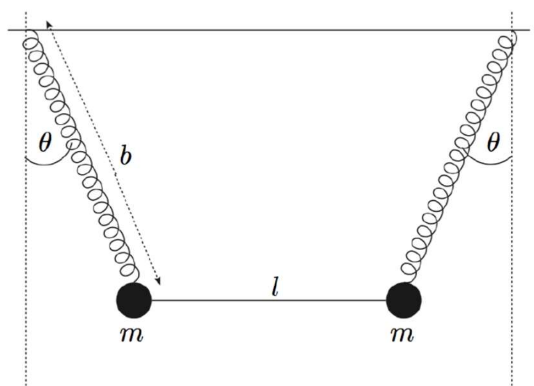
Physics 220 Fall 2024

Problem Set # 7 (due on Thu Nov 21th)

1) Consider two masses with mass *m* which each are connected to massless springs with spring constant *k* and equilibrium length *b*. The mass points are connected by a massless rigid rod of length l (see Figure). Assume the the motion is just in the plane and neglect gravity.
   a) Find appropriate generalized coordinates and express the Lagrangian in terms of them.
   b) Calculate the characteristic frequencies of the small oscillations.
   c) Find the normal modes and sketch the motion they correspond to.
   d) What happens to the characteristic frequencies if I increase the value of the spring constants ?



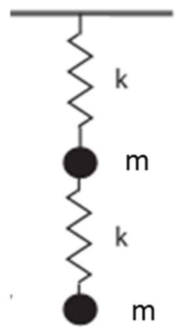
2) Three masses with mass m are connected along a line by four identical massless springs with spring constant k. The left and right springs are connected to walls. Consider only the linear motion. The coordinates of the i-th mass is denoted yi(t)

   a) Express the Lagrangian in terms of the displacement from equilibrium x_i = yi(t) - y(0)
   b) Find the matrices m_{ij} and v_{ij} for the small oscillations (here this approximation is actually exact).
   c) Find the normal frequencies, normal modes, and identify the type of motion associated with them.
   d) More difficult: Solve the analogous problem for N masses and show that the eigen-frequencies are given by

$$\omega_j = 2\sqrt{\frac{k}{m}}\left|\sin\frac{j\pi}{2(N+1)}\right| \qquad\qquad j = 1,2,\cdots N$$

Hint: Solve the characteristic equation in the form D_N = det $|m\omega^2$- v| = 0 by expanding the determinant and derive a recursion relation for the determinants D_N for lower N. Solve the recursion with an ansatz $D = \beta^N$.

3) Two equal masses are connected as shown with two identical mass springs of spring constant k.



a) Considering only motion in the vertical direction, calculate the angular frequencies of the two normal modes.
b) Find the ratio of amplitudes of the two masses in each separate mode

4) Consider a particle with mass *m* in the following potential

$$V(x) = \frac{m\omega^2 x^2}{2} + \frac{\lambda \cos\gamma t\, x^3}{3}$$

where $\gamma \sim \omega$ and we assume that $r = \frac{\gamma}{\omega}$ is generic i.e. not rational. Here we assume that λ is small and we use perturbation theory to arrive at an approximate solution. For the initial conditions we take x(t = 0) = A and v(t = 0) = 0.

a) Here we will work with dimensionful quantities. However, for the series expansion of the solution $x(t) = x0+\varepsilon x1+\ldots$, we will need a dimensionless small constant parameter. What is ε? (Feel free to ignore any numerical prefactors).

b) Solve this problem to first order, that is, find x(t) to first order in the small parameter.
c) What are the new frequencies introduced in first order?
d) Show that the correction vanishes in the $\gamma \gg \omega$ limit. Give a physical argument for your result.