| | |
|---|---|
| Title | A Design Method for Morphological Filters with Approximations of Min/Max Operators |
| Author(s) | Nakashizuka, Makoto |
| Citation | Proceedings : APSIPA ASC 2009 : Asia-Pacific Signal and Information Processing Association, 2009 Annual Summit and Conference: 841-844 |
| Issue Date | 2009-10-04 |
| DOI | |
| Doc URL | http://hdl.handle.net/2115/39819 |
| Right | |
| Type | proceedings |
| Additional Information | |
| File Information | WA-P2-6.pdf |

# A Design Method for Morphological Filters with Approximations of Min/Max Operators

Makoto NAKASHIZUKA
Osaka University, 1-3 Machikaneyama, Toyonaka, Osaka 560-8531, Japan
E-mail: nkszk@sys.es.osaka-u.ac.jp

***Abstract*—This paper presents a novel design method for the structuring elements of morphological filters. The morphological filters, which have been applied to low-level image processing tasks, are implemented with max and min functions. Due to the non-differentiability of the max and min functions, it is difficult to apply any gradient-based optimization technique to design of the morphological filters. In this paper, we introduce approximating functions of the max and min functions to approximate the morphological filters. Since the proposed approximating functions are differentiable, the outputs of the approximated morphological filters are also differentiable. The gradient based optimization technique can be easily applied to the proposed approximated morphological filters. We approximate the dilation and opening filters with the approximated max and min functions, and design the quasi-optimum structuring elements for the both filters. In experiments, we demonstrate several design examples and applications to image denoising.**

## I. Introduction

The mathematical morphology is a methodology of signal and image processing based on set operations. Originally, the morphological filters have been proposed for analysis of binary images. Later, the morphological filters have been extended to gray-level image processing methods and contribute to low-level image processing tasks including image enhancement and denoising[1]. The gray-level morphological filters are constructed with two basic operations, dilation and erosion. The dilation and erosion filters are implemented as the max and min filters that output the maximum and minimum of the intensities within local windows, respectively. A structuring element[1] (SE) of the dilation and erosion specifies the shape of the window and the coefficients that are added to the input pixels within the window. For image denoising, closing and opening filters are implemented as cascade connections of the dilation and erosion filters. The closing filter is realized by the dilation and successive erosion filtering and fills the pits and valleys of the intensity surface of an input image. The opening filter is realized by the erosion and successive dilation and eliminates peaks and ridges of the intensity surface. In the morphological filters, the image local structures that are eliminated or preserved by filtering are specified by the SE. Hence, the design of the SE is an important topic in the morphological image processing. However, very few approaches exist for design of the SE.

In Ref. [2], the design method for the stack filters, which is a class of non-linear filters including the morphological filters, has been proposed. However, this approach is limited to the filters that process quantized signals due to the threshold decomposition. In Ref. [6][7][8], stochastic optimization approaches have been proposed. These approach requires huge number of trials and a long time to convergence of the SE. The major difficulty of the design of the SE is that the min and max functions are not differentiable. It is difficult to apply gradient-based optimization techniques to design of the SE. In order to remedy this problem, the rank functions that include the max and min functions are approximated as the differentiable functions in Ref. [3][4]. By this approximation, the cost function that is defined for the SE is also approximated as a differentiable function. A gradient-based optimization technique can be applied to this approximated cost function. This method can be applied to design the dilation and erosion filters, however, it can not be applied to the opening and closing filters that are implemented as cascade connections of the erosion and dilation filter directly. For same purpose, the generalized mean functions were introduced to approximate the max and min functions in Ref. [5]. This approximation is limited to nonnegative SEs and signals and is not capable to approximate the general class of the morphological filters.

In this paper, we propose a novel design method for the SEs of the morphological filters. We introduce approximating functions of the max and min functions to approximate the morphological filters. The approximating functions consist of exponent and log functions. By using these approximating functions, any morphological filter can be approximated as a filter, the outputs of which are differentiable with respect to an element of the SE. In the next section, the gray-level morphological filters are defined with the max and min functions. In Sect. 3, the approximating functions are introduced to the morphological filters. In Sect. 4, the cost function for the design of the SE is defined as the squared error between an input and an ideal image. The gradient of the cost function is also derived for the minimization. Finally, some examples of the SE design are demonstrated. The denoising result using the optimized SE is also shown to demonstrate the advantage of the optimization.

## II. Morphological Filters for Gray Scale Images

The morphological filters for gray-level images can be implemented with "min" and "max" function. The "max" function $\max(\mathbb{X})$ is defined for the set of the values $\mathbb{X} = \{x_1, x_2, \cdots x_N\}$ where $x_i \in \mathbb{R}$. This function returns the maximum value of the elements in $\mathbb{X}$. The "min" function

$\min(\mathbb{X})$ returns the minimum of the values of the elements in $\mathbb{X}$. By using the max function, the morphological dilation for a gray-level image $\{f_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{I}}$ is defined as

$$Df_{\mathbf{x}} = \max\left(\{f_{\mathbf{x}+\mathbf{y}} + s_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{C}}\right) \tag{1}$$

where $\mathbf{x}$ and $\mathbf{y}$ denote two-dimensional coordinates of the image pixels. $\mathbb{I}$ is a set of the coordinate of the pixels of the image. $\mathbb{C}$ is a set of two-dimensional coordinates that are supported by the SE $s_{\mathbf{y}}$. The morphological erosion for a gray-level image is defined with the min function as

$$Ef_{\mathbf{x}} = \min\left(\{f_{\mathbf{x}-\mathbf{y}} - s_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{C}}\right). \tag{2}$$

Basically, the morphology filters for the gray-level images are constructed with the dilation and erosion operations. The opening filter is defined as the dilation after the erosion:

$$Of_{\mathbf{x}} = \max\left(\{Ef_{\mathbf{x}+\mathbf{y}} + s_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{C}}\right). \tag{3}$$

By this operation, the image is approximated as a union of the SE, each of them is generated by shifting of the coordinate and intensities of $\{s_{\mathbf{x}}\}_{\mathbf{x}\in\mathbb{C}}$. The closing filter is defined as the erosion follows the dilation:

$$Cf\mathbf{x} = \min\left(\{Df_{\mathbf{x}-\mathbf{y}} - s_{\mathbf{y}}\}_{\mathbf{y}\in\mathbb{C}}\right). \tag{4}$$

The closing filter is the dual of the opening filter and approximates the complement of the image with the SE[1]. In this paper, we present the design methods for the dilation and opening filters. It is obvious that the design methods for the dilation and opening filters can be applied to the erosion and closing filters by duality of the mathematical morphology.

## III. Approximations of Morphological Filters

In this section, we introduce approximating functions of the min and max functions to the morphological filters. By introducing the approximating functions, the outputs of the dilation and closing filters are approximated as differentiable functions with respect to the elements of the SE. First, the approximating function for the max function is defined for the set of values $\mathbb{X} = \{x_1, \cdots x_N\}$ as

$$\mu(\mathbb{X}) = T \log \sum_{i=1}^{N} e^{x_i/T} \tag{5}$$

where $T$ is a scaling parameter for $x_i$. The output of this function mainly depends on the maximum of the values in the set $\mathbb{X}$. When all values of the elements of the set $\mathbb{X}$ are bounded, then the difference between the maximum and its approximation is bounded as

$$0 < \mu(\mathbb{X}) - \max(\mathbb{X}) \leq T \log N. \tag{6}$$

Thus, the approximation error decreases along with the decrement of the scale parameter $T$. When the all values in the set $\mathbb{X}$ are same, the error becomes the maximum $T \log N$. The approximating function for the min function is also defined as

$$\nu(\mathbb{X}) = -T \log \sum_{i=1}^{N} e^{-x_i/T}. \tag{7}$$

The error between the approximation and the minimum of the values of the set $\mathbb{X}$ is bounded as

$$-T \log N \leq \nu(\mathbb{X}) - \min(\mathbb{X}) < 0 \tag{8}$$

When the all values in the set $\mathbb{X}$ are same, the difference becomes the minimum $-T \log N$.

By introducing the approximating functions, the dilation and erosion filters can be approximated as

$$\hat{D}f_{\mathbf{x}} = T \log \sum_{\mathbf{y}\in\mathbb{C}} e^{(f_{\mathbf{x}+\mathbf{y}} + s_{\mathbf{y}})/T} \tag{9}$$

and

$$\hat{E}f_{\mathbf{x}} = -T \log \sum_{\mathbf{y}\in\mathbb{C}} e^{-(f_{\mathbf{x}-\mathbf{y}} - s_{\mathbf{y}})/T}, \tag{10}$$

respectively. By using above approximation filters, the opening filter is approximated as

$$\hat{O}f_{\mathbf{x}} = T \log \sum_{\mathbf{y}\in\mathbb{C}} e^{\hat{E}f_{\mathbf{x}+\mathbf{y}} + s_{\mathbf{y}}}. \tag{11}$$

The both dilation and opening are approximated as functions that are differentiable by using the approximating functions of max and min functions.

## IV. Optimization of Structuring Elements

For the optimization of the SEs, we suppose that an example of an input and an ideal image is obtained. The cost function is defined as the squared error between the input and the ideal image. The cost function

$$Q_D = \frac{1}{2} \sum_{\mathbf{x}\in\mathbb{I}} (Df_{\mathbf{x}} - g_{\mathbf{x}})^2 \tag{12}$$

is defined for the optimization of the SE of the dilation filter and

$$Q_O = \frac{1}{2} \sum_{\mathbf{x}\in\mathbb{I}} (Of_{\mathbf{x}} - g_{\mathbf{x}})^2 \tag{13}$$

for the opening filter. $g_{\mathbf{x}}$ is the ideal output of the dilation and opening filters. To apply the gradient-based optimization, the dilation and opening are replaced with its approximation shown in (9) and (11). The approximation of the cost functions are obtained as

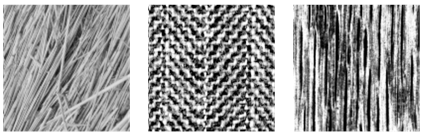
$$\hat{Q}_D = \frac{1}{2} \sum_{\mathbf{x}\in\mathbb{I}} (\hat{D}f_{\mathbf{x}} - g_{\mathbf{x}})^2 \tag{14}$$

and

$$\hat{Q}_O = \frac{1}{2} \sum_{\mathbf{x}\in\mathbb{I}} (\hat{O}f_{\mathbf{x}} - g_{\mathbf{x}})^2 \tag{15}$$

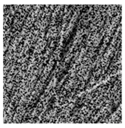
for the dilation and opening filters, respectively. Let us suppose that $s_{\mathbf{y}_i}$ is any element of the SE $\{s_{\mathbf{y}}\}_{\mathbf{y}\in C}$. When $T = 1$, the partial differential of the cost function of the dilation filter is

$$\frac{\partial \hat{Q}_D}{\partial s_{\mathbf{y}_i}} = \sum_{\mathbf{x}\in\mathbb{I}} \left(\hat{D}f_{\mathbf{x}} - g_{\mathbf{x}}\right) \frac{e^{f_{\mathbf{x}+\mathbf{y}_i} + s_{\mathbf{y}_i}}}{e^{\hat{D}f_{\mathbf{x}}}}. \tag{16}$$



(a) (b) (c)

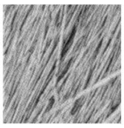
Fig. 1. Texture images for experiments.



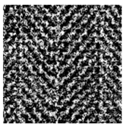
MSE: 6955

| | | | | |
|---|---|---|---|---|
| −115 | −107 | −65 | −19 | −100 |
| −112 | −87 | −19 | −35 | −106 |
| −104 | −62 | 0 | −82 | −117 |
| −93 | −29 | −22 | −104 | −121 |
| −77 | −20 | −79 | −117 | −121 |

(a) D15



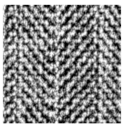
MSE: 299



MSE: 5678

| | | | | |
|---|---|---|---|---|
| −185 | −178 | −160 | −175 | −195 |
| −189 | −134 | −46 | −108 | −173 |
| −187 | −94 | 6 | −63 | −168 |
| −192 | −132 | −44 | −139 | −185 |
| −189 | −172 | −151 | −186 | −192 |

(b) D16



MSE: 1367

MSE: 5954

| | | | | |
|---|---|---|---|---|
| −210 | −179 | −30 | −183 | −231 |
| −208 | −174 | −11 | −179 | −231 |
| −206 | −176 | 2 | −181 | −231 |
| −207 | −166 | −11 | −183 | −231 |
| −207 | −173 | −29 | −188 | −231 |

(c) D68

MSE: 453

Fig. 2. Design examples of dilation filters. left: Noisy images, center: resultant SEs and right: opening results

For the cost function of the opening filter, the partial differential is

$$\frac{\partial \hat{Q}_O}{\partial s_{\mathbf{y}_i}} = \sum_{\mathbf{x}\in\mathbb{I}} (\hat{O} f_{\mathbf{x}} - g_{\mathbf{x}}) e^{-\hat{O} f_{\mathbf{x}}} \, (e^{\hat{E} f_{\mathbf{x}+\mathbf{y}_i}} e^{s_{\mathbf{y}_i}} + \sum_{\mathbf{y}\in\mathbb{C}} \frac{\partial e^{\hat{E} f_{\mathbf{x}+\mathbf{y}}}}{\partial s_{\mathbf{y}_i}} e^{s_{\mathbf{y}}} \,) \tag{17}$$

where

$$\frac{\partial e^{\hat{E} f_{\mathbf{x}}}}{\partial s_{\mathbf{y}_i}} = -\frac{e^{-f_{\mathbf{x}+\mathbf{y}_i}} e^{s_{\mathbf{y}_i}}}{(\sum_{\mathbf{y}\in\mathbb{C}} e^{-f_{\mathbf{x}+\mathbf{y}}} e^{s_{\mathbf{y}}})^2}. \tag{18}$$

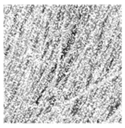
We apply the gradient descent method by using the gradient that are given by (16) and (18) to minimize the squared error (14) and (15), respectively.

## V. Design Examples

In this section, some examples of the design of the SEs by using the approximations of the morphological filters. In order to minimize the squared error between an ideal image and a morphological filter output in (14) or (15), we apply the gradient descent algorithm with a line search. The vector of the SE $\mathbf{s} = (s_{\mathbf{y}_1} \cdots s_{\mathbf{y}_N})^{\mathrm{T}}$, where $N$ is the number of the elements in the SE, is updated with the rule

$$\mathbf{s} \leftarrow \mathbf{s} - t\nabla\hat{Q} \tag{19}$$

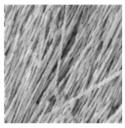
where the elements of the gradient $\nabla\hat{Q}$ is given by (16) for the dilation filter, (18) for the opening filter, respectively. $t$ is specified by using the line search at each iteration. The iteration is stopped when the decrement rate $(Q_i - Q_{i+1})/Q_i$, where $Q_i$ is the cost of the i-th iteration, is less than $10^{-4}$.



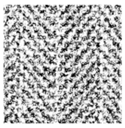
MSE: 3977

| | | | | |
|---|---|---|---|---|
| −46 | 36 | 63 | −7 | −47 |
| −8 | 69 | 54 | −37 | −37 |
| 39 | 75 | 25 | −54 | −45 |
| 64 | 58 | −26 | −58 | −44 |
| 62 | 9 | −53 | −46 | −45 |

(a) D15



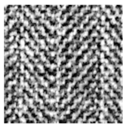
MSE: 231



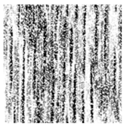
MSE: 5250

| | | | | |
|---|---|---|---|---|
| −75 | −65 | 25 | 57 | −68 |
| −61 | −1 | 128 | 122 | −66 |
| −67 | 69 | 145 | 131 | −20 |
| −81 | 10 | 105 | 83 | −50 |
| −93 | −64 | −39 | −45 | −79 |

(b) D16



MSE: 1083



MSE: 4936

| | | | | |
|---|---|---|---|---|
| 130 | −11 | −37 | −48 | −35 |
| 145 | −8 | −49 | −52 | −42 |
| 153 | −8 | −50 | −54 | −43 |
| 146 | −14 | −50 | −54 | −46 |
| 133 | 14 | −31 | −50 | −31 |

(c) D68

MSE: 610

Fig. 3. Design examples of opening filters. left: Noisy images, center: resultant SEs and right: opening results

(a) Noisy Input
(MSE:5661)

(b) Ideal output

| | | | | |
|---|---|---|---|---|
| −71 | −28 | 32 | 13 | −71 |
| −49 | 0 | 50 | 47 | −25 |
| −51 | 39 | 51 | 48 | 0 |
| −44 | 39 | 51 | 45 | −17 |
| −54 | 25 | 45 | 4 | −78 |

(c) Optimum SE

(d) Opening with optimum SE
(MSE: 107)

Fig. 4. Design of opening filter for Lenna image.

For the approximation of the morphological filters, the scale parameter $T$ have to be specified. The approximation precisions of the morphological filters depend on the scale parameter $T$. As seen in (6) and (8), the absolute of the difference between the true and the approximation of the min or max function decreases along with the decrement of $T$. However, smaller $T$ tends to cause slow convergence rate since the range of the exponents expands with $T$ decreases. $T$ is empirically specified as $4.0$ for gray-level images, intensities of which in the range $[0, 255]$. The images that are employed for the experiments are shown in Fig. 1. Each image contains $128 \times 128$ pixels and is extracted from the $512 \times 512$ image of the Brodatz texture database. Each image consists of a texture that can be characterized with a simple local structure and is appropriate to be modeled by mathematical morphology with single SE.

First, we demonstrate the design of the dilation filters, which

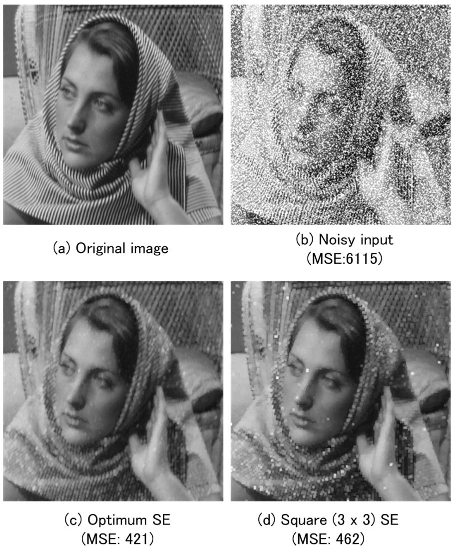



Fig. 5. Denoising results obtained by opening filters.

minimizes the error between the original images shown in Fig. 1 and the filter outputs. The dilation filter can fill the pits of the intensity surface of an image. Under this limitation, the noise is generated by adding a random value in range $[-255, 0]$ to each pixel with occurrence probability $0.5$. The noisy images of the texture image in Fig. 1 are shown in the left column of Fig. 2. The iteration of the gradient descent starts with the initial SE that are specified as a flat $5 \times 5$ square SE for all pairs of a clean and a noisy image. The SEs after convergence of the minimization are shown in the second column of Fig. 2. The shape of the SE depends on the texture, to which the SE is adapted. Each SE is similar to the local structure of its corresponding texture image. The outputs of the dilation filters with the optimized SEs are shown in the third column of Fig. 3. These results are given by the dilation with the true max function. We note that the significant difference between the true and approximated dilation filter can not be observed in all results. Actually, the difference of MSE (Mean Squared Error) between the approximated and the true dilation filter is less than $1\%$ of the MSE obtained by the true filter and is negligible. As seen in Fig. 3, the noises are well suppressed in the outputs of the dilation filters. Simultaneously, the local structures of the textures are preserved using the optimized SEs.

The design examples for the opening filters are shown in Fig. 3. On the contrary of the dilation filter, the opening filter removes small peaks of the images. Thus, the opening filter is used for the removal of the noises, values of which are positive. Under this limitation, the noise is generated by adding a random value in range $[0, 255]$ to each pixel with occurrence probability $0.5$. The iteration of the gradient descent starts with the initial SE that are specifies as a flat $5 \times 5$ square SE. The noisy images that are corrupted by the positive noises are shown in the first column of Fig. 3. The converged SEs after the minimization are shown in the second column of Fig. 3. The results of the opening filters with the optimized SEs are shown in the third column of Fig. 3. As well as the dilation filter, the SE that is optimized to preserve the texture reflects the local structure of image.

In the results shown in Fig. 2 and 3, the texture images are employed for demonstration of the SE design. Moreover, the target ideal images are obtained. In next example, the SE of the opening filter is optimized to minimize the error between the noisy and ideal images, and the resultant SE is applied to denoising of a different image. For the design of the SE, the pair of an ideal and noisy images shown in Fig. 4(a) and (b) is employed. The optimum SE for the opening of Fig. 4(a) is shown in Fig. 4(c). This optimum SE is applied to the denoising of the noisy image shown in Fig. 5(b). The denoising result obtained with the optimum SE in Fig. 4(c) is shown in Fig. 5(c). For comparison, the result obtained with a $3 \times 3$ flat square SE, which is widely utilized for morphological image analysis, is shown in Fig. 5(d). Compared with the square SE, the noise is well suppressed while preserving the fine textures in the denoising result obtained with the SE which is designed by the proposed method.

## VI. Conclusions

In this paper, we propose the SE design method for the morphological filters. In order to introduce the gradient-based optimization, we approximate the morphological filters with the approximating function of the min and max functions. In experiments, we show that the SEs that are obtained by the proposed approximation well approximate the local structures of the texture images while eliminating noise components.

In our approach, the cost function for the SE is defined by the pair of the noisy and ideal image. In actual processing, the ideal image is not obtained. Hence, the SE should be designed from the noisy image without the ideal image. The cost function that does not include the ideal image is a topic of the future research.

## References

[1] P. Maragos, “Morphological filtering for image enhancement and feature detection, ” Chapter 3. 3 for the Book: The Image and Video Processing Handbook, A. C. Bovik Ed., Elsevier Academic Press, pp. 135-156, 2005.

[2] E. J. Coyle and J. H. Lin, “Stack filters and the mean absolute error criterion, ” IEEE Trans. on ASSP, vol. 35, no. 8, pp. 1244-1254, Aug. 1988.

[3] P. Salembier, “Adaptive rank order based filters, ” Signal Processing, vol. 27, pp. 1-25, 1992.

[4] L. F. C. Pessoa and P. Maragos, “MRL-Filters: a general class of nonlinear systems and their optimal design for image processing, ” IEEE Trans. on Image Processing, vol. 7, no. 7, pp. 966-978, July 1998.

[5] Y. Won, “Shift-invariant fuzzy-morphology neural network for automatic target recognition, ” IEICE Trans. on Fundamentals, vol. E81-A, no. 6, pp. 1119-1127, June 1998.

[6] N. R. Harvey and S. Marshall, “The use of genetic algorithms in morphological filter design, ” Signal Processing: Image Communication, vol. 8, pp. 55-71, 1996.

[7] A. Asano, T. Syudoh and S. Yokozeki, “Efficient algorithm for learning optimization of morphological filters, ” Optical Review, vol. 2, no. 3, pp. 159-162, 1995.

[8] A. Asano, Y. Kobayashi, C. Muraki and M. Muneyasu, “Optimization of gray scale morphological opening for noise removal in texture images, ” Proc, on The 47th IEEE Int. Midwest Symposium of Circuits and Systems, vol. I, pp. 313-316, July 2004.