

# 1 Jacobian-vector products

Consider the parameterized ODE initial value problem

$$\dot{y} = f(t, y, a), \quad y(0, a) = y_0(a), \quad (1)$$

by which we mean

$$\partial_0 y(t, a) = f(t, y(t, a), a). \quad y(0, a) = y_0(a) \quad (2)$$

for all  $t$  and  $a$  in some domains. We want to understand how the solution to the ODE changes (e.g. at particular values of  $t$ ) for small perturbations of  $a$ . That is, we want to be able to compute the Jacobian-vector product

$$(a, v) \mapsto \partial_1 y(t, a)[v] \quad (3)$$

at any particular values of  $t$  and  $a$ , where  $v$  can be interpreted as a small perturbation to the value of  $a$ .

Since the ODE holds true for all values of  $a$  (or at least those close to a particular  $a_0$  in which we are interested), we can view both sides as functions of  $a$ , and assuming differentiability we can differentiate both sides with respect to  $a$  to find a new equation that must be satisfied:

$$\partial_1 \partial_0 y(t, a) = \partial_2 f(t, y(t, a), a) + \partial_1 f(t, y(t, a), a) \circ \partial_1 y(t, a), \quad (4)$$

or, applied to a particular perturbation vector (and using the fact that partial derivatives commute),

$$\partial_0((t, a) \mapsto \partial_1 y(t, a)[v]) = \partial_2 f(t, y(t, a), a)[v] + \partial_1 f(t, y(t, a), a)[\partial_1 y(t, a)[v]].$$

We can identify  $z(t, a) \triangleq \partial_1 y(t, a)[v]$  as a new state vector to write a joint ODE system

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} f(t, y, a) \\ g(t, y, z, a) \end{bmatrix}, \quad \begin{bmatrix} y(0, a) \\ z(0, a) \end{bmatrix} = \begin{bmatrix} y_0(a) \\ \partial y_0(a)[v] \end{bmatrix}, \quad (5)$$

$$g(t, y, z, a) = \partial_1 f(t, y, a)[z] + \partial_2 f(t, y, a)[v]. \quad (6)$$

Notice that the dynamics on the  $z$  component are linear/affine in  $z$  (and  $v$ !).

# 2 Vector-Jacobian products

Consider the parameterized linear/affine ODE IVP

$$\partial_0 z(t) = A(t)z(t) + B(t)v, \quad z(0) = Cv. \quad (7)$$

On the vector space of solutions  $\mathcal{Z}$  consider a nice linear functional  $\mathcal{D} : \mathcal{Z} \rightarrow \mathbb{R}$ . This induces a linear function on the vector space of possible perturbations  $v$ .

Take the special case of the evaluation functional  $\mathcal{D}[z] = d^T z(1)$  ... maybe.