1 Jacobian-vector products

Consider the parameterized ODE initial value problem

$$\dot{y} = f(t, y, a), \qquad y(0, a) = y_0(a), \tag{1}$$

by which we mean

$$\partial_0 y(t, a) = f(t, y(t, a), a). \qquad y(0, a) = y_0(a). \tag{2}$$

We want to understand how the solution to the ODE changes (e.g. at particular values of t) for small perturbations of a. That is, we want to compute the Jacobian-vector product

$$(a, v) \mapsto \partial_1 y(t, a)[v] \tag{3}$$

where v is a small perturbation to a.

Since the ODE holds true for all values of a (or at least those close to a particular a_0 in which we are interested), we can view both sides as functions of a, and assuming differentiability we can differentiate both sides with respect to a to find a new equation that must be satisfied:

$$\partial_1 \partial_0 y(t, a) = \partial_2 f(t, y(t, a), a) + \partial_1 f(t, y(t, a), a) \circ \partial_1 y(t, a). \tag{4}$$

Using the fact that partial derivatives commute, we can identify $z(t, a) \triangleq \partial_1 y(t, a)[v]$ as a new state vector to write a joint ODE system

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} f(t, y, a) \\ g(t, y, z, a) \end{bmatrix}, \qquad \begin{bmatrix} y(0, a) \\ z(0, a) \end{bmatrix} = \begin{bmatrix} y_0(a) \\ \partial y_0(a)[v] \end{bmatrix}, \tag{5}$$

$$g(t, y, z, a) = \partial_1 f(t, y, a)[z] + \partial_2 f(t, y, a)[v]. \tag{6}$$

Notice that the dynamics on the z component are linear/affine in z (and v!).

2 Vector-Jacobian products

Consider the parameterized linear/affine ODE IVP

$$\partial_0 z(t) = A(t) z(t) + B(t) v, \qquad z(0) = Cv. \tag{7}$$

On the vector space of solutions $\mathcal{Y}$ consider a nice linear functional $\mathcal{D} : \mathcal{Y} \to \mathbb{R}$. This induces a linear function on the vector space of possible perturbations v.

Take the special case of the evaluation functional $\mathcal{D}[y] = d^{\mathsf{T}} y(1)$.