Lecture # 17

Tuesday, November 26, 2024 10:02 AM

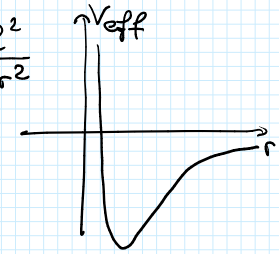
Central force problem

$$L = \frac{1}{2}\mu\left(\dot r^2 + r^2\dot\phi^2\right) - V(r)$$

reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$

$\mu r^2 \dot\phi = \ell$ constant p_ϕ $\qquad \dot\phi = \frac{\ell}{\mu r^2}$

$V_{eff} = V(r) + \frac{\ell^2}{2\mu r^2}$



$$\frac{d}{dt}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} = 0 \Rightarrow \mu \ddot r = -\frac{\partial V}{\partial r} + \mu r \dot\phi^2$$

$V = -\frac{\alpha}{r} \qquad \alpha = GM\mu = Gm_1m_2$

$u = 1/r \qquad dt = \frac{\mu r^2}{\ell} d\phi \qquad \left(-\frac{u^2\ell^2}{\mu}\right)\frac{d^2u}{d\phi^2} = \frac{\ell^2}{\mu}u^3 + F(1/u) \qquad F(r) = -\frac{\partial V}{\partial r}$

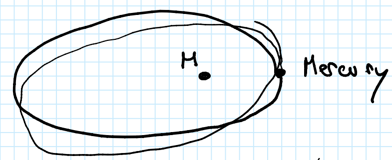
$-\alpha u^2$ is Kepler case

$$\frac{d^2u}{d\phi^2} + u = -\frac{\mu}{\ell^2 u^2}F(1/u) = \frac{\mu\alpha}{\ell^2}$$

$$\frac{1}{r} = u = \frac{\mu\alpha}{\ell^2} + A\cos\phi$$

elliptical closed orbits

Precession of the perihelion



angular separation between two consecutive r_{min} $= 0$ for grav. potential

$574''$/century

– sun oblateness

dealt with assuming 2 spherical suns of mass $M/2$ displaced by a small amount δ

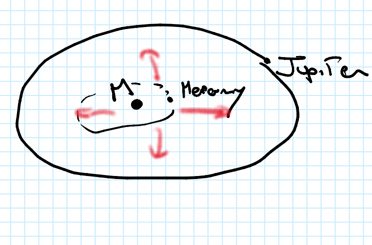
$$F \propto \frac{-M/2}{(r+\delta)^2} - \frac{M/2}{(r-\delta)^2} = -\frac{M}{2r^2}\left[\frac{1}{(1-\delta/r)^2} + \frac{1}{(1+\delta/r)^2}\right]$$

Taylor expansion $\frac{1}{(1+x)^2} \simeq 1 - 2x + 3x^2 + \dots$

$$-\frac{M}{r^2}\left(1 + 3\frac{\delta^2}{r^2}\right) \Rightarrow F(1/u) = -\alpha u^2\left(1 + 3\delta^2 u^2\right)$$

– presence of all other planets

potential of a ring of mass (M_J) at radius R_J



$$U(r) = -G\mu\int\frac{\rho\, d\ell}{|R-r|} = -G\mu\frac{M}{2\pi R}\int_0^{2\pi}\frac{R\,d\vartheta}{\sqrt{R^2+r^2-2Rr\cos\vartheta}}$$

$$= -\frac{GM\mu}{2\pi R}\int_0^{2\pi}\frac{d\vartheta}{\sqrt{1+\frac{r^2}{R^2}-2\frac{r}{R}\cos\vartheta}} \sim 1+\frac{r}{R}\cos\vartheta-\frac{1}{2}\frac{r^2}{R^2}+\frac{3}{8}\frac{4r^2}{R^2}\cos^2\vartheta$$

$$= -\frac{GM\mu}{2\pi R}\left(1+\frac{1}{4}\frac{r^2}{R^2}\right)$$

For $-\frac{1}{r^2}+\sum_i \alpha r$ summing up all contributions

$\cong 540''$/century precession.

— General relativity

$$U(r) = -\frac{\alpha}{r}\left(1+O\left(\frac{\overset{r\dot\phi}{v^2}}{c^2}\right)\right)$$

correction $$-\frac{\alpha}{r}\frac{v^2}{c^2} \simeq -\frac{\alpha}{r}\frac{\left(r\frac{\ell}{\mu r^2}\right)^2}{c^2} \propto 1/r^3$$

$$F(r) = -\frac{\partial U}{\partial r} = -\frac{\alpha}{r^2}-\frac{3\alpha\ell^2}{\mu^2c^2r^4} = -\alpha u^2-\frac{3\alpha\ell^2}{\mu^2c^2}u^4$$

go back to trajectory equation

$$\frac{d^2u}{d\phi^2}+u = -\frac{\mu}{\ell^2u^2}F(1/u)$$

$$= \frac{\mu\overset{=GM\mu}{\alpha}}{\ell^2}+\frac{\cancel{\mu}}{\cancel{\ell^2}u^2}\frac{3\alpha\cancel{\ell^2}}{\cancel{\mu^2}c^2}u^4$$

$$= \frac{\mu\alpha}{\ell^2}+\frac{3GM}{c^2}u^2 = \frac{\mu\alpha}{\ell^2}+\delta u^2$$

Use perturbation theory

at zeroth order $$u = u_0(1+\varepsilon\cos\phi)$$

at first order $$\frac{d^2u_1}{d\phi^2}+u_1 = \delta\Big(u_0^2+2u_0^2\varepsilon\cos\phi+u_0^2\varepsilon^2\underbrace{\cos^2\phi}_{=\frac{1+\cos2\phi}{2}}\Big)$$

$$u_1 = \delta\left(u_0^2+\frac{u_0^2\varepsilon^2}{2}\right)-\frac{\delta u_0^2\varepsilon^2\cos2\phi}{6}+\varepsilon u_0^2\delta\phi\sin\phi$$

collect all secular contributions

$$u \simeq u_0(1+\varepsilon\cos(\phi-\delta u_0\phi)) \simeq u_0(1+\varepsilon\cos\phi)+u_0\varepsilon\delta u_0\phi\sin\phi$$

$$\phi-\delta u_0\phi = 2\pi$$

$\phi = 2\pi$ …

$\varphi - \delta u_0 \varphi = 2\pi$

$$\phi = \frac{2\pi}{1-\delta u_0} \simeq 2\pi\left(1+\delta u_0\right) \qquad \Delta\phi = 2\pi\delta u_0$$

$$= 2\pi \frac{3GM}{c^2}\frac{\mu a}{\ell^2} = \frac{6\pi a^2}{c^2 \ell^2} \simeq 43''/\text{century}$$

Lagrange equilibrium points

reduced three body problem $\qquad m_1, m_2 \gg m$

satellites in Earth/moon system, planetoids/asteroids in Sun/Jupiter and so on

for m_1, m_2 this is classic two body problem

restrict discussion to case $\varepsilon = 0$ circular orbit in ecliptic plane

$$\vec{r}_1 = \begin{pmatrix} r_1 \cos\omega t \\ r_1 \sin\omega t \\ 0 \end{pmatrix} \qquad \vec{r}_2 = \begin{pmatrix} -r_2\cos\omega t \\ -r_2 \sin\omega t \\ 0 \end{pmatrix}$$

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0$$

$$GM = a^3\frac{(2\pi)^2}{T^2}$$

comes from $u_0 = \frac{1}{a} = \frac{\mu a}{\ell^2} = \frac{GM\cancel{\mu^2}}{\cancel{\mu^2} a^2\omega^2 a^2} \Rightarrow a^3\omega^2 = GM$

for small satellite

$$L = \frac{1}{2}m\dot{\vec{x}}^2 + \frac{Gmm_1}{|\vec{x}-\vec{r}_1|} + \frac{Gmm_2}{|\vec{x}-\vec{r}_2|}$$

use rotating reference frame $\quad \vec{x} = \begin{pmatrix} x\cos\omega t - y\sin\omega t \\ x\sin\omega t + y\cos\omega t \\ z \end{pmatrix}$

$$\vec{x}-\vec{r}_1 = \begin{pmatrix} x\cos\omega t - y\sin\omega t - r_1\cos\omega t \\ x\sin\omega t + y\cos\omega t - r_1 \sin\omega t \\ 0 \end{pmatrix} \qquad |\vec{x}-\vec{r}_1| = \sqrt{(x-r_1)^2+y^2+z^2}$$

$$L = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) + \frac{1}{2}m\omega^2(x^2+y^2) + \frac{1}{2}m\,2\omega\underbrace{(x\dot{y}-y\dot{x})}_{m\omega\,(x_L+\eta_1)\dot{\eta}_2} + \frac{Gmm_1}{\sqrt{(x-r_1)^2+y^2+z^2}} + \frac{Gmm_2}{\sqrt{(x+r_2)^2+y^2+z^2}}$$

$$\frac{\partial V}{\partial z} = 0$$

$\Rightarrow z=0$ is equilibrium $\qquad V(z) \propto -\frac{1}{\sqrt{D+z^2}} \Rightarrow \left.\frac{\partial^2 V}{\partial z^2}\right|_{z=0} > 0$

stable equilibrium at $z=0$

$\rightarrow z=0$ ... $\sqrt{D+z^2}$ ... $\partial z^2\big|_{z=0}$

stable equilibrium at $z=0$

effective potential

$$V_{eff} = -\frac{1}{2}\cancel{m}\omega^2(x^2+y^2) - \frac{G\cancel{m}m_1}{\sqrt{}} - \frac{G\cancel{m}m_2}{\sqrt{}}$$

$$\frac{\partial V_{eff}}{\partial x} = 0 \qquad \frac{\partial V_{eff}}{\partial y} = 0$$

$$\omega^2 x = \frac{Gm_1(x-r_1)}{(\quad)^{3/2}} + \frac{Gm_2(x+r_2)}{(\quad)^{3/2}}$$

$$\omega^2 y = \frac{Gm_1 y}{(\quad)^{3/2}} + \frac{Gm_2 y}{(\quad)^{3/2}}$$

2 cases

$y=0$ collinear equilibrium points. satellites are on line from m_1 to m_2

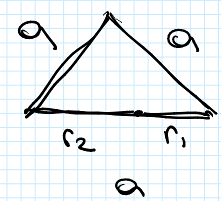
$y \neq 0$

$$\omega^2 = \frac{Gm_1}{(\quad)^{3/2}} + \frac{Gm_2}{(\quad)^{3/2}} = \frac{G(m_1+m_2)}{a^3}$$

$$-\frac{Gm_1 r_1}{((x-r_1)^2+y^2)^{3/2}} + \frac{Gm_2 r_2}{((x+r_2)^2+y^2)^{3/2}}$$

These can be solved by

$$(x+r_2)^2+y^2 = a^2$$
$$(x-r_1)^2+y^2 = a^2$$



equilateral triangle condition

Look at stability of L4 and L5

$$x_L = \frac{1}{2}(r_1-r_2) \qquad y_L = \frac{\sqrt{3}}{2}(r_1+r_2)$$

$$x_L = \frac{1}{2}(r_1 - r_2) \qquad y_L = \frac{\sqrt{3}}{2}(r_1 + r_2)$$

$$x = x_L + \eta_1 \qquad y = y_L + \eta_2$$

$$\begin{cases} \ddot{\eta}_1 - 2\omega\dot{\eta}_2 - \frac{3}{4}\omega^2\eta_1 + \lambda\omega^2\eta_2 = 0 \\ \ddot{\eta}_2 + 2\omega\dot{\eta}_1 - 9/4\,\omega^2\eta_2 + \lambda\omega^2\eta_1 = 0 \end{cases} \qquad \lambda = \frac{3\sqrt{3}}{4}\left(\frac{m_1 - m_2}{m_1 + m_2}\right)$$

ansatz $e^{i\bar{\omega}t}$

$$\begin{vmatrix} -\bar{\omega}^2 - \frac{3}{4}\omega^2 & -2i\omega\bar{\omega} + \lambda\omega^2 \\ +2i\omega\bar{\omega} + \lambda\omega^2 & -\bar{\omega}^2 - \frac{9}{4}\omega^2 \end{vmatrix} = 0$$

$$- \bar{\omega}^4 + \omega^2\bar{\omega}^6 + \left(-\frac{27}{16} + \lambda^2\right)\omega^4 = 0$$

$$\bar{\omega}^2 = \frac{\omega^2 \pm \sqrt{\omega^4 - 4\left(+\frac{27}{16} - \lambda^2\right)}}{2}$$

$$= \omega^2 \quad \frac{1}{2} \pm \sqrt{-\frac{23}{16} + \frac{27}{16}\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2}$$

for $m_1 >> m_2$

frequencies are real

$$-\frac{23}{16} + \frac{27}{16}\left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 > 0$$