Ensemble Kalman filter & nonlinear problems

EnKF algorithm: $\{x_j^i\}$ forecast ensemble

(localization & Inflation)

analysis ensemble.

A different perspective

Simplified set up:

$$x_1 = Mx_0, \qquad x_0 \sim N(\mu, P_0)$$
$$y_1 = Hx_1 + \eta, \qquad \eta \sim N(0, R)$$

Assumptions: $E[x^t - x_1] = 0$
(unbiased)

$E[y - Hx^t] = 0$

x_0, η uncorrelated.

Goal: find estimate such that average error is small.

$$\min_{\hat{x}} \frac{1}{n} \sum_{i=1}^{n} (x^t_i - \hat{x}_i)^2 \left(\Leftrightarrow \min_{\hat{x}} \sum_{i=1}^{n} (x_i^t - \hat{x}_i)^2 \right)$$

$$\Rightarrow \min_{\hat{x}} E\left[(x^t - \hat{x})^T (x^t - \hat{x})\right]$$

average over components

Note: $y^T x = (y_1 \cdots y_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = y_1 x_1 + \cdots + y_n x_n$

$$y x^T = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}(x_1 \cdots x_n) = \begin{pmatrix} y_1 x_1 & y_1 x_2 & \cdots \\ & y_2 x_2 & \vdots \\ & & \ddots \\ & & & y_n x_n \end{pmatrix}$$

$\hookrightarrow y^T x = tr(y x^T)$

$$\Rightarrow \arg\min_{\hat{x}} E[(x^t - \hat{x})^T (x^t - \hat{x})]$$

$$= \arg\min_{\hat{x}} tr\left(E[(x^t - \hat{x})(x^t - \hat{x})^T]\right)$$

$$= \arg\min_{\hat{x}} tr(P)$$

$$\underline{\text{if}} \quad E[(x^t - \hat{x})] = 0$$

(unbiased).

"Best unbiased estimate":

an $\hat{x}$ such that $E[x^t - \hat{x}] = 0$

and that minimizes $tr(P)$

"Best linear unbiased estimate" (BLUE)

an $\hat{x}$ such that: $E[x^t - \hat{x}] = 0$ (unbiased)

minimizes $tr(P)$ (best)

$\hat{x} = x^b + G(y - Hx^b)$ (linear fct of obs.)

(i) Check unbiasedness under our assumptions.

(I) $E[x^t - x_t] = 0$

(II) $E[y - Hx_t] = 0$

$$E[x^t - \hat{x}] = E[x^t - x_g - G(y - Hx_g)]$$

$$= \underbrace{E[x^t - x_g]}_{=0} - GE[y - Hx_g]$$

$$= -GE[y - H(x_g + x_t - x_t)]$$

$$= -G\underbrace{E[y - Hx_t]}_{=0} - GH\underbrace{E[x_g - x_t]}_{=0} \quad = 0 \checkmark$$

↳ under our assumptions, estimate is unbiased!

(ii) Find G that minimizes $tr(P)$, where P is cov. of

$$\hat{x} = x_g + G(y - Hx_g)$$

$$= (I - GH)x_g + Gy$$

$$P = (I - GH)P_g(I - GH)^T + G^T R G^T$$

$$= (P_g - GHP_g)(I - GH)^T + G^T R G^T$$

$$= P_g - P_g H^T G^T - GHP_g + GHP_g H^T G^T + GRG^T$$

to minimize tr(P):

compute $\frac{\partial}{\partial G} tr(P) = tr\left(\frac{\partial}{\partial G} P\right)$

$\hookrightarrow \frac{\partial}{\partial G} P = -P_f H^T - P_f H^T \, 2G\,(HP_fH^T + R) = 0$

$$G = P_f H^T (HP_fH^T + R)^{-1}$$

Kalman Gain.

check 2nd derivative: $HP_fH^T + R$ is SPD

$\rightsquigarrow$ G is the minimizer!

$\Rightarrow$ KF is BLUE for linear problems.

But we can relax these assumptions.
What we need is:

(I) $E[x^t - Mx_0] = 0$ // very good model

(II) $E[y - Hx^t] = 0$ // very good observations

Under (I) & (II): EnKF gives best linear unbiased estimate, even if model is nonlinear.

$\Rightarrow$ EnKF makes sense for nonlinear models as well.

EnKF for nonlinear models:

$\{x_k^j\}$ $j = 1 \dots N_e$, ensemble at time k

$x_f^j = f(x_k^j)$ is forecast ensemble.

$\uparrow$ non-linear model

$\Rightarrow$ Rest is exactly as before.

$\Rightarrow$ What you get (at least in $N_e \to \infty$ limit) is the BLUE.