Sakurai 3.1

Sakurai 3.1

$$\exp\left(\frac{-iS_z\phi}{\hbar}\right) S_x \exp\left(\frac{+iS_z\phi}{\hbar}\right)$$

$$S_x = \frac{\hbar}{2}\left(|+\rangle\langle -| + |-\rangle\langle +|\right)$$

$$\exp\left(-\frac{i\phi}{2}\right) S_x = \frac{\hbar}{2} e^{-i\phi/2} |+\rangle\langle +| \left(|+\rangle\langle -| + |-\rangle\langle +|\right)$$

$$+ \frac{\hbar}{2} e^{+i\phi/2} |-\rangle\langle -| \left(|+\rangle\langle -| + |-\rangle\langle +|\right)$$

$$= \frac{\hbar}{2}\left(e^{-i\phi/2}(|+\rangle\langle -|) + e^{+i\phi/2} |-\rangle\langle +|\right)$$

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i\phi}{2}\right)^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^n = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} -\frac{i\phi}{2} & 0 \\ 0 & \frac{i\phi}{2} \end{pmatrix}^n = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}$$

$$\exp\left(\frac{-i\phi}{2}\right) S_x \exp\left(\frac{i\phi}{2}\right) = \frac{\hbar}{2}\left(e^{-i\phi/2} |+\rangle\langle -| + e^{+i\phi/2} |-\rangle\langle +|\right)$$

$$\left(e^{+i\phi/2} |+\rangle\langle +| + e^{-i\phi/2} |-\rangle\langle -|\right) = \frac{\hbar}{2}\left(e^{-i\phi} |+\rangle\langle -| + e^{i\phi} |-\rangle\langle +|\right)$$

Sakurai 3.4

$$A = -a_0\hat{I} \pm (a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)$$

$$A = \begin{pmatrix} a_0 + ia_3 & ia_1 + a_2 \\ ia_1 - a_2 & a_0 - ia_3 \end{pmatrix} \quad A^\dagger = \begin{pmatrix} a_0 - ia_3 & -ia_1 - a_2 \\ -ia_1 + a_2 & a_0 + ia_3 \end{pmatrix}$$

$U = A(A^\dagger)^{-1}$ using Mathematica, $\det(A) = a_0^2 + a_1^2 + a_2^2 + a_3^2 = \alpha^2$

$\det(A^\dagger) = a_0^2 + a_1^2 + a_2^2 + a_3^2 = \alpha^2$

$$\det U = \det(A(A^\dagger)^{-1}) = \det(A)\det((A^\dagger)^{-1}) = \frac{\det(A)}{\det(A^\dagger)} = 1$$

$$\cos\left(\frac{\phi}{2}\right) = Re(a) = \frac{a_0 - \vec{a}^2}{\alpha^2} \qquad \sin\left(\frac{\phi}{2}\right) = \sqrt{(1-\cos^2(\phi/2))} = \frac{2a|\vec{a}|}{\alpha^2}$$

$$U = \frac{1}{\alpha^2}\begin{pmatrix} a_0 - |\vec{a}|^2 + 2ia_0a_3 & 2a_0a_2 + 2ia_0a_1 \\ -2a_0a_2 + 2ia_0a_1 & a_0 - a^2 - 2ia_0a_3 \end{pmatrix}$$

From general rotation matrix

$$\exp\left(\frac{-i\hat{\sigma}\cdot\hat{n}\phi}{2}\right) = \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) - in_z\sin\left(\frac{\phi}{2}\right) & (-in_x - n_y)\sin\left(\frac{\phi}{2}\right) \\ (-in_x + n_y)\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) + in_z\sin\left(\frac{\phi}{2}\right) \end{pmatrix}$$

$$\frac{a_0 - |\vec{a}|^2}{\alpha^2} = \cos\left(\frac{\phi}{2}\right)$$

$$-n_x\sin(\phi/2) = 2a_0a_1$$

$$-2a_0a_2 = n_y\sin\left(\frac{\phi}{2}\right)$$

$$2a_0a_3 = n_z\sin(\phi/2)$$

Sakurai 3.10

$$\exp\left(\frac{-i\hat{\sigma}\cdot\hat{n}\phi}{2}\right) = \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) - in_z \sin\left(\frac{\phi}{2}\right) & (-in_x - n_y)\sin\left(\frac{\phi}{2}\right) \\ (-in_x + n_y)\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) + in_z \sin\left(\frac{\phi}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right) & -e^{-i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) \\ e^{i(\alpha-\gamma)/2}\sin\left(\frac{\beta}{2}\right) & e^{i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right) \end{pmatrix}$$

$$\cos\left(\frac{\phi}{2}\right) - in_z \sin\left(\frac{\phi}{2}\right) = e^{-i(\alpha+\gamma)/2}\cos\left(\frac{\beta}{2}\right)$$

$$2\cos\left(\frac{\phi}{2}\right) = 2\sin\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)$$

$$\frac{\phi}{2} = \arccos\left(\sin\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\right)$$

$$\phi = 2\arccos\left(\sin\left(\frac{\alpha+\gamma}{2}\right)\cos\left(\frac{\beta}{2}\right)\right)$$

Sakurai 3.20

$$J_- = \hbar\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_+ = \hbar\sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_x = \frac{J_+ + J_-}{2} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow \lambda = -\hbar, 0, +\hbar$$

$$J_y = \frac{1}{2i}(J_+ - J_-) \quad \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$J_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\lambda_1 = -\hbar$: $\quad |\lambda_1\rangle = \frac{1}{4}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{4}\left(|+\rangle - \sqrt{2}\,|0\rangle + |-\rangle\right)$

$\lambda_2 = 0$: $\quad |\lambda_2\rangle = \frac{1}{\sqrt{2}}\left(|-\rangle - |+\rangle\right)$

$\lambda_3 = +\hbar$: $\quad |\lambda_3\rangle = \frac{1}{4}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{4}\left(|+\rangle + \sqrt{2}\,|0\rangle + |-\rangle\right)$

$$J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$\lambda_1 = -\hbar \qquad |\lambda_1\rangle = \frac{1}{4}\left(-|+\rangle + i\sqrt{2}\,|0\rangle + |-\rangle\right)$

$\lambda_2 = 0 \qquad |\lambda_2\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$

$\lambda_3 = +\hbar \qquad |\lambda_3\rangle = \frac{1}{4}\left(-| - i\sqrt{2}\,|0\rangle + |-\rangle\right)$

Sakurai 3.23

$$\psi(\vec{x}) = (x+y+3z)f(r)$$

$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

(a) $$\hat{L}^2\psi = -\hbar^2 f(r)\, r\left((-6\sin\theta\cos\theta + (\cos\phi+\sin\phi)(\cos^2\theta-\sin^2\theta) - \frac{\cos\phi+\sin\phi}{\sin\theta}\right)$$

$$= 2\hbar^2 f(r)\, r(\sin\theta\cos\phi + \sin\theta\sin\phi + 3\cos\theta)$$

$$= 2\hbar^2\psi(\vec{x}) = l(l+1)\hbar^2\psi(\vec{x}) \longrightarrow l=1$$

(b) $$\langle x|\hat{L}_z|\psi\rangle = \hat{L}_z\psi(\vec{x})$$

$$Y_{+1}^{\pm 1} = \sqrt{\frac{3}{8\pi}}\frac{(x\pm iy)}{r} \qquad Y_1^{+1} + Y_1^{-1} = \sqrt{\frac{3}{8\pi}}\frac{1}{r}(x+\cancel{iy}+x-\cancel{iy})$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad = \sqrt{\frac{3}{2\pi}}\,x$$

$$x = \sqrt{\frac{2\pi}{3}}\, r\,(Y_1^{-1} - Y_1^{+1})$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\frac{z}{r} \qquad -i(Y_1^{-1} - Y_1^{+1}) = \sqrt{\frac{3}{8\pi}}\frac{1}{r}(2y)$$

$$z = \sqrt{\frac{4\pi}{3}}\, r\, Y_1^0 \qquad y = i\sqrt{\frac{2\pi}{3}}\, r\,(Y_1^{-1} + Y_1^{+1})$$

$$\psi(\vec{x}) = \sqrt{\frac{2\pi}{3}}\, r f(r)\left(3\sqrt{2}\,Y_1^0 + (1+i)Y_1^{-1} + (1-i)Y_1^{+1}\right)$$

$$|A|^2(18+2+2) = 1 \rightarrow N = \frac{1}{\sqrt{22}}$$

$$\psi(\vec{x}) = \sqrt{\frac{\pi}{33}}\, r f(r)\left(3\sqrt{2}\,Y_1^0 + (1+i)Y_1^{-1} + (1-i)Y_1^{+1}\right)$$

$$P(m=0) = \frac{9}{11}$$
$$P(m=1) = \frac{1}{11}$$
$$P(m=-1) = \frac{1}{11}$$

Sakurai 3.23

(c) $$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}(rf(r)) + \frac{2}{r}\frac{\partial}{\partial r}(r\,f(r)) - \frac{2}{r^2}(r\,f(r))\right) + V(r)r\,f(r)Y_\ell^m$$

$$= E\,r\,f(r)Y_\ell^m$$

$$V(r) = \frac{\hbar^2}{2m}\left(\frac{rf''(r)+4f'(r)}{rf(r)}\right)$$

Sakurai 3.24

$$\hat{J}_x|j,m\rangle = \frac{1}{2}(J_+ + J_-)|j,m\rangle = \frac{\hbar}{2}\sqrt{(j-m)(j+m+1)}\;|j,m+1\rangle$$
$$+\frac{\hbar}{2}\sqrt{(j+m)(j-m+1)}\;|j,m-1\rangle$$

$$J_y|j,m\rangle = \frac{1}{2i}(J_+ - J_-) = \frac{\hbar}{2i}\left[\cdots|j,m+1\rangle - \cdots|j,m-1\rangle\right]$$

$$\rightarrow \langle l,m|\hat{L}_x|l,m\rangle = \frac{1}{2}\langle l,m|(\hat{L}_+ + \hat{L}_-)|l,m\rangle$$

$$= \frac{1}{2}\left[\langle l,m|\frac{\hbar}{2}\cancel{\cdots}|l,m+1\rangle + \langle l,m|\frac{\hbar}{2}\cancel{\cdots}|l,m-1\rangle\right]$$

$$= 0$$

$$\rightarrow \langle l,m|\hat{L}_y|l,m\rangle = \frac{1}{2i}\left[\langle l,m|(\hat{L}_+ - \hat{L}_-)|l,m\rangle\right]$$

$$= \frac{1}{2i}\left[\langle l,m|\frac{\hbar}{2}\cdots|l,m+1\rangle - \langle l,m|\frac{\hbar}{2}\cdots|l,m-1\rangle\right]$$

$$= 0$$

$$\langle L_x^2\rangle = \left\langle\frac{1}{4}(L_+ + L_-)(L_+ + L_-)\right\rangle = \frac{1}{4}\left(\cancel{\langle L_+^2\rangle} + \langle L_+L_-\rangle + \langle L_-L_+\rangle\right.$$
$$\left.+\cancel{\langle L_-^2\rangle}\right) = \frac{1}{4}\left(\langle L_+L_-\rangle + \langle L_-L_+\rangle\right)$$

$$\langle L_y^2\rangle = \left\langle -\frac{1}{4}(L_+ - L_-)(L_+ - L_-)\right\rangle$$

$$= -\frac{1}{4}\left(\cancel{\langle L_+^2\rangle} - \langle(L_+L_-)\rangle - \langle(L_-L_+)\rangle + \cancel{\langle L_-^2\rangle}\right)$$

$$= \frac{1}{4}\left(\langle L_+L_-\rangle + \langle L_-L_+\rangle\right) = \langle L_x^2\rangle =$$

$$J_\pm J_\mp = J^2 - J_z^2 \pm \hbar J_z$$

$$\rightarrow \frac{1}{4}(L_+L_- + L_-L_+) = \frac{1}{4}\left(J^2 - J_z^2 + \cancel{\hbar J_z} + J^2 - J_z^2 - \cancel{\hbar J_z}\right)$$

$$= (J^2 - J_z^2)/2 = [l(l+1) - m^2]\hbar^2/2$$

Sakurai 3.24

$$\langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{1}{2}\left[l(l+1) - m^2 \right]\hbar^2$$

Sakurai 3.33

(3.335) $\quad m = m_1 + m_2$

(3.339) $\quad S^2 = S_1^2 + S_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$

$$|++\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |+-\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad |-+\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$|--\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$S^2|++\rangle = (S_1^2 + S_2^2 + 2S_{1z}S_{2z})|++\rangle$$

$$\left(\tfrac{1}{2}(\tfrac{1}{2}+1)\hbar^2 + \tfrac{1}{2}(\tfrac{1}{2}+1)\hbar^2 + 2\tfrac{1}{2}\hbar\cdot\tfrac{1}{2}\hbar\right)|++\rangle$$

$$= \left(\tfrac{3}{4}+\tfrac{3}{4}+\tfrac{1}{2}\right)\hbar^2|++\rangle = 2\hbar^2|++\rangle$$

$$= l(l+1)\hbar^2|l,l\rangle$$

$$S_{1+}|--\rangle = S_{1+}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = |+-\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$S_{1+}|-+\rangle = S_{1+}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$S_{1+} = \begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$S_{2+}|+-\rangle = \sqrt{\left(\tfrac{1}{2}+\tfrac{1}{2}\right)}\,|++\rangle$$

$$S_{2+}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad S_{2+} = \begin{pmatrix} & 0 & \\ & 0 & \\ 0 & 1\;0 & 1 \\ & 0 & \end{pmatrix}$$

$$S_{2+}|--\rangle = |-+\rangle$$

$$S_{1-}|+-\rangle = |--\rangle \longrightarrow \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \longrightarrow \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$S_{1-}|++\rangle = |-+\rangle \longrightarrow \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \longrightarrow \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{pmatrix}$$

$$S_{2-}|++\rangle = |+-\rangle \qquad \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \longrightarrow \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$S_{2-}|-+\rangle = |--\rangle \qquad \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \longrightarrow \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$S_{2-} = \begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}$$

$$S_{1z}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = +\frac{1}{2}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$S_{1z}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad S_{1z} = \begin{pmatrix}1/2&&&\\&+\frac{1}{2}&&\\&&-1/2&\\&&&-\frac{1}{2}\end{pmatrix}$$

$$S_{1z}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = +\frac{1}{2}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad S_{2z} = \begin{pmatrix}\frac{1}{2}&&&\\&-\frac{1}{2}&&\\&&1/2&\\&&&-\frac{1}{2}\end{pmatrix}$$

$$S_{1z}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$S^2 = \frac{1}{2}(S_+S_- + S_-S_2) + S_z^2$$

$$\rightarrow S_1^2 = S_2^2 = \frac{3}{4}\hbar^2\hat{I} \qquad (\text{take } \hbar = 1)$$

$$S_1^2 + S_2^2 + 2S_{z1}S_{z2} + S_{1+}S_{2-} + S_{1-}S_{2+}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$\lambda = 1, 2,$

$$\lambda = 2 \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = 1 \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Sakurai 3.34

$$J_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\, |j, m \pm 1\rangle$$

find 9 states from $j = 0, 1, 2$ in $|j, m\rangle$ formed by adding
$j_1 = 1,\ j_2 = 1$

$|0, 0\rangle = \quad |+-\rangle \quad |0\ 0\rangle \quad |-+\rangle$
$|1, 0\rangle = \quad |+-\rangle \quad |-+\rangle$
$|1, -1\rangle = \quad |0-\rangle \quad |-0\rangle$
$|1, +1\rangle = \quad |0+\rangle \quad |+0\rangle$
$|2, -2\rangle = |--\rangle$
$|2, -1\rangle =$
$|2, 0\rangle =$
$|2, +1\rangle$
$|2, +2\rangle$

$$\langle j_1\ j_2\ m_1\ m_2 | J_\pm | j, m\rangle$$
$$= \hbar\sqrt{(j_1 \pm m_1)(j_1 \mp m_1 + 1)}\ \langle j_1, j_2, m_1 \mp 1, m_2 | j, m\rangle$$
$$+ \hbar\sqrt{(j_2 \pm m_2)(j_2 \mp m_2 + 1)}\ \langle j_1\ j_2\ m_1\ m_2 \mp 1 | j, m\rangle$$

$$J_+|2, -2\rangle = \hbar\sqrt{(2-(-2))(2-2+1)}\ \ |2, -1\rangle = 2\hbar|2, -1\rangle$$
$$(J_{1+} + J_{2+})|2, -2\rangle = (J_{1+} + J_{2+})|--\rangle = J_{1+}|--\rangle + J_{2+}|--\rangle$$
$$= \hbar\sqrt{(1-(-1))(1-1+1)}\big(|+-\rangle + |-+\rangle\big)$$
$$= \hbar\sqrt{2}\,(|0-\rangle + |-0\rangle) \longrightarrow |2, -1\rangle = \frac{1}{\sqrt{2}}(|0-\rangle + |-0\rangle)$$

Sakurai 3.34

$$J_+|2,-1\rangle = \hbar\sqrt{(2-(-1))(2+(-1)+1)}\,|2,0\rangle = \hbar\sqrt{6}\,|2,0\rangle$$

$$= \frac{\hbar}{\sqrt{2}}\left[J_{1+}|0-\rangle + J_{1+}|-0\rangle + J_{2+}|0-\rangle + J_{2+}|-0\rangle\right]$$

$$= \frac{\hbar}{\sqrt{2}}\left(\sqrt{(1-0)(1+0+1)}\,|+-\rangle + \sqrt{(2)(1)}\,|00\rangle + \sqrt{2}\,|00\rangle + \sqrt{2}\,|-+\rangle\right)$$

$$= |+-\rangle + 2|00\rangle + |-+\rangle$$

$$\rightarrow |2,0\rangle = \frac{1}{\sqrt{6}}|+-\rangle + \sqrt{\frac{2}{3}}|00\rangle + \frac{1}{\sqrt{6}}|-+\rangle$$

$$\hbar\sqrt{(2-0)(2+0+1)}\,|2,1\rangle = \hbar\sqrt{6}\,|2,1\rangle$$

$$J_+|2,1\rangle = \frac{1}{\sqrt{6}}J_{2+}|+-\rangle + \sqrt{\frac{2}{3}}(J_{1+}+J_{2+})|00\rangle + \frac{1}{\sqrt{6}}J_{1+}|-+\rangle$$

$$\frac{1}{\sqrt{6}}\sqrt{(1-(-1))(1+(-1)+1)}\;|+0\rangle + \sqrt{\frac{2}{3}}(\sqrt{2}|+0\rangle + \sqrt{2}|0+\rangle) + \frac{1}{\sqrt{3}}|0+\rangle$$

$$= \frac{1}{\sqrt{3}}|+0\rangle + \sqrt{\frac{4}{3}}|+0\rangle + \sqrt{\frac{4}{3}}|0+\rangle + \frac{1}{\sqrt{3}}|0+\rangle$$

$$= \hbar\frac{3}{\sqrt{3}}(|+0\rangle + |0+\rangle) \qquad |2,1\rangle = \frac{3}{3\sqrt{2}}(|+0\rangle + |0+\rangle) = \frac{1}{\sqrt{2}}(|+0\rangle + |0+\rangle)$$

$$\hbar\sqrt{(2-1)(2+1+1)}\,|2,2\rangle = 2\hbar|2,2\rangle = \frac{\hbar}{\sqrt{2}}\left(\sqrt{(1-0)(2)}\right)(|++\rangle + |++\rangle)$$

$$2\hbar|2,2\rangle = 2\hbar|++\rangle$$

$$|2,2\rangle = |++\rangle$$

In summary, for the $J = 2$ states:

$$|2,2\rangle = |++\rangle$$

$$|2,1\rangle = \frac{1}{\sqrt{2}}(|+0\rangle + |0+\rangle)$$

$$|2,0\rangle = \frac{1}{\sqrt{6}}|+-\rangle + \sqrt{\frac{2}{3}}|00\rangle + \frac{1}{\sqrt{6}}|-+\rangle$$

$$|2,-1\rangle = \frac{1}{\sqrt{2}}(|0-\rangle + |-0\rangle)$$

$$|2,-2\rangle = +|--\rangle$$

Sakurai 3.34

$|2;1\rangle$ to derive $|1,1\rangle = a|+0\rangle + b|0+\rangle$

$$\langle 2,1|1,1\rangle = \frac{a+b}{\sqrt{2}} = 0 \qquad aa^* + bb^* = 1$$

$$a = -b \longrightarrow a = \frac{1}{\sqrt{2}} \quad b = -\frac{1}{\sqrt{2}}$$

$$\therefore |1,1\rangle = \frac{1}{\sqrt{2}}(|+0\rangle - |0+\rangle)$$

$$J_-|1,1\rangle = \sqrt{(1+1)(1-1+1)}\,|1,0\rangle$$

$$= \frac{1}{\sqrt{2}}(J_{1-}|+0\rangle + J_{2-}|+0\rangle - J_{1-}|0+\rangle - J_{2-}|0+\rangle)$$

$$\sqrt{2}\,|1,0\rangle = \frac{1}{\sqrt{2}}(\sqrt{2}\cancel{|00\rangle} + \sqrt{2}|+-\rangle - \sqrt{2}|-+\rangle - \sqrt{2}\cancel{|00\rangle})$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$$

$$J_-|1,0\rangle = \sqrt{2}|1,-1\rangle = \frac{1}{\sqrt{2}}(J_{1-}|+-\rangle + J_{2-}|+-\rangle - J_{1-}|-+\rangle - J_{2-}|-+\rangle)$$

$$\sqrt{2}|1,-1\rangle = \frac{1}{\sqrt{2}}(\sqrt{2}|0-\rangle - \sqrt{2}|-0\rangle)$$

$$|1,-1\rangle = \frac{1}{\sqrt{2}}(|0-\rangle - |-0\rangle)$$

$$\langle 2,0|0,0\rangle = \langle 1,0|0,0\rangle = 0$$

$$|2,0\rangle = \frac{1}{\sqrt6}|+-\rangle + \sqrt{\frac{2}{3}}|0\,0\rangle + \frac{1}{\sqrt6}|-+\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt2}(|+-\rangle - |-+\rangle)$$

$$|0,0\rangle = a|+-\rangle + b|00\rangle + c|--\rangle$$

$$\langle 2,0|0,0\rangle = \frac{a}{\sqrt6} + \sqrt{\frac{2}{3}}b + \frac{c}{\sqrt6} = 0$$

$$\langle 1,0|0,0\rangle = \frac{1}{\sqrt2}(a-c) = 0 \longrightarrow a = c$$

$$\frac{2}{\sqrt6}a + \sqrt{\frac{2}{3}}b = 0 \rightarrow a = \frac{\sqrt6}{2}\cdot -\sqrt{\frac{2}{3}}b = -b$$

$$a^2+b^2+c^2=1 \longrightarrow 2a^2+b^2=1 \longrightarrow 2a^2+a^2=1 \quad a=\sqrt{\frac{1}{3}}$$

$$b=-\sqrt{\frac{1}{3}}$$

$$c=+\sqrt{\frac{1}{3}}$$

$$\rightarrow |0,0\rangle = \frac{1}{\sqrt3}(|+-\rangle - |0\,0\rangle + |-+\rangle)$$

$$\boxed{\begin{aligned}
|2,2\rangle &= |++\rangle\\
|2,1\rangle &= \frac{1}{\sqrt2}(|+0\rangle + |0+\rangle)\\
|2,0\rangle &= \frac{1}{\sqrt6}|+-\rangle + \sqrt{\frac{2}{3}}|0\,0\rangle + \frac{1}{\sqrt6}|-+\rangle\\
|2,-1\rangle &= \frac{1}{\sqrt2}(|0-\rangle + |-0\rangle)\\
|2,-2\rangle &= +|--\rangle\\
|0,0\rangle &= \frac{1}{\sqrt3}(|+-\rangle - |0\,0\rangle + |-+\rangle)\\
|1,1\rangle &= \frac{1}{\sqrt2}(|+0\rangle - |0+\rangle)\\
|1,0\rangle &= \frac{1}{\sqrt2}(|+-\rangle - |-+\rangle)\\
|1,-1\rangle &= \frac{1}{}(|0-\rangle - |-0\rangle)
\end{aligned}}$$