Extremely rapid review of probability and r.v.

Ω - Sample space ⇝ all possible outcomes of an experiment
E - Event ⇝ subset of Ω
$\mathcal{B}$ ~ set of all events we want to consider
↳ used to construct a σ-algebra

P - probability measure
↳ function that assigns probability to events in $\mathcal{B}$

$$P : \mathcal{B} \to \mathbb{R}$$

(i) $P(\Omega) = 1$

(ii) $0 \leq P(A) \leq 1$

(iii) $A \cap B \neq 0 \to P(A \cup B) = P(A) + P(B)$

intersection empty — union of sets

$(\Omega, \mathcal{B}, P) \to$ Probability space.

A random variable is a fctn $X : \Omega \to \mathbb{R}$ such that

$$A_\eta = \{\omega \in \Omega : X(\omega) \leq \eta\}$$

is an element of $\mathcal{B}$

⇝ we can assign probabilities to all events $X(\omega) \leq \eta$

⇝ we can assign probabilities to the values of the r.v.

Probability distribution fct (PDF) of a r.v. X.

$$F_X(\eta) = P(\{\omega \in \Omega : X(\omega) \leq \eta\})$$

Properties:

- $p_x(\eta) \geq 0$ ($F_x(\eta)$ is non-decreasing)
- $F_x(\eta) = \int_{-\infty}^{\eta} p_x(y)\,dy$
- $\int_{-\infty}^{\infty} p_x(y)\,dy = 1$
- $F_x(\eta + d\eta) - F_x(\eta) \approx p_x(\eta) \cdot \text{Volume}$.

(this is how you can figure out what pdfs of r.v. are, e.g. $x, y \to x+y$)

For purposes of this class:

- Probability space usually continuous
- we work with assumed probability densities!
- we will almost exclusively deal with multivariate case (40 for hw, code should work with hundreds, should be scalable to hundreds of millions)
- Notation is a mess

Examples:

Uniform: $X \sim U[a, b]$

$p_x(\eta) = \text{const}$

(volume of the cube with sides $[a, b]$)

Gaussian $X \sim N(\mu, \Sigma)$

If x is r.v. and g is continuous monotonically increasing fcn then $y = g(x)$ is also a r.v.

"change of variables"

Prob that x is between a and b

$$\int_a^b p_x(x)\,dx = \int_{g(a)}^{g(b)} p_x(g^{-1}(y)) \left(\frac{dx}{dy}\right) dy$$

Prob that x is between a and b = Prob that y is between $g(a)$ and $g(b)$ $= \int_{g(a)}^{g(b)} p_y(y)\,dy$

Compare integrals: $\quad p_y(y) = p_x(g^{-1}(y)) \dfrac{dx}{dy}$

More general multivariate case:

$$p_y(y) = p_x(g^{-1}(y))\ |J|$$

J = Jacobian of map from $x \to y$

$$\det\left[\frac{dx}{dy}\right]$$

<u>Example:</u> $\quad x \sim \mathcal{N}(0, I) \qquad y = \mu + L\cancel{x} \Leftrightarrow x = L^{-1}(y-\mu)$

More examples of r.v.

<u>χ - distribution:</u>

$$y = \left(\sum_{i=1}^{k} \left(\frac{(x_i - \mu_i)}{\sigma_i} \right)^2 \right)^{1/2} \qquad x_i \sim N(\mu_i, \sigma_i) \quad k > 0$$

Note that $y > 0$! Good for modeling non-negative quantities such as densities or permeabilities or concentrations

$$p(y) \propto y^{k-1} e^{-y^2/2}$$

<u>χ^2 - distribution:</u>

$$y = \sum_{i=1}^{k} \left(\frac{(x_i - \mu_i)}{\sigma_i} \right)^2$$

$$p(y) \propto y^{k/2 - 1} e^{-y/2}$$

<u>Multivariate t</u>

$$x \sim N(0, \Sigma) \qquad u \sim \chi^2_\nu$$

$y = \left(\frac{\nu}{u}\right)(x - \mu)$ is multivariate t

$$p(y) \propto \left[1 + \tfrac{1}{\nu}(x-\mu)^T \Sigma^{-1} (x-\mu)\right]^{-\frac{\nu+p}{2}}$$

p is dim y

Expected Values & Moments

η is r.v., f is nice enough fcn.

$$E[f(\eta)] = \int_{-\infty}^{\infty} f(x)\, p_\eta(x)\, dx$$

(See text for more general definition using Stieltjes integral)

$\bar{\eta} = E[\eta]$ = expected value

$E[(\eta-\bar{\eta})^n]$ = nth centered moment

2^{nd} centered moment is variance

$\sqrt{Var} = \sigma$ → standard deviation

Multivariate:

$Cov[(\eta-\bar{\eta})(\eta-\bar{\eta})^T]$

is covariance of η,

is $n \times n$ SPD matrix.

Example: η is a r.v. What is best estimate of its value?

$$\min_c E((\eta - c)^2)$$

$$E[(\eta-c)^2] = \underbrace{E[\eta^2] - 2E[\eta]c - c^2}_{}$$

Expectation is linear
$E[\text{Const}] = \text{Const}.$

Derivative: $-2E[\eta] - 2c = 0 \Rightarrow c = E[\eta]$

↳ expected value is best estimate of η!

Independence:

Two events are independent if $P(A,B) = P(A)P(B)$

Uncorrelated vs. independent:

x, y are uncorrelated if $\quad cov(x,y) = E[(x-\bar{x})(y-\bar{y})] = 0$

$\bar{x} = E[x], \quad \bar{y} = E[y]$

HW shows: $\quad x, y$ independent $\Rightarrow$ uncorrelated

x, y uncorrelated $\not\Rightarrow$ independent

x, y Gaussian: independent $\Leftrightarrow$ uncorrelated.

"earlier?"

Marginals: $\quad x, y$ jointly distributed $p_{xy}(x,y)$.

Marginal: $\quad p_x(x) = \int_{-\infty}^{\infty} p_{x,y}(x,y)\,dy$

describes probabilities for values of x while y can take on any value.

Chebychev inequality: $\quad \eta$ is r.v.

g is non-negative, non-decreasing fcn

Take every a:

$$P(\eta \geq a) \leq \frac{E[g(\eta)]}{g(a)}$$

Example: $\quad \eta$ is r.v.

$\bar{\eta} = E[\eta]$