Physics 220 Fall 2024

Problem Set #1 (due on Thu Oct 3rd)

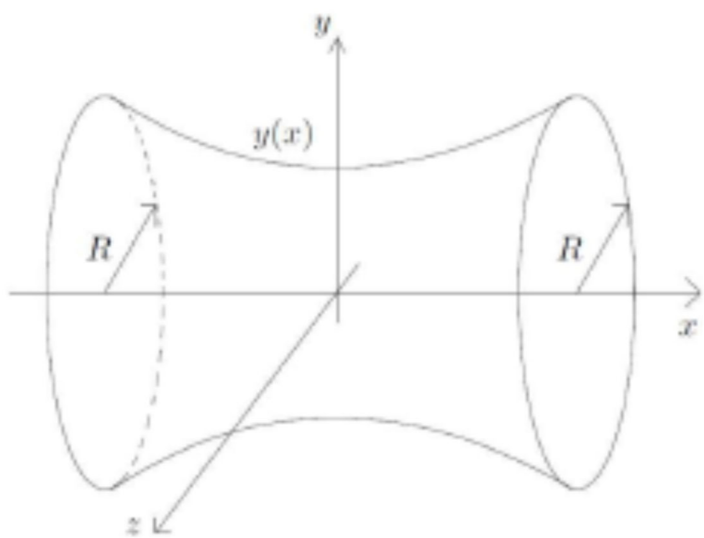
1) Consider a soap-film spanned between to rings of radius R located at $x = -x_0$ and $x = + x_0$. Assume that the film is rotationally symmetric around the x-axis (see Figure).

   a) Show that the area functional can be expressed in terms of the location y(x) of the soap film at the cross section z = 0; y > 0 and is given by

$$A[y] = 2\pi \int_{-x_0}^{x_0} dx\, y(x)\sqrt{1 + y'(x)^2}$$

   b) Find the Euler-Lagrange equations

   c) Find the solution which minimizes the area of the soap film.



2) In this problem you will show that for small times in a one dimensional system the solution of the Euler-Lagrange equations minimizes the action. Consider an action of the form

$$S[x] = \int_0^T dt\left(\frac{1}{2}m\dot{x}^2 - V(x)\right)$$

a) Show that for a variation $\delta x(0) = \delta x(T) = 0$ the change in the action at first order can be written as

$$\Delta S = -\int_0^T dt\left(m\ddot{x} + \frac{\partial V}{\partial x}\right)\delta x + o(\delta x^2)$$

b) Show that for an $x(t)$ that satisfies the Euler-Lgrange equations, the second variation has the form

$$\delta S = \int_0^T dt(m(\delta\dot{x})^2 - g(t)\delta x^2) + o(\delta x^3)$$

and show that $g(t) = \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x(t)}$

c) Using the boundary condition $\delta x(0) = \delta x(T) = 0$ and estimating the value of $g(t)$ show that one can choose a time T small enough that the order δx^2 term in the variation of S is always positive

3) Show that the Euler Lagrange equations take the same form regardless of the coordinate system used. Consider a coordinate transformation $r_j = r_j(q_1, q_2 \ldots q_N, t)$ which is invertible (i.e. $\frac{\partial r_j}{\partial q_i} = J_{ij}$ with $\det J \neq 0$). Show that in the motion satisfies

$$\frac{\partial L'}{\partial r_j} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{r}_j} = 0$$

where $L'(r_j, \dot{r}_j, t)$ is the Lagrangian written in terms of the new coordinates.

4) Consider a situation where the Lagrangian depends, in addition to the generalized coordinate and its derivative also on its second derivative, that is $L(q, \dot{q}, \ddot{q}, t)$

   a) Using the variation principle, find the Euler Lagrange equation (see Problem 2.10 in Goldstein) and note the boundary condition at the end points

   b) Consider

$$L = \alpha \ddot{x}^2 + \frac{m}{2}\dot{x}^2 - V(x)$$

   such that α is a constant. Derive the equation of motion in this case.

5) (optional) One of the open questions in astrophysics is that of galaxy rotation. It appears that far from the galaxy center the gas travels "too fast", that is, according to the observed matter profile, far from the center the velocity should scale like *1/r*, but instead it is measured to be constant in *r*. This led to the idea that there is dark matter in the galaxy and we are still searching to find out what it is. Yet, there is one more logical possibility to explain this discrepancy which requires modifying the laws of physics as we know them. For example, what if the second Newton law is not valid for distances as large as the size of the galaxy? This alternative mechanism to dark matter is called Modified Newtonian Dynamics (MOND, you can look for the acronym on Wikipedia). The basic idea is that instead of $F = ma$, we have

$$F = ma * f\left(\frac{a}{a_0}\right) \qquad (1)$$

such that a_0 us a new constant of Nature which can fit the observation if f satisfies

$$\lim_{x\to\infty} f(x) = 1, \text{ and } \quad \lim_{x\to 0} f(x) = x \qquad \text{with } a_0 \sim 10^{-10}\,\frac{m}{s^2}$$

   a) Show that the relation in (1) cannot be derived from a standard Lagrangian function which only depends on coordinates and their derivatives.

   b) One way to try to get to (1) is to assume that L is a function of second order derivatives, but in general we will get terms in the equation of motions proportional to $\dot{a}$ which are absent from Eq. (1). In the specific case we care (galaxy rotation) we can set $\dot{a} = 0$ and we further assume that the Lagrangian has the following form

$$L(x, v, a) = T(v, a) - V(x)$$

   What is the form of $T(v, a)$ in the limit that the acceleration is much larger than a_0 ?

c) Defining $g(v, \alpha) = \frac{1}{m}\frac{\partial^2 T}{\partial v^2}$ where $\alpha = a/a_0$ show that in order to retrieve Eq. (1) as equation of motion, the following conditions have to be satisfied: i) g has to be independent of v; ii) in the large α limit, $g(\alpha) = f(\alpha) + c\alpha$; iii) in the small α limit $g(x) - xg'(x) = x$. Note that you don't have to find $g(\alpha)$ explicitly.

d) Find an explicit solution for $T(v, a)$