1) $U(x) = x - \frac{\alpha x^2}{2}$

$$\mathbb{E}[U(x)] = \mathbb{E}[x] - \mathbb{E}\left[\frac{\alpha x^2}{2}\right]$$

where $x \sim N(\mu, \sigma^2)$

$$= \mathbb{E}[x] - \frac{\alpha}{2}\mathbb{E}[x^2]$$

$$\sigma^2 = \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

$$\sigma^2 = \mathbb{E}[x^2] - \mu^2$$

$$\mathbb{E}[x^2] = \sigma^2 + \mu^2$$

$$\boxed{\mathbb{E}[U(x)] = \mu - \frac{\alpha(\mu^2+\sigma^2)}{2}}$$

$$X_{CE} = U^{-1}(\mathbb{E}[U(x)])$$

$$X_{CE} = U^{-1}\left(\mu - \frac{\alpha(\mu^2-\sigma^2)}{2}\right)$$

x s.t.: $x - \frac{\alpha x^2}{2} = \mu - \frac{\alpha(\mu^2-\sigma^2)}{2} = \mu - \frac{\alpha\mu^2}{2} + \frac{\alpha\sigma^2}{2}$

$$\boxed{X_{CE} = \frac{1 + \sqrt{1 + (\mu^2+\sigma^2)\alpha^2 - 2\alpha\mu}}{\alpha}}$$

$$\tilde{\Pi}_A = \mathbb{E}[x] - x_{CE}$$

$$\boxed{\tilde{\Pi}_A = \mu - \frac{1 + \sqrt{1 + (\mu^2+\sigma^2)\alpha^2 - 2\alpha\mu}}{\alpha}}$$

Portfolio Return $= zx + (1-z)r$

Maximize Expected utility: $\max_z \mathbb{E}\left[U(zx + (1-z)r)\right]$

$$\Rightarrow \boxed{z = -\frac{\alpha r \mu - \alpha r^2 - \mu + r}{\alpha(\mu^2 - 2\mu r + r^2 + \sigma^2)}}$$

> $U(x) := x - \frac{\alpha \cdot x^2}{2}$:

> $U(z \cdot x + (1-z) \cdot r)$

$$zx + (1-z)r - \frac{\alpha(zx + (1-z)r)^2}{2} \tag{1}$$

> $expand(\%)$

$$zx - rz + r - \frac{1}{2}\alpha r^2 z^2 + \alpha r x z^2 - \frac{1}{2}\alpha x^2 z^2 + \alpha r^2 z - \alpha r x z - \frac{1}{2}\alpha r^2 \tag{2}$$

Note: E[x] = mu, E[x^2] = $\sigma^2 + \mu^2$

> $Expectation := z\mu - rz + r - \frac{1}{2}\alpha r^2 z^2 + \alpha r \mu \cdot z^2 - \frac{1}{2}\alpha \cdot (\sigma^2 + \mu^2) z^2 + \alpha r^2 z - \alpha r \mu z - \frac{1}{2}\alpha r^2$

$$Expectation := z\mu - rz + r - \frac{\alpha r^2 z^2}{2} + \alpha r \mu z^2 - \frac{\alpha(\mu^2 + \sigma^2) z^2}{2} + \alpha r^2 z - \alpha r \mu z - \frac{\alpha r^2}{2} \tag{3}$$

> $\frac{\partial}{\partial z}(Expectation)$

$$\mu - r - \alpha r^2 z + 2\alpha r \mu z - \alpha(\mu^2 + \sigma^2) z + \alpha r^2 - \alpha r \mu \tag{4}$$

> $solve(\% = 0, z)$

$$-\frac{\alpha r \mu - \alpha r^2 - \mu + r}{\alpha(\mu^2 - 2\mu r + r^2 + \sigma^2)} \tag{5}$$

```
> r := 0.05 :
  μ := 0.11 :
  sigma := 0.03 :
> optimal
```

$$-\frac{222.2222222\,(0.0030\,\alpha - 0.06)}{\alpha}$$

```
> solve(optimal = 0, α)
```

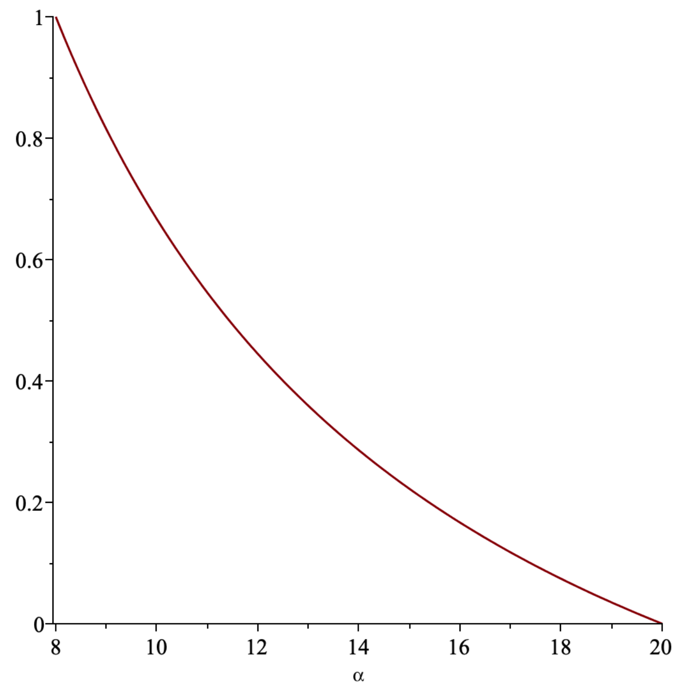
$$20.$$

```
> solve(optimal = 1, α)
```

$$8.000000000$$

```
> plot(optimal, α = 8 ..20)
```

z



Alpha is your level of risk-aversion. As alpha increases, you become more risk averse, and at a certain level, you put all of your portfolio into the risk less investment. As you decrease your risk aversion, you increase your allocation to the risky asset (z increases as alpha decreases).

3)

$$\begin{cases} x(1-\alpha & \text{w.p.} \quad p \\ x(1-\beta & \text{w.p.} \quad 1-p \end{cases}$$

$$U(w) = \log(w)$$

$$\rightarrow \quad w_1 = \begin{cases} w_0(1-f) + f w_0(1+\alpha) & \text{w.p.} \quad p \\ = w_0(1+\alpha f) & \\ w_0(1-f) + f w_0(1-\beta) & \text{w.p.} \quad 1-p \\ = w_0(1-\beta f) & \end{cases}$$

$$\rightarrow \log(w_1) = \begin{cases} \log(w_0(1+\alpha f)) & \text{w.p.} \quad p \\ \log(w_0(1-\beta f)) & \text{w.p.} \quad 1-p \end{cases}$$

$$\mathbb{E}[\log(W)] = p\log(W_0(1+\alpha f)) + (1-p)\log(W_0(1-\beta f))$$

$$\frac{\partial}{\partial f} = \frac{p\alpha}{\alpha f+1} - \frac{(1-p)\beta}{-bf+1} = 0$$

$$\Rightarrow \boxed{f^* = \frac{p\alpha - (1-p)\beta}{\alpha\beta}}$$

$$\frac{\partial^2}{\partial f^2} = \frac{-p\alpha^2}{(1+\alpha f)^2} - \frac{(1-p)\beta^2}{(1-\beta f)^2} < 0$$

$\rightarrow$ Concave since $\frac{\partial^2}{\partial f^2} < 0$. So f^* is a max.

The formula makes sense since the higher the probability of a positive return, the more you bet. At the extremes, if p=1, then you bet everything. If p=0, you bet nothing, since this is a guaranteed loss. For fixed beta, increasing alpha means you increase your bet. Likewise, for fixed alpha, increasing beta means you decrease your bet.