Physics 220 Fall 2024

Problem Set # 9 (due on Sun Dec 8th)

1) Consider a particle of mass m moving in the following potential $V(r) = -\frac{\alpha}{r}, \alpha > 0$.
   a) Show that when the energy E < 0 (corresponding to bound elliptical orbits), the relation between time t and radius r is given by
   $$t = \sqrt{\frac{\mu}{2|E|}} \int dr \frac{r}{\sqrt{a^2\varepsilon^2 - (r-a)^2}}$$
   In so doing, define what the constants a and ε are in terms of the initial conditions, as well as physically. You may treat the initial angular momentum L and initial energy E as the initial conditions.
   b) By making the change of variables $r = a(1 - \varepsilon \cos\theta)$ in the integral, deduce that the relation between t and r can be written in the parametric form
   $$t(\theta) = \sqrt{\frac{\mu a^3}{\alpha}}(\theta - \varepsilon \sin\theta)$$

2) Consider here the classic hard sphere scattering problem: a point particle of mass m1 and velocity v1 (measured in the lab) scatters off a sphere of mass m2 with an infinitely rigid potential. This means that $V(|\vec{r_1} - \vec{r_2}|) = V(r) = 0$ for $r > a$ and $V(r) \to \infty$ for $r < a$, where r is defined to be zero at the sphere's center. Assume the sphere is initially at rest in the lab system.
   a) As usual, we start in the CoM system. Without doing any calculations, figure out how the scattering must work geometrically. Draw a picture in the CoM frame for an arbitrary impact parameter which shows the trajectory of m1 as it bounces off the sphere.
   b) Again, without doing any calculations, does the scattering cross section in the CoM depend on the incident velocity of the particles? Why or why not?
   c) In the CoM system, calculate the differential cross section $\frac{d\sigma}{d\Omega}$. Integrate the differential cross section over all angles to find out the total cross section.
   d) Assuming m1<m2, calculate the differential cross section in the lab frame.

3) By adding to the gravitational potential energy $V(r) = -\frac{\alpha}{r}, \alpha > 0$ a small perturbation $\delta U(r)$, the trajectories become no longer closed orbits and at each turn the perihelion is shifted by a small angle $\delta\varphi$. Calculate this apsidal distance for $\delta U(r) = \beta/r^2$ and $\delta U(r) = \gamma/r^3$

4) A particle is launched from afar, with an impact parameter b, toward an attracting center creating the potential

$$V(r) = -\frac{\alpha}{r^n}, \quad \text{with} \quad n > 2 \text{ and } \alpha > 0$$

a) For the case when the initial kinetic energy E of the particle is barely sufficient for escaping its capture by this attracting center, express the minimum approach distance via b and n.

b) Calculate the capture's total cross-section and explore its limit at n approaches 2.