1 Jacobian-vector products

Consider the parameterized ODE initial value problem

$$\dot{y} = f(t, y, a), \qquad y(0, a) = y_0(a), \tag{1}$$

by which we mean

$$\partial_0 y(t, a) = f(t, y(t, a), a), \qquad y(0, a) = y_0(a), \tag{2}$$

for all t and a in some domains. We want to understand how the solution to the ODE changes (e.g. at particular values of t) for small perturbations of a. That is, we want to be able to compute the Jacobian-vector product

$$(a, v) \mapsto \partial_1 y(t, a)[v] \tag{3}$$

at any particular values of t and a, where v can be interpreted as a small perturbation to the value of a.

Since the ODE holds true for all values of a (or at least those close to a particular a_0 in which we are interested), we can view both sides as functions of a, and assuming differentiability we can differentiate both sides with respect to a to find a new equation that must be satisfied:

$$\partial_1 \partial_0 y(t, a) = \partial_2 f(t, y(t, a), a) + \partial_1 f(t, y(t, a), a) \circ \partial_1 y(t, a). \tag{4}$$

Applying both sides to a particular perturbation vector v and using the fact that partial derivaives commute, we have

$$\partial_0(\,(t, a) \mapsto \partial_1 y(t, a)[v]\,) = \partial_2 f(t, y(t, a), a)[v] + \partial_1 f(t, y(t, a), a)[\,\partial_1 y(t, a)[v]\,].$$

We can identify $z(t, a) \triangleq \partial_1 y(t, a)[v]$ as a new state vector to write a joint ODE system

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} f(t, y, a) \\ g(t, y, z, a) \end{bmatrix}, \qquad \begin{bmatrix} y(0, a) \\ z(0, a) \end{bmatrix} = \begin{bmatrix} y_0(a) \\ \partial y_0(a)[v] \end{bmatrix}, \tag{5}$$

$$g(t, y, z, a) = \partial_1 f(t, y, a)[z] + \partial_2 f(t, y, a)[v]. \tag{6}$$

Notice that the dynamics on the z component are linear/affine in z (and v!).

2 Vector-Jacobian products

Consider the parameterized linear/affine ODE IVP

$$\partial z(t) = A(t) z(t) + B(t) v, \qquad z(0) = Cv. \tag{7}$$

On the vector space of solutions $\mathcal{Z}$ consider a nice linear functional $\mathcal{D} : \mathcal{Z} \to \mathbb{R}$. This induces a linear function on the vector space of possible perturbations v.

Take the special case of the evaluation functional $\mathcal{D}[z] = d^{\mathsf{T}} z(1)$ ... maybe.