Physics 221A Homework 7

3.1, 3.4, 3.10, 3.20, 3.23, 3.24, 3.33, 3.34

- **3.1** Use the specific form of S_x given by (3.25) to evaluate (3.23) and show that S_x rotates as expected through an angle ϕ about the z-axis.
- **3.4** Consider the 2×2 matrix defined by

$$U = \frac{a_0 + i\boldsymbol{\sigma} \cdot \mathbf{a}}{a_0 - i\boldsymbol{\sigma} \cdot \mathbf{a}},$$

where a_0 is a real number and **a** is a three-dimensional vector with real components.

- a. Prove that U is unitary and unimodular.
- b. In general, a 2×2 unitary unimodular matrix represents a rotation in three dimensions. Find the axis and angle of rotation appropriate for U in terms of a_0, a_1, a_2 , and a_3 .
- **3.10** Consider a sequence of Euler rotations represented by

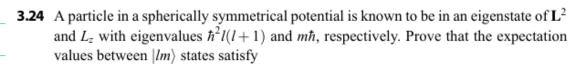
$$\begin{split} \mathscr{D}^{(1/2)}(\alpha,\beta,\gamma) &= \exp\left(\frac{-i\sigma_3\alpha}{2}\right) \exp\left(\frac{-i\sigma_2\beta}{2}\right) \exp\left(\frac{-i\sigma_3\gamma}{2}\right) \\ &= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}. \end{split}$$

Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a *single* rotation about some axis by an angle θ . Find θ .

- **3.20** Construct the matrix representations of the operators J_x and J_y for a spin 1 system, in the J_z basis, spanned by the kets $|+\rangle \equiv |1,1\rangle$, $|0\rangle \equiv |1,0\rangle$, and $|-\rangle \equiv |1,-1\rangle$. Use these matrices to find the three analogous eigenstates for each of the two operators J_x and J_y in terms of $|+\rangle$, $|0\rangle$, and $|-\rangle$.
- **3.23** The wave function of a particle subjected to a spherically symmetrical potential V(r) is given by

$$\psi(\mathbf{x}) = (x+y+3z)f(r).$$

- a. Is ψ an eigenfunction of L^2 ? If so, what is the l-value? If not, what are the possible values of l we may obtain when L^2 is measured?
- b. What are the probabilities for the particle to be found in various m_l states?
- c. Suppose it is known somehow that $\psi(\mathbf{x})$ is an energy eigenfunction with eigenvalue E. Indicate how we may find V(r).



$$\langle L_x \rangle = \langle L_y \rangle = 0, \quad \langle L_x^2 \rangle = \langle L_y^2 \rangle = \frac{[l(l+1)\hbar^2 - m^2\hbar^2]}{2}.$$

Interpret this result semiclassically.

- **3.33** Carry through the argument outlined on p. 208 for adding two spin $\frac{1}{2}$ particles by diagonalizing the 4×4 matrix corresponding to the operator S^2 given in (3.339). That is, construct the matrix representation of S^2 in the $|\pm\pm\rangle$ basis, and find the eigenvalues and eigenvectors. Your result should agree with (3.335).
 - **3.34** Find all nine states $|j,m\rangle$ for j=2, 1, and 0 formed by adding $j_1=1$ and $j_2=1$. Use a simplified notation, where $|j,m\rangle$ is explicit and \pm , 0 stand for $m_{1,2}=\pm 1,0$, respectively, for example

$$|1,1\rangle = \frac{1}{\sqrt{2}}|+0\rangle - \frac{1}{\sqrt{2}}|0+\rangle.$$

You may also want to make use of the ladder operators J_{\pm} , or recursion relations, as well as orthonormality. Check your answers by finding a table of Clebsch–Gordan coefficients for comparison; see Appendix E.