Lecture 6: Transformation matrices

project 2: 3 manipulations via matrix multiplication:

- move: translate data
- Stretch: Scale data
- rotate data

Matrix transformation: Matrix multiplication involving a transformation matrix M and data matrix A.

manipulated data

numpy matrix multiply operator

data matrix A

$$\underbrace{A'} = \underbrace{M} @ \underbrace{A}$$

transformation matrix always go on left

always go on the right

☆ $\underset{A'}{\text{shape}(\text{output})} = \underset{A}{\text{shape}(\text{input})}$

Example: Temperature data

append col of 1s.

$$A = \begin{array}{c} \\ \text{day 1} \\ \text{day 2} \end{array} \begin{array}{c} \begin{array}{cccc} \text{ME} & \text{FL} & \text{DC} & \end{array} \\ \begin{bmatrix} 32 & 105 & 50 & 1 \\ 6 & 85 & 40 & 1 \end{bmatrix} \end{array}$$

shape : (N, M)

: $(2, 3)$

$N = \#$ data samples $= 2$

$M = \#$ vars $= 3$ $\Rightarrow$ shape w/ col of 1s

$\Rightarrow (2, 4)$

$$M = \begin{bmatrix} 1 & 0 & 0 & -32 \\ 0 & 1 & 0 & -80 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

want : $A' = \underbrace{M}_{(4,4)} @ \cancel{A}$

$(4,4) \times (2,4)$

mismatch!
Can't do!

1) Check shapes:

Work around: Try transposing A:

$$A' = \underbrace{M}_{(4,4)} @ A.T$$

$$(4, 4) \times (4, 2)$$

we're good!

$$A.T = \begin{bmatrix} 32 & 6 \\ 105 & 85 \\ 50 & 40 \\ 1 & 1 \end{bmatrix}$$

2) Figure Out output shape:

$$\text{Shape}(A') = (4, 4) \times (4, 2) = (4, 2)$$

3) Do the math

$$\underbrace{\overset{A'}{\begin{bmatrix} *1 & *2 \\ *3 & *4 \\ *5 & *6 \\ *7 & *8 \end{bmatrix}}}_{(4,2)} = \underbrace{\overbrace{\begin{bmatrix} 1 & 0 & 0 & -32 \\ 0 & 1 & 0 & -80 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{M}}_{(4,4)} \underbrace{\overbrace{\begin{bmatrix} 32 & 6 \\ 105 & 85 \\ 50 & 40 \\ 1 & 1 \end{bmatrix}}^{A.T}}_{(4,2)}$$

$$*1 = 1 \cdot 32 + \cancel{0 \cdot 105 + 0 \cdot 50} + -32 \cdot 1$$
$$= 32 - 32 = 0$$

$$*2 = 1 \cdot 6 + \cancel{0 \cdot 85 + 0 \cdot 40} + -32 \cdot 1$$
$$= 6 - 32 = -26$$

$$*3 = \cancel{0 \cdot 32} + 1 \cdot 105 + \cancel{0 \cdot 50} + -80 \cdot 1$$
$$= 105 - 80 = 25$$

$$*4 = \cancel{0 \cdot 6} + 1 \cdot 85 + \cancel{0 \cdot 40} + -80 \cdot 1$$
$$= 85 - 80 = 5$$

$*5 = 1 \cdot 50 + 10 \cdot 1 = 60$

$*6 = 1 \cdot 40 + 10 \cdot 1 = 50$

$*7 = 1$

$*8 = 1$

need to transpose

shape: $(4, 2) \Longrightarrow (2, 4)$

$$A' = \begin{bmatrix} 0 & -26 \\ 25 & 5 \\ 60 & 50 \\ 1 & 1 \end{bmatrix}$$

<u>wanted</u>: shape: $(2, 4)$

answer

$$A'.T = \begin{bmatrix} 0 & 25 & \boxed{60} & 1 \\ -26 & 5 & \boxed{50} & 1 \end{bmatrix}$$

Compare w/ original A:

$$A = \begin{array}{c} \\ \text{day 1} \\ \text{day 2} \end{array} \begin{array}{c} \begin{array}{cccc} \text{ME} & \text{FL} & \text{DC} & \end{array} \\ \begin{bmatrix} 32 & 105 & 50 & 1 \\ 6 & 85 & 40 & 1 \end{bmatrix} \end{array}$$

<u>ME col</u>: How does output values (A'.T) compare to those in A? Values are 32 less than original

<u>FL col</u>: Values are 80 less than before

<u>DC col</u>: Values are 10 more than before

$$\underbrace{\left[\begin{matrix}\overbrace{\begin{matrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{matrix}}^{M} & \begin{matrix}-32\\-80\\10\\1\end{matrix}\end{matrix}\right]}_{\text{Translation matrix } T}$$

shift in 1st variable (ME)

shift in 2nd var (FL)

shift in 3rd var (DL)

$$T = \underset{\substack{\uparrow \\ \text{For 3 vars}}}{T_3}\left(\overbrace{t_x, t_y, t_z}^{\text{shifts in each var}}\right) = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$t_x = -32$

$t_y = -80$

$t_z = 10$

What if we wanted to translate X variable by +6 and leave the rest alone?

$t_x = 6 \quad t_y = 0 \quad t_z = 0$

☒ Translation Summary

To translate 3 variables $\vec{f} = (f_x, f_y, f_z)$:

$$\boxed{A' = (T @ A.T).T}$$

identity

Implementation tip:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$= np.eye(4)$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \end{bmatrix}$$

$\vec{f}$:

$$\begin{bmatrix} f_x \\ f_y \\ f_z \\ 1 \end{bmatrix} \begin{array}{l} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{translation amounts} \\ \}\ \text{1 appended} \end{array}$$

col of ones = homogeneous coordinate