Lecture # 3

Wednesday, October 2, 2024 10:22 PM

Lagrangian formalism deals well with constraints

holonomic $\quad \phi_J(q_1, \dots q_N, t) = 0 \quad J = 1, \dots s$

can use method of Lagrange multipliers

$$L' = L + \sum_J \lambda_J \phi_J$$

$$\lambda_J : \quad \phi_J = 0$$

$$q_i : \quad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_i} + \sum_J \lambda_J \frac{\partial \phi_J}{\partial q_i} = 0$$

$$\frac{\partial L'}{\partial q_i} - \frac{d}{dt}\frac{\partial L'}{\partial \dot q_i} = 0$$

for holonomic constraints . forces orthogonal to surface

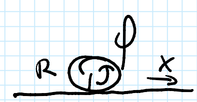
$$\vec F_{const} \cdot \delta \vec x = 0$$

D'Alembert principle

" derive Lagrangian formalism

Non holonomic : velocity dependent / inequalities

example : rolling without slipping in 1D



$$\dot x = -R\dot\phi$$

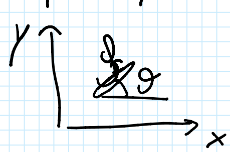
but this can be rearranged $\quad \frac{d}{dt}(x + R\phi) = 0$

$$x + R\phi = x_0 \qquad x + R\phi - x_0 = 0$$

back to holonomic

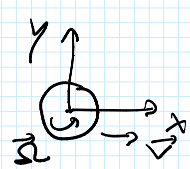
Not always possible

- motion of unicyclist



$$\dot x = R\dot\phi\cos\vartheta \qquad \dot y = \dot x \tan\vartheta$$
$$\dot y = R\dot\phi\sin\vartheta$$

- Sphere rolling without slipping on a plane



$$\vec V - R(\vec\Omega \times \hat n) = 0$$

Non holonomic constraints linear in velocities

$$\phi_r(q,\dot q,t) = \sum_i \psi^{(r)}(q_i)\dot q_i \qquad \psi^{(r)} = \frac{\partial \Omega^{(r)}}{\partial q_i}$$

$$= \frac{d}{dt}\Omega^{(r)}$$

Then we can consider $L' = L + \sum_r \lambda_r \Omega^{(r)}$

following recipe and writing E-L equations λ_r: $\Omega^{(r)} = 0$

$$q_i \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot q_i} - \frac{\partial L}{\partial q_i} - \sum \lambda_r \underbrace{\frac{\partial \Omega^{(r)}}{\partial q_i}}_{=\psi^{(r)}} = 0$$

Let us continue exploring Lagrangian mechanics

what happens if we add total time derivative to Lagrangian? $\Rightarrow$ Equivalent Lagrangian

$$\frac{d}{dt}\Lambda(q,t)$$

$$L' = L(q,\dot q,t) + \frac{d}{dt}\Lambda(q,t) = L(q,\dot q,t) + \frac{\partial \Lambda}{\partial q}\dot q + \frac{\partial \Lambda}{\partial t}$$

$$S'[q] = S[q] + \int_{t_1}^{t_2} \frac{d}{dt}\Lambda(q,t)\,dt = S[q] + \Lambda(q(t_2),t_2) - \Lambda(q(t_1),t_1)$$

$\delta S' = \delta S$ so E-L equations will not change

$$\frac{\partial L'}{\partial q_i} - \frac{d}{dt}\frac{\partial L'}{\partial \dot q_i} = 0 = \frac{\partial L}{\partial q_i} + \cancel{\frac{\partial}{\partial q_i}\frac{d\Lambda}{dt}} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_i} - \frac{d}{dt}\frac{\partial}{\partial \dot q_i}\left(\frac{\partial \Lambda}{\partial q_i}\dot q_i + \cancel{\frac{\partial \Lambda}{\partial t}}\right)$$

$$\cancel{-\frac{d}{dt}\frac{\partial \Lambda}{\partial q_i}}$$

$$= \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_i}$$

$$L = \frac{1}{2}m\dot{\vec x}^2 - q\phi + q\dot{\vec x}\cdot\vec A \qquad \vec E = -\vec\nabla\phi - \frac{\partial \vec A}{\partial t}$$

Lagrangian of charged particle $\qquad \vec B = \vec\nabla \times \vec A$

Lagrangian of charged particle in EM field

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

eq. of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{x}}} = \frac{d}{dt}\left(m\dot{\vec{x}} + q\vec{A}\right) = \frac{\partial L}{\partial \vec{x}} = -q\vec{\nabla}\phi + q\,\vec{\nabla}\left(\dot{\vec{x}}\cdot\vec{A}\right)$$

Now use
$$\vec{\nabla}\left(\dot{\vec{x}}\cdot\vec{A}\right) = \dot{\vec{x}}\times\left(\vec{\nabla}\times\vec{A}\right) + \left(\dot{\vec{x}}\cdot\vec{\nabla}\right)\vec{A}$$

$$\frac{d}{dt}q\vec{A}(\vec{x},t) = q\frac{\partial\vec{A}}{\partial t} + q\frac{\partial A}{\partial x}\dot{x}$$

$$m\ddot{\vec{x}} + \cancel{q\left(\dot{\vec{x}}\cdot\vec{\nabla}\right)\vec{A}} + q\frac{\partial\vec{A}}{\partial t} = -q\vec{\nabla}\phi + q\,\dot{\vec{x}}\times\left(\vec{\nabla}\times\vec{A}\right) + \cancel{q\left(\dot{\vec{x}}\cdot\vec{\nabla}\right)\vec{A}}$$

$$m\ddot{\vec{x}} = -q\vec{\nabla}\phi - q\frac{\partial\vec{A}}{\partial t} + q\,\dot{\vec{x}}\times\left(\vec{\nabla}\times\vec{A}\right) = q\vec{E} + q\,\vec{v}\times\vec{B}$$

Lorentz force

Consider a gauge transformation

$$\phi' = \phi - \frac{\partial\Lambda}{\partial t} \qquad \vec{A}' = \vec{A} + \vec{\nabla}\Lambda$$

$$L' = L + q\frac{\partial\Lambda}{\partial t} + q\left(\dot{\vec{x}}\cdot\vec{\nabla}\right)\Lambda = L + q\frac{d\Lambda}{dt}$$

Symmetry and conservation laws

Some conservation laws directly follow from E-L equations

- coordinate q_i is cyclic if it does not appear in Lagrangian $L(\dot{q}_i \dots)$

$$\Rightarrow \frac{\partial L}{\partial q_i} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

$\frac{\partial L}{\partial \dot{q}_i}$ conjugate momentum of q_i is a conserved quantity

Examples:

$$L = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - V(x_1)$$ flat dimension in potential

x_2 is cyclic $\Rightarrow \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2$ is conserved

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - V(r)$$ central potential . angular momentum

$L = \frac{1}{2} m(\dot{r}^2 + r^2 \dot{\varphi}^2) - V(r)$ central potential . angular momentum

φ is cyclic $\Rightarrow \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi}$ is conserved

if L does not depend on time $\left(\frac{\partial L}{\partial t} = 0\right)$

$$H = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \quad \text{is conserved}$$

proof using E-L equations

$$\frac{dH}{dt} = \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i\right) - \frac{d}{dt} L(q, \dot{q})$$

$$\cancel{\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right) \dot{q}_i} + \cancel{\frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i} - \cancel{\frac{\partial L}{\partial q_i} \dot{q}_i} - \cancel{\frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i} = 0$$

because of E-L

for example

$$L = \frac{1}{2} m \dot{\vec{q}}^2 - V(\vec{q})$$

$$H = m \dot{\vec{q}} \cdot \dot{\vec{q}} - L = m \dot{q}^2 - \frac{1}{2} m \dot{q}^2 + V(q) = \frac{1}{2} m \dot{q}^2 + V = T + V = E$$

(conservation of H is more general)

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what are commonalities in all these examples?

- we use E-L equations to prove conservation
- there is some invariance (symmetry) of Lagrangian
- continuous transformations

cyclic variable $q' \to q + q_0$

time invariance $t' = t + t_0$

Noether's theorem generalizes all of this, and provide a formula to calculate conserved charge

Define a one parameter family of infinitesimal transformations

Define a one parameter family of infinitesimal transformations

$$q_i'(t,\varepsilon) = q_i(t) + \varepsilon \delta q_i + O(\varepsilon^2)$$

can always build up finite transformations adding up infinitesimal steps

Then consider

$$L'(q_i', \dot{q}_i', t) - L(q_i, \dot{q}_i, t)$$

$q_i'(t,\varepsilon)$ is a symmetry of Lagrangian if difference is 0 at order ε

(slightly more generally, Lagrangians can be equivalent or differ by $\frac{d}{dt}\Lambda$)

Mathematically

$$\frac{\partial}{\partial \varepsilon} L'(q_i', \dot{q}_i', t)\Big|_{\varepsilon=0} = \frac{d}{dt}\Lambda(q,t)$$

example: rotations

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - V(x_1^2 + x_2^2)$$

$$x_1' = x_1 + \varepsilon x_2$$
$$x_2' = x_2 - \varepsilon x_1$$

$$L' = \frac{1}{2}m\left[(\dot{x}_1 + \varepsilon \dot{x}_2)^2 + (\dot{x}_2 - \varepsilon \dot{x}_1)^2\right] - V\left((x_1 + \varepsilon x_2)^2 + (x_2 - \varepsilon x_1)^2\right)$$

$$\frac{\partial L'}{\partial \varepsilon} = \frac{1}{2}m(2\dot{x}_1\dot{x}_2 - 2\dot{x}_2\dot{x}_1) - \frac{\partial V}{\partial r}(2x_1x_2 - 2x_2x_1) = 0$$

Noether's theorem:

for each continuous symmetry of Lagrangian there exists a conserved quantity $Q = \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta q_i - \Lambda$