Homework 3

Problem 1

(a) $x, \dot{x}, t$

$$L = e^{\frac{\gamma t}{m}}\left(\frac{m}{2}\dot{x}^2 - \frac{k}{2}x^2\right)$$

$$\frac{\partial L}{\partial x} = -e^{\frac{\gamma t}{m}} kx \longrightarrow$$

$$\frac{\partial L}{\partial \dot{x}} = e^{\frac{\gamma t}{m}} m\dot{x}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = -e^{\frac{\gamma t}{m}} kx - \left(m\dot{x}\frac{d}{dt}(e^{\gamma t/m}) + e^{\gamma t/m}\frac{d}{dt}(m\dot{x})\right)$$

$$= -e^{\frac{\gamma t}{m}}kx - \left(m\dot{x}\frac{\gamma}{m}e^{\gamma t/m} + m\ddot{x}e^{\gamma t/m}\right)$$

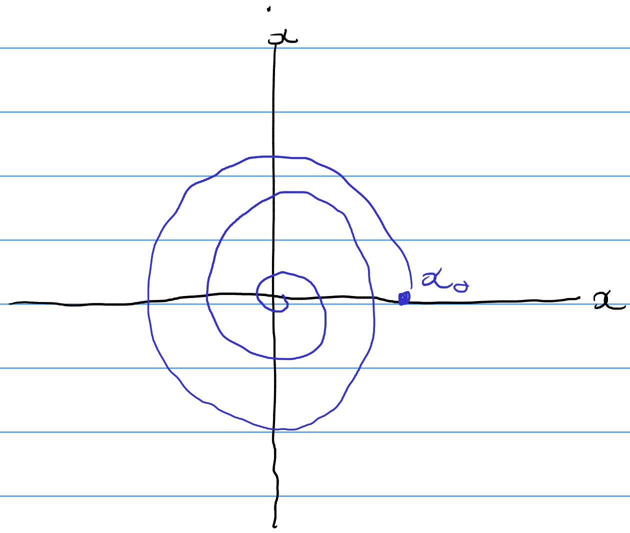
$$= e^{\gamma t/m}\left(-kx - \gamma\dot{x} + m\ddot{x}\right) = 0$$

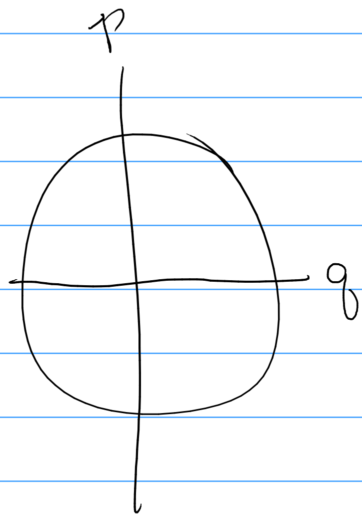
$$\longrightarrow \ddot{x} - \frac{\gamma}{m}\dot{x} - kx = 0 \longrightarrow \quad x(t) = \begin{cases} e^{-\frac{1}{2}at}\left[c_1 e^{\frac{1}{2}\lambda t} + c_2 e^{-\frac{1}{2}\lambda t}\right] & \lambda^2 > 0 \\ e^{-\frac{1}{2}at}\left[c_1 \sin(\frac{1}{2}\lambda t) + c_2\cos(\frac{1}{2}\lambda t)\right] & \lambda^2 = 4b - a^2 > 0 \\ e^{-\frac{1}{2}at}(c_1 x + c_2) & a^2 = 4b \end{cases}$$

$a = -\frac{\gamma}{m}$ $\quad b = -k$

$\lambda = a^2 - 4b$

(b)





(c) $H = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = \frac{\partial L}{\partial \dot{x}} \dot{x} - L = e^{\gamma t/m} m\dot{x}^2 - e^{\gamma t/m}\left(\frac{m\dot{x}^2}{2} - \frac{kx^2}{2}\right) = e^{\gamma t/m}\left(\frac{m\dot{x}^2}{2} + \frac{kx^2}{2}\right)$

(d) Hamiltonian flow preserves symplectic structure

$MJM^T = J$

$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$M = \begin{pmatrix} 1 + dt\frac{\partial^2 H}{\partial q \partial p} & dt\frac{\partial^2 H}{\partial p^2} \\ -dt\frac{\partial^2 H}{\partial q^2} & 1 - dt\frac{\partial^2 H}{\partial p \partial q} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -e^{\gamma t/m}k & 1 \end{pmatrix}$$

$\frac{\partial^2 H}{\partial q \partial p} = \frac{\partial}{\partial p}\left(e^{\gamma t/m} kx\right) = 0$

$\frac{\partial^2 H}{\partial q^2} =$

$$\begin{pmatrix} 1 & 0 \\ -e^{\gamma t/m}k & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -e^{\gamma t/m}k \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -e^{\gamma t/m}k & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & e^{\gamma t/m}k \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\vec{A}\cdot\dot{\vec{r}} = A_x\dot{x} + A_y\dot{y} + A_z\dot{z}$$
$$A_x = 0$$

Problem 2

$$\leftarrow A_y\dot{y}$$

(a) $$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + e(\vec{A}\cdot\dot{\vec{r}})$$

$$\frac{\partial L}{\partial x} = -e\frac{\partial A_y}{\partial x}\dot{y} \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \frac{\partial L}{\partial x} - \frac{d}{dt}(m\dot{x}) = e\frac{\partial A_y}{\partial x}\dot{y} + m\ddot{x} = 0$$

$$\frac{\partial L}{\partial y} = 0 \quad \frac{\partial L}{\partial \dot{y}} = m\dot{y} + eA_y \quad \frac{d}{dt}(m\dot{y} + eA_y) = 0 \quad m\dot{y} = -eA_y$$

$$\frac{\partial L}{\partial z} = e\frac{\partial A_y}{\partial z}\dot{y} \quad \frac{\partial L}{\partial \dot{z}} = m\dot{z} \quad m\ddot{z} - e\frac{\partial A_y}{\partial z}\dot{y} = 0$$

$$H = \sum_i p_i\dot{q}_i - L = m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - eA_y\dot{y}$$
$$= \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - eA_y\dot{y}$$
$$= \frac{p_0^2}{2m} - eA_y p_0\cos\theta$$

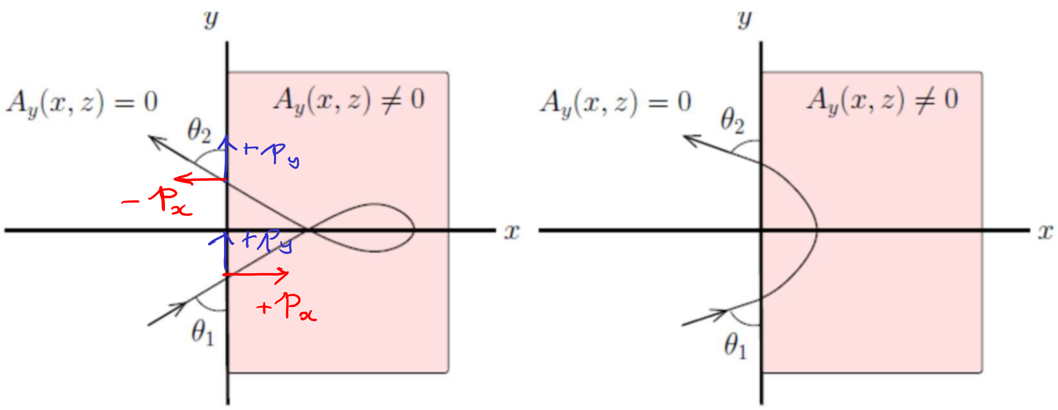
$$\vec{p} = m\dot{x}\hat{x} + (m\dot{y} + eA_y)\hat{y} + (m\dot{z})\hat{z}$$

(b) $$\frac{\partial L}{\partial z} = e\frac{\partial A_y}{\partial z}\dot{z}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \longrightarrow B_z = (\vec{\nabla}\times\vec{A})_z = (\partial_x A_y - \partial_y A_x)$$
$$B_y = \cancel{\partial_x A_z} - \cancel{\partial_z A_x} = 0$$
$$B_x = \cancel{\partial_y A_z} - \cancel{\partial_z A_y} = 0 \longrightarrow$$

at $z=0$ $\partial A_y/\partial z = 0$, so $\partial L/\partial z = 0$, so momentum along z is conserved.

(c)



Problem 2

(c) $m\ddot{x} + e\frac{\partial A_y}{\partial x}\dot{y} = 0$

$$\int_0^t \left[m\ddot{x} + e\frac{\partial A_y}{\partial x}\dot{y}\right] dt = m\dot{x} + e\frac{\partial A_y}{\partial x} y = 0$$

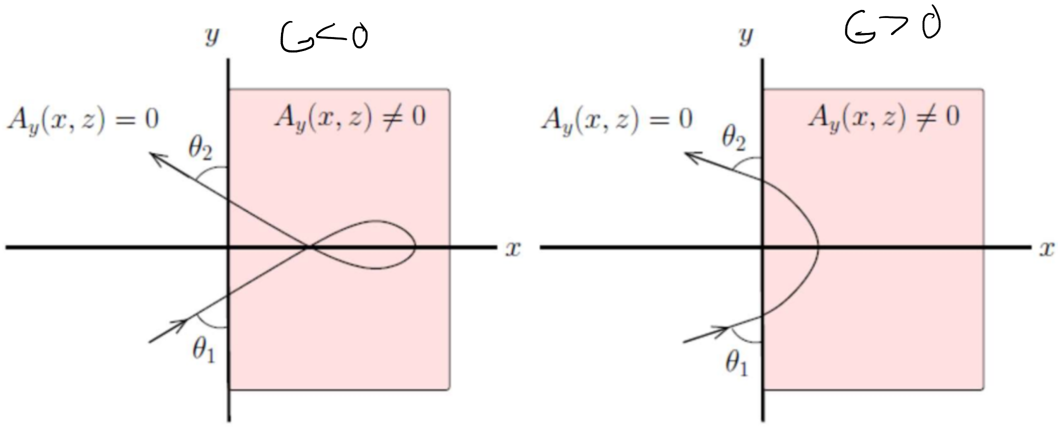
$$m\dot{x} = -e\frac{\partial A_y}{\partial x} y \qquad m\dot{x} = 0 \text{ when } y = 0$$

The electron will have reflected across the x axis by the time it has escaped. This means it will be the same distance from the x-axis. mxdot will have the same magnitude velocity along the x direction but of opposite sign.

(d) $G((\hat{x}z - \hat{z}x))$

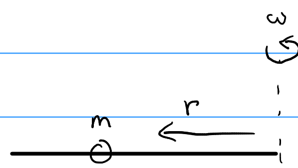
$$B_x = \hat{x}(\partial_y A_z - \partial_z A_y)$$
$$Gz = -\partial_z A_y$$
$$A_y = -Gz^2 + c$$
$$-Gx\hat{z} = \hat{z}(\partial_x A_y - \cancel{\partial_y A_x}) = \hat{z}\partial_x A_y$$
$$A_y = -Gx^2 + c$$
$$A_y = -G(x^2 + z^2) + c$$

$$\dot{y} = -\frac{eA_y}{m} = \frac{e}{m}\left[G(x^2+z^2) + c\right]$$



Problem 3

(a)



$L=\frac{m}{2}[(r\dot\theta)^2+\dot r^2] \longrightarrow L=\frac{m}{2}[(r'(\dot\theta'+\omega))^2+\dot r'^2]$

in the frame w/ θ'. $\dot\theta'=0$ because we are rotating w/ the frame.

$\theta=\theta'+\omega t \qquad \dot\theta=\dot\theta'+\omega$

$r=r' \qquad \dot r=\dot r'$

$\longrightarrow L=\frac{m}{2}[(r\omega)^2+\dot r^2]$

(b) $H=\sum_i(p_i\dot q_i)-L = \cancel{\frac{\partial L}{\partial\dot\theta}\dot\theta} + \frac{\partial L}{\partial\dot r}\dot r - L = m\dot r^2-\frac{m}{2}[(r\omega)^2+\dot r^2]=$

$m\dot r^2-\frac{m}{2}[(r\omega)^2+\dot r^2] = \frac{m}{2}\dot r^2-\frac{m}{2}(r^2\omega^2)$

$\frac{d}{dt}H=\frac{\partial}{\partial t}H=0 \longrightarrow$ Hamiltonian is constant

(c) $L=T=\frac{1}{2}m(r^2\omega^2+\dot r^2)$

$L=T-V \qquad V=0 \rightarrow L=T$

$H=T+V \qquad V=0 \rightarrow H=T \qquad L=H$

$-\frac{\partial L}{\partial t}=\frac{\partial H}{\partial t}=0$

Problem 4

(a) $\Lambda(q,t)$ so Lagrangian has no $\ddot{q}$ term

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} + \frac{\partial}{\partial q}\left(\frac{d\Lambda}{dt}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} + \cancel{\frac{\partial}{\partial \dot{q}}\left(\frac{d\Lambda}{dt}\right)}\right) = 0$$

$$\frac{d}{dt}\Lambda = \frac{\partial \Lambda}{\partial t} + \frac{\partial \Lambda}{\partial q}\dot{q}$$

$$\frac{\partial}{\partial q}\left(\frac{d\Lambda}{dt}\right) = \frac{\partial}{\partial q}\cancel{\frac{\partial \Lambda}{\partial t}} + \frac{\partial \Lambda}{\partial q}\cancel{\dot{q}}$$

$$\frac{\partial L}{\partial q} = \frac{\partial L}{\partial q} + \frac{\partial}{\partial q}\frac{\partial \Lambda}{\partial t} + \frac{\partial^2 \Lambda}{\partial q^2}\dot{q}$$

- The coordinates do not change
- momentum has an additional term $\frac{\partial \Lambda}{\partial q}\dot{q}$

(b) $H = p\dot{q} - L$

$$p = \frac{\partial L}{\partial \dot{q}} + \frac{\partial \Lambda}{\partial q}\dot{q} \rightarrow H = \frac{\partial L}{\partial \dot{q}}\dot{q} + \frac{\partial \Lambda}{\partial q}\dot{q}^2 - L(q,\dot{q},t) - \frac{d\Lambda}{dt}$$

$$H' = H + \frac{\partial \Lambda}{\partial q}\dot{q} - \frac{d\Lambda}{dt}$$

(c) $$\{A,B\}_{q^I,p^I} = \frac{\partial A}{\partial q^I}\frac{\partial B}{\partial p^I} - \frac{\partial A}{\partial p^I}\frac{\partial B}{\partial q^I} = \frac{\partial A}{\partial q^I}\,\partial q^I$$

$$\frac{\partial q^I}{\partial q} = 1 \quad \frac{\partial p^I}{\partial p} = 1$$