Translation Operator

Momentum is a generator of momentum

deBroglie relation

$$T(d\vec{x}) = 1 - \frac{i\,\vec{p}\cdot d\vec{x}}{\hbar} = 1 - i\vec{k}\cdot d\vec{x} \longrightarrow \vec{p} = \hbar\vec{k}$$

$$\hat{T}(-dx) = 1 + \frac{i\,dx}{\hbar}\hat{p}$$

$$\langle x|\hat{T}(dx) = \langle x-dx|$$

$$\langle x|\hat{T}(-dx)|\psi\rangle = \langle x+dx|\psi\rangle = \psi(x+dx)$$

$$\langle x|T(-dx)|\psi\rangle =$$

$$= \langle x|\left(1 - -\frac{i\,dx}{\hbar}\hat{p}\right)|\psi\rangle$$

$$= \psi(x) + \frac{i\,dx}{\hbar}\langle x|\hat{p}|\psi\rangle \quad (3)$$

$$\langle x|\hat{p}|\psi\rangle = \left[\frac{\psi(x+dx) - \psi(x)}{dx}\right]\frac{\hbar}{i} = -i\hbar\frac{\partial}{\partial x}\psi$$

as lim $dx \to 0$

$$\hat{p} = -i\hbar\frac{d}{dx}$$

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int e^{-ipx/\hbar}\psi(x)\,dx$$

$$|\psi'\rangle = \hat{x}|\psi\rangle$$

$\phi'(p)$ in terms of $\phi(p)$

$$\phi'(p) = \frac{1}{\sqrt{2\pi\hbar}}\int e^{-ipx/\hbar}\psi'(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}}\int e^{-ipx/\hbar}x\psi(x)\,dx$$

$$\phi'(p) = i\hbar\frac{d}{dp}\left[\frac{1}{2\pi\hbar}\int e^{-ipx/\hbar}\psi(x)\,dx\right]$$

$$\phi'(p) = i\hbar\frac{d}{dp}\phi(p) \longrightarrow \hat{x} = +i\hbar\frac{d}{dp}$$

$$\phi(\vec{p}) = \langle\hat{p}|\psi\rangle \longrightarrow \hat{\vec{r}} = +i\hbar\nabla$$