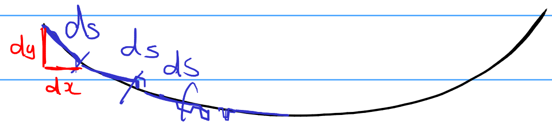
Physics 220 HW #1
Problem #1

a) surface area = integral of "ribbons" w/ radii $y(x) \rightarrow 2\pi\, y(x)\, ds$

$$\rightarrow A[y] = \int 2\pi\, y(x)\, ds \longrightarrow ds^2 = dx^2 + dy^2 \longrightarrow ds = \sqrt{dx^2+dy^2}$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$



$$\rightarrow A[y] = 2\pi \int_{-x_0}^{+x_0} y(x)\sqrt{1+y'(x)^2}\, dx$$

b) $$A[y+\delta y] - A[y] = 2\pi \int_{-x_0}^{+x_0} \left[(y+\varepsilon\eta)\sqrt{1+(y'+\varepsilon\eta')^2} - y\sqrt{1+y'^2}\right] dx$$

$$F(x, y, y') = y\sqrt{1+y'^2} \qquad \frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\rightarrow \sqrt{1+y'^2} - \frac{\partial}{\partial x}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) = \sqrt{1+y'^2} - \frac{(y'^2+yy'')\sqrt{1+y'^2} - \frac{yy'^2y''}{\sqrt{1+y'^2}}}{1+y'^2} = 0$$

$$(1+y'^2)^{3/2} = (y'^2+yy'')\sqrt{1+y'^2} - \frac{yy'y''}{\sqrt{1+y'^2}}$$

$$(1+\cancel{y'^2}) = (\cancel{y'^2}+yy'') - \frac{yy'y''}{1+y'^2}$$

$$(1-yy'')(1+y'^2) = -yy'^2y''$$

$$1+y'^2 - \cancel{yy'^2}y'' - yy'' = -\cancel{yy'^2}y''$$

$$1+y'^2 - yy'' = 0$$

$$yy'' - y'^2 - 1 = 0 \longrightarrow \frac{A}{A}\cosh^2\left(\frac{x}{A}\right) - \sinh^2\left(\frac{x}{A}\right) - 1 = 0$$

$$f(x) = 0 \quad g(x) = \frac{1}{y^2}$$

$$u = \frac{y'_x}{y}$$

$$u'_x + \frac{1}{y^2} = 0 \qquad y = A\cosh\left(\frac{x}{A}\right) \qquad y'' = \frac{1}{A}\cosh\left(\frac{x}{A}\right)$$

$$y' = \sinh\left(\frac{x}{A}\right)$$

Problem 1 cont'd

$$y(-x_0) = A\cosh\left(\frac{x_0}{A}\right) = R \longrightarrow A\frac{e^{-x_0/A} + e^{+x_0/A}}{2} = R$$

$$\frac{x_0}{A} = \cosh$$

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Problem 2

a) $$S[x+\delta x] = S[x] + \frac{dS}{dx}\delta x + \frac{1}{2}\frac{d^2S}{dx^2}\delta x^2 + O(x^3)$$

$$S[x] = \int_0^T dt\left(\frac{1}{2}m\dot{x}^2 - V(x)\right)$$

$\rightarrow x+\delta x =$

$$\dot{x}+\delta\dot{x} = \dot{x} + \frac{d}{dx}[\dot{x}]\delta x \quad \frac{d}{dx}\left[\frac{1}{2}m\dot{x}^2\right] = \frac{d}{dt}\frac{dt}{dx}\left[\frac{1}{2}m\dot{x}^2\right]$$

$$\rightarrow = \frac{m}{2\dot{x}}\frac{d}{dt}\left[\dot{x}^2\right] = \frac{m}{\cancel{2\dot{x}}}\cancel{2\dot{x}}\ddot{x} = m\ddot{x}$$

$$\rightarrow \dot{x}+\delta\dot{x} = m\ddot{x}\,\delta x$$

why is there a (−) sign in the 1st order KE term

$$\Delta S = S[x+\delta x] - S[x] = \int_0^T dt\left[\left(\cancel{\frac{1}{2}m\dot{x}^2} + m\dot{x}\,\delta x - \cancel{V(x)} - \frac{\partial V}{\partial x}\delta x\right) - \left(\cancel{\frac{1}{2}m\dot{x}^2} - \cancel{V(x)}\right)\right] = \int_0^T dt\left[\left(m\dot{x} - \frac{\partial V}{\partial x}\right)\delta x\right]$$

$$\Delta S = S[x+\delta x] - S[x] = \int_0^T dt\left[\left(\frac{1}{2}m\left(\frac{d}{dt}(x+\delta x)\right)^2 - V(x+\delta x)\right) - \left(\frac{1}{2}m\dot{x}^2 - V(x)\right)\right] = \int_0^T dt\left[\left(\frac{1}{2}m\left(\dot{x}+\frac{d}{dt}(\delta x)\right)^2 - \left(V(x) + \frac{\partial V}{\partial x}\delta x\right)\right]$$

$$= \int_0^T dt\left[\left(\frac{1}{2}\cancel{m}\left(\dot{x}^2 + 2\dot{x}\frac{d}{dt}(\delta x) + \cancel{\left(\frac{d}{dt}(\delta x)\right)^2}\right) - \left(\cancel{V(x)} + \frac{\partial V}{\partial x}\delta x\right)\right) - \left(\frac{1}{2}\cancel{m}\dot{x}^2 - \cancel{V(x)}\right)\right] = \int_0^T dt\left[m\dot{x}\frac{d}{dt}(\delta x) - \frac{\partial V}{\partial x}\delta x\right] =$$

$$\int u\,dv = uv - \int v\,du \qquad = \cancel{m\dot{x}\,\delta x}\Big|_0^T - \int_0^T dt\,(m\ddot{x}) - \int_0^T dt\,\frac{\partial V}{\partial x}\delta x$$

$$= -\int_0^T dt\left(m\ddot{x} + \frac{\partial V}{\partial x}\delta x\right)$$

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Problem 2

b)

$$\tilde{x}(t) = x(t) + \varepsilon h(t)$$
$$\dot{\tilde{x}}(t) = \dot{x}(t) + \varepsilon \dot{h}(t)$$
$$\frac{\partial \tilde{x}}{\partial \varepsilon} = h(t)$$
$$\frac{\partial \dot{\tilde{x}}}{\partial \varepsilon} = \dot{h}(t)$$

$$\left.\frac{\partial^2 S}{\partial \varepsilon^2}\right|_{\varepsilon=0} = \left.\frac{\partial^2}{\partial \varepsilon^2}\int_0^T dt\left[\frac{1}{2}m\dot{\tilde{x}}^2 - V(\tilde{x})\right]\right|_{\varepsilon=0} = \left.\frac{\partial^2}{\partial \varepsilon^2}\int_0^T dt\left(\frac{1}{2}m(\dot{x}+\varepsilon\dot{h})^2 - V(x+\varepsilon h)\right)\right|_{\varepsilon=0}$$

$$= \frac{\partial^2}{\partial \varepsilon^2}\int_0^T dt\left(\frac{1}{2}m(\dot{x}^2 + 2\varepsilon\dot{x}\dot{h} + \varepsilon^2\dot{h}^2) - V(x+\varepsilon h)\right)$$

$$= \int_0^T dt\left(m\dot{h}^2 - \frac{\partial^2 V}{\partial x^2}h^2\right) = \int_0^T dt\left(m\delta\dot{x}^2 - \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x(t)}\delta x^2\right)$$

$$\frac{\partial^2}{\partial \varepsilon^2}V = \frac{\partial}{\partial \varepsilon}\left(\frac{\partial V}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial \varepsilon}\right) = \frac{\partial}{\partial \varepsilon}\left(\frac{\partial V}{\partial x}h\right) = \frac{\partial^2 V}{\partial \tilde{x}^2}h^2$$

$h \to \delta x \quad \dot{h} \to \delta\dot{x}$

as $\lim_{\varepsilon\to 0}\tilde{x} = \lim_{\varepsilon\to 0} x+\varepsilon h = x \qquad \left.\frac{\partial^2}{\partial \varepsilon^2}V\right|_{\varepsilon=0} = \frac{\partial^2 V}{\partial x^2}\delta x^2$

$$\int_0^T m(\delta\dot{x})^2\,dt = \cancel{m\,\delta\dot{x}\,\delta x\Big|_0^T} - \int_0^T m\,\delta\ddot{x}\,\delta\dot{x}\,dt$$

$$\int_0^T \left(m(\delta\dot{x})^2 - g(t)\,\delta x^2\right)dt =$$

"L"

$$\left(\frac{d}{dt}\delta x\right)^2$$

Change of variables to isolate T

$$L + \delta L \sim L + \frac{dL}{dT}\delta T + \frac{1}{2}\frac{d^2L}{dT^2}\delta T^2$$

$$\frac{d}{dt}(\delta x)^2 = 2\,\delta\dot{x}\,\delta x$$

$\delta\dot{x}^2$

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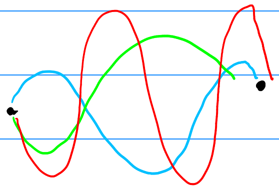
$$\delta^2 S = \int_0^T \left(m(\delta\dot{x})^2 - g(t)\delta x^2 \right) dt$$

$$g(t) = \frac{\partial^2 V}{\partial x^2}\bigg|_{x=x(t)} \quad ; \quad \delta x(0) = \delta x(T) = 0$$

$$\lim_{T\to 0} \delta^2 S = \lim_{T\to 0} \int_0^T dt \left(m\frac{\delta x^2}{T^2} - \frac{\partial^2 V}{\partial x^2}\bigg|_{x=x(t)} \delta x^2 \right) = \lim_{T\to 0} \int_0^T dt \left(\frac{m}{T^2} - \frac{\partial^2 V}{\partial x^2} \right) \delta x^2$$

$$\frac{d}{dt}(\delta x) = \delta\dot{x}$$



$$\delta x^2 = \sum_n a_n \sin\left(\frac{n\pi t}{T}\right) + b_n \cos\left(\frac{n\pi t}{T}\right)$$

$$\delta x(0) = 0 \longrightarrow \cos\left(\frac{n\pi \cdot 0}{T}\right) \neq 0 \quad \therefore\ b_n = 0$$

$$\delta x(T) = 0 \longrightarrow \cos\left(\frac{n\pi T}{T}\right) \neq 0$$

$$\delta x = \sum_n a_n \sin\left(\frac{n\pi t}{T}\right) \qquad -\int_0^T dt \frac{1}{2} f(t)\, \delta x^2 \geq -c\int_0^T dt \frac{1}{2}\delta x^2$$

$$\delta^2 S = \int_0^T dt \left[m \left(\sum_n \frac{a_n n\pi}{T} \cos\left(\frac{n\pi t}{T}\right) \right) - c\left(\sum_n a_n \sin\left(\frac{n\pi t}{T}\right) \right)^2 \right]$$

$$\geq \frac{T}{4} \sum_{n=1}^{\infty} \left[m \left(\frac{n\pi}{T}\right)^2 - c \right] a_n^2$$

C is constant. We can pick a small T s.t T will dominate C term. Thus $\delta^2 S \geq 0$ is possible and there is a path that minimizes action

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Problem 3

$$r_j(q_1, \ldots, q_N) \longrightarrow \dot{r}_j = \sum_{i=1}^{N} \frac{\partial r_j}{\partial q_i} \dot{q}_i$$

$$L(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N, t) \longmapsto L'(r_j, \dot{r}_j, t)$$

$$\frac{\partial L'}{\partial r_j} = \sum_i \left(\frac{\partial L'}{\partial q_i} \frac{\partial q_i}{\partial r_j} + \frac{\partial L'}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial r_j} \right)$$

$$\frac{\partial \dot{q}_i}{\partial r_j} = \left(\sum_k \frac{\partial q_i}{\partial r_j \partial r_k} \dot{r}_k \right) + \frac{\partial q_i}{\partial r_j \partial t}$$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \longmapsto \frac{\partial L}{\partial r_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_j} \right) = 0$$

$$\sum_i \left(\frac{\partial L'}{\partial q_i} \frac{\partial q_i}{\partial r_j} + \frac{\partial L'}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial r_j} \right) - \frac{d}{dt}\left(\sum_i \frac{\partial L'}{\partial q_i} \cancel{\frac{\partial q_i}{\partial \dot{r}_j}} + \frac{\partial L'}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial \dot{r}_j} \right) = 0$$

$$\sum_i \left(\frac{\partial L'}{\partial q_i} \frac{\partial q_i}{\partial r_j} + \frac{\partial L'}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial r_j} \right) \qquad q_i = Q(r_1, \ldots, r_N, t)$$

$$- \frac{d}{dt}\left(\sum_i \frac{\partial L'}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial \dot{r}_j} \right) =$$

$$\frac{\partial q_i}{\partial r_j \partial r_k} \dot{r}_k = \frac{\partial Q(r_1, \ldots, r_N, t)}{\partial r_j \, \partial r_k} \frac{\partial r_k}{\partial t}$$

$$\sum_i \left(\frac{\partial L'}{\partial q_i} \frac{\partial q_i}{\partial r_j} + \frac{\partial L'}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial r_j} \right) -$$

$$\sum_i \left[\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i} \right) \frac{\partial \dot{q}_i}{\partial \dot{r}_j} + \frac{d}{dt}\left(\frac{\partial \dot{q}_i}{\partial \dot{r}_j} \right) \frac{\partial L'}{\partial \dot{q}_i} \right]$$

q_i is a smooth, continuous function

$$= \qquad \frac{d}{dt}\left(\frac{\partial \dot{q}_i}{\partial \dot{r}_j} \right) = \frac{d}{dt}\left(\frac{\partial q_i}{\partial r_i} + \cancel{\frac{\partial q_i}{\partial r_j \partial t}} \right) = \frac{d}{dt}\left(\frac{\partial q_i}{\partial r_i} \right) = \frac{\partial}{\partial r_i}(\dot{q}_i)$$

$$\sum_i \left(\frac{\partial L'}{\partial q_i} \frac{\partial q_i}{\partial r_j} \right) + \sum_i \left(\cancel{\frac{\partial L'}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial r_j}} \right) - \sum_i \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i} \right) \frac{\partial q_i}{\partial r_j} - \sum_i \cancel{\frac{\partial L'}{\partial \dot{q}_i} \frac{\partial \dot{q}_i}{\partial r_j}}$$

$$= \sum_i \left(\frac{\partial L'}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i} \right) \right) \cancel{\frac{\partial q_i}{\partial r_j}} = 0 \longrightarrow \therefore \sum_i \left(\frac{\partial L'}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i} \right) \right) = 0$$

Problem 4

$$\delta S = \left.\frac{\partial S}{\partial \varepsilon}\right|_{\varepsilon=0} = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{\partial L}{\partial \ddot{q}}\delta \ddot{q} \right) + O(\varepsilon^2)$$

$$= \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q}\varepsilon h + \frac{\partial L}{\partial \dot{q}}\varepsilon \dot{h} + \frac{\partial L}{\partial \ddot{q}}\varepsilon \ddot{h} \right)$$

$$= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q} h + \frac{\partial L}{\partial \dot{q}}\dot{h} + \frac{\partial L}{\partial \ddot{q}}\ddot{h} \right) = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q} h\right) + \cancel{\left.\left(\frac{\partial L}{\partial \dot{q}} h\right)\right|_{t_1}^{t_2}} - \int_{t_1}^{t_2} dt \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) h \right)$$

$$+ \left.\left(\frac{\partial L}{\partial \ddot{q}}\dot{h}\right|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right)\dot{h}\right)\right) = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right) h + \left.\frac{\partial L}{\partial \ddot{q}}\dot{h}\right|_{t_1}^{t_2}$$

$$- \left[\cancel{\left.\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right) h\right|_{t_1}^{t_2}} - \int_{t_1}^{t_2} dt \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) h \right] = \left.\cancel{\frac{\partial L}{\partial \ddot{q}}\dot{h}}\right|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right]$$

assume $\dot{h}(t_1) = \dot{h}(t_2) = 0$

$$\delta S = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right] h(t) = 0$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0$$

b) $$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) = 0$$

$\partial_x L = -\partial V/\partial x$

$\partial_{\dot{x}} L = 2m\dot{x}$ $\longrightarrow -\frac{\partial V}{\partial x} - \frac{d}{dt}(2m\dot{x}) + \frac{d^2}{dt^2}(2\alpha\ddot{x}) = 0$

$\partial_{\ddot{x}} L = 2\alpha\ddot{x}$

$$\boxed{-\frac{\partial V}{\partial x} - 2m\ddot{x} + 2\alpha x^{(4)} = 0}$$