Problem Set 4: Reductions

Due: Tuesday 11/23 11:59PM

Feel free to work in groups of at most 4 for these - if you have a group of more than 4, please run it by me first. If you do work in a group, please include the names of those that you worked with. **However: each student should submit a separate copy where the solutions have been written by yourself.**

1. (10 points) A certain table tennis league is organized so that each season the n players play against each other in a round-robin format, so that each player plays against each other player. The player who wins the most games is declared the winner: ties are handled by tossing a coin. A certain player has come to us mid season, worrying that they may have been technically eliminated. Our goal is to determine in a polynomial amount of time whether they can still win or not. Show that you can solve this problem with a runtime that is polynomial in n. Specifically, I want a polynomial time reduction to the flow decision problem. With even more clarity:

   Elimination Game (EG)

   Input: Players $P_1, ..., P_n$, where player P_i has won w_i games so far, and remaining games $G_1, ..., G_m$.
   Output: True if P_n cannot possibly finish the most wins out of $P_1, ..., P_n$.

   Flow

   Input: A graph G with special vertices s and t, capacities c_e for every edge e, and a number k.
   Output: True if there is a valid flow on G with value k and False otherwise.

   For full credit for this problem, I'm asking you to show: EG $\leq_p$ Flow.

   As an example, suppose this league has 8 players $P_1, ..., P_8$, where we are trying to determine if P_8 has been eliminated. We are given that the only game left to be played is between P_6 and P_7. We are also given that P_6, P_7, and P_8 each have 5 wins in total. Then no matter the results of the remaining games (note that every game individually cannot end in a tie), P_8 cannot be the winner of the league.

2. (10 points) Google has a list $A_1, ..., A_n$ of advertisers that want to show ads to a group of users $u_1, ..., u_m$. Specifically, each advertiser A_i has a list of users G_i (a subcollection of $u_1, ..., u_m$) that it would like to show ads to (these groups may overlap); however, different advertisers have purchased different plans from Google, so Google will only show r_i users the advertisement from A_i. Additionally, Google has run some analytics over their userbase and knows for each u_j the number c_j of ads u_j can see before u_j will get fed up and install an adblock. Your goal is to determine whether Google can display ads so that r_i of advertiser A_i's ads get shown to users in G_i, no user u_j gets more than c_j ads, and no user gets shown the same ad multiple times.

> **ADVERTISEMENT PROBLEM (ADS)**
>
> **Input:** Advertisers $A_1, ..., A_n$ and users $u_1, ..., u_m$, groups of users $G_1, ..., G_n$, numbers $r_1, ..., r_n$ and $c_1, .., c_m$
> **Output:** TRUE if there is a way to show users ads so that each advertiser has its ads shown to r_i users in G_i and no user gets shown more than 1 ad and FALSE otherwise.

Specifically, I'm asking you to show that ADS $\leq_p$ FLOW.

3. I've mentioned a few times how decision problems are easier to conceptualize for reduction type problems rather than optimization, but it's good to check along the way that these problems really are the "same", at least in terms of difficulty. Let's check for the Clique problem:

> **CLIQUE DECISION (CD)**
>
> **Input:** A graph G and number k
> **Output:** TRUE if there are k vertices in G that are all adjacent to each other and FALSE otherwise.

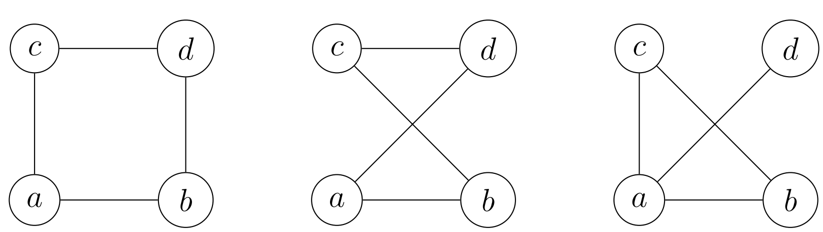
> **CLIQUE OPTIMIZATION (CO)**
>
> **Input:** A graph G
> **Output:** The largest collection of vertices in G that are all adjacent to each other.

Show that

(a) (5 points) if you have a polynomial time algorithm for CO then you can design a polynomial time algorithm for CD

(b) (10 points) if you have a polynomial time algorithm for CD then you can design a polynomial time algorithm for CO

4. Often times we look at problems over graphs which we assume are either directed or undirected, but maybe we start wondering about which of these types of graphs are harder to deal with. Consider the following graph isomorphism problem: we are given two graphs, and we want to see if one is simply a relabeling of the other. For example, consider the following graphs:



The first two are just relabellings of the vertices, but there is no way to relabel the vertices of the last one to become the same as the first two.

Let n be the number of vertices and m be the number of edges.

(a) (5 points) Show that if you have a $O(n+m)$ time algorithm for the graph isomorphism problem for directed graphs, then you can design a $O(n+m)$ time algorithm for the graph isomorphism problem on undirected graphs.

(b) (10 points) Show that if you have a $O(n+m)$ time algorithm for the graph isomorphism problem for undirected graphs, then you can design a $O(n+m)$ time algorithm for the graph isomorphism problem on directed graphs.