Physics 221a HW #5
2.12, 2.14, 2.16, 2.19, 2.23

2.12 A one-dimensional simple harmonic oscillator with natural frequency ω is in initial state

$$|\alpha\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\delta}}{\sqrt{2}}|1\rangle$$

where δ is a real number.

a. Find the time-dependent wave function $\langle x'|\alpha;t\rangle$ and evaluate the (time-dependent) expectation values $\langle x\rangle$ and $\langle p\rangle$ in the state $|\alpha;t\rangle$, i.e. in the Schrödinger picture.
b. Now calculate $\langle x\rangle$ and $\langle p\rangle$ in the Heisenberg picture and compare the results.

2.14 Consider a particle subject to a one-dimensional simple harmonic oscillator potential. Suppose at $t = 0$ the state vector is given by

$$\exp\left(\frac{-ipa}{\hbar}\right)|0\rangle,$$

where p is the momentum operator, a is some number with dimension of length, and the state $|0\rangle$ is the one for which $\langle x\rangle = 0 = \langle p\rangle$. Using the Heisenberg picture, evaluate the expectation value $\langle x\rangle$ for $t \geq 0$.

2.16 Consider a one-dimensional simple harmonic oscillator.

a. Using

$$\left.\begin{matrix} a \\ a^\dagger \end{matrix}\right\} = \sqrt{\frac{m\omega}{2\hbar}}\left(x \pm \frac{ip}{m\omega}\right), \qquad \left.\begin{matrix} a|n\rangle \\ a^\dagger|n\rangle \end{matrix}\right\} = \left\{\begin{matrix} \sqrt{n}|n-1\rangle \\ \sqrt{n+1}|n+1\rangle, \end{matrix}\right.$$

evaluate $\langle m|x|n\rangle$, $\langle m|p|n\rangle$, $\langle m|\{x,p\}|n\rangle$, $\langle m|x^2|n\rangle$, and $\langle m|p^2|n\rangle$.

b. Translated from classical physics, the virial theorem states that

$$\left\langle \frac{\mathbf{p}^2}{m} \right\rangle = \langle \mathbf{x}\cdot\nabla V\rangle \quad \text{(3D)} \qquad \text{or} \qquad \left\langle \frac{p^2}{m} \right\rangle = \langle x\frac{dV}{dx}\rangle \quad \text{(1D)}$$

Check that the virial theorem holds for the expectation values of the kinetic and the potential energy taken with respect to an energy eigenstate.

2.19 Consider again a one-dimensional simple harmonic oscillator. Do the following algebraically, that is, without using wave functions.

a. Construct a linear combination of $|0\rangle$ and $|1\rangle$ such that $\langle x\rangle$ is as large as possible.
b. Suppose the oscillator is in the state constructed in (a) at $t = 0$. What is the state vector for $t > 0$ in the Schrödinger picture? Evaluate the expectation value $\langle x\rangle$ as a function of time for $t > 0$ using (i) the Schrödinger picture and (ii) the Heisenberg picture.
c. Evaluate $\langle(\Delta x)^2\rangle$ as a function of time using either picture.

2.23 Make the definitions

$$J_\pm \equiv \hbar a_\pm^\dagger a_\mp, \qquad J_z \equiv \frac{\hbar}{2}(a_+^\dagger a_+ - a_-^\dagger a_-), \qquad N \equiv a_+^\dagger a_+ + a_-^\dagger a_-$$

where $a_\pm$ and $a_\pm^\dagger$ are the annihilation and creation operators of two *independent* simple harmonic oscillators satisfying the usual simple harmonic oscillator commutation relations. Also make the definition

$$\mathbf{J}^2 \equiv J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+).$$

Prove

$$[J_z, J_\pm] = \pm\hbar J_\pm, \qquad [\mathbf{J}^2, J_z] = 0, \quad \mathbf{J}^2 = \left(\frac{\hbar^2}{2}\right) N\left[\left(\frac{N}{2}\right) + 1\right].$$