Problem 1

1) First of all, remember that in general if we have a map

$$f: \mathbb{R}^n \to \mathbb{R}^n$$

$$x_i \mapsto x_i' = f_i(\vec{x})$$

then it sends a volume element

$$dV = dx_1 \cdots dx_n$$

to a volume element:

$$dV' = dx_1' \cdots dx_n' = \left| \det\left(\frac{\partial f_i}{\partial x_j}\right)\right| dx_1 \cdots dx_n$$

$$= \left| \det\left(\frac{\partial f_i}{\partial x_j}\right)\right| dV$$

so it is volume-preserving if and only if

$$\left| \det\left(\frac{\partial f_i}{\partial x_j}\right)\right| = 1$$

In our case, the map f is the time-evolution by a time t, call it U_t, defined as:

$$U_t : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$\vec{x}_0 \longmapsto U_t(\vec{x}_0) \equiv \vec{x}\,(t; \vec{x}_0)$$

where by the notation $\vec{x}\,(t; \vec{x}_0)$ we emphasize that the position at time t depends on the initial position $\vec{x}_0$. The thing we need to show is:

$$\left| \det(M) \right| = 1$$

where the matrix M is defined as:

$$M_{ij} = \frac{\partial U_t^{\,i}}{\partial x_0^{\,j}} = \frac{\partial}{\partial x_0^{\,j}} \left(x^i (t; \vec{x}_0) \right)$$

Now, by definition

$$\frac{\partial}{\partial t} \left(\vec{x}(t, \vec{x}_0) \right) = \vec{v}\left(\vec{x}(t, \vec{x}_0)\right)$$

so

$$\frac{\partial M_{ij}}{\partial t} = \frac{\partial}{\partial t} \frac{\partial}{\partial x_0^j} x^i (t, \vec{x}_0)$$

$$= \frac{\partial}{\partial x_0^j} \frac{\partial}{\partial t} x^i (t, \vec{x}_0)$$

$$= \frac{\partial}{\partial x_0^j} v^i \left(\vec{x}(t, \vec{x}_0) \right)$$

chain rule

$$= \frac{\partial v^i}{\partial x^k} \frac{\partial}{\partial x_0^j} x^k (t, \vec{x}_0) = \frac{\partial v^i}{\partial x^k} \cdot M_{kj}$$

We now observe the following: for any matrix $A(t)$, we have

$$\frac{d}{dt} \det(A(t)) = \det(A) \cdot \operatorname{tr}\left[A^{-1} \frac{dA}{dt} \right]$$

(a proof is provided for reference at the end of the solution). So:

$$\frac{\partial}{\partial t} \det(M) = \det(M) \cdot \operatorname{tr}\left[M^{-1}(t) \frac{\partial M}{\partial t} \right]$$

$$= \det(M) \cdot (M^{-1})_{ji} \frac{\partial M_{ij}}{\partial t}$$

$$= \det(M) \cdot (M^{-1})_{ji} \frac{\partial v^i}{\partial x^k} M_{kj}$$

$$= \det(M) \frac{\partial v^i}{\partial x^k} (M \cdot M^{-1})_{ki}$$

$$= \det(M) \cdot \frac{\partial v^i}{\partial x^k} \delta_{ki}$$

$$= \det(M) \cdot (\vec{\nabla} \cdot \vec{v})$$

hence, if $\vec{\nabla} \cdot \vec{v} = 0$,

$$\frac{\partial}{\partial t} \det M = 0$$

so $\det M$ is constant. But:

$$M_{ij}(t=0) = \frac{\partial}{\partial x_0^j} x^i(t=0, \vec{x}_0) = \frac{\partial}{\partial x_0^j} x_0^i = \delta_{ij}$$

so $M(t=0) = \mathbb{1}$ and $\det(M(t)) = \det(M(0)) = \det(\mathbb{1}) = 1$,

which concludes.

b) We denote indices ranging from 1 to n as greek letters, while those ranging from 1 to $2n$ as latin letters, so for example:

$$\vec{\nabla} \cdot \vec{v} = \sum_{i=1}^{2n} \frac{\partial v_i}{\partial x_i} = \sum_{\alpha=1}^{n} \left(\frac{\partial v_\alpha}{\partial q_\alpha} + \frac{\partial v_{n+\alpha}}{\partial p_\alpha} \right)$$

now, $J = \begin{pmatrix} \mathbb{0} & \mathbb{1} \\ -\mathbb{1} & \mathbb{0} \end{pmatrix}$, which means that for $\alpha = 1, \dots, n$

we have:

$$v_\alpha = \frac{\partial H}{\partial p_\alpha} \qquad v_{n+\alpha} = -\frac{\partial H}{\partial q_\alpha}$$

so:

$$\vec{\nabla} \cdot \vec{v} = \sum_{\alpha=1}^{n} \underbrace{\left(\frac{\partial}{\partial q_\alpha} \frac{\partial H}{\partial p_\alpha} - \frac{\partial}{\partial p_\alpha} \frac{\partial H}{\partial q_\alpha} \right)}_{0 \text{ by symmetry of partial derivatives}} = 0$$

Proof that $\frac{d}{dt}\det(A) = \det(A)\cdot \mathrm{tr}\left(A^{-1}\frac{dA}{dt}\right)$

$$\frac{d}{dt}\det\left(A(t)\right) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\det\left(A(t+\varepsilon)\right) - \det\left(A(t)\right)\right]$$

but

$$A(t+\varepsilon) \simeq A(t) + \frac{dA}{dt}\cdot\varepsilon + \mathcal{O}(\varepsilon^2)$$

$$= A(t) + A(t)\cdot A^{-1}(t)\cdot\frac{dA}{dt}\cdot\varepsilon + \mathcal{O}(\varepsilon^2$$

$$= A\left(\mathbb{1} + \varepsilon A^{-1}\frac{dA}{dt} + \mathcal{O}(\varepsilon^2)\right)$$

$$\Rightarrow \det\left(A(t+\varepsilon)\right) \simeq \det\left[A\left(\mathbb{1} + \varepsilon A^{-1}\frac{dA}{dt}\right)\right]$$

by $\det(AB) = \det A\,\det B$

$$= \det A \cdot \det\left(\mathbb{1} + \varepsilon A^{-1}\frac{dA}{dt}\right)$$

so:

$$\frac{d}{dt}\det(A(t)) = \det(A)\cdot\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\det\left(\mathbb{1} + \varepsilon A^{-1}\frac{dA}{dt}\right) - 1\right]$$

now, for any matrix M,

$$\det M = \frac{1}{n!}\,\varepsilon_{i_1\ldots i_n}\,\varepsilon_{j_1\ldots j_n}\,M_{i_1 j_1}\cdots M_{i_n j_n}$$

(⚠: $\varepsilon_{i_1\ldots i_n}$ here is the Levi-Civita symbol, it has nothing to do with the small parameter also called ε)

so if $M = \mathbb{1} + \varepsilon B$, at lowest order we obtain:

$$\det M = \frac{1}{n!} \varepsilon_{i_1 \dots i_n} \varepsilon_{j_1 \dots j_n} \prod_{k=1}^{n} \left(\delta_{i_k j_k} + \varepsilon B_{i_k j_k} \right)$$

$$= \frac{1}{n!} \varepsilon_{i_1 \dots i_n} \varepsilon_{j_1 \dots j_n} \left(\delta_{i_1 j_1} \cdots \delta_{i_n j_n} + \varepsilon \sum_{k=1}^{n} \delta_{i_1 j_1} \cdots B_{i_k j_k} \cdots \delta_{i_n j_n} \right)$$

$$= \det \mathbb{1} + \varepsilon \cdot \frac{1}{n!} \cdot \sum_{k=1}^{n} \underbrace{\varepsilon_{i_1 \dots i_n} \varepsilon_{i_1 \dots i_{k-1} j_k i_{k+1} \dots i_n} B_{i_k j_k}}$$

by antisymmetry of ε's, this is

$$= \varepsilon_{i_1 \dots i_{n-1} i} \varepsilon_{i_1 \dots i_{n-1} j} B_{ij}$$

$$= 1 + \varepsilon \frac{1}{n!} n \cdot \varepsilon_{i_1 \dots i_{n-1} i} \varepsilon_{i_1 \dots i_{n-1} j} B_{ij}$$

now, by representation theory arguments:

$$\varepsilon_{i_1 \dots i_{n-1} i} \varepsilon_{i_1 \dots i_{n-1} j} = \alpha \delta_{ij}$$

contracting i and j:

$$\varepsilon_{i_1 \dots i_n} \varepsilon_{i_1 \dots i_n} = \alpha \cdot \delta_{ii}$$

but $\varepsilon_{i_1 \dots i_n} \varepsilon_{i_1 \dots i_n}$ is 1 when the i_k's are distinct and 0 otherwise, so it is $n!$, while $\delta_{ii} = n$, hence:

$$n! = \alpha n \implies \alpha = (n-1)!$$

so:

$$\det M \simeq 1 + \varepsilon \frac{1}{n!} n (n-1)! \delta_{ij} B_{ij} = 1 + \varepsilon \operatorname{tr} B$$

Hence finally

$$\frac{d}{dt}\det(A(t)) = \det(A)\cdot\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\det\left(\mathbb{1}+\varepsilon A^{-1}\frac{dA}{dt}\right)-1\right]$$

$$= \det(A)\cdot\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[1+\varepsilon\,\mathrm{tr}\left(A^{-1}\frac{dA}{dt}\right)-1\right]$$

$$= \det A \;.\; \mathrm{tr}\left(A^{-1}\frac{dA}{dt}\right)$$

as required

Problem 2

Remember that we can write cross products in components as:

$$(\vec{a} \times \vec{b})_i = \varepsilon_{ijk} a_j b_k$$

and that we have the identity

$$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

corresponding to the triple product identity $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

a) We start by observing $L_j = \varepsilon_{jkl} r_k p_l$ and computing

$$\{r_i, L_j\} = \varepsilon_{jkl} \{r_i, r_k p_l\} =$$

$$= \varepsilon_{jkl} \left(\underbrace{\{r_i, r_k\}}_{0} p_l + r_k \underbrace{\{r_i, p_l\}}_{\delta_{il}} \right)$$

$$= \varepsilon_{jkl} r_k \delta_{il}$$

$$= \varepsilon_{jki} r_k = \varepsilon_{ijk} r_k$$

and similarly:

$$\{p_i, L_j\} = \varepsilon_{jkl} \{p_i, r_k p_l\} =$$

$$= - \varepsilon_{jkl} \delta_{ik} p_l$$

$$= \varepsilon_{ijl} p_l$$

This yields the angular momentum algebra:

$$\{L_i, L_j\} = 2\, \varepsilon_{ijk} L_k \qquad \text{(see at the end for a proof using } \varepsilon_{ijk}\text{'s)}$$

Now, in components:

$$K_j = \frac{1}{m} \varepsilon_{jlm} p_l L_m - \frac{1}{r} r_j$$

so:

$$\{L_i, K_j\} = \frac{1}{m} \varepsilon_{jlm} \{L_i, p_l L_m\} - \{L_i, \frac{1}{r} r_j\}$$

The first term is

$$\frac{1}{m} \varepsilon_{jlm} \{L_i, p_l L_m\} = \frac{1}{m} \varepsilon_{jlm} \left(\{L_i, p_l\} L_m + \{L_i, L_m\} p_l \right)$$

$$= \frac{1}{m} \varepsilon_{jlm} \left(\varepsilon_{ilk} p_k L_m + \varepsilon_{imk} L_k p_l \right)$$

$$= \frac{1}{m} \left(\varepsilon_{lmj} \varepsilon_{lki} p_k L_m + \varepsilon_{mjl} \varepsilon_{mki} L_k p_l \right)$$

$$= \frac{1}{m} \left(\delta_{ij} p_k L_k - L_i p_j + L_j p_i - \delta_{ij} L_k p_k \right)$$

$$= \frac{1}{m} \left(p_i L_j - L_j p_i \right)$$

The other term is:

$$- \{L_i, \frac{1}{r} r_j\} = - \{L_i, \frac{1}{r}\} r_j - \frac{1}{r} \underbrace{\{L_i, r_j\}}_{\varepsilon_{ijk} r_k}$$

We explicitly compute:

$$\{L_i, \frac{1}{r}\} = m \, \varepsilon_{ijk} \left(\frac{\partial}{\partial r_l} (r_j p_k) \underbrace{\frac{\partial}{\partial p_l} \frac{1}{r}}_{0} - \frac{\partial}{\partial p_l} (r_j p_k) \frac{\partial}{\partial r_l} \frac{1}{r} \right)$$

$$= - m \, \varepsilon_{ijk} \, r_j \, \delta_{kl} \left(- \frac{r_l}{r^3} \right)$$

$$= + m \frac{1}{r^3} \varepsilon_{ijl} r_j r_l = 0$$

Putting it all together

$$\{L_i, K_j\} = \frac{1}{m}\left(p_i L_j - p_j L_i\right) - \frac{1}{r} \varepsilon_{ijk} r_k$$

while:

$$\varepsilon_{ijk} K_k = \frac{1}{m} \varepsilon_{ijk} \varepsilon_{klm} p_l L_m - \frac{1}{r} \varepsilon_{ijk} r_k$$

$$= \frac{1}{m} \varepsilon_{kij} \varepsilon_{klm} p_l L_m - \frac{1}{r} \varepsilon_{ijk} r_k$$

$$= \frac{1}{m} \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) p_l L_m - \frac{1}{r} \varepsilon_{ijk} r_k$$

$$= \frac{1}{m}\left(p_i L_j - p_j L_i\right) - \frac{1}{r} \varepsilon_{ijk} r_k$$

proving $\{L_i, K_j\} = \varepsilon_{ijk} K_k$

b) We rewrite:

$$K_i = \frac{1}{m} \varepsilon_{ijk} p_j L_k - \frac{1}{r} r_i$$

$$= \frac{1}{m} \varepsilon_{ijk} p_j \varepsilon_{klm} r_l p_m - \frac{1}{r} r_i$$

$$= \frac{1}{m} \varepsilon_{kij} \varepsilon_{klm} p_j r_l p_m - \frac{1}{r} r_i$$

$$= \frac{1}{m}\left(r_i p_j p_j - p_i r_j p_j\right) - \frac{1}{r} r_i$$

$$= \frac{1}{m}\left(r_i |\vec{p}|^2 - p_i (\vec{r}\cdot\vec{p})\right) - \frac{1}{r} r_i$$

This gives:

$$\frac{\partial K_i}{\partial r_k} = \frac{1}{m}\left(\delta_{ik} |\vec{p}|^2 - p_i p_k\right) - \frac{1}{r}\delta_{ik} + \frac{1}{r^3} r_i r_k$$

$$= \frac{1}{m} |\vec{p}|^2 \left(\delta_{ik} - \frac{p_i p_k}{|\vec{p}|^2}\right) - \frac{1}{r}\left(\delta_{ik} - \frac{r_i r_k}{r^2}\right)$$

$$\frac{\partial K_i}{\partial p_k} = \frac{1}{m}\left(2 r_i p_k - \delta_{ik} r_l p_l - p_i r_k\right)$$

Defining $\Pi^{(v)}_{ij} = \delta_{ij} - \frac{v_i v_j}{|\vec{v}|^2}$ we have

$$\Pi^{(v)}_{ij} v_j = \delta_{ij} v_j - v_i \frac{v_j v_j}{|\vec{v}|^2} = v_i - v_i = 0$$

and

$$\frac{\partial K_i}{\partial r_k} = \frac{1}{m} |\vec{p}|^2 \Pi^{(p)}_{ik} - \frac{1}{r} \Pi^{(r)}_{ik}$$

So, finally:

$$\{K_i, K_j\} = \frac{\partial K_i}{\partial r_k} \frac{\partial K_j}{\partial r_k} - (i \leftrightarrow j)$$

$$= \left(\frac{1}{m} |\vec{p}|^2 \Pi^{(p)}_{ik} - \frac{1}{r}\Pi^{(r)}_{ik}\right) \frac{1}{m}\left(2 r_j p_k - \delta_{jk} r_l p_l - p_j r_k\right) - (i \leftrightarrow j)$$

$$= \Bigg[\frac{|\vec{p}|^2}{m^2}\left(2 \overbrace{\Pi^{(p)}_{ik} p_k}^{0} r_j - \Pi^{(p)}_{ij} r_l p_l - p_j \Pi^{(p)}_{ik} r_k\right)$$
$$- \frac{1}{m}\frac{1}{r}\left(2 r_j \Pi^{(r)}_{ik} p_k - \Pi^{(r)}_{ij} r_l p_l - p_j \underbrace{\Pi^{(r)}_{ik} r_k}_{0}\right)\Bigg] - (i \leftrightarrow j)$$

now, Π_{ij} and δ_{ij} are symmetric, so the corresponding terms in $-(i \leftrightarrow j)$ cancel them, so we are left with:

$$= -\frac{2}{m}\left[\left(\frac{|\bar{p}|^2}{2m} p_j \, \Pi_{ik}^{(p)} r_k + \frac{1}{r} r_j \, \Pi_{ik}^{(r)} p_k\right) - (i \leftrightarrow j)\right]$$

$$= -\frac{2}{m}\left\{\left[\frac{|p|^2}{2m}\left(p_j r_i - \underbrace{p_i p_j}_{\substack{\text{cancels w/} \\ i\leftrightarrow j \text{ term}}} \frac{\vec{r}\cdot\vec{p}}{|p|^2}\right) + \frac{1}{r}\left(r_j p_i - \underbrace{r_j r_i}_{\substack{\text{cancels w/} \\ i\leftrightarrow j \text{ term}}} \frac{\vec{r}\cdot\vec{p}}{|r|^2}\right)\right] - (i \leftrightarrow j)\right\}$$

$$= -\frac{2}{m}\left[\frac{|p|^2}{2m}\left(p_j r_i - p_i r_j\right) + \frac{1}{r}\left(r_j p_i - r_i p_j\right)\right]$$

$$= -\frac{2}{m}\left(\frac{|p|^2}{2m} - \frac{1}{r}\right)\underbrace{\left(r_i p_j - p_j r_i\right)}_{\varepsilon_{ijk} L_k}$$

$$= -\frac{2}{m}\left(\frac{|p|^2}{2m} - \frac{1}{r}\right)\varepsilon_{ijk} L_k$$

c) Observe:

$\{L_i, H\} = 0$ by rotational invariance

$$\{p_i, H\} = \{p_i, V(r)\} = \frac{\partial p_i}{\partial r_j}\underbrace{\frac{\partial V}{\partial p_j}}_{0} - \frac{\partial p_i}{\partial p_j}\frac{\partial V}{\partial r_j} =$$

$$= -\delta_{ij}\frac{\partial V}{\partial r_j} = -\frac{\partial V}{\partial r_i} = -V'\frac{\partial |\bar{r}|}{\partial r_j}$$

$$= -V'\frac{r_j}{r}$$

$$\left\{\frac{1}{r} r_i, H\right\} = \left\{\frac{1}{r} r_i, \frac{p^2}{2m}\right\} = \frac{\partial}{\partial r_k}\left(\frac{r_i}{r}\right)\frac{\partial}{\partial p_k}\left(\frac{p^2}{2m}\right) =$$

$$= \frac{1}{2m}\left(\frac{\delta_{ik}}{r} - \frac{r_i r_k}{r^3}\right)\cdot 2 p_k$$

$$= \frac{1}{m}\frac{1}{r}\left(\delta_{ik} - \frac{r_i r_k}{r^2}\right) p_k = \frac{1}{m}\frac{1}{r}\,\Pi^{(r)}_{ik}\, p_k$$

So:

$$\{K_i, H\} = \frac{1}{m}\varepsilon_{ijk}\{p_j L_k, H\} - \left\{\frac{1}{r} r_i, H\right\}$$

$$= \frac{1}{m}\varepsilon_{ijk}\Big(p_j \underbrace{\{L_k, H\}}_{0} + \{p_j, H\} L_k\Big) - \frac{1}{m}\frac{1}{r}\,\Pi^{(r)}_{ik}\, p_k$$

$$= \frac{1}{m}\varepsilon_{ijk}\left(-V'\frac{r_j}{r} L_k\right) - \frac{1}{m}\frac{1}{r}\,\Pi^{(r)}_{ik}\, p_k$$

and

$$\varepsilon_{ijk} r_j L_k = \varepsilon_{ijk} r_j \varepsilon_{klm} r_l p_m = \varepsilon_{kij}\varepsilon_{klm} r_j r_l p_m =$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) r_j r_l p_m$$

$$= \left(r_i r_m - \delta_{im} r_j r_j\right) p_m$$

$$= -|r|^2\,\Pi^{(r)}_{im}\, p_m$$

So:

$$\{K_i, H\} = \frac{1}{m}\left(V'|\bar{r}|\,\Pi^{(r)}_{im}\, p_m - \frac{1}{r}\,\Pi^{(r)}_{ik}\, p_k\right)$$

$$= \frac{1}{m}\left(\frac{-1 + |r|^2 V'}{|r|^3}\right)|r|^2\,\Pi^{(r)}_{ik}\, p_k$$

$$= \frac{1}{m}\left(\frac{-1 + |r|^2 V'}{|r|^3}\right)\left(-\varepsilon_{ijk} r_j L_k\right)$$

$$= \frac{1}{m}\left(\frac{-1 + |r|^2 V'}{|r|^3} \right) \left(- \vec{r} \times \hat{L} \right)_i$$

$$= \frac{1}{m}\left(\frac{-1 + |r|^2 V'}{|r|^3} \right) \left(\vec{r} \times (\vec{p} \times \vec{r}) \right)_i$$

as required

Proof of the angular momentum algebra

$$\{L_i, L_j\} = \{\varepsilon_{ikl} r_k p_l , L_j\} =$$

$$= \varepsilon_{ikl} \left(r_k \{p_l, L_j\} + \{r_k, L_j\} p_l \right)$$

$$= \varepsilon_{ikl} \left(r_k \, \varepsilon_{ljm} \, p_m + \varepsilon_{kjm} \, r_m \, p_l \right)$$

$$= r_k p_m \, \varepsilon_{lik} \varepsilon_{ljm} + r_m p_l \, \varepsilon_{kli} \varepsilon_{kjm}$$

$$= \delta_{ij} r_k p_k - p_i r_j + r_i p_j - \delta_{ij} r_k p_k$$

$$= r_i p_j - r_j p_i$$

vice-versa

$$\varepsilon_{ijk} L_k = \varepsilon_{ijk} \varepsilon_{klm} r_l p_m = \varepsilon_{kij} \varepsilon_{klm} r_l p_m$$

$$= \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) r_l p_m$$

$$= \left(r_i p_j - r_j p_i \right)$$

Problem 3

a) In general:

$$L = \frac{1}{2} m |\dot{x}|^2 - e\Phi + e\,\dot{\vec{x}}\cdot\vec{A} \qquad \left(\text{assume } \Phi = 0 \text{ up to gauge}\right)$$

so the <u>canonical</u> momenta are:

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + eA_i$$

we then have

$$\{m\dot{x}_i, m\dot{x}_j\} = \{p_i - eA_i, p_j - eA_j\}$$

since A only depends on x:

$$= -e\{p_i, A_j\} - e\{A_i, p_j\}$$

$$= -e\left(\{p_i, A_j\} - \{p_j, A_i\}\right)$$

now:

$$\{p_\ell, A_m\} = \underbrace{\frac{\partial p_\ell}{\partial x_k}\frac{\partial A_m}{\partial p_k}}_{0} - \frac{\partial p_\ell}{\partial p_k}\frac{\partial A_m}{\partial x_k} = -\frac{\partial A_m}{\partial x_\ell}$$

so:

$$\{m\dot{x}_i, m\dot{x}_j\} = -e\left(-\frac{\partial A_j}{\partial x_i} + \frac{\partial A_i}{\partial x_j}\right) = e\left(\partial_i A_j - \partial_j A_i\right)$$

which, using $\vec{B} = \vec{\nabla}\times\vec{A}$

$$B_i = \varepsilon_{ijk}\,\partial_j A_k$$

$$\varepsilon_{ijk} B_k = \varepsilon_{ijk} \varepsilon_{klm} \partial_l A_m = (\varepsilon_{kij} \varepsilon_{klm}) \partial_l A_m$$

$$= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_l A_m$$

$$= \partial_i A_j - \partial_j A_i$$

so:

$$\{m\dot{x}_i, m\dot{x}_j\} = e\, \varepsilon_{ijk} B_k$$

Similarly

$$\{m\dot{x}_i, x_j\} = \{p_i - eA_i, x_j\} = \{p_i, x_j\} = -\delta_{ij}$$

b) We have:

$$J_i = \varepsilon_{ijk} x_j m\dot{x}_k - e g_m \frac{1}{|x|} x_i$$

while:

$$H = p_i \dot{x}_i - L = (m\dot{x}_i + eA_i)\dot{x}_i - \frac{1}{2}\dot{x}_i(m\dot{x}_i + eA_i)$$

$$= \frac{1}{2m}(m\dot{x}_i)^2 = \frac{1}{2m}(p_i - eA_i)^2$$

so:

$$\{J_i, H\} = \frac{1}{2m}\{J_i, (m\dot{x}_l)^2\} = \frac{1}{2m} 2 m\dot{x}_l \{J, m\dot{x}_l\} =$$

$$= \dot{x}_l \{J_i, m\dot{x}_l\} = \dot{x}_l \left(\varepsilon_{ijk}\{x_j m\dot{x}_k, m\dot{x}_l\} - eg\{\frac{x_i}{|x|}, m\dot{x}_l\}\right)$$

$$= \dot{x}_l \left[\varepsilon_{ijk}\left(x_j e\, \varepsilon_{klm} B_m - m\dot{x}_k \delta_{jl}\right) + eg\{m\dot{x}_l, \frac{1}{|x|} x_i\}\right]$$

now, using

$$\{m\dot{x}_e, f(\vec{x})\} = -\frac{\partial f}{\partial x_e}$$

we get:

$$\left\{m\dot{x}_e, \frac{1}{|x|}x_i\right\} = -\frac{\partial}{\partial x_e}\left(\frac{x_i}{|x|}\right) = -\frac{\delta_{ie}}{|x|} + \frac{x_i x_e}{|x|^3}$$

so:

$$\{J_i, H\} = \dot{x}_e\left[\varepsilon_{kij}\left(x_j e\, \varepsilon_{kem} B_m - m\dot{x}_k \delta_{je}\right) - \frac{eg}{|x|}\left(\delta_{ie} - \frac{x_i x_e}{|x|^2}\right)\right]$$

$$= \dot{x}_e\left[\delta_{ie}\, e(\vec{x}\cdot\vec{B}) - eB_i x_e - m x_k \varepsilon_{kie} - \frac{eg}{|x|}\left(\delta_{ie} - \frac{x_i x_e}{|x|^2}\right)\right]$$

$$= e\dot{x}_i(\vec{x}\cdot\vec{B}) - eB_i(\vec{x}\cdot\dot{\vec{x}}) - m\underbrace{(\dot{\vec{x}}\times\dot{\vec{x}})_i}_{0} - \frac{eg}{|x|}\left(\dot{x}_i - x_i\frac{\vec{x}\cdot\dot{\vec{x}}}{|x|^2}\right)$$

using $\vec{B} = g\frac{\vec{x}}{|x|^3}$ we get

$$= eg\frac{\dot{x}_i}{|x|} - eg\, x_i\frac{\vec{x}\cdot\dot{\vec{x}}}{|x|^3} - eg\frac{\dot{x}_i}{x} + eg\, x_i\frac{\vec{x}\cdot\dot{\vec{x}}}{|x|^3}$$

$$= 0$$

c) Some angular momentum is stored in the electromagnetic field.

$$\vec{L} \sim \int \vec{r}\times(\vec{E}\times\vec{B})$$

Problem 4

a) Compute:

$$\frac{\partial Q}{\partial q} = \frac{1}{1+\left(\frac{q}{p}\right)^2}\,\frac{1}{p} = \frac{p}{p^2+q^2}$$

$$\frac{\partial Q}{\partial p} = \frac{1}{1+\left(\frac{q}{p}\right)^2}\left(-\frac{q}{p^2}\right) = -\frac{q}{p^2+q^2}$$

$$\frac{\partial P}{\partial q} = q$$

$$\frac{\partial P}{\partial p} = p$$

Now, $\{P,P\} = \{Q,Q\} = 0$ is automatic. Moreover:

$$\{Q,P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \frac{p^2}{p^2+q^2} + \frac{q^2}{p^2+q^2} = 1$$

b) Again:

$$\frac{\partial Q}{\partial q} = 2pq \qquad \frac{\partial Q}{\partial p} = q^2$$

$$\frac{\partial P}{\partial q} = -\frac{1}{q^2} \qquad \frac{\partial P}{\partial p} = 0$$

$$\{Q,P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 2pq\cdot 0 - q^2\left(-\frac{1}{q^2}\right) = 1$$

c) Finally

$$\frac{\partial Q}{\partial q} = \frac{1}{1+\sqrt{q}\cos p}\,\frac{1}{2\sqrt{q}}\cos p$$

$$\frac{\partial Q}{\partial p} = - \frac{1}{1+\sqrt{q}\cos p} \sqrt{q} \sin p$$

$$\frac{\partial P}{\partial q} = 2 \left(\frac{1}{2\sqrt{q}} + \cos p \right) \sin p$$

$$\frac{\partial P}{\partial p} = -2 \sqrt{q}^{2} \sin^2 p + 2\sqrt{q} \left(1+\sqrt{q}\cos p\right) \cos p$$

$$= 2\sqrt{q} \left(-\sqrt{q}\sin^2 p + \left(1+\sqrt{q}\cos p\right)\cos p \right)$$

So:

$$\{Q, P\} = \frac{1}{1+\sqrt{q}\cos p} \frac{\cos p}{2\sqrt{q}} 2\sqrt{q} \left(-\sqrt{q}\sin^2 p + \left(1+\sqrt{q}\cos p\right)\cos p \right)$$

$$- \left(- \frac{1}{1+\sqrt{q}\cos p} \sqrt{q} \sin p \right) 2 \left(\frac{1}{2\sqrt{q}} + \cos p \right) \sin p$$

$$= \frac{1}{1+\sqrt{q}\cos p} \left(-\sqrt{q}\sin^2 p \cos p + \cos^2 p + \sqrt{q}\cos^3 p + \sin^2 p + 2\sqrt{q}\sin^2 p \cos p \right)$$

$$= \frac{1}{1+\sqrt{q}\cos p} \left(1 + \sqrt{q} \cos p \left(\sin^2 p + \cos^2 p \right) \right)$$

$$= \frac{1}{1+\sqrt{q}\cos p} \left(1 + \sqrt{q} \cos p \right) = 1$$