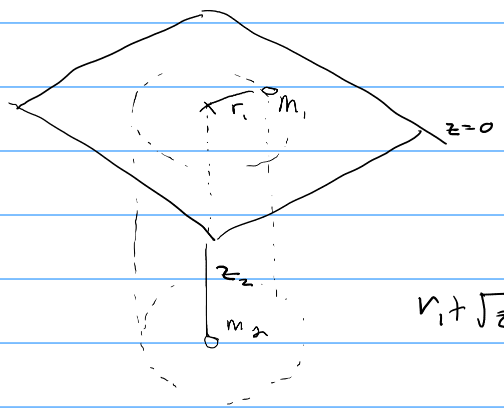
Homework 2

Problem 1

We want a cylindrical basis $\longrightarrow$ (r, θ, z)

a) λ: $r_1 + r_2 = L \longrightarrow r_1 + r_2 - L = 0 \rightarrow \phi(r_1, r_2, L, t) = 0$

$$L = T - V + \sum_j \lambda_j \frac{\partial \phi_j}{\partial q_i}$$



$r_1 + \sqrt{z_2^2 + r_2^2} - L = 0$

$r_1, \theta_1, z_1 = 0$

r_2, θ_2, z_2

$\phi_1 = r_1 + \sqrt{r_2^2 + z_2^2} - L = 0$

$\phi_2 = z_1 = 0$

$T_1 = \frac{1}{2} m_1 (r_1 \dot{\theta}_1^2 + \dot{r}_1^2 + \dot{z}_1^2) = \frac{1}{2} m_1 (r_1 \dot{\theta}_1^2 + \dot{r}_1^2) \qquad (z_1 = 0)$

$T_2 = \frac{1}{2} M_2 (r_2 \dot{\theta}_2^2 + \dot{r}_2^2 + \dot{z}_2^2)$

$V_2 = m_2 g z_2$

$V_1 = m_1 g z_1 = 0$

$L' = T_1 + T_2 - V_2 + \sum_j \lambda_j \phi_j = \frac{1}{2} m_1 (r_1 \dot{\theta}_1^2 + \dot{r}_1^2) + \frac{1}{2} m_2 (r_2 \dot{\theta}_2^2 + \dot{r}_2^2 + \dot{z}_2^2) - m_2 g z_2$

$+ \lambda_1 (r_1 + \sqrt{r_2^2 + z_2^2} - L) + \lambda_2 (z_1)$

Problem 1

$$\frac{\partial L'}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) + \sum_j \lambda_j \frac{\partial \phi_j}{\partial q_i} = 0$$

$$r_1 + \sqrt{r_2^2 + z_2^2} - L = 0 \longrightarrow \dot{r}_1 + \frac{\frac{d}{dt}(r_2^2 + z_2^2)}{2\sqrt{r_2^2 + z_2^2}} = \dot{r}_1 + \frac{\dot{r}_2 r_2 + \dot{z}_2 z_2}{\sqrt{r_2^2 + z_2^2}} = 0$$

$$\dot{r}_1 = -\frac{\dot{r}_2 r_2 + \dot{z}_2 z_2}{\sqrt{r_2^2 + z_2^2}} \qquad r_1^2 + \sqrt{r_2^2 + z_2^2} - L =$$

$$\frac{\partial L'}{\partial r_1} = \frac{1}{2} m_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 \frac{\partial}{\partial r_1}(r_2 \dot{\theta}_2) + \lambda_1 \frac{\partial}{\partial r_1}\left(r_1 + \sqrt{r_2^2 + z_2^2} - L\right)$$

$$\frac{1}{2} m_1 \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{\theta}_2^2 \frac{r_1 - L}{\sqrt{L^2 - 2Lr_1 + r_1^2 - z_2^2}} + \lambda_1 = 0$$

$$r_2 = \sqrt{(L - r_1 + z_2)(L - r_1 - z_2)} \qquad z_2 = \sqrt{(L - r_1 + r_2)(L - r_1 - r_2)}$$

$$\frac{\partial r_2}{\partial r_1} = \frac{r_1 - L}{\sqrt{L^2 - 2Lr_1 + r_1^2 - z_2^2}} \longrightarrow \frac{\partial r_2}{\partial r_1} = \frac{\sqrt{L^2 - 2Lr_1 + r_1^2 - z_2^2}}{r}$$

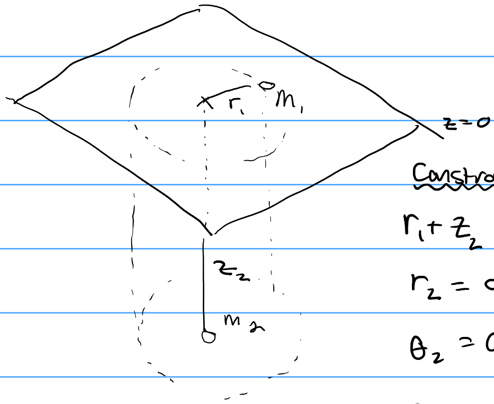
$$\theta_1 : \frac{\partial L'}{\partial \theta_1} = 0$$

$$z_1 : \frac{\partial L'}{\partial z_1} = 0$$

$$r_2 : \frac{\partial L'}{\partial r_2} =$$

Problem 1

(a)



Constraints:

$r_1 + z_2 - L = 0$ $\qquad$ $_1(r_1, r_2; L) = 0$

$r_2 = 0 \rightarrow \phi_2(r_2) = r_2 = 0$

$\theta_2 = 0 \rightarrow \phi_3(\theta_2) = \theta_2 = 0$

$z_1 = 0 \rightarrow \phi_4(z_1) = z_1 = 0$

$$L' = T - V + \sum_i \lambda_i \phi_i(\cdots) = \frac{1}{2} m_1 [(r_1 \dot\theta_1)^2 + \dot r_1^2] + \frac{1}{2} m_2 \dot z_2^2 - m_2 g (L - z_2)$$
$$+ \lambda_1 (r_1 + z_2 - L) + \underbrace{\lambda_2 (r_2)}_{=0} + \underbrace{\lambda_3 (\theta_2)}_{=0} + \underbrace{\lambda_4 (z_1)}_{=0}$$

(b) $$\frac{\partial L'}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot q_i}\right) + \sum_i \lambda_i \frac{\partial \phi_i}{\partial q_i} = 0$$

$\dot r_1 + \dot z_2 = 0 \longrightarrow \dot r_1 = -\dot z_2$ $\qquad$ $V_2(z) = m_2 g (L - z)$

$$\frac{\partial L'}{\partial r_1} = \frac{1}{2} m_1 \left[\dot\theta_1^2 \frac{\partial}{\partial r_1}(r_1^2) + r_1^2 \cancel{\frac{\partial}{\partial r_1}(\dot\theta_1^2)} + \cancel{\frac{\partial}{\partial r_1}(\dot r_1^2)} \right] + \frac{1}{2} m_2 \cancel{\frac{\partial}{\partial r_1}(\dot r_1^2)} - m_2 g \frac{\partial}{\partial r_1}(L - (L - r_1))$$
$$+ \lambda_1$$
$$= m_1 r_1 \dot\theta_1^2 - m_2 g + \lambda_1$$

$$\frac{\partial L'}{\partial \dot r_1} = m_1 \dot r_1 + m_2 \dot r_1$$

$$\frac{\partial L'}{\partial r_1} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot r_1}\right) = m_1 r_1 \dot\theta_1^2 - m_2 g - (m_1 + m_2) \ddot r_1 + \lambda = 0$$

$\frac{\partial L}{\partial \theta_1} = 0 \longrightarrow \frac{d}{dt}\left[\frac{\partial L}{\partial \dot\theta_1}\right] = 0$ $\qquad$ $\lambda_1 = (m_1 + m_2) \ddot r_1 + m_2 g - m_1 r_1 \dot\theta_1^2$

$\frac{\partial L}{\partial \dot\theta_1} = \frac{1}{2} m_1 [2 r_1^2 \dot\theta_1] \longrightarrow \frac{d}{dt}(m_1 r_1^2 \dot\theta_1) = 0$ $\qquad$ conservation of angular momentum.

$$\frac{\partial L'}{\partial z_2} = \frac{\partial}{\partial z_2}\left[\frac{1}{2} m_1 \left(((L - z_2)\dot\theta_1)^2 + \dot z_2^2 \right) + \frac{1}{2} m_2 \dot z_2^2 - m_2 g (L - z_2) + \lambda (r_1 + z_2 - L) \right]$$

$$= \frac{1}{2} m_1 \frac{\partial}{\partial z_2}\left((\cancel{L^2} - 2 L z_2 + z_2^2) \dot\theta_1^2 + \cancel{\dot z_2^2} \right) + m_2 g + \lambda =$$

$$= \frac{1}{2} m_1 (-2L + 2 z_2) \dot\theta_1^2 + m_2 g + \lambda = -m_1 L \dot\theta_2^2 + m_1 z_2 \dot\theta_2^2 + m_2 g + \lambda_1 = 0$$

Problem 1

$r_1: m_1 r_1 \dot\theta_1^2 - m_2 g - (m_1+m_2)\ddot r_1 + \lambda_1 = 0$

$z_2: -m_1 L \dot\theta_2^2 + m_1 z_2 \dot\theta_1^2 + m_2 g - (m_1+m_2)\ddot z_2 + \lambda_1 = 0$

$-m_1 \cancel{L}\dot\theta_1^2 + m_1(\cancel{L} - r_1)\dot\theta_1^2 + m_2 g - (m_1+m_2)\ddot z_2 + \lambda_1 = 0$

$-m_1 r_1 \dot\theta_1^2 + m_2 g + (m_1+m_2)\ddot r_1 + \lambda_1 = 0$

$\rightarrow +2m_1 r_1 \dot\theta_1^2 - 2m_2 g - 2(m_1+m_2)\ddot r_1 = 0$

$m_1 r_1 \dot\theta_1^2 - m_2 g - (m_1+m_2)\ddot r_1 = 0 \longrightarrow \ddot r_1 = \dfrac{m_1 r_1 \dot\theta_1^2 - m_2 g}{m_1+m_2}$

$\frac{d}{dt}(r_1^2 \dot\theta_1) = 2r_1 \dot r_1 \dot\theta_1 + r_1^2 \ddot\theta_1 = 0 \rightarrow 2\dot r_1 \dot\theta_1 = -r_1 \ddot\theta_1 \longrightarrow r_1 = -\dfrac{2\dot r_1 \dot\theta_1}{\ddot\theta_1}$

$$\boxed{-\frac{2m_1 \dot r_1 \dot\theta_1^3}{\ddot\theta_1} - m_2 g - (m_1+m_2)\ddot r_1 = 0}$$

(d) Since $r_1^2 \dot\theta_1$ is constant, angular acceleration $\dot r_1 = 0$ if the m_1 moves in a circle. $\ddot r_1 = 0$

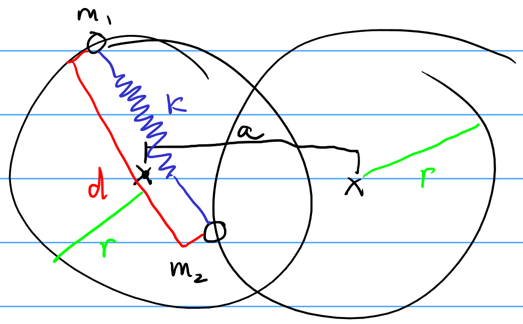
$m_1 r_1 \dot\theta_1^2 - m_2 g - (m_1+m_2)\ddot r_1 + \lambda_1 = 0 \longrightarrow \lambda_1 = (m_1+m_2)\ddot r_1 + m_2 g - m_1 r \dot\theta_1^2$

$m_1 r_1 \dot\theta_1^2 - m_2 g = 0$

$r_1 = \dfrac{m_2 g}{m_1 \omega^2}$

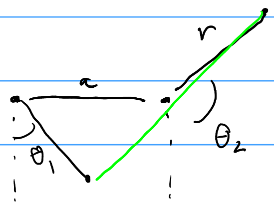
Tension $m_2 g$

Problem 2



a) Coordinate θ_1, θ_2 correspond to the angular positions of m_1 and m_2, respectively

$$L = T - V = \frac{1}{2} m_1 (r\dot{\theta}_1)^2 + \frac{1}{2} m_2 (r\dot{\theta}_2)^2 - \frac{1}{2} k d^2$$



$$\vec{d} = \vec{a} + \vec{r}_2 - \vec{r}_1$$

$$\vec{a} = a\hat{x}$$

$$\vec{r}_2 = r(\sin\theta_2 \hat{x} - \cos\theta_2 \hat{y})$$

$$\vec{r}_1 = r(\sin\theta_1 \hat{x} - \cos\theta_1 \hat{y})$$

$$\vec{d}(\theta_1 = 0, \theta_2 = 0) = a\hat{x}$$

$$\vec{d} = (a + r(\sin\theta_2 - \sin\theta_1))\hat{x} - r(\cos\theta_2 - \cos\theta_1)\hat{y}$$

$$d^2 = (a + r(\sin\theta_2 - \sin\theta_1))^2 + r(\cos\theta_2 - \cos\theta_1)^2$$

$$L = T - V = \frac{1}{2} m_1 (r\dot{\theta}_1)^2 + \frac{1}{2} m_2 (r\dot{\theta}_2)^2$$
$$- \frac{1}{2} k \left[(a + r(\sin\theta_2 - \sin\theta_1))^2 + r(\cos\theta_2 - \cos\theta_1)^2 \right]$$

$$\phi_1 = r_1 = 0$$

$$\phi_2 = r_2 = 0$$

Problem 2

(b) variables present, θ_1, $\dot{\theta}_1$, θ_2, $\dot{\theta}_2$

$$\frac{\partial L}{\partial \theta_1} = kr[r(\cos\theta_2 \sin\theta_1 - \sin\theta_2 \cos\theta_1) - a\cos\theta_1]$$
$$= kr[r\sin(\theta_1 - \theta_2) - a\cos\theta_1]$$

$$\frac{\partial L}{\partial \theta_2} = kr(a\cos\theta_2 - r\sin(\theta_1 - \theta_2)]$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 r^2 \dot{\theta}_1$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 r^2 \dot{\theta}_1$$

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}_1}\right] = 0$$

$$\begin{cases} kr(r\sin(\theta_1 - \theta_2) - a\cos\theta_1) - m_1 r^2 \ddot{\theta}_1 = 0 \\ kr(a\cos\theta_2 - r\sin(\theta_1 - \theta_2)) - m_2 r^2 \ddot{\theta}_2 = 0 \end{cases}$$

(c) $\rightarrow$ let $a = 0$ $\rightarrow$ $\begin{cases} kr(r\sin(\theta_1 - \theta_2) - \cancel{a\cos\theta_1}) - m_1 r^2 \ddot{\theta}_1 = 0 \\ kr(\cancel{a\cos\theta_2} - r\sin(\theta_1 - \theta_2)) - m_2 r^2 \ddot{\theta}_2 = 0 \end{cases}$

$\rightarrow \begin{cases} kr^2\sin(\theta_1 - \theta_2) - m_1 r^2 \ddot{\theta}_1 = 0 \\ -kr^2\sin(\theta_1 - \theta_2) - m_2 r^2 \ddot{\theta}_2 = 0 \end{cases}$ $\longrightarrow$ add together $\longrightarrow$ $m_1 r^2 \ddot{\theta}_1 + m_2 r^2 \ddot{\theta}_2 = 0$

lack of θ_1, θ_2 implies a rotational symmetry $\rightarrow$ total angular momentum conserved

(d) $m_1 r^2 \ddot{\theta}_1 = -m_2 r^2 \ddot{\theta}_2 \longrightarrow \ddot{\theta}_1 = -\frac{m_2}{m_1}\ddot{\theta}_2$

Problem 3

$\tilde{x} = x + \varepsilon h$

$\dot{\tilde{x}} = \dot{x} + \varepsilon \dot{h}$

(a) $L_1' = \frac{1}{2}m\left(\dot{\vec{x}} + \varepsilon\dot{\vec{h}}\right)^2 + q\vec{E}\cdot(\vec{x}+\varepsilon\vec{h}) = \frac{1}{2}m(\dot{\vec{x}}^2 + 2\varepsilon\dot{\vec{x}}\cdot\dot{\vec{h}} + \varepsilon^2\dot{\vec{h}}^2) + q(\vec{E}\cdot\vec{x} + \varepsilon\vec{E}\cdot\vec{h})$

$L_1' = \frac{1}{2}m \quad + q\vec{E}\cdot\vec{x} + \varepsilon\, m\dot{x}\dot{h} + \varepsilon^2\left(\frac{1}{2}m\dot{h}^2\right) + \varepsilon\vec{E}\cdot\vec{h}$

$L_1' = L_1 + \varepsilon\left(m\dot{x}\dot{h} + \vec{E}\cdot\vec{h}\right) + O(\varepsilon^2)$

$\left.\frac{\partial L_1'}{\partial \varepsilon}\right|_{\varepsilon=0} = m\dot{x}\dot{h} + \vec{E}\cdot\vec{h} + \cancel{O(\varepsilon^2)} = m\dot{x}\,\delta\dot{x} + \vec{E}\cdot\delta\vec{x} = \frac{d\Lambda}{dt} \therefore \Lambda(x,\dot{x},t)$

$\dot{x}(t)$ $\qquad$ $\delta\vec{x} = \left.\frac{\partial\vec{x}(t,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} = \delta\vec{x}(t)$

(b) $Q_1 = \sum_i \frac{\partial L_1}{\partial \dot{q}_i}\delta q_i - \Lambda$

$\frac{\partial L_1}{\partial \dot{\vec{x}}} = \frac{1}{2}m\frac{\partial}{\partial\dot{x}}(\dot{\vec{x}}\cdot\dot{\vec{x}}) = \frac{1}{2}m\left(\dot{\vec{x}} + \dot{\vec{x}}\right) = m\dot{\vec{x}}$

$Q_1 = m\dot{\vec{x}}\cdot\delta\vec{x} - \int\left(m(\dot{\vec{x}}\cdot\delta\dot{\vec{x}}) + q\vec{E}\cdot\delta\vec{x}\right)dt$

$= \cancel{m\dot{\vec{x}}\cdot\delta\vec{x}} - \left[\cancel{m\dot{\vec{x}}\cdot\delta\vec{x}} - \int m(\ddot{\vec{x}}\cdot\delta\vec{x})dt\right] - \int(q\vec{E}\cdot\delta\vec{x})dt$

$= \int\left[m(\ddot{\vec{x}}\cdot\delta\vec{x}) - q\vec{E}\cdot\delta\vec{x}\right]dt$

$\frac{dQ_1}{dt} = 0 \longrightarrow m(\ddot{\vec{x}}\cdot\delta\vec{x}) - q\vec{E}\cdot\delta\vec{x} = 0 \longrightarrow \boxed{q\vec{E}\cdot\delta\vec{x} = m\ddot{\vec{x}}\cdot\delta\vec{x}}$

The infinitesimal symmetry is the acceleration due to $\vec{E}$.

Problem 3

(c) $L_2 = \frac{1}{2}m(\dot{\vec{x}}\cdot\dot{\vec{x}}) - q(\vec{E}\cdot\vec{x})t$

$\frac{\partial L_2}{\partial \vec{x}} = 0 \rightarrow$ invariant w/ translation

from (b), L_1 is also invariant w/ translation

(d) $Q_2 = \frac{\partial L_2}{\partial \dot{\vec{x}}}\cdot\delta\vec{x} - \Lambda_2 \qquad \Lambda_2 = \int \frac{\partial L_2}{\partial \varepsilon}\Big|_{\varepsilon=0} dt$

$\frac{\partial L_2}{\partial \dot{\vec{x}}} = m\dot{\vec{x}} - qt(\cancel{\dot{\vec{x}}\cdot\frac{\partial}{\partial\dot{\vec{x}}}}\vec{E} + \vec{E}\cdot\frac{\partial}{\partial\dot{\vec{x}}}\dot{\vec{x}}) = m\dot{\vec{x}} - q\vec{E}t$

$\frac{\partial L_2}{\partial \dot{\vec{x}}}\cdot\delta\vec{x} = m\dot{\vec{x}}\cdot\delta\vec{x} - q(\vec{E}\cdot\delta\vec{x})t$

$L_2' = \frac{1}{2}m(\dot{\vec{x}}+\varepsilon\dot{\vec{h}})\cdot(\dot{\vec{x}}+\varepsilon\dot{\vec{h}}) - q(\vec{E}\cdot(\dot{\vec{x}}+\varepsilon\vec{h}))t$

$= \frac{1}{2}m(\dot{\vec{x}}\cdot\dot{\vec{x}} + 2\varepsilon(\dot{\vec{x}}\cdot\dot{\vec{h}}) + \varepsilon^2\dot{\vec{h}}\cdot\dot{\vec{h}}) - q[\vec{E}\cdot\dot{\vec{x}} + \varepsilon(\vec{E}\cdot\vec{h})]t$

$L_2' = L_2 + \varepsilon[m(\dot{\vec{x}}\cdot\dot{\vec{h}}) + q(\vec{E}\cdot\dot{\vec{h}})t] + O(\varepsilon^2)$

$\frac{\partial L_2'}{\partial \varepsilon}\Big|_{\varepsilon=0} = m(\dot{\vec{x}}\cdot\delta\dot{\vec{x}}) + q(\vec{E}\cdot\delta\dot{\vec{x}})t \qquad \int u\,dv = uv - \int v\,du$

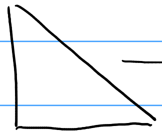
$\Lambda_2 = \int[m(\dot{\vec{x}}\cdot\delta\dot{\vec{x}}) + q(\vec{E}\cdot\delta\dot{\vec{x}})t]\,dt = m[\cancel{\dot{\vec{x}}\cdot\delta\vec{x}} - \cancel{\int\ddot{\vec{x}}\cdot\delta\vec{x}\,dt}]$
$+ \cancel{q(\vec{E}\cdot\delta\vec{x})t} - \cancel{\int q(\vec{E}\cdot\delta\vec{x})dt}$

$= 0$

$\rightarrow Q_2 = m\dot{\vec{x}}\cdot\delta\vec{x} - q(\vec{E}\cdot\delta\vec{x})t \longrightarrow \frac{dQ_2}{dt} = m\ddot{\vec{x}}\cdot\delta\vec{x} - q(\vec{E}\cdot\delta\vec{x}) = 0$

Problem 4

(a)



$\longrightarrow x = \frac{1}{2} a t^2 \quad \ddot{x} = a$

$y = y_0 - \tan(\alpha) x$

$\tan(\alpha) x' + y' - y_0 = 0 \quad \phi$

$\downarrow$

$x = x' + x_0(t) = x' + \frac{1}{2} a t^2 \qquad y = y' \qquad \tan(\alpha)\dot{x} + \dot{y} = 0$

$\dot{x} = \dot{x}' + at$

$L(x, \dot{x}, t) \longrightarrow L'(x', \dot{x}', t) \qquad x'(x, t) = x - \frac{1}{2} a t^2$

$\dot{x}'(\dot{x}, t) = \dot{x} - at$

$\ddot{x}' = \ddot{x} - a$

$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy\sin\alpha$

$L' = \frac{1}{2} m[(\dot{x}' + at)^2 + \dot{y}'^2] - mgy'\sin\alpha - \tan(\alpha)(x' + \frac{1}{2} a t^2)$

$= \frac{1}{2} m[(\dot{x}'^2 + 2\dot{x}' at + a^2 t^2) + \dot{y}'^2] - mgy'\sin\alpha$

$\frac{\partial L'}{\partial x'} = 0 \quad \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}'}\right) = 0 \longrightarrow \frac{\partial L'}{\partial \dot{x}'} = m\dot{x}' + ma \longrightarrow \frac{d}{dt}(m\dot{x}' + ma) = m\ddot{x}' = 0$

$= m(\ddot{x} - a) = 0$

$m\ddot{x} = ma$

$\frac{\partial L'}{\partial y'} = -mg\sin\alpha \quad \frac{\partial L'}{\partial \dot{y}'} = m\dot{y}'$

$\frac{\partial L'}{\partial x'} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}'}\right) = 0 \longrightarrow \frac{\partial L}{\partial x'} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}'}\right) + \lambda_1 \tan(\alpha)$

$m\ddot{x}' + \lambda_1 \tan(\alpha) = 0$

$\frac{\partial L}{\partial y'} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}'}\right) + \lambda_1 = 0$

$-mg\sin\alpha - m\ddot{y}' + \lambda_1 = 0$

$mg\sin\alpha\tan(\alpha) + m\ddot{y}'\tan(\alpha) + m\ddot{x}' = 0$

$\ddot{y} = -\tan(\alpha)\ddot{x}$

$mg\sin\alpha\tan(\alpha) + (-m\tan(\alpha) + m)\ddot{x}' = 0$

$mg\sin\alpha\tan\alpha - m(\tan\alpha - 1)(\ddot{x} - a) = 0$

$$-\cancel{M}(\tan\alpha-1)(\ddot{x}-a)=-\cancel{M}g\sin\alpha\tan\alpha$$

$$(\tan\alpha-1)(\ddot{x}-a)=g\sin\alpha\tan\alpha$$

$$\ddot{x}=\frac{g\sin\alpha\tan\alpha}{\tan\alpha-1}+a$$