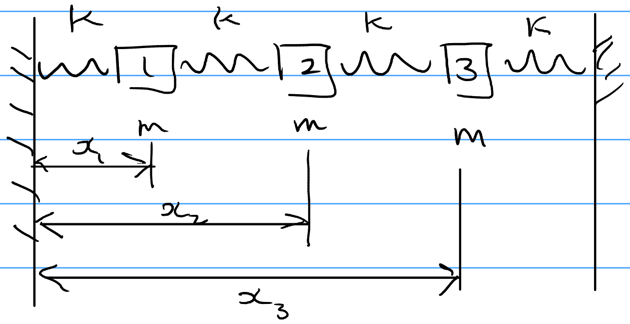
Physics 220 Homework 7 Problem 2

a)



$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \right) - \frac{k}{2} \left(x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + x_3^2 \right)$$

$$\omega_0^2 V - \omega^2 M = 0 \qquad \omega^2 = \lambda \omega_0^2 = \lambda \left(\frac{k}{m} \right)$$

$$V - \frac{\omega^2}{\omega_0^2} M = 0 \longrightarrow V - \lambda M = 0$$

b)

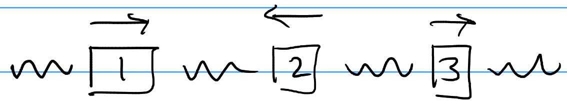
$$M = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad V = \begin{pmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{pmatrix}$$

$$V - \lambda M = \begin{pmatrix} 2-\lambda & -1 & \\ -1 & 2-\lambda & -1 \\ & -1 & 2-\lambda \end{pmatrix} \longrightarrow \det(V - \lambda M) = 0$$

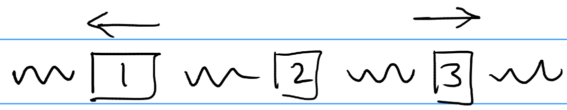
$$\lambda = 2+\sqrt{2},\ 2,\ 2-\sqrt{2}$$

c)

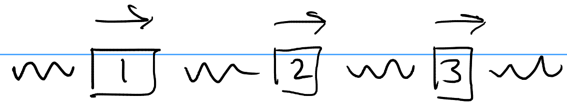
$$\lambda = 2+\sqrt{2} \longrightarrow \vec{\gamma}_1 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$



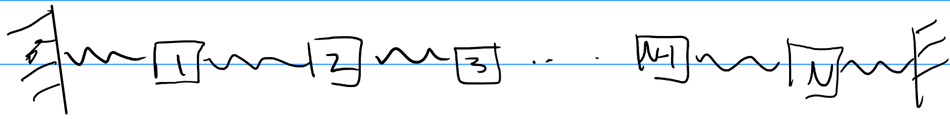
$$\lambda = 2 \qquad \vec{\gamma}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$



$$\lambda = 2-\sqrt{2} \qquad \vec{\gamma}_3 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$



d)



$$\frac{1}{2}m\left(\sum_{i=1}^{N}\dot{x}_i^2\right) - \frac{1}{2}k\left(x_1^2 + \sum_{i=1}^{N-1}(x_{i+1}-x_i)^2 + x_N^2\right)$$

$$x_2^2 - 2x_1x_2 + x_1^2$$

$$-x_1x_2 - x_2x_1$$

$$V = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & \\ & & & \ddots \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

$$D_N = (2-\lambda)D_{N-1} - 4D_{N-2}$$

$$V - \lambda M = \begin{pmatrix} 2-\lambda & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2-\lambda \end{pmatrix} \longrightarrow D_N = \det(V-\lambda M)$$

$$D_N = \sum_{j=1}^{N}(-1)^{i+j}M_{ij}a_{ij} = (2-\lambda)D_{n-1} - (-1)\underbrace{\begin{vmatrix} -1 & -1 & 0 & 0 \\ 0 & 2-\lambda & -1 & \\ 0 & -1 & \ddots & \\ 0 & & & \ddots \end{vmatrix}}_{}$$

$$= (-1)D_{N-2}$$

$$\begin{pmatrix} \boxed{2-\lambda} & \boxed{-1} & & & \\ -1 & 2-\lambda & -1 & & \\ & -1 & 2-\lambda & -1 & \\ & -1 & & \ddots & -1 \\ & & & -1 & 2-\lambda \end{pmatrix}$$

$$D_N = (2-\lambda)D_{N-1} - D_{N-2}$$

$D_N = \beta^N$ $\quad\quad$ $D_N = a D_{N-1} + b D_{N-2}$

$\beta^N = a\beta^{N-1} + b\beta^{N-2}$

$\beta^2 = a\beta + b$

$\beta^2 = (2-\lambda)\beta - 1 \longrightarrow \beta^2 - (2-\lambda)\beta + 1 = 0$

$\beta = \frac{1}{2}(2 \pm \sqrt{\lambda - 4})$