$\textsc{ma}274$, Fall 2021 — Problem Set 6

Matthew Bass

1. Each of the statements in exercises 3.3.8 and 3.3.15 in NSSM can be translated into the language of unions and intersections of sets. Give all eight translations. (No need to explain, just give the statements.) Are the translated statements easier to understand than the original versions?

3.3.8

1. $x \in A$ and $x \in B$

$A \cap B$

2. $x \in A$ or $x \in B$

$A \cup B$

3. For all $A \in \mathscr{A}, a \in A$.

$\bigcap_{A \in \mathscr{A}} A$

4. There exists $A \in \mathscr{A}$, such that $a \in A$.

$\bigcup_{A \in \mathscr{A}} A$

3.3.15

1. For all $x \in X$, there exists $H \in \mathscr{H}$, such that $x \in H$.

$X \cap \bigcup_{H \in \mathscr{H}} H$

2. For all $x \in X$ and for all $H \in \mathscr{H}$, we have $x \in H$.

$X \cap \bigcap_{H \in \mathscr{H}} H$

3. There exists $x \in X$ such that for all $H \in \mathscr{H}$, $x \in H$.

$$X \cup \bigcap_{H \in \mathscr{H}} H$$

4. There exists $x \in X$ and there exists $H \in \mathscr{H}$, such that $x \in H$.

$$X \cup \bigcup_{H \in \mathscr{H}} H$$

2. Read Example 7.0.2 in NSSM. Are $\frac{1}{3}$ and $\frac{5}{15}$ the same? Why or why not?

Proof: Suppose that $=$ is a relationship. To show that it is an equivalent relationship we must show that $=$ is reflexive, symmetric, and transitive.

Reflexive: For all x, $x = x$, so $=$ is reflexive.

Symmetric: For all x and y, if $x = y$, then $y = x$, so $=$ is symmetric.

Transitive: For all x, y and z, if $x = y$ and $y = z$, then $x = z$, so $=$ is transitive.

Because $=$ is reflexive, symmetric, and transitive, it is an equivalence relationship.

Therefore, $\frac{1}{3}$ and $\frac{5}{15}$ are not the same but they are equivalent. $\square$

3. Let $\mathscr{L}$ be the set of all lines on the plane. Define a relation by saying, tor all $a, b \in \mathscr{L}$, that aRb if and only if a and b are parallel. Is this relation reflexive? Is it symmetric? Is it transitive?

Reflexive: Let $a \in \mathscr{L}$

Based on the definition of parallel lines, any two lines $a, b \in \mathscr{L}$ are parallel if they have the same slope.

Since a line always has the same slope as itself, that means aRa.

Thus the relationship is reflexive.

Symmetric: Let $a, b \in \mathscr{L}$

If aRb, then a is parallel with b.

Based on the definition of parallel lines, any two lines $a, b \in \mathscr{L}$ are parallel if they have the same slope.

That means that if a is parallel with b then they have the same slopes meaning b is parallel with a also, so if aRb then bRa.

Thus the relationship is symetric.

Transitive: Let $a, b, c \in \mathscr{L}$

If aRb, then a is parallel with b. So that means the slope of $a =$ the slope of b.

Then if bRc, then b is parallel with c. So that means the slope of $b =$ the slope of c.

Since we showed in question 2 of the problem set that $=$ is an equivelence relationship, that means the slope of $a =$ the slope of c.

Therefore a is parallel with c, meaning aRc.

Thus the relationship is transitive.

4. NSSM, Exercise 7.2.7, items 1–5

***7.2.7.1*:** The real numbers $\mathbb{R}$ with the relation $\leq$ (the usual "less than or equal to")

Reflexive: Let x be any $x \in \mathscr{R}$

If $x \leq X$, that means $(x < x) \vee (x = x)$, and $x = x$ is always true so xRx.

Thus the real numbers $\mathbb{R}$ with the relation $\leq$ is reflexive.

Symmetric: We know $1, 2 \in \mathscr{R}$

$1 \leq 2$ however $2 \not\leq 1$

Thus the real numbers $\mathbb{R}$ with the relation $\leq$ is not symmetric.

Also more generally for any $x, y \in \mathscr{R}$, if $x \leq y$, that means $(x < y) \vee (x = y)$.

If $x < y$ then $y \leq x$ can not be true because that would means $(y < x) \vee (y = x)$ both of which can not be true.

Thus further proving, the real numbers $\mathbb{R}$ with the relation $\leq$ is not symmetric.

Anti-Symmetric: Let $x, y \in \mathscr{R}$, if $x \leq y$ and $y \leq x$ that means $\big((x < y) \vee (x = y)\big) \wedge \big((y < x) \vee (y = x)\big)$.

The only way for that logical statement to hold true is if $x = y$

Thus proving, the real numbers $\mathbb{R}$ with the relation $\leq$ is anti-symmetric.

Transitive: Let $x, y, z \in \mathscr{R}$

If $x \leq y$, that means $(x < y) \vee (x = y)$

If $y \leq z$, that means $(y < z) \vee (y = z)$

Therefore, xRz, thus proving, the real numbers $\mathbb{R}$ with the relation $\leq$ is transitive.

***7.2.7.2*:** The set $\mathscr{P}(\mathbb{R}^2)$ with the relation $\subset$.

Reflexive: The relation is reflexive because we have proved that for any set X, $X \subset X$.

Symmetric: Let $A = \{(0,0)\}$ and $B = \{(0,0),(1,1)\}$
We know $A, B \in \mathscr{P}(\mathbb{R}^2)$

$A \subset B$ but $B \not\subset A$

Therefore, the relation $\subset$ with the set $\mathscr{P}(\mathbb{R}^2)$ is not symmetric

Anti-Symmetric: The relation $\subset$ with the set $\mathscr{P}(\mathbb{R}^2)$ is anti-symmetric because we have proved that for any sets X, Y, if $X \subset Y$ and $Y \subset X$, then $X = Y$.

Transitive: For any sets A, B, C. By the definition of subsets.

If $A \subset B$ then $\forall a \in A[a \in B]$.

Also if $B \subset C$ then $\forall b \in B[b \in C]$.

Which implies $\forall a \in A[a \in C]$. There by definition of subsets that means $A \subset C$.

Thus proving that the relation $\subset$ with the set $\mathscr{P}(\mathbb{R}^2)$ is transitive.

***7.2.7.3*:** The set

$$N = \left\{0, \{0\}, \{0, \{0\}\}, \left\{0, \{0\}, \{0, \{0\}\}\right\}, \dots\right\}$$

with the relation $\in$ (meaning "is an element of").

Reflexive: The relation $\in$ with set N is not reflexive because any set is not an element of itself , and any element is not an element.

Symmetric: We know $0, \{0\} \in N$

$0 \in \{0\}$ but $\{0\} \notin 0$

Therefore, the relation $\in$ with the set N is not symmetric

Anti-Symmetric: For any $x, y \in N$.

It is true that $(x \in y) \vee (y \in x)$ but it will never be true that $(x \in y) \wedge (y \in x)$.

Therefore it is vacuously true that the relation $\in$ with set N is anti-symmetric.

Transitive: The relation $\in$ with set N is transitive because every element in N contains every other preceding element in N

So for $a, b, c \in N$. If $a \in b$ and $b \in c$, then $a \in c$ based off of the definition of set N.

***7.2.7.4*:** The set $\left\{0, \{0\}, \{\{0\}\}\right\}$ with the relation $\in$ (meaning "is an element of").

Reflexive: The relation $\in$ with set $\left\{0, \{0\}, \{\{0\}\}\right\}$ is not reflexive because any set is not an element of itself , and any element is not an element.

Symmetric: We know $0, \{0\} \in \left\{0, \{0\}, \{\{0\}\}\right\}$

$0 \in \{0\}$ but $\{0\} \notin 0$

Therefore, the relation $\in$ with the set $\left\{0, \{0\}, \{\{0\}\}\right\}$ is not symmetric.

Anti-Symmetric: For any $x, y \in \left\{0, \{0\}, \{\{0\}\}\right\}$.

It is true that $(x \in y) \vee (y \in x)$ but it will never be true that $(x \in y) \wedge (y \in x)$.

Therefore it is vacuously true that the relation $\in$ with set $\Big\{0, \{0\}, \{\{0\}\}\Big\}$ is anti-symmetric.

Transitive: We know $0, \{0\}, \{\{0\}\} \in \Big\{0, \{0\}, \{\{0\}\}\Big\}$

$0 \in \{0\}$ and $\{0\} \in \{\{0\}\}$.

But $0 \notin \{\{0\}\}$.

Therefore, the relation $\in$ with the set $\Big\{0, \{0\}, \{\{0\}\}\Big\}$ is transitive.

***7.2.7.5*:** The positive real numbers $\mathbb{Q}^+$ with the relation $\sim$ defined by $a \sim b$ if and only if $\frac{a}{b} \leq 1$.

Reflexive: For any $a \in \mathbb{Q}^+$, $\frac{a}{a} = 1$, and $1 \leq 1$ meaning $a \sim a$.

Thus the relation $\sim$ with the positive real numbers $\mathbb{Q}^+$ is reflexive.

Symmetric: We know $2, 3 \in \mathbb{Q}^+$

$\frac{2}{3} \leq 1$ but $\frac{2}{3} > 1$

Therefore, the relation $\sim$ with the positive real numbers $\mathbb{Q}^+$ is not symmetric.

Anti-Symmetric: For any $a, b \in \mathbb{Q}^+$.

If $a \sim b$ and $b \sim a$ that means $\frac{a}{b} \leq 1$ and $\frac{b}{a} \leq 1$.

If a is greater than b, $\frac{b}{a} < 1$ but $\frac{a}{b} > 1$ and visa-versa if $b > a$.

Therefore that means if $a \sim b$ and $b \sim a$ then $\frac{a}{b} = 1$ and $\frac{b}{a} = 1$, which means $a = b$.

Therefore it is true that the relation $\sim$ with the positive real numbers $\mathbb{Q}^+$ is anti-symmetric.

Transitive: For any $a, b, c \in \mathbb{Q}^+$.

If $a \sim b$, then $\frac{a}{b} \leq 1$, which means $a \leq b$.

Also if $b \sim c$, then $\frac{b}{c} \leq 1$, which means $b \leq c$.

Therefore, $a \leq b \leq c$, which means $a \leq c$, so $\frac{a}{c} \leq 1$.

Thus, $a \sim c$, proving the relation $\sim$ for all positive real numbers $\mathbb{Q}^+$ is transitive.

5. NSSM, Exercise 7.2.7, items 6–11

***7.2.7.6*:** The integers $\mathbb{Z}$ with the relation $\equiv_5$, defined by declaring $x \equiv_5 y$ if and only if $x - y = 5k$ for some $k \in \mathbb{Z}$.

Reflexive: For any $x \in \mathbb{Z}$, $x - x = 5(0)$ and since $0 \in \mathbb{Z}$, $x \equiv_5 x$

Thus the relation $\equiv_5$ with the integers $\mathbb{Z}$ is reflexive.

Symmetric: Let $x, y \in \mathbb{Z}$ be arbitrary.

If $x \equiv_5 y$ then, $x - y = 5(k)$

Multiplying both sides of $x - y = 5(k)$ by -1 gives us $y - x = 5(-k)$, meaning that $y \equiv_5 x$

Therefore, the relation $\equiv_5$ with the integers $\mathbb{Z}$ is symmetric.

Anti-Symmetric: We know $1, 6 \in \mathbb{Z}$.

$1 - 6 = 5(-1)$ meaning $1 \equiv_5 6$ and $6 - 1 = 5(1)$ meaning $6 \equiv_5 1$

But $1 \neq 6$. Therefore, the relation $\equiv_5$ with the integers $\mathbb{Z}$ is not anti-symmetric.

Transitive: For any $x, y, z \in \mathbb{Z}$.

If $x \equiv_5 y$, then $x - y = 5(k_1)$ and if $y \equiv_5 z$, then $y - z = 5(k_2)$

Adding the two equations together we get $(x - y) + (y - z) = 5(k_1) + 5(k_2)$

Simplifying the equation we get $x - z = 5(k_1 + k_2)$, showing that $x \equiv_5 z$

Thus, proving the relation $\equiv_5$ for integers $\mathbb{Z}$ is transitive.

***7.2.7.7*:** The set $\mathbb{R}^2\backslash\{(0,0)\}$ with the relation $\sim$ defined by declaring $(x, y) \sim (a, b)$ if and only if there exists $k \in \mathbb{R}\backslash\{0\}$ such that $(x, y) = (ka, kb)$.

Reflexive: For any $(x, y) \in \mathbb{R}^2\backslash\{(0,0)\}$, $(x, y) = \big((1)x, (1)y\big)$ and since $1 \in \mathbb{R}\backslash\{0\}$, $(x, y) \sim (x, y)$

Thus, the relation $\sim$ with the set $\mathbb{R}^2\backslash\{(0,0)\}$ is reflexive.

Symmetric: For any $(x, y), (a, b) \in \mathbb{R}^2\backslash\{(0,0)\}$

If $(x, y) \sim (a, b)$ that means, $(x, y) = \big((k)a, (k)b\big)$

So $x = (k)a$ and $y = (k)b$

Solving for a and b we get $a = x\frac{1}{k}$ and $b = y\frac{1}{k}$ a

That means $(a, b) = \big((\frac{1}{k})x, (\frac{1}{k})y\big)$ and also since $\frac{1}{k} \in \mathbb{R}\backslash\{0\}$, $(a, b) \sim (x, y)$

Therefore, the relation sim with the set $\mathbb{R}^2\backslash\{(0,0)\}$ is symmetric.

Anti-Symmetric: We know $(1, 1), (2, 2) \in \mathbb{R}^2\backslash\{(0,0)\}$.

$(1, 1) \sim (2, 2)$ since $0.5 \in \mathbb{R}\backslash\{0\}$ and $(1, 1) = \big((0.5)2, (0.5)2\big)$

Also, $(2, 2) \sim (1, 1)$ since $2 \in \mathbb{R}\backslash\{0\}$ and $(2, 2) = \big((2)1, (2)1\big)$

But $(1,1) \neq (2,2)$. Therefore, the relation *sim* with the set $\mathbb{R}^2 \backslash \{(0,0)\}$ is not anti-symmetric.

Transitive: For any $(x,y),(a,b),(c,d) \in \mathbb{R}^2 \backslash \{(0,0)\}$

If $(x,y) \sim (a,b)$ that means, $(x,y) = \big((k_1)a,(k_1)b\big)$

Then if $(a,b) \sim (c,d)$ that means, $(a,b) = \big((k_2)c,(k_2)d\big)$

So $a = (k_2)c$ and $b = (k_2)d$

Then subbing in those values, $(x,y) = \big((k_1)(k_2)c,(k_1)(k_2)d\big)$

That means $(x,y) \sim (c,d)$

Therefore, the relation *sim* with the set $\mathbb{R}^2 \backslash \{(0,0)\}$ is transitive.

***7.2.7.8*:** The set $\mathbb{Z} \times \mathbb{N}$ with the relation $\sim$ defined by declaring $(a,b) \sim (c,d)$ if and only if $ad = bc$.

Reflexive: For any $(a,b) \in \mathbb{Z} \times \mathbb{N}$, based off of what we know about multiplication ($N \subset Z$ so axiom [MC]) $ab = ba$, therefore $(a,b) \sim (a,b)$

Thus, the relation $\sim$ with the set $\mathbb{Z} \times \mathbb{N}$ is reflexive.

Symmetric: For any $(a,b),(c,d) \in \mathbb{Z} \times \mathbb{N}$

If $(a,b) \sim (c,d)$ that means, $ad = bc$.

Based off of what we know about multiplication ($N \subset Z$ so axiom [MC]) $ad = da$ and $bc = cb$.

Therefore, $da = cb$, meaning $(c,d) \sim (a,b)$

Thus, the relation $\sim$ with the set $\mathbb{Z} \times \mathbb{N}$ is symmetric.

Anti-Symmetric: We know $(1,2),(2,4) \in \mathbb{Z} \times \mathbb{N}$.

$(1,2) \sim (2,4)$ since $1(4) = 4 = 2(2)$

Also, $(2,4) \sim (1,2)$ since $2(2) = 4 = 1(4)$

But, $(1,2) \neq (2,4)$

Thus, the relation $\sim$ with the set $\mathbb{Z} \times \mathbb{N}$ is not anti-symmetric.

Transitive: For any $(a,b),(c,d),(x,y) \in \mathbb{Z} \times \mathbb{N}$

If $(a,b) \sim (c,d)$ that means, $ad = bc$, which can be rewritten as $\frac{a}{b} = \frac{c}{d}$.

Also, if $(c,d) \sim (x,y)$ that means, $cy = dx$, which can be rewritten as $\frac{c}{d} = \frac{x}{y}$.

Therefore, $\frac{a}{b} = \frac{c}{d} = \frac{x}{y}$, so $\frac{a}{b} = \frac{x}{y}$.

$\frac{a}{b} = \frac{x}{y}$ can be rewritten as $ay = bx$, meaning $(a,b) \sim (x,y)$.

Thus, the relation $\sim$ with the set $\mathbb{Z} \times \mathbb{N}$ is transitive.

***7.2.7.9*:** Let a and b be real numbers ($a, b \in \mathbb{R}$). Define $a \sim b$ if and only if $|a-b| \leq 1$.

Reflexive: We know for any $a \in \mathbb{R}$

$|a-a| = |0| = 0 \leq 1$

Therefore, $a \sim a$

Thus, the relation $\sim$ with the real numbers $\mathbb{R}$ is reflexive.

Symmetric: For any $a, b \in \mathbb{R}$

$|a-b| = |b-a|$

So if $a \sim b$ then that means $|a-b| \leq 1$

Which means $|b-a| \leq 1$, so $b \sim a$

Thus, the relation $\sim$ with the real numbers $\mathbb{R}$ is symmetric.

Anti-Symmetric: We know $3, 4 \in \mathbb{R}$

$3 \sim 4$ since $|3 - 4| = |(-1)| = 1 \leq 1$

Also, $4 \sim 3$ since $|4 - 3| = |(1)| = 1 \leq 1$

$3 \neq 4$ thus, the relation $\sim$ with the real numbers $\mathbb{R}$ is not anti-symmetric.

Transitive: We know $3, 4, 5 \in \mathbb{R}$

$3 \sim 4$ since $|3 - 4| = |(-1)| = 1 \leq 1$

Also, $4 \sim 5$ since $|4 - 5| = |(-1)| = 1 \leq 1$

But $|3 - 5| = |(-2)| = 2 > 1$, so $3 \nsim 5$

Thus, the relation $\sim$ with the real numbers $\mathbb{R}$ is not transitive.

***7.2.7.10*:** Let a and b be points in $\mathbb{R}^2$. Define $a \sim b$ if and only if $d(a, b) \leq 1$. (Here $d(a, b)$ is the distance from a to b.)

Reflexive: We know for any $a \in \mathbb{R}^2$

$d(a, a) = 0 \leq 1$

Therefore, $a \sim a$

Thus, the relation $\sim$ with the set of pairs of real numbers $\mathbb{R}^2$ is reflexive.

Symmetric: We know for any $a, b \in \mathbb{R}^2$

$d(a, b) = d(b, a)$

So if $a \sim b$ then that means $d(a, b) \leq 1$

Which means $d(b, a) \leq 1$, so $b \sim a$

Thus, the relation $\sim$ with the set of pairs of real numbers $\mathbb{R}^2$ is symmetric.

Anti-Symmetric: We know $(0,1),(0,2) \in \mathbb{R}^2$

$(0,1) \sim (0,2)$ since $d\big((0,1),(0,2)\big) = 1 \leq 1$

Also, $(0,2) \sim (0,1)$ since $d\big((0,2),(0,1)\big) = 1 \leq 1$

$(0,1) \neq (0,2)$ thus, the relation $\sim$ with the set of pairs of real numbers $\mathbb{R}^2$ is not anti-symmetric.

Transitive: We know $(0,1),(0,2),(0,3) \in \mathbb{R}^2$

$(0,1) \sim (0,2)$ since $d\big((0,1),(0,2)\big) = 1 \leq 1$

Also, $(0,2) \sim (0,3)$ since $d\big((0,2),(0,3)\big) = 1 \leq 1$

But $d\big((0,1),(0,3)\big) = 2 > 1$, so $(0,1) \nsim (0,3)$

Thus, the relation $\sim$ with the set of pairs of real numbers $\mathbb{R}^2$ is not transitive.

***7.2.7.11*:** A group G having operation o and identity $\mathbb{1}$. The relation $\sim$ is defined by declaring $g_1 \sim g_2$ if and only if there exists $h \in G$ such that $g_1 = h^{-1} \circ g_2 \circ h$

Reflexive: We know for any $g \in G$, since G is a group

By the identity axiom (G2) $g = g \circ \mathbb{1}$

By the inverses axiom (G3) we know $\exists h, h^{-1} \in G[h \circ h^{-1} = \mathbb{1}]$

So plugging in $h \circ h^{-1}$ for $\not\models$ in $g = g \circ \mathbb{1}$

It gives us $g = g \circ (h \circ h^{-1})$

Then by the axiom of associativity for groups (G4) $g = g \circ (h \circ h^{-1}) = h \circ g \circ h^{-1}$

Therefore,$g = h \circ g \circ h^{-1}$ meaning, $g \sim g$

Thus, the relation $\sim$ with group G having operation o and identity $\mathbb{1}$, is reflexive.

Symmetric: For any $g_1, g_2 \in G$.

If $g_1 \sim g_2$ then $g_1 = h^{-1} \circ g_2 \circ h$

Operate on each side of the equation operate with h^{-1} on the right, giving us $g_1 \circ h^{-1} = h^{-1} \circ g_2 \circ h \circ h^{-1}$.

Due to the group axioms of associativity (G4) and inverses (G3) we know $g_1 \circ h^{-1} = h^{-1} \circ g_2 \circ h \circ h^{-1} = h^{-1} \circ g_2 \circ (h \circ h^{-1}) = h^{-1} \circ g_2 \circ \mathbb{1}$.

Then due to the identity axiom (G2), $g_1 \circ h^{-1} = h^{-1} \circ g_2 \circ \mathbb{1} = h^{-1} \circ g_2$.

Then operate on each side of the equation operate with h on the left, giving us $h \circ g_1 \circ h^{-1} = h \circ h^{-1} \circ g_2$.

Due to the group axioms of associativity (G4) and inverses (G3) we know $h \circ g_1 \circ h^{-1} = h \circ h^{-1} \circ g_2 = (h \circ h^{-1}) \circ g_2 = \mathbb{1} \circ g_2$.

Then due to the identity axiom (G2), $h \circ g_1 \circ h^{-1} = \mathbb{1} \circ g_2 = g_2$.

Thus by the axiom of associativity (G4) $g_2 = h^{-1} \circ g_1 \circ h$ meaning $g_2 \sim g_1$

Thus, the relation $\sim$ with group G having operation o and identity $\mathbb{1}$, is symmetric.

Anti-Symmetric: We know $g_1, g_2 \in G$

$g_1 \sim g_2$ since $g_1 = h^{-1} \circ g_2 \circ h = g_2$

Also, $g_2 \sim g_1$ since $g_2 = h^{-1} \circ g_1 \circ h = g_1$

Since when $g_1 \sim g_2$, $g_1 = g_2$ and when $g_2 \sim g_1$, $g_2 = g_1$, it can be concluded that the relation $\sim$ with group G having operation o and identity $\mathbb{1}$, is anti-symmetric.

Transitive: We know $g_1, g_2, g_3 \in G$

$g_1 \sim g_2$ since $g_1 = h^{-1} \circ g_2 \circ h = g_2$

Also, $g_2 \sim g_3$ since $g_2 = k^{-1} \circ g_3 \circ k = g_3$ for $k, k^{-1} \in G$

Since when $g_1 \sim g_2$, $g_1 = g_2$ and when $g_2 \sim g_3$, $g_2 = g_3$, it can be concluded that $g_1 = g_3$ which can then be expanded as we have done earlier to $g_1 = j^{-1} \circ g_3 \circ j$ for $j, j^{-1} \in G$

Therefore, $g_1 \sim g_3$, thus the relation $\sim$ with group G having operation o and identity $\mathbb{1}$, is transitive.

6. NSSM, Example 7.2.10: check that the relations defined in 2, 3, and 4 are indeed equivalence relations.

***7.2.10.2*:** For real numbers x and y, define $x \sim y$ if and only if $x - y$ is a rational number. Then $\sim$ is an equivalence relation on $\mathbb{R}$.

Reflexive: We know for any $x \in \mathbb{R}$, $x - x = 0$ and $0 \in \mathbb{Q}$.

Thus $x \sim x$, therefore the relation $\sim$ with the real numbers $\mathbb{R}$ is reflexive.

Symmetric: We know for any $x, y \in \mathbb{R}$

If $x \sim y$ then $(x - y) \in \mathbb{Q}$.

Multiply $(x - y)$ by -1 and we get $(x - y)(-1) = y - x$.

Since $-1 \in Q$ and we also know the product of any two rational numbers is also a rational number, that means $(y - x) \in \mathbb{Q}$

Thus $y \sim x$, therefore the relation $\sim$ with the real numbers $\mathbb{R}$ is symmetric.

Transitive: We know for any $x, y, z \in \mathbb{R}$

If $x \sim y$ then $(x - y) \in \mathbb{Q}$.

We can say $x - y = a$ for some $a \in \mathbb{Q}$

And if $y \sim z$ then $(y - z) \in \mathbb{Q}$.

We can say $y - z = b$ for some $b \in \mathbb{Q}$

Solving for y we get $y = z + b$

Plugging in the new value for y into $x-y = a$ we get $x-(z+b) = x - z - b = a$

Then solving for $x - z$ we get $x - z = a + b$, and since $b \in Q$ and we also know the sum of any two rational numbers is also a rational number, that means $(x - z) \in \mathbb{Q}$

Thus $x \sim z$, therefore the relation $\sim$ with the real numbers $\mathbb{R}$ is transitive.

Since the relation has been proven to be reflexive,symmetric, and transitive, it is an equivalence relation.

***7.2.10.3*:** Let U be a set and let $A \subset U$ be a subset. Define $x \sim y$ if and only if either both x and y are elements of A or $x = y$. Then $\sim$ is an equivalence relation on U.

Reflexive: We know for any $x \in U$, $x = x$

Thus $x \sim x$, therefore the relation $\sim$ with the set U and $A \subset U$ is reflexive.

Symmetric: We know for any $x, y \in U$

If $x \sim y$ then $(x, y \in A) \vee (x = y)$.

Which implies $(y, x \in A) \vee (y = x)$, thus $y \sim x$

Therefore the relation $\sim$ with the set U and $A \subset U$ is symmetric.

Transitive: We know for any $x, y, z \in U$

If $x \sim y$ then $(x, y \in A) \vee (x = y)$.

Also if $y \sim z$ then $(y, z \in A) \vee (y = z)$.

There are two cases that make $y \sim z$, $(y, z \in A)$ and $y = z$

Case 1: If $y, z \in A$, because $x \sim y$, no matter what $x, z \in A$. This is because if $x, y \in A$, then obviously $x \in A$
If $x = y$ because $y \in A$ that implies $x \in A$ also.
Thus no matter what since $x, z \in A$, $x \sim z$.

Case 2: If $y = z$, because $x \sim y$, no matter what $x \sim z$. This is because if $x, y \in A$, then since $y = z$ that implies $x, z \in A$.
If $x = y$ because $y = z$ that means $x = z$ also.
Thus no matter what if $y = z$, $x \sim z$

Thus since it doesn't matter how $x \sim y$ and $y \sim z$ as long as they do relate then $x \sim z$.

Therefore the relation $\sim$ with the set U and $A \subset U$ is transitive.

Since the relation has been proven to be reflexive,symmetric, and transitive, it is an equivalence relation.

7.2.10.4: Let $C([0, 1])$ be the set of all continuous real-valued functions on the interval $[0, 1] \subset \mathbb{R}$. For $f, g \in C([0, 1])$, define $f \sim g$ if and only if

$$\int_0^1 f dx = \int_0^1 g dx$$

Then $\sim$ is an equivalence relation.

Reflexive: We know for any $f \in C([0, 1])$,

$\int_0^1 f dx = \int_0^1 f dx$ thus $f \sim f$,

Therefore the relation $\sim$ with $C([0, 1])$, the set of all continuous real-valued functions on the interval $[0, 1] \subset \mathbb{R}$ is reflexive.

Symmetric: We know for any $f, g \in C([0,1])$,

If $f \sim g$ then $\int_0^1 f dx = \int_0^1 g dx$

This means $\int_0^1 g dx = \int_0^1 f dx$, so $g \sim f$

Therefore the relation $\sim$ with $C([0,1])$, the set of all continuous real-valued functions on the interval $[0,1] \subset \mathbb{R}$ is symmetric.

Transitive: We know for any $f, g, h \in C([0,1])$,

If $f \sim g$ then $\int_0^1 f dx = \int_0^1 g dx$

Also if $g \sim h$ then $\int_0^1 g dx = \int_0^1 h dx$

This means $\int_0^1 f dx = \int_0^1 g dx = \int_0^1 h dx$, so $f \sim g$

Therefore the relation $\sim$ with $C([0,1])$, the set of all continuous real-valued functions on the interval $[0,1] \subset \mathbb{R}$ is transitive.

Since the relation has been proven to be reflexive,symmetric, and transitive, it is an equivalence relation.

7. NSSM, Exercise 7.3.5, items 2, 4, and 5.

For the following sets X, equivalence relations $\sim$, and elements $x \in X$, determine the equivalence class for the given element.

***7.3.5.2*:** Let $X = \mathscr{P}(\{1,2,3,4,5\})$. For $A, B \in X$, define $A \sim B$ if and only if A and B have the same number of elements.

(a) Let $A = \{1,2\}$. Find $[A]$.

$[A] =$ all sets with 2 elements in X

(b) Let $A = \{2, 3\}$. Find $[A]$.

$[A] =$ all sets with 2 elements in X

(c) Let $A = \{1\}$. Find $[A]$.

$[A] =$ all sets with 1 element in X

(d) Let $A = \{2\}$. Find $[A]$.

$[A] =$ all sets with 1 element in X

(e) Let $A = \{1, 2, 3, 4, 5\}$. Find $[A]$.

$[A] =$ all sets with 5 element in X ie just set $A = \{1, 2, 3, 4, 5\}$.

***7.3.5.4*:** Let $X = C^1(\mathbb{R})$ be the set of all real-valued functions f on $\mathbb{R}$ such that f is differentiable and f' is continuous. Define $f \sim g$ if and only if $f'(t) = g'(t)$ for every $t \in \mathbb{R}$.

(a) Let $f(t) = t^2$ for all $t \in \mathbb{R}$. Find $[f]$.

$f(t) = t^2$ so $f'(t) = 2t$

$f \sim g$ means $g'(t) = 2t$

So $g(t) = \int 2t dt = t^2 + c$

Thus, $[f] = \{g(t) = t^2 + c : c \in \mathbb{R}\}$

(b) Let $f(t) = e^{2t}$ for all $t \in \mathbb{R}$. Find $[f]$.

$f(t) = e^{2t}$ so $f'(t) = 2e^{2t}$

$f \sim g$ means $g'(t) = 2e^{2t}$

So $g(t) = \int 2e^{2t} dt = e^{2t} + c$

Thus, $[f] = \{g(t) = e^{2t} + c : c \in \mathbb{R}\}$

(c) Let $f \in X$ be arbitrary. Find $[f]$.

$f \sim g$ means $f' = g'$

So $g = \int f' = f + c$

Thus, $[f] = \{g = f + c : c \in \mathbb{R}\}$

***7.3.5.5*:** Let $X = \mathbb{R}^4$. Let $W = \{(x, y, z, w) \in X : x + y + z + w = 0\}$. For $\mathbf{v}, \mathbf{w} \in X$, define $\mathbf{v} \sim \mathbf{w}$ if and only if $\mathbf{v} - \mathbf{w} \in W$

(a) Let $\mathbf{v} = (0, 0, 0, 0)$. Find $[\mathbf{v}]$

If $\mathbf{v} \sim \mathbf{w}$, then $\mathbf{v} - \mathbf{w} \in W$

That means $(0, 0, 0, 0) - (x, y, z, w) \in W$, thus $(-x, -y, -z, -w) \in W$.

Therefore, $-x - y - z - w = -1(x + y + z + w) = 0$, so $(x + y + z + w) = 0$

Thus$[\mathbf{v}] = W$

(b) Let $\mathbf{v} = (1, 0, -1, 0)$. Find $[\mathbf{v}]$.

If $\mathbf{v} \sim \mathbf{w}$, then $\mathbf{v} - \mathbf{w} \in W$

That means $(1, 0, -1, 0) - (x, y, z, w) \in W$, thus $(1 - x, -y, -1 - z, -w) \in W$.

Therefore, $(1 - x) + (-y) + (-1 - z) + (-w) = -1(x + y + z + w) = 0$, so $(x + y + z + w) = 0$

Thus$[\mathbf{v}] = W$

(c) Let $\mathbf{v} = (3, 3, 3, 3)$. Find $[\mathbf{v}]$.

If $\mathbf{v} \sim \mathbf{w}$, then $\mathbf{v} - \mathbf{w} \in W$

That means $(3, 3, 3, 3) - (x, y, z, w) \in W$, thus $(3 - x, 3 - y, 3 - z, 3 - w) \in W$.

Therefore, $(3 - x) + (3 - y) + (3 - z) + (3 - w) = 12 + -1(x + y + z + w) = 0$, so $(x + y + z + w) = 12$

Thus$[\mathbf{v}] = \{x, y, z, w \in \mathbb{R}^4 : x + y + z + w = 12\}$

(d) Let $\mathbf{v} = (4, 3, 2, 3)$. Find $[\mathbf{v}]$.

If $\mathbf{v} \sim \mathbf{w}$, then $\mathbf{v} - \mathbf{w} \in W$

That means $(4, 3, 2, 3) - (x, y, z, w) \in W$, thus $(4 - x, 3 - y, 2 - z, 3 - w) \in W$.

Therefore, $(4 - x) + (3 - y) + (2 - z) + (3 - w) = 12 + -1(x + y + z + w) = 0$, so $(x + y + z + w) = 12$

Thus$[\mathbf{v}] = \{x, y, z, w \in \mathbb{R}^4 : x + y + z + w = 12\}$

8. (This should sound familiar!) Fix an integer $n \in \mathbb{Z}$ and define a relation $\equiv$ on $\mathbb{Z}$ by the rule that $a \equiv b$ if and only if $b - a$ is a multiple of n. Show that this is an equivalence relation.

Proof: To show that if we fix an integer $n \in \mathbb{Z}$ and define a relation $\equiv$ on $\mathbb{Z}$ by the rule that $a \equiv b$ if and only if $b - a$ is a multiple of n, is an equivalence relationship we will show that it is reflexive, symmetric, and transitive.

Reflexive: We know for any $a \in \mathbb{Z}$, $a - a = 0$

Then we know that 0 is a multiple for any $k \in \mathbb{Z}$,$(n|0)$

Thus $a \equiv a$, showing that the relationship $\equiv$ on $\mathbb{Z}$ is reflexive.

Symmetric: We know for any $a, b \in \mathbb{Z}$

If $a \equiv b$, then $(b - a)|n$, which means $b - a = nk$ for some $k \in \mathbb{Z}$

Multiplying -1 to $b - a = nk$ results in $a - b = n(-k)$, and since $-k \in \mathbb{Z}$, $(a - b)|n$

Thus $b \equiv a$, showing that the relationship $\equiv$ on $\mathbb{Z}$ is symmetric.

Transitive: We know for any $a, b, c \in \mathbb{Z}$

If $a \equiv b$, then $(b-a)|n$, which means $b-a = nk_1$ for some $k_2 \in \mathbb{Z}$

Adding the two equations together we get $(c-b)+(b-a) = nk_2 + nk_1$

Which simplified gets us $c-a = n(k_2+k_1)$ and since $(k_2+k_1) \in \mathbb{Z}$, we know $(c-a)|n$

Thus $a \equiv c$, showing that the relationship $\equiv$ on $\mathbb{Z}$ is transitive.

Since we have shown that the relationship that the relationship $\equiv$ on $\mathbb{Z}$ is reflexive, symmetric, and transitive, we can say it is an equivalence relationship. $\square$

9. With the same situation as in the previous problem, describe the partition of $\mathbb{Z}$ corresponding to the equivalence relation $\equiv$. (This just means: describe the equivalence classes under $\equiv$.) In particular, find a transversal for $\equiv$.

$\mathbb{Z}\backslash \equiv_n = \{[0],[1],...[n-1]\}$

10. NSSM, Exercise 7.1.5.

For each of the following statements, find a partition of $\mathbb{N}$ satisfying the stated requirement. You will create different partitions for each of the requirements.

1. Every room of the partition has exactly two inhabitants;

$\{2k-1, 2k : k \in \mathbb{N}\}$

2. There are exactly three rooms in the partition;

$\{3n, 1+3n, 2+3n : n \in \mathbb{N}\}$

3. There are infinitely many rooms in the partition and each room has infinitely many inhabitants.