Lecture # 2

Tuesday, October 3, 2023 10:01 AM

$$S = \int L(q_i, \dot{q}_i, t)\, dt \qquad i = 1, \dots N$$

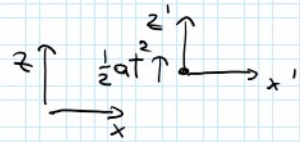
$\delta S = 0 \iff$ E-L equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

More examples:

Non inertial reference frames

a) Constant acceleration



$$x = x'$$
$$y = y'$$
$$z = z' + \frac{1}{2}at^2 \quad \Rightarrow \quad \dot{z} = \dot{z}' + at$$

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

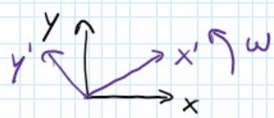
$$L' = \frac{1}{2}m\left[\dot{x}'^2 + \dot{y}'^2 + (\dot{z}' + at)^2\right]$$

E-L

$$m\ddot{x}' = 0$$
$$m\ddot{y}' = 0$$
$$m\ddot{z}' + ma = 0 \qquad m\ddot{z}' = -ma$$

equivalence principle

b) rotating frame



$$\begin{cases} x = x'\cos\omega t - y'\sin\omega t \\ y = y'\cos\omega t + x'\sin\omega t \\ z = z' \end{cases}$$

$$\dot{x} = \dot{x}'\cos\omega t - \dot{y}'\sin\omega t - x'\omega\sin\omega t - y'\omega\cos\omega t$$
$$\dot{y} = \dot{y}'\cos\omega t + \dot{x}'\sin\omega t - y'\omega\sin\omega t + x'\omega\cos\omega t$$
$$\dot{z} = \dot{z}'$$

$$L = \frac{1}{2}m\left[(\dot{x}' - \omega y')^2 + (\dot{y}' + \omega x')^2 + \dot{z}'^2\right] \overset{\text{drop prime}}{=} \frac{1}{2}m\left(\dot{\vec{r}} + \vec{\omega}\times\vec{r}\right)^2 \qquad \vec{\omega} = (0, 0, \omega)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} = \frac{d}{dt}\, m(\dot{\vec{r}} + \vec{\omega}\times\vec{r}) = m\ddot{\vec{r}} + m(\omega \times \dot{\vec{r}})$$

$$\frac{\partial L}{\partial \vec{r}} = \frac{\partial}{\partial \vec{r}}\left[m\dot{\vec{r}}\cdot(\vec{\omega}\times\vec{r}) + \frac{1}{2}m(\vec{\omega}\times\vec{r})^2\right] = m(\dot{\vec{r}}\times\vec{\omega}) - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$\frac{\partial L}{\partial \vec{r}} = \frac{\partial}{\partial \vec{r}}\left[m\dot{\vec{r}}\cdot(\vec{\omega}\times\vec{r})+\frac{1}{2}m(\vec{\omega}\times\vec{r})^2\right] = m(\dot{\vec{r}}\times\vec{\omega}) - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

use $\dot{\vec{r}}\cdot(\vec{\omega}\times\vec{r}) = \vec{r}\cdot(\dot{\vec{r}}\times\vec{\omega})$

$$(\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r}) = ((\vec{\omega}\times\vec{r})\times\vec{\omega})\cdot\vec{r}$$

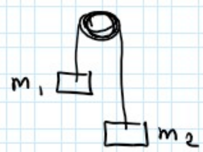
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} - \frac{\partial L}{\partial \vec{r}} = 0 = m\ddot{\vec{r}} + 2m\,\vec{\omega}\times\dot{\vec{r}} + m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

Coriolis   centrifugal

Lagrangian handling of constraints

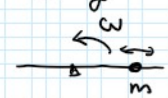
Examples:

- Atwood machine



- double pendulum

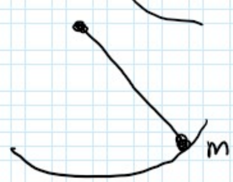
- bead on rotating wire



- motion on surface

- rolling w/o slipping

- spherical pendulum



- mass sliding a sphere

Mathematical formulation

consider a particle in 2D constrained to move $x(\lambda), y(\lambda)$

we can imagine this as due to $\phi(x,y)\begin{cases} = 0 & x(\lambda), y(\lambda) \\ = \infty & \text{otherwise}\end{cases}$

examples:

$$\begin{cases} x^2+y^2=R^2 & \phi(x^2+y^2-R^2)=0 \\ ax+by+c=0 & \end{cases}$$

Lagrangian mechanics different cases

holonomic $\begin{cases}\text{scleronomic} & \text{- independent of time} \quad \phi(q_1,\dots q_N)=0 \\ \text{rheonomic} & \text{- time dependent} \quad \phi(q_1,\dots q_N,t)=0\end{cases}$

– time dependent $\qquad \phi(q_1, \dots q_N, t) = 0$

non holonomic $\left\{ \begin{array}{l} \text{velocity dependent} \\ \text{inequalities} \end{array} \right.$

for multiple constraints, independent of each other

$$\left\{ \begin{array}{l} \phi_1(q_1 \dots q_N, t) = 0 \\ \vdots \\ \phi_s(q_1, \dots q_N, t) = 0 \end{array} \right. \qquad s\text{- holonomic constraints}$$

Two ways to deal with them

① Introduce s new auxiliary variables $\lambda_J(t)$ $\quad J = 1 \dots s$

$$S[q, \lambda] = \int_{t_1}^{t_2} dt \quad L(q_i, \dot{q}_i, t) + \sum_J \lambda_J \phi_J(q_1 \dots q_N, t) = L'(q, \lambda, \dot{q}, t)$$

E-L equations

$$\lambda: \qquad \frac{\partial L'}{\partial \lambda_i} = 0 = \phi_i(q_i, \dots q_N, t) = 0 \qquad s$$

$$q: \qquad \frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_J \lambda_J \frac{\partial \phi_J}{\partial q_i} = 0 \qquad N$$

$N + s$ system of equations which can be solved to find

N generalized coordinates, s Lagrange multipliers (λ_J)

$\frac{\partial \phi_J}{\partial q_i}$ has the role of force due to constraint scaled by λ_J

② Pick $N-s$ generalized coordinates which parametrize constraint surface

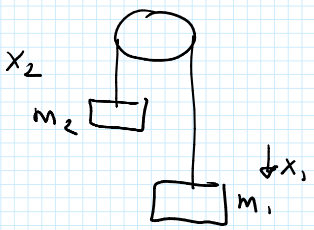
$\tilde{q}_J \qquad J = 1, \dots N-s$

$L'(\tilde{q}, \phi_r(\tilde{q}))$ $\qquad \tilde{q}$ automatically satisfy constraints

$$L'\left(\tilde{q}, \phi_r(\tilde{q})\right) \qquad \tilde{q} \text{ automatically satisfy constraints}$$

$$\frac{\partial L}{\partial \tilde{q}_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\tilde{q}}_i} = 0$$

Example: Atwood machine



$$L = \frac{1}{2} m_1 \dot{x}_1^2 - mgx_1 + \frac{1}{2} m_2 \dot{x}_2^2 - m_2 g x_2$$

$$x_1 + x_2 = \ell \qquad x_1 - x_2 - \ell = 0$$

$$L' = L + \lambda(x_1 + x_2 - \ell)$$

$$\lambda: \quad x_1 + x_2 - \ell = 0 \qquad \ddot{x}_2 = -\ddot{x}_1$$

$$x_1: \quad m_1 \ddot{x}_1 + m_1 g - \lambda = 0$$

$$x_2: \quad m_2 \ddot{x}_2 + m_2 g - \lambda = 0$$

$$(m_1 + m_2)\ddot{x}_1 + (m_1 - m_2) g = 0 \qquad \ddot{x}_1 = -\frac{(m_1 - m_2)}{m_1 + m_2} g$$

$$\lambda = m_1 \ddot{x}_1 + m_1 g = -m_1 \frac{m_1 - m_2}{m_1 + m_2} g + m_1 g = \frac{m_1 m_2}{m_1 + m_2} g \qquad \text{Tension}$$

force due to constraint

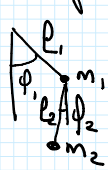
use other strategy

$$x_2 = -x_1 + \ell$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_1^2 - m_1 g x_1 - m_2 g(\ell - x_1)$$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 - (m_1 - m_2) g x_1 - m_2 g \ell$$

Double pendulum



$$L = \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{z}_1^2\right) - m_1 g z_1 + \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{z}_2^2\right) - m_2 g z_2$$

$$x_1 = \ell_1 \sin\phi_1 \qquad x_2 = \ell_1 \sin\phi_1 + \ell_2 \sin\phi_2$$

$$z_1 = -\ell_1 \cos\phi_1 \qquad z_2 = -\ell_1 \cos\phi_1 - \ell_2 \cos\phi_2$$

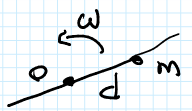
$$x_1 = \ell_1 \sin\phi_1 \qquad x_2 = \ell_1 \sin\phi_1 + \ell_2 \sin\phi_2$$

$$z_1 = -\ell_1 \cos\phi_1 \qquad z_2 = -\ell_1 \cos\phi_1 - \ell_2 \cos\phi_2$$

$$L = \frac{1}{2}(m_1+m_2)\ell_1^2 \dot\phi_1^2 + \frac{1}{2} m_2 \ell_2 \dot\phi_2^2 + \frac{1}{2} m_2 \ell_1 \ell_2 \dot\phi_1 \dot\phi_2 \cos(\phi_2 - \phi_1)$$

$$+ (m_1+m_2) g \ell_1 \cos\phi_1 + m_2 \ell_2 g \cos\phi_2$$

Bead on rotating string



$$L = \frac{1}{2} m (\dot x^2 + \dot y^2)$$

$$x = d \cos\omega t$$

$$y = d \sin\omega t$$

$$L = \frac{1}{2} m \left(\dot d \cos\omega t - \omega d \sin\omega t\right)^2 + \left(\dot d \sin\omega t + \omega d \cos\omega t\right)^2$$

$$= \frac{1}{2} m \dot d^2 + \frac{1}{2} m \omega^2 d^2$$

E·L equation $\quad m \ddot d - m\omega^2 d = 0$

$$d = d_0 \cosh\omega(t - t_0)$$