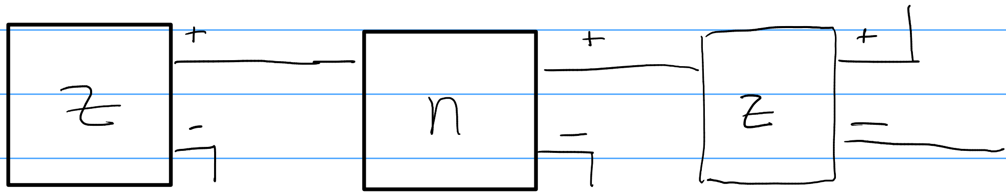
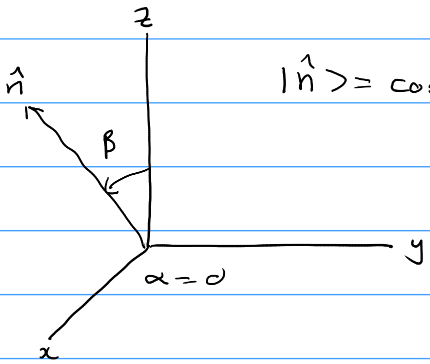
Homework 2

1.15, 1.28, 3.11, 3.12, 3.14

1.15

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$





$$|\hat{n}\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)e^{i\alpha}|-\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right)|-\rangle$$

$$|\langle +|\hat{n}\rangle|^2 = \cos^2(\beta/2)$$

$$|\langle\hat{n}|+\rangle|^2 \, |\langle -|\hat{n}\rangle|^2 = \left(\left|\langle +|+\rangle \cos\left(\frac{\beta}{2}\right)\right|^2 + \left|\langle -|\cancel{+}\rangle \sin\left(\frac{\beta}{2}\right)\right|^2\right)$$

$$\left(\left|\langle -|\cancel{+}\rangle \cos(\beta/2)\right|^2 + \left|\langle -|-\rangle e^{i\alpha} \sin\left(\frac{\beta}{2}\right)\right|^2\right) = \cos^2\left(\frac{\beta}{2}\right) \sin^2\left(\frac{\beta}{2}\right)$$

$$= \frac{4\cos^2\left(\frac{\beta}{2}\right)\sin^2(\beta/2)}{4} = \frac{\sin^2(\beta)}{4}$$

1.28

$$U = \left(|+\rangle_x \langle +|_z + |-\rangle_x \langle -|_z\right) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}\right]$$

$$U = \left[\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\0 & -1\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

$$\hat{U}|+\rangle_z = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

3.11

(1.147) $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$

(3.100) $\rho \equiv \sum w_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}|$

Prove $\quad 0 \leq tr(\rho^2) \leq 1$

$$\rho^2 = \left(\sum_i w_i |\alpha^{(i)}\rangle\langle\alpha^{(i)}|\right)\left(\sum_j w_j |\alpha^{(j)}\rangle\langle\alpha^{(j)}|\right)$$

$$= \sum_{ij} w_i w_j |\alpha^{(i)}\rangle\underbrace{\langle\alpha^{(i)}|\alpha^{(j)}\rangle}_{=\delta_{ij}}\langle\alpha^{(j)}| = \sum_{ij} w_i w_j \delta_{ij} |\alpha^{(i)}\rangle\langle\alpha^{(j)}|$$

$w_i \leq 1$ $\rightarrow$ $w_i w_j \leq 1$ if a term

$w_j \leq 1$

$$= \sum_i w_i^2 |\alpha^{(i)}\rangle\langle\alpha^{(i)}| \qquad tr(\rho^2) = \sum_i w_i^2 \qquad 0 \leq \sum_i w_i^2 \leq 1 \rightarrow 0 \leq tr(\rho^2) \leq 1$$

3.12

$$|\psi\rangle = \sqrt{a}\,|+\rangle_z + \sqrt{1-a}\,|-\rangle_y$$

$$[A] = \sum_i w_i \langle \alpha^{(i)} | \hat{A} | \alpha^{(i)} \rangle$$

$$[\hat{S}_x] = a\langle +|_z \hat{S}_x |+\rangle_z + (1-a)\langle -|_y \hat{S}_x |-\rangle_y$$

$$= a(1\ 0)\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + (1-a)\frac{1}{2}(1+i)\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{\hbar a}{2}(1\ 0)\begin{pmatrix}0\\1\end{pmatrix} + (1-a)\frac{\hbar}{4}(1+i)\begin{pmatrix}-i\\1\end{pmatrix}$$

$$= (1-a)\frac{\hbar}{4}(-i+i) = 0$$

$$[\hat{S}_y] = a\langle +|_z \hat{S}_y |+\rangle_z + (1-a)\langle -|_y \hat{S}_y |-\rangle_y$$

$$= a(1\ 0)\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + (1-a)\frac{1}{2}(1+i)\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix}$$

$$= \frac{\hbar a}{2}(1\ 0)\begin{pmatrix}0\\i\end{pmatrix} + (1-a)\frac{\hbar}{4}(1+i)\begin{pmatrix}-1\\i\end{pmatrix} = 0 + \frac{\hbar}{4}(1-a)(-1-1)$$

$$= \frac{\hbar}{4}(1-a)\cdot -2 = -\frac{\hbar}{2}(1-a)$$

$$[\hat{S}_z] = a\langle +|_z \hat{S}_z |+\rangle_z + (1-a)\langle -|_y \hat{S}_z |-\rangle_y$$

$$= \frac{\hbar}{2}a(1\ 0)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + (1-a)\frac{\hbar}{4}(1+i)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix}$$

$$= \frac{\hbar}{2}a + (1-a)\frac{\hbar}{4}(1+i)\begin{pmatrix}1\\i\end{pmatrix} = \frac{\hbar}{2}a$$

3.14

$$\hat{H} = \frac{\vec{p}\cdot\vec{p}}{2m} + \frac{1}{2}m\omega^2 r^2$$

$$[A] = tr(\rho A)$$

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|\alpha\rangle + |\beta\rangle + |2\rangle) = \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) + |2\rangle\right)$$

$$= \frac{1}{\sqrt{6}}(|0\rangle + |1\rangle) + \frac{1}{\sqrt{6}}|1\rangle + \frac{1}{\sqrt{6}}|2\rangle + \frac{1}{\sqrt{3}}|2\rangle$$

$$= \frac{1}{\sqrt{6}}|0\rangle + \frac{2}{\sqrt{6}}|1\rangle + \frac{1+\sqrt{2}}{\sqrt{6}}|2\rangle$$

$$\hat{\rho} \equiv \begin{pmatrix} \frac{1}{6} & 0 & 0 & \\ 0 & 2/3 & 0 & \\ 0 & 0 & \frac{3+2\sqrt{2}}{6} & \\ & & & 0 \\ & & & & \ddots \end{pmatrix}$$

$$E_0 = \frac{\hbar\omega}{2} \quad E_1 = \frac{3}{2}\hbar\omega \quad E_2 = \frac{5}{2}\hbar\omega$$

$$[\hat{E}] = tr(\rho E) = \frac{1}{6}E_0 + \frac{2}{3}E_1 + \frac{3+2\sqrt{2}}{6}E_2$$

$$= \frac{1}{6}\cdot\frac{1}{2}\hbar\omega + \frac{2}{3}\cdot\frac{3}{2}\hbar\omega + \frac{3+2\sqrt{2}}{6}\cdot\frac{5}{2}\hbar\omega = \left(\frac{1}{12} + 1 + \frac{15+10\sqrt{2}}{12}\right)\hbar\omega$$

$$= \frac{13+15+10\sqrt{2}}{12}\hbar\omega = \frac{14+5\sqrt{2}}{6}\hbar\omega \approx 3.52\hbar\omega$$