Physics 221A Homework 7

3.1, 3.4, 3.10, 3.20, 3.23, 3.24, 3.33, 3.34

3.1 Use the specific form of S_x given by (3.25) to evaluate (3.23) and show that S_x rotates as expected through an angle ϕ about the z-axis.

3.4 Consider the 2×2 matrix defined by

$$U = \frac{a_0 + i\boldsymbol{\sigma} \cdot \mathbf{a}}{a_0 - i\boldsymbol{\sigma} \cdot \mathbf{a}},$$

where a_0 is a real number and $\mathbf{a}$ is a three-dimensional vector with real components.

a. Prove that U is unitary and unimodular.

b. In general, a 2×2 unitary unimodular matrix represents a rotation in three dimensions. Find the axis and angle of rotation appropriate for U in terms of a_0, a_1, a_2, and a_3.

3.10 Consider a sequence of Euler rotations represented by

$$\mathscr{D}^{(1/2)}(\alpha,\beta,\gamma) = \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right)$$
$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}.$$

Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a *single* rotation about some axis by an angle θ. Find θ.

3.20 Construct the matrix representations of the operators J_x and J_y for a spin 1 system, in the J_z basis, spanned by the kets $|+\rangle \equiv |1,1\rangle$, $|0\rangle \equiv |1,0\rangle$, and $|-\rangle \equiv |1,-1\rangle$. Use these matrices to find the three analogous eigenstates for each of the two operators J_x and J_y in terms of $|+\rangle$, $|0\rangle$, and $|-\rangle$.

3.23 The wave function of a particle subjected to a spherically symmetrical potential $V(r)$ is given by

$$\psi(\mathbf{x}) = (x + y + 3z)f(r).$$

a. Is ψ an eigenfunction of $\mathbf{L}^2$? If so, what is the l-value? If not, what are the possible values of l we may obtain when $\mathbf{L}^2$ is measured?

b. What are the probabilities for the particle to be found in various m_l states?

c. Suppose it is known somehow that $\psi(\mathbf{x})$ is an energy eigenfunction with eigenvalue E. Indicate how we may find $V(r)$.

3.24 A particle in a spherically symmetrical potential is known to be in an eigenstate of $\mathbf{L}^2$ and L_z with eigenvalues $\hbar^2 l(l+1)$ and $m\hbar$, respectively. Prove that the expectation values between $|lm\rangle$ states satisfy

$$\langle L_x \rangle = \langle L_y \rangle = 0, \quad \langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{[l(l+1)\hbar^2 - m^2\hbar^2]}{2}.$$

Interpret this result semiclassically.

3.33 Carry through the argument outlined on p. 208 for adding two spin $\frac{1}{2}$ particles by diagonalizing the 4×4 matrix corresponding to the operator $\mathbf{S}^2$ given in (3.339). That is, construct the matrix representation of $\mathbf{S}^2$ in the $|\pm\pm\rangle$ basis, and find the eigenvalues and eigenvectors. Your result should agree with (3.335).

3.34 Find all nine states $|j,m\rangle$ for $j = 2$, 1, and 0 formed by adding $j_1 = 1$ and $j_2 = 1$. Use a simplified notation, where $|j,m\rangle$ is explicit and $\pm, 0$ stand for $m_{1,2} = \pm 1, 0$, respectively, for example

$$|1,1\rangle = \frac{1}{\sqrt{2}}|+0\rangle - \frac{1}{\sqrt{2}}|0+\rangle.$$

You may also want to make use of the ladder operators $J_\pm$, or recursion relations, as well as orthonormality. Check your answers by finding a table of Clebsch–Gordan coefficients for comparison; see Appendix E.