Problem 1

a) $$\frac{\partial L}{\partial x} = -e^{\gamma t/m} \kappa x$$

$$\frac{\partial L}{\partial \dot{x}} = e^{\gamma t/m} m\dot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = e^{\gamma t/m}\left(m\ddot{x} + m\dot{x}\frac{\gamma}{m}\right)$$

So:

$$e^{\gamma t/m}\left(m\ddot{x} + \gamma\dot{x}\right) + e^{\gamma t/m}\kappa x = 0$$

$$m\ddot{x} + \gamma\dot{x} + \kappa x = 0$$

Let $x = \mathrm{Re}\left(Ae^{i\tilde{\omega}t}\right)$, then

$$-m\tilde{\omega}^2 + i\gamma\tilde{\omega} + \kappa = 0$$

$$\tilde{\omega}^2 - i\frac{\gamma}{m}\tilde{\omega} + \frac{\kappa}{m} = 0$$

define $\omega_0^2 \equiv \frac{\kappa}{m}$, so:

$$\tilde{\omega}^2 - i\frac{\gamma}{m}\tilde{\omega} + \omega_0^2 = 0$$

$$\tilde{\omega} = \frac{1}{2}\left(i\frac{\gamma}{m} \pm \sqrt{4\omega_0^2 - \frac{\gamma^2}{m^2}}\right) =$$

$$= i\frac{\gamma}{2m} \pm \sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2}$$

We have three cases:

1) $\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2 > 0$. Then defining $\Omega = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2}$ we have:

$$x(t) = \mathrm{Re}\left(A \cdot e^{-\frac{\gamma}{2m}t} \, e^{\pm i\Omega t}\right)$$

if $A = A_0 \, e^{i\phi}$,

$$x(t) = A_0 \, e^{-\frac{\gamma}{2m}t} \cos\left(\Omega t + \phi\right)$$

2) $\omega_0^2 = \frac{\gamma}{4m}$. Then one solution is:

$$x(t) = A \, e^{-\frac{\gamma}{2m}t}$$

but we also have:

$$x(t) = B t \, e^{-\frac{\gamma}{2m}t}$$

indeed:

$$\dot{x}(t) = B\left(1 - \frac{\gamma}{2m}t\right) e^{-\frac{\gamma}{2m}t}$$

$$\ddot{x}(t) = -B\frac{\gamma}{2m} e^{-\frac{\gamma}{2m}t} - B\frac{\gamma}{2m}\left(1 - \frac{\gamma}{2m}t\right) e^{-\frac{\gamma}{2m}t}$$

$$= -B\frac{\gamma}{2m}\left(2 - \frac{\gamma}{2m}t\right) e^{-\frac{\gamma}{2m}t}$$

Plugging in to the EoM:

$$-mB\frac{\gamma}{2m}\left(2 - \frac{\gamma}{2m}t\right) e^{-\frac{\gamma}{2m}t} + \gamma B\left(1 - \frac{\gamma}{2m}t\right) e^{-\frac{\gamma}{2m}t} + m\frac{\gamma^2}{4m^2} B t \, e^{-\gamma/2m} =$$

$$= B\left(-\gamma + \frac{\gamma^2}{4m}t + \gamma - \frac{\gamma^2}{2m}t + \frac{\gamma^2}{4m}t\right) e^{-\gamma/2m} = 0$$

3) $\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2 < 0$. Then defining $\sqrt{\omega_0^2 - \left(\frac{\gamma}{2m}\right)^2} = i\Gamma/2m$

$$\Gamma = \sqrt{\gamma^2 - 4m^2\omega_0^2}$$

we have

$$x(t) = A \exp\left(-\frac{\gamma}{2m} t \pm \frac{\Gamma}{2m} t\right)$$

$$= A \exp\left(-\frac{1}{2m}(\gamma \pm \Gamma) t\right)$$

since $\Gamma < \gamma$, both solutions go to 0 as $t \to \infty$ and are thus allowed, so in general:

$$x(t) = e^{-\frac{\gamma}{2m} t}\left(A e^{-\Gamma t} + B e^{\Gamma t}\right)$$

b) Assume we are in the underdamped case $\omega_0^2 > \frac{\gamma^2}{4m^2}$. Then

$$x(t) = A_0 e^{-\frac{\gamma}{2m} t} \cos(\Omega t + \phi)$$

$$\dot{x}(t) = -A_0 e^{-\frac{\gamma}{2m} t} \Omega \sin(\Omega t + \phi)$$

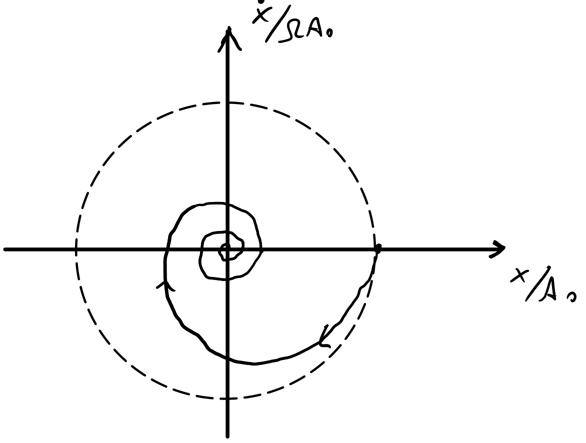
In terms of normalized variables:

$$\frac{x}{A_0} = e^{-\frac{\gamma}{2m} t} \cos(\Omega t + \phi)$$

$$\frac{\dot{x}}{\Omega A_0} = -e^{-\frac{\gamma}{2m} t} \sin(\Omega t + \phi)$$

so we have a clockwise inspiral towards $x = 0$, $\dot{x} = 0$.

Choosing $\phi = 0$



c) $$p = \frac{\partial L}{\partial \dot{x}} = e^{\gamma t/m} \, m\dot{x}$$

$$H = p\dot{x} - L = e^{\gamma t/m}\left(\frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2\right) =$$

$$= e^{-\gamma t/m}\frac{p^2}{2m} + e^{+\gamma t/m}\frac{1}{2} k x^2$$

d) We can write

$$x(t) = A e^{-\frac{\gamma}{2m}t}\cos(\Omega t) + B e^{-\frac{\gamma}{2m}t}\sin(\Omega t)$$

$$\dot{x}(t) = -A e^{-\frac{\gamma}{2m}t}\,\Omega\sin(\Omega t) + B e^{-\frac{\gamma}{2m}}\cos(\Omega t)$$

$$\Rightarrow p(t) = -m\Omega A e^{+\frac{\gamma}{2m}t}\sin(\Omega t) + m\Omega B e^{+\frac{\gamma}{2m}t}\cos(\Omega t)$$

In particular $A = x_0$ and $B = \frac{p_0}{m\Omega}$. We can write this as:

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-\frac{\gamma}{2m}t}\cos(\Omega t) & \frac{1}{m\Omega} e^{-\frac{\gamma}{2m}}\sin(\Omega t) \\ -m\Omega\, e^{\frac{\gamma}{2m}t}\sin(\Omega t) & e^{\frac{\gamma}{2m}}\cos(\Omega t) \end{pmatrix}}_{U(t)} \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}$$

Observe:

$$\det\left(U(t)\right) = \cos^2(\Omega t) + \sin^2(\Omega t) = 1$$

hence time evolution preserves phase space volume.

Problem 2

a) $L = \frac{1}{2} m |\dot{\vec{x}}|^2 - q\,\Phi(\vec{x}) + q\,\vec{A}(\vec{x}) \cdot \dot{\vec{x}}$ $\qquad (q = -|e|)$

To avoid confusion, denote the mechanical momentum by $\vec{p}$ and the canonical momentum by $\vec{\pi}$. Then:

$$\pi_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + q\,A_i(x) \quad \Rightarrow \quad \vec{\pi} = m\dot{\vec{x}} + q\vec{A}$$

$$H = \pi_i \dot{x}_i - L = m\,\dot{x}_i\,\dot{x}_i + q\,\dot{x}_i A_i - \frac{1}{2} m\,\dot{x}_i\,\dot{x}_i + q\,\Phi - q\,A_i\,\dot{x}_i$$

$$= \frac{1}{2} m\,|\dot{\vec{x}}|^2 + q\,\Phi = \frac{1}{2m}\left(\vec{\pi} - q\vec{A}\right)^2 + q\,\Phi$$

In our case $\Phi = 0$, $\vec{A} = A_y(x,z)\,\hat{y}$

$$\pi_x = m\dot{x} = p_x$$

$$\pi_y = m\dot{y} + q\,A_y(x,z) = p_y + q\,A_y$$

$$\pi_z = m\dot{z}$$

And

$$H = \frac{1}{2m}\left[p_x^2 + p_z^2 + \left(\pi_y - q A_y\right)^2\right]$$

Hamilton's equations for z and p_z are:

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = \frac{1}{m}\left(\pi_y - q A_y\right)\left(-q\,\frac{\partial A_y}{\partial z}\right)$$

Now, $\vec{B} = \vec{\nabla} \wedge \vec{A}$, hence:

$$B_x = \partial_y A_z - \partial_z A_y = -\partial_z A_y$$

so:

$$\dot{p}_z = \frac{q}{m} (\pi_y - q A_y) B_x$$

since B_x is 0 at $z=0$, $\dot{p}_z = 0$, hence z is constant and remains 0.

c) The y coordinate is cyclic, so π_y is conserved

$$\frac{d\pi_y}{dt} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{\partial L}{\partial y} = 0$$

Moreover, A_y does not explicitly depend on t, so neither does L, and thus the Hamiltonian is conserved.

Now, π_y is conserved, but in the $x<0$ region $\pi_y = p_y$, so if the electron starts and ends there then $p_{y,i} = p_{y,f}$.

If, moreover, we are in the $z=0$ plane, then $p_z = 0$ (see (b)), so

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right)$$

since H is conserved and $p_{y,f} = p_{y,i}$, we must have

$$p_{x,f} = \pm p_{x,i}$$

Since the electron has to be moving left at the end, $p_{x,f} = -p_{x,i}$.

Hence $\vec{p_f}$ is the reflection of $\vec{p_i}$ with respect to the yz plane.

d) At the maximum depth, $p_x = 0$, so:

$$H = \frac{1}{2m}\left(\pi_y - q A_y(x_{max}, 0)\right)^2$$

or:

$$A_y(x_{max}, 0) = \frac{1}{q}\left(\pi_y \pm \sqrt{2mH}\right)$$

where the RHS only contains conserved quantities.

In our case

$$\vec{B} = G\left(z\hat{x} - x\hat{z}\right)$$

corresponding to $A_y(x,z) = -\frac{1}{2}G\left(x^2 + z^2\right)$, giving:

$$-\frac{1}{2}G x_{max}^2 = \frac{1}{q}\left(\pi_y \pm \sqrt{2mH}\right)$$

or:

$$x_{max} = \sqrt{\frac{2}{qG}\left(\pi_y \pm \sqrt{2mH}\right)}$$

where we chose the positive root since $x_{max} > 0$. We have:

$$\pi_y = \pi_y(0) = p_0 \cos\theta$$

$$H = H(0) = \frac{1}{2m} p_0^2$$

so (using $q = -|e|$)

$$x_{max} = \sqrt{-\frac{2}{|e|G}\left(p_0\cos\theta \pm p_0\right)} = \sqrt{\frac{2p_0}{|e|G}\left(\cos\theta \pm 1\right)}$$

If $p_y(0) \geq 0$, $\cos\theta \in [0, 1)$, so

$$\cos\theta + 1 \in [1,2) > 0$$

$$\cos\theta - 1 \in [-1,0) < 0$$

Hence

- If $G > 0$, we must take the $+$ sign, and

$$x_{max} = \sqrt{\frac{2}{|e| \cdot |G|} \cdot (\cos\theta + 1)}$$

- If $G < 0$, we must take the $-$ sign, and

$$x_{max} = \sqrt{\frac{2}{|e||G|} (1 - \cos\theta)}$$

Since $\cos\theta + 1 > 1 - \cos\theta$, the greater x_{max} is reached for $G > 0$, hence the left figure is $G > 0$ and the right is $G < 0$.

This can also be checked by $\vec{F} = q\vec{v} \times \vec{B}$ and right-hand rule

Problem 3

a) The Lagrangian in polar coordinates is:

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right)$$

since $\dot{\theta} = \omega$, we get

$$L = \frac{1}{2} m \left(\dot{r}^2 + \omega^2 r^2 \right)$$

b) $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$

$$\Rightarrow H = p_r \dot{r} - L = \frac{1}{2} m \dot{r}^2 - \frac{1}{2} m \omega^2 r^2$$

Since L does not explicitly depend on time, H is conserved by Noether's theorem

c) $K = \frac{1}{2} m \left(\dot{r}^2 + \omega^2 r^2 \right) = L \neq H$. Since H is conserved and H and L differ by $p_r \dot{r} = m\dot{r}^2$, which is <u>not</u> conserved, $L = K$ is <u>not</u> conserved.

Problem 4

a) The coordinates do not change. The moments change as:

$$p_i' = \frac{\partial}{\partial \dot{q}_i}\left(L + \frac{d\Lambda}{dt}\right) = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i}\frac{d\Lambda}{dt} = p_i + \frac{\partial}{\partial \dot{q}_i}\frac{d\Lambda}{dt}$$

Since L cannot contain $\ddot{q}$, Λ has to depend only on q and t, so:

$$\frac{d\Lambda}{dt} = \frac{\partial \Lambda}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial \Lambda}{\partial t} = \frac{\partial \Lambda}{\partial q_j}\dot{q}_j + \frac{\partial \Lambda}{\partial t}$$

so:

$$\frac{\partial}{\partial \dot{q}_i}\frac{d\Lambda}{dt} = \frac{\partial \Lambda}{\partial q_j}\delta_{ij} = \frac{\partial \Lambda}{\partial q_i}$$

so!

$$p_i' = p_i + \frac{\partial \Lambda}{\partial q_i}$$

b) $$H' = p_i' \dot{q}_i - L' = \left(p_i + \frac{\partial \Lambda}{\partial q_i}\right)\dot{q}_i - \left(L + \frac{d\Lambda}{dt}\right) =$$

$$= \left(p_i \dot{q}_i - L\right) - \left(\frac{d\Lambda}{dt} - \frac{\partial \Lambda}{\partial q_i}\dot{q}_i\right)$$

$$= H - \left(\frac{\partial \Lambda}{\partial q_i}\dot{q}_i + \frac{\partial \Lambda}{\partial t} - \frac{\partial \Lambda}{\partial q_i}\dot{q}_i\right)$$

$$= H - \frac{\partial \Lambda}{\partial t}$$

c) $Q_i = q_i$, $\quad P_i = p_i + \frac{\partial \Lambda}{\partial q_i}(q,t)$

We have:

$$\{Q_i, Q_j\} = \frac{\partial Q_i}{\partial q_k} \underbrace{\frac{\partial Q_j}{\partial p_k}}_{0} - \underbrace{\frac{\partial Q_i}{\partial p_k}}_{0} \frac{\partial Q_j}{\partial q_k} = 0$$

$$\{Q_i, P_j\} = \frac{\partial Q_i}{\partial q_k} \frac{\partial P_j}{\partial p_k} - \underbrace{\frac{\partial Q_i}{\partial p_k}}_{0} \frac{\partial P_j}{\partial q_k} = \delta_{ik} \frac{\partial P_j}{\partial p_k}$$

$$= \delta_{ik} \delta_{jk} = \delta_{ij}$$

$$\{P_i, P_j\} = \frac{\partial P_i}{\partial q_k} \frac{\partial P_j}{\partial p_k} - \frac{\partial P_i}{\partial p_k} \frac{\partial P_j}{\partial q_k} =$$

$$= \frac{\partial^2 \Lambda}{\partial q_k \partial q_i} \delta_{jk} - \delta_{ik} \frac{\partial^2 \Lambda}{\partial q_k \partial q_j}$$

$$= \frac{\partial^2 \Lambda}{\partial q_j \partial q_i} - \frac{\partial^2 \Lambda}{\partial q_i \partial q_j} = 0$$

by symmetry of partial derivatives