Lecture # 4

Tuesday, October 8, 2024 9:58 AM

$L(q_i, \dot{q}_i, t)$ E-L equations $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

Noether's Theorem:

for each continuous symmetry of the Lagrangian

there exists a conserved quantity $Q = \sum_i \frac{\partial L}{\partial \dot{q}_i}\delta q_i - \Lambda$

$$q_i'(t,\varepsilon) = q_i(t) + \varepsilon \delta q_i + o(\varepsilon^2)$$

$$\frac{\partial}{\partial \varepsilon} L(q_i, \dot{q}_i, t)\Big|_{\varepsilon=0} = \frac{d}{dt}\Lambda(q,t)$$

$$\frac{\partial L}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i = \frac{d\Lambda}{dt}$$

E-L equations

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i - \Lambda\right) = 0 \qquad Q = \frac{\partial L}{\partial \dot{q}_i}\delta q_i - \Lambda \text{ is conserved}$$

Note: existence of symmetry (δq_i) has to hold without E-L equations

conservation of Q is based on E-L equations

Examples:

- cyclic variable $q_i' = q_i + \varepsilon c$ $\delta q_i = c$ $\Lambda = 0$

  by Noether's Theorem $Q = \frac{\partial L}{\partial \dot{q}_i}\delta q_i - \Lambda$ is conserved

  $= c\,\frac{\partial L}{\partial \dot{q}_i}$ conjugate momentum of cyclic variable is conserved

- time translation $q_i' = q_i(t+\varepsilon) = q_i + \varepsilon \dot{q}_i \Rightarrow \delta q_i = \dot{q}_i$

$[L(q_i + \varepsilon \dot{q}_i, \dot{q} + \varepsilon \ddot{q}, t) - L(q_i, \dot{q}, t)]/\varepsilon$

• time translation $\quad q_i' = q_i(t+\varepsilon) = q_i + \varepsilon \dot{q}_i \quad \Rightarrow \delta q_i = \dot{q}_i$

$[L(q_i+\varepsilon\dot{q}_i, \dot{q}_i+\varepsilon\ddot{q}_i, t) - L(q_i, \dot{q}_i, t)]/\varepsilon$

$$\left.\frac{\partial L}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \frac{dL}{dt} - \frac{\partial L}{\partial t}$$

but if L is not explicitly time dependent $\frac{\partial L}{\partial t} = 0$

and I have a total time differential

$\Rightarrow \Lambda = L$

$$Q = \frac{\partial L}{\partial \dot{q}_i}\delta q_i - \Lambda$$

$$\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$$

$$\frac{dL}{dt} = \frac{d\,L(q,\dot{q},t)}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial t}$$

• rotational symmetry

$$L = \frac{1}{2}m\dot{\vec{x}}^2 - V(|\vec{x}|)$$

$$\delta\vec{x} = (\vec{\omega}\times\vec{x})$$

$x_1' = x_1 + \varepsilon x_2 \qquad \Leftarrow \begin{cases} \cos\vartheta\, x + \sin\vartheta\, y \\ -\sin\vartheta\, x + \cos\vartheta\, y \end{cases}$

$x_2' = x_2 - \varepsilon x_1$

$\Lambda = 0$

$$Q = \frac{\partial L}{\partial \dot{\vec{x}}}\delta\vec{x} = m\dot{\vec{x}}\cdot(\vec{\omega}\times\vec{x}) = \vec{\omega}\cdot(\vec{x}\times m\dot{\vec{x}}) = \vec{\omega}\cdot(\vec{x}\times\vec{p})$$

$\vec{L} = \vec{x}\times\vec{p}$ is conserved quantity associated with rotational symmetry

---

Hamiltonian mechanics

- turn 2nd order differential equations into systems of first order equations
- More general transformations mixing coordinates and momenta
- geometric structure of dynamic flow, Liouville's theorem and symplectic dynamics
- Poisson brackets $\rightarrow$ quantum mechanics

Setup starts from $L(q,\dot{q},t)$ and E-L equations $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$

2nd order ODE

for each generalized coordinate q_i

$dt\,\partial q \quad \partial q$

2nd order ODE

for each generalized coordinate q_i

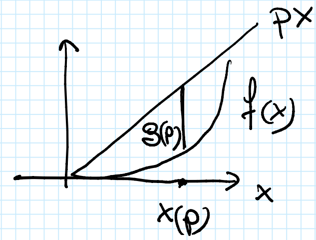
we can define a generalized momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$

perform a Legendre Transformation

$$\{L, q, \dot{q}\} \rightarrow \{H, q, p\}$$

Useful in many field of physics (Thermodynamic potentials, capacitors, algebraic geometric, etc.)

Consider a function $f(x)$, convex if $f''(x) > 0$



Then Legendre Transformation $g(p)$ is obtained by

$$g(p) = \sup_{x \in \mathbb{R}} px - f(x)$$

use $F(p, x) = px - f(x)$ and look for max w.r.t. x

$\frac{\partial F}{\partial x} = 0$ defines $x(p)$ Then $g(p) = F(p, x(p))$

convexity is important for uniqueness

example: 1. $f(x) = a x^2 \quad a > 0$

$$F(p,x) = px - ax^2 \qquad \frac{\partial F}{\partial x} = 0 = p - 2ax \Rightarrow x = p/2a$$

$$g(p) = F(p, x(p)) = p \cdot \frac{p}{2a} - a\left(\frac{p}{2a}\right)^2 = \frac{p^2}{2a} - \frac{p^2}{4a} = \frac{p^2}{4a} \qquad \frac{\partial f}{\partial x} = 2ax = p$$

$$\frac{\partial g}{\partial p} = \frac{p}{2a} = x$$

2. $f(x) = \frac{x^a}{a}$

$$F(x,p) = px - \frac{x^a}{a} \qquad \frac{\partial F}{\partial x} = p - x^{a-1} \Rightarrow x = p^{1/a-1}$$

$$g(p) = F(x(p), p) = p \cdot p^{\frac{1}{a-1}} - \frac{1}{a} p^{\frac{a}{a-1}} = \frac{a-1}{a} p^{a/a-1} = \frac{1}{\beta} p^\beta$$

$$\beta = \frac{a}{a-1} \quad \text{or} \quad \frac{1}{\beta} = 1 - \frac{1}{a} \Rightarrow \frac{1}{a} + \frac{1}{\beta} = 1$$

Legendre's Transform is an involution $\Leftrightarrow$ squares to identity

Legendre's transform is an involution $\iff$ squares to identity

Young inequality $\quad px \leq f(x) + g(p)$

useful to generate inequalities $\quad px \leq \frac{x^\alpha}{\alpha} + \frac{p^\beta}{\beta}$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$

Simple relationship of Legendre's transform

$f' = a f(x) \qquad g' = a g(p/a)$

$f' = f(\alpha x) \qquad g' = g(p/\alpha)$

$f' = f(x) + c \qquad g' = g(p) - c$

$f' = f(x+y) \qquad g' = g(p) - py$

Higher dimensionality $\quad \bar{x} = (x_1, \ldots x_N) \qquad f(\bar{x})$ is convex in $\mathbb{R}^N$

Then $F(\bar{p},\bar{x}) = \bar{p}^T\bar{x} - f(\bar{x}) \qquad \frac{\partial^2 f}{\partial x_i \partial x_j} = H_{ij}$ positive definite

$g(\bar{p}) = \sup_{\bar{x}\in\mathbb{R}^N} F(\bar{p},\bar{x}) = F(\bar{p}, \bar{x}(\bar{p}))$ where $\bar{x}(\bar{p})$ is found by $\bar{p} = \frac{\partial f}{\partial \bar{x}}$

$\frac{\partial F}{\partial \bar{x}} = 0$

Apply Legendre's transform w.r.t. $\dot{q}_i$ to Lagrangian mechanics

$L(q,\dot{q},t) \to H(q,p,t) \qquad$ Hamiltonian mechanics

$H(q_i,p_i,t) = \sum_i \dot{q}_i p_i - L(q_i,\dot{q}_i,t)$ where $\dot{q}_i$ are defined by $p_i = \frac{\partial L}{\partial \dot{q}_i}$

how do q_i and p_i change or evolve in time?

$\dot{q}_i = \frac{\partial H}{\partial p_i}$

$\dot{p}_i = -\frac{\partial H}{\partial q_i}$

Hamilton equations

Proof. consider the total differential of $H = p\dot{q} - L$

Proof. Consider the total differential of $H = pq - L$

$$dH = \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt = \sum_i \cancel{p_i d\dot{q}_i} + \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \cancel{\frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i} - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$+\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$\frac{\partial H}{\partial t} = \frac{\partial L}{\partial t}$ $\quad$ dt term gives relation between explicit time dependencies of L, H

Note that $\frac{dH}{dt} = \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial t} = \cancel{\dot{q}\dot{p}} - \cancel{\dot{p}\dot{q}} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$

So if $\frac{\partial H}{\partial t} = 0$ (Lagrangian not explicitly dependent on time)

$\Rightarrow$ H is conserved quantity

Examples:

1D $\quad L = \frac{1}{2}m\dot{q}^2 - V(q) \qquad p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \Rightarrow \dot{q} = \frac{p}{m}$

$$H = \dot{q}p - L = \frac{p}{m}p - \frac{1}{2}m\left(\frac{p}{m}\right)^2 + V(q) = \frac{p^2}{2m} + V(q) \quad \text{total energy}$$

More generally $\quad L = \frac{1}{2}\sum_{ij} \dot{q}_i M_{ij} \dot{q}_j - V(\vec{q})$

M is a $N \times N$ symmetric matrix with all positive eigenvalues to guarantee uniqueness of Legendre's Transform

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = M_{ij}\dot{q}_j \quad \Rightarrow \dot{q}_i = M^{-1}_{ij} p_j$$

$$H = p_j M^{-1}_{ij} p_i - \frac{1}{2} p_j M^{-1}_{ji} M_{ik} M^{-1}_{kl} p_l + V(\bar{q})$$

$$= \frac{1}{2} p_j M^{-1}_{ij} p_i + V(\bar{q})$$

easy to apply if M is diagonal

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\sin^2\vartheta\,\dot{\varphi}^2\right) \qquad M = \begin{pmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & mr^2\sin^2\vartheta \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{mr^2} & 0 \\ & & \end{pmatrix} \qquad H = \frac{1}{\ldots}\left(\ldots^2 \quad \ldots^2 \quad \ldots^3\right)$$

$$M^{-1} = \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{mr^2} & 0 \\ 0 & 0 & \frac{1}{mr^2\sin^2\theta} \end{pmatrix}$$

$$H = \frac{1}{2}\left(\frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_\phi^3}{mr^2\sin^2\theta}\right)$$

p_r , $p_\theta = mr^2\dot{\theta}$, $p_\phi = mr^2\sin^2\theta\,\dot{\phi}$