Expected values & moments

Definition Let $(\Omega, \mathcal{B}, P)$ be a probability space

Let η be a r.v.

$$E[f(\eta)] = \int_{\Omega} f(\eta(\omega))\, dP = \int_{-\infty}^{\infty} f(\eta)\, dF(\eta)$$

is the expected value of $f(\eta)$.

Interpretation via Stieltjes integral (see text)

If F is differentiable:

$$E[f(\eta)] = \int_{-\infty}^{\infty} f(x)\, p(x)\, dx$$

Important special cases:

$E[\eta^n]$ = nth moment

$E[(\eta-\mu)^n]$ = nth centered moment, $\mu = E[\eta]$

2nd centered moment is variance, $Var(\eta)$

$\sigma = \sqrt{Var(\eta)}$ is the standard deviation.

Recall: Two events A, B are independent if

$P(A \cap B) = P(A) \cdot P(B)$

Two r.v.s are independent if the events

$\{\omega \in \Omega : \eta_1(\omega) \le x_1\}$ & $\{\omega \in \Omega : \eta_2(\omega) \le x_2\}$

are independent for all x_1, x_2.

It follows that

$$F_{\eta_1\eta_2}(x, y) = F_{\eta_1}(x) F_{\eta_2}(y)$$

Fact: joint pdf of two independent rvs factorizes.

$$\eta_1 \eta_2 \sim p_{\eta_1\eta_2}(x,y)$$

$$p_{\eta_1\eta_2}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{\eta_1\eta_2}(x,y)$$

$$F_{\eta_1\eta_2}(x,y) = F_{\eta_1}(x)\, F_{\eta_2}(y)$$

$$(P(A \cap B) = P(A)P(B))$$

$$\Rightarrow p_{\eta_1\eta_2}(x,y) = p_{\eta_1}(x)\, p_{\eta_2}(y)$$

Note: $p_{\eta_1\eta_2}(x,y)$ describes joint probabilities of what happens to r.v.s η_1 and η_2.

What if we are only interested in η_1 but don't care about the value of η_2?

↳ prob $x < \eta_1 < x+dx$, η_2 can have any value.

$$\hookrightarrow P(\{\omega \in \Omega : x < \eta_1 < x+dx,\ -\infty < \eta_2 < \infty\})$$

$$\approx p_{\eta_1\eta_2}(x,y) \times \text{Volume}$$

$$= \int_{-\infty}^{x} \underbrace{\int_{-\infty}^{\infty} p_{\eta_1\eta_2}(\cancel{x},\cancel{t})\, d\cancel{y}}_{} \, d\cancel{t} = \int_{-\infty}^{x} p_{\eta}(\epsilon)\, d\epsilon$$

$p_{\eta}(\epsilon) = \int_{-\infty}^{\infty} p_{\eta_1\eta_2}(\cancel{x}\,\epsilon)\, d\cancel{t}$ is called the marginal.

Functions of random variables

x is r.v.

g is continuous monotonically increasing fct.

$y = g(x)$ is a r.v.

what is $p_y(\cdot)$?

Prob. that x is between a and b

$$\int_a^b p_x(x)\,dx \underset{\substack{\uparrow \\ \text{change of} \\ \text{vars.}}}{=} \int_{g(a)}^{g(b)} p_x(g^{-1}(y)) \left(\frac{dy}{dx}\right)^{-1} dy$$

$$= \text{Prob that } y \text{ is between } g(a) \text{ and } g(b)$$

$$= \int_{g(a)}^{g(b)} p_y(y)\,dy$$

$$\Rightarrow \quad p_y(y) = p_x(g^{-1}(y)) \left(\frac{dy}{dx}\right)^{-1}$$

Example: $x \sim N(0,1)$ $\quad g(x) = ax \quad \frac{dy}{dx} = a$

$y = ax, \; a > 0 \quad g^{-1}(y) = \frac{1}{a} y$

$$p_y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y}{a}\right)^2\right) \cdot \frac{1}{a}$$

$$\rightsquigarrow y \sim N(0, a^2)$$

More general multivariate case:

$$p_x(x) = p_y(y)\,|J|$$

$$J = \det\left(\frac{dx}{dy}\right) \quad \nwarrow \text{ "Jacobian of map } x \to y\text{"}$$

Definition: Covariance of two r.v. η_1 and η_2

$$\text{cov}(\eta_1, \eta_2) = E[(\eta_1 - \bar{\eta}_1)(\eta_2 - \bar{\eta}_2)]$$

$$\bar{\eta}_1 = E[\eta_1], \bar{\eta}_2 = E[\eta_2]$$

If $\text{cov}(\eta_1, \eta_2) = 0$, η_1 η_2 are "uncorrelated".

More general: x is multivariate r.v.

$$\text{cov}(x) = E[(x - \bar{x})(x - \bar{x})^T] \text{ is a matrix}$$

Describes correlations of the various elements of x.

For a Gaussian : $x \sim \mathcal{N}(\mu, \Sigma)$

$$\Sigma_{ij} = \text{cov}(x_i, x_j)$$

Fact: Independent $\Rightarrow$ uncorrelated

Uncorrelated $\not\Rightarrow$ independent.

If x is Gaussian: uncorrelated $\Leftrightarrow$ independent.

(see HW).

More properties you can prove over land and which will come up.

$$E[x+y] = E[x] + E[y]$$

If x, y are independent: $\text{var}(x+y) = \text{var}(x) + \text{var}(y)$

Cheby chef: η is r.v.

g is non-negative, non-decreasing fcn.

for every a

$$P(\eta \geq a) \leq \frac{E[g(\eta)]}{g(a)}$$

Application of Chebyshev:

η is r.v. $\quad \bar{\eta} = E[\eta]$

$\xi = |\eta - \bar{\eta}|$ is a r.v.

$g(y) = y^2$

$$P(|\eta - \bar{\eta}| \geq a) \leq \frac{E[|\eta - \bar{\eta}|^2]}{a^2} = \frac{\mathrm{Var}(\eta)}{a^2} = \frac{\sigma^2}{a^2}$$

pick $k\sigma = a$

$$P(|\eta - \bar{\eta}| \geq k\sigma) \leq \frac{1}{k^2}$$

→ It is unlikely to see η take a values that are farther away from $\bar{\eta}$ than a few σ's.