11/21/24 HW will be due Sunday, Dec. 1st!

Recap: rigid bodies have 6 degrees of freedom, 3 rotational degrees of freedom

↳ helpful to describe the motion of the body in the body reference frame because of this

inertia tensor: $\sum_\alpha m_\alpha (x^{\alpha 2} \delta_{ij} - x_i^\alpha x_j^\alpha) = I_{ij}$

principal axes of inertia: diagonalize the inertia tensor!

$KE = \frac{1}{2} \vec{\omega} \overleftrightarrow{I} \vec{\omega}$, $\vec{L} = \overleftrightarrow{I} \vec{\omega}$

→ describes the body in its own reference frame

EOM give us Euler's eqn: We discussed the no-torque case last class, so now we're looking at what happens when torque is added!

Examining the case of - "rigid body in an external grav. potential": We can no longer look at the body in its own reference frame! → need to specify the position of the body in the external reference frame

many different ways of handling this! →

Euler Angles [Convention] → see posted notes for diagram

ϕ - rotation about $\hat{z}$

w, e_1', "line of nodes" - θ

ψ - rotation around e_3 (helps you line up x, y, z with e_1, e_2, e_3...)

This is the x-convention, which is one of the more common ones!

$R = R(\psi)R(\theta)R(\phi)$ → build the transformation by multiplying matrices! → this is the matrix s.t. $R^T\dot{R}$ = antisymmetric!

$$(R^T\dot{R})_{ij} = \epsilon_{ijk}w_k = \begin{pmatrix} 0 & \dot{\psi}+\cos\theta\dot{\phi} & -\sin\psi\dot{\theta}+\sin\theta\cos\psi\dot{\phi} \\ & 0 & \cos\psi\dot{\theta}+\sin\theta\sin\psi\dot{\phi} \\ & & 0 \end{pmatrix}$$

see online notes for fully written out R matrix!

$$= \begin{pmatrix} 0 & w_1 & w_2 \\ & 0 & w_3 \\ & & 0 \end{pmatrix}$$

→ kind of an algebra-intensive way to find w_1, w_2, w_3, but it works

$$\dot{\phi}\hat{z} = (\cos\theta\,\hat{e}_3 + \sin\theta\sin\psi_1\,\hat{e}_1 + \sin\theta\cos\psi\,\hat{e}_2)\,\dot{\phi}$$

$$\dot{\theta}\hat{e}_1' = \dot{\theta}(\cos\psi_1\,\hat{e}_1 - \sin\psi\,\hat{e}_2)$$

$$\dot{\psi}\hat{e}_3 = \dot{\psi}(\hat{e}_3)$$

coefficients of e_i! →

$w_1 = \cos\psi\dot{\theta} + \sin\theta\sin\psi\dot{\phi}$

$w_2 = -\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi$

$w_3 = \cos\theta\dot{\phi} + \dot{\psi}$

We want to write the KE in terms of the Euler angles — symmetric heavy top $(I_1 = I_2)$

$T = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2$

$V = mgR\cos\theta$

"heavy" implies that the potential [only] depends on the position in z?

We want to use a Lagrangian approach (will see some cyclic variables).

doesn't depend explicitly on time — energy is also conserved!

$\Rightarrow L = \frac{1}{2} I_1 (\dot\theta^2 + \sin^2\theta\, \dot\phi^2) + \frac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)^2 - mgR\cos\theta$

ϕ, ψ are cyclic so $\underbrace{p_\phi, p_\psi}_{\text{angular momentum components around } \hat z \text{ \& } \hat e_3}$ are conserved

not surprising since the torque $= R\hat e_3 \times mg\hat z$

$\frac{\partial L}{\partial \dot\phi} = p_\phi = \dot\phi I_1 \sin^2\theta + \frac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)\cos\theta$

$\frac{\partial L}{\partial \dot\psi} = p_\psi = I_3(\dot\psi + \dot\phi\cos\theta) = I_3\omega_3$

$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} \Rightarrow I_1\ddot\theta = I_1\dot\phi^2 \sin\theta\cos\theta - I_3(\dot\psi + \dot\phi\cos\theta)\dot\phi\sin\theta + mgR\sin\theta$

can use conserved quantities to get rid of $\dot\phi$ & $\dot\psi$!

Then look for solutions where θ is constant

$\dot\phi = \Omega$ (const. precession around $\hat z$)

$I_1\Omega^2\cos\theta\sin\theta - I_3\omega_3\Omega\sin\theta + mgR\sin\theta = 0$

can factor out $\sin\theta$ to get a quadratic eqn $(\theta = \text{const for } \pi, 0)$

$$\Omega = \frac{I_3\omega_3 \pm \sqrt{(I_3\omega_3)^2 - 4mgR I_1\cos\theta}}{2I_1\cos\theta}$$

Let's assume that $(I_3\omega_3)^2 \gg 4mgR I_1\cos\theta$

$$\approx \left(\frac{2I_3\omega_3}{2I_1\cos\theta} \right. \quad \& \quad \frac{\cancel{I_3\omega_3}\; 2mgR\cancel{I_1\cos\theta}}{(I_3\omega_3)\cancel{2I_1\cos\theta}} < \frac{mgR}{I_3\omega_3}$$

depends on whether you take + or −

$\frac{I_3 \omega_3}{2 I \cos\theta}$ → wobbling / precession can occur even in the absence of outside forces! (see Earth's wobble)

Different view – from energy conservation

Energy is conserved (L does not depend on time!)

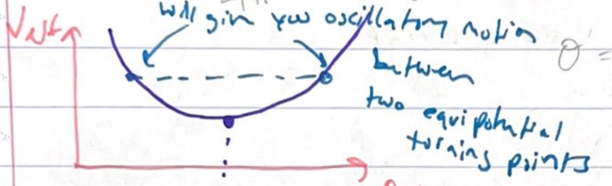
$E = \frac{1}{2} I_1 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) + \frac{1}{2} I_3 \omega_3^2 + mgR\cos\theta$

we eliminate $\dot{\phi}$ $\left(\dot{\phi} = \frac{p_\phi - \cos\theta\, p_\psi}{I_1 \sin^2\theta}\right)$ → from definitions of p_ϕ, p_ψ!

$\dot{\psi} = \frac{p_\psi}{I_3}$ → then plug into eqn for E, simplify

$E = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} \frac{(p_\phi - p_\psi \cos\theta)^2}{I_1 \sin^2\theta} + \frac{1}{2} I_3 \omega_3^2 + mgR\cos\theta$

$V_{eff}(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + mgR\cos\theta + \text{const.}$



$\theta = 0, \pi \Rightarrow V_{eff} \to \infty$

minimum of the potential gives you constant θ

θ_0 → regular precession happens here!

nutation motion [axis can wobble & even change signs?] – lecture notes have a picture that might help w/ understanding / intuition!

Zoe almost imagine [drawn loops] nutation like someone spinning an umbrella at a low angle? (see simulation if available)

for 1D motion w/ no time-dependent constraints or velocity-dependent potentials – Lagrangian can be very general but still be reduced to 1 degree of freedom

$E = \frac{1}{2} M(q) \dot{q}^2 + V(q)$ is conserved

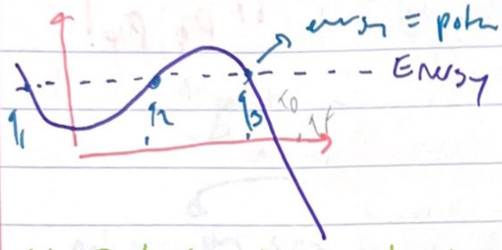
$\dot{q} = \sqrt{\frac{2}{M(q)}} \sqrt{E - V(q)}$ we can invert! (to solve for q)

calculating small squares → obtain by adding them up

Known as <u>solution by quadrature</u>

$$t = \int \frac{dq}{\sqrt{\frac{2}{M(q)}\left(E - V(q)\right)}} + \text{const}$$ depending on I.C.'s!

For an arbitrary potential



$q_1 < q < q_2$ oscillations

$q_2 < q < q_3$ classically forbidden

$q > q_3$ unbound motion!

if @ $t=0$ $q_f > q_3$, when does it get to q_f?

$$\int_0^{t_f} dt = \int \frac{dq}{\sqrt{\frac{2}{M(q)}\left(E - V(q)\right)}}$$ if $\dot{q} < 0$ then $\int_{q_3}^{q_0} \frac{dq}{\sqrt{\quad}} + \int_{q_3}^{q_f} \frac{dq}{\sqrt{\quad}}$

For the oscillatory case, the integral can be used to calculate the period

$$T(E) = 2\int_{q_1}^{q_2} \frac{dq}{\sqrt{\frac{2}{M(q)}\left(E - V(q)\right)}}$$

bc of return trip

Ex. Harmonic oscillator $V = \frac{1}{2}kq^2$ $\quad T(E) = 2\int^{q_0} \frac{dq}{\sqrt{\frac{2}{m}\frac{1}{2}k\left(q_0^2 - q^2\right)}}$

$E = \frac{1}{2}kq_0^2$ ← boundary when $q = q_0$, potential = all energy!

$$= 2\sqrt{\frac{m}{k}} \int_{-q_0}^{q_0} \frac{dq}{\sqrt{q_0^2 - q^2}} \qquad u = q/q_0$$

$$= 2\sqrt{\frac{m}{k}} \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = \boxed{2\pi\sqrt{\frac{m}{k}}}$$ → period of SHO!

↳ does not depend on energy (∴ does not depend on q_0!)

Ex, Pendulum $\quad V = mgl(1-\cos\theta) \quad E = mgl(1-\cos\theta_m)$

$$T(E) = 2\int_{-\theta_m}^{\theta_m} \frac{d\theta}{\sqrt{\frac{2}{ml^2} mgl(1-\cos\theta_m - 1+\cos\theta)}}$$

specifically Taylor expand the potential

$$= 2\sqrt{\frac{l}{g}}\int_{-\theta_m}^{\theta_m} \frac{d\theta}{\sqrt{2(\cos\theta_m - \cos\theta)}}$$

→ this does depend on θ_m! Taylor expand to find out how

$$= 2\sqrt{\frac{l}{g}}\int_{-\theta_m}^{\theta_m} \frac{d\theta}{\sqrt{\theta_m^2 - \theta^2 - \frac{\theta_m^4}{12} + \frac{\theta^4}{12}}} = 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_m^2}{16} + \dots\right)$$

SHO for small angles, but this gives us our corrections — when it starts to depend on θ_m!