MA274, Fall 2021 — Writing Assignment 1

This writing assignment is due on **Friday, September 17**. Please read the instructions and follow them carefully.

When faced with a mathematical problem, we start out not knowing how to begin. So we explore: try out examples, make some wild guesses, write out fragments of proofs. Eventually, if we are persistent, we discover a pathway and prove a theorem (maybe not the one we set out to prove).

This assignment is about that process. I will give you something to think about (see over). You will have a week and a half to work on it. The problem is elementary but not easy. It is open-ended, so there isn't a single way to "solve" it. As you explore it, keep a diary, recording both what you do each day and what the experience is like.

At the end, write a summary of the process and its outcome. If you prove some theorems, write them out and give proofs'. If you don't succeed in proving somthing, record what you think is true, how far you got towards proving it, and what got in the way.

Your summary will be very boring if it boils down to "I googled it and here is the answer." So don't do that. Try it on your own and with your friends, but don't look it up. If a friend looks it up, don't let them tell you.

Your final write-up should be between two and four pages using the LaTeX article class with 12-point fonts. (No exceptions!) I prefer to receive it in one-sided printed form.

Beatty Sequences

Suppose we take an irrational number $\alpha > 1$ and look at the sequence

$$\alpha,\ 2\alpha,\ 3\alpha,\ \ldots n\alpha,\ \ldots$$

None of these numbers will be integers. (Do you see why?) We can, however, round them down to the nearest integer; that will give a sequence of positive integers.

Let's do it with $\alpha = \sqrt{2}$ and *SageMath*:

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sage: a=sqrt(2)
sage: for n in (1..21): print(floor(n*a))
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We get

$$1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, 21, 22, 24, 25, 26, 28, 29\ldots$$

Notice that we do not get all integers. Here is a list of the missing ones up to 30:

$$3, 6, 10, 13, 17, 20, 23, 27, 30\ldots$$

The first sequence, $(\lfloor n\alpha \rfloor)$, is called the *Beatty Sequence* attached to α. The second is its *complementary sequence*. You are invited to prove things about Beatty sequences. Here are some things to explore. You should not try to answer all of them; just investigate the ones that interest you.

a. There are clearly infinitely many integers in any Beatty sequence. Show that the complementary sequence is infinite as well.

b. Can we recover α from its Beatty sequence? Does

$$1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, 21, 22, 24, 25, 26, 28\,29\ldots$$

somehow say $\sqrt{2}$?

c. Is the complementary sequence also a Beatty sequence? That is, is there an irrational number $\beta > 1$ so that the integer parts of $n\beta$ give the second sequence? If so, how is β related to α?

d. In our example so far, the number in the second sequence is always bigger than the corresponding number in the first. The differences are

$$2, 4, 6.8, 10, \ldots$$

Will this pattern continue? Will it be the same for all Beatty sequences?

e. Why did I require $\alpha > 1$? What goes wrong otherwise?

f. Why did I require α to be irrational?

g. Suppose I wrote down an increasing sequence of integers. Will it be a Beatty sequence? For example, could I get the sequence of all odd numbers that way? Is there a way to decide whether a given sequence is a Beatty sequence?