeat for different choice of α, r.

Call (α, r) which yields lowest RMSE "optimal".

→ This is "grid-based" optimization.
You pick values for (α, r), evaluate RMSE over that grid, that's it.

→ This is parallelizable, that's why we want to use HPC.

How to pick the initial ensemble for EnKF.

1) Do a "long" simulation.

   Store only last state, which is hopefully near the attractor/has shaken off all weirdness of first choice of IC's.

2) Do a long simulation starting from last state of previous simulation.

   Initial ensemble are states chosen at random from this simulation.

   ⇝ Initial ensemble is typically quite good, hence, spin up.