Physics 220 Fall 2024

Problem Set # 4 (due on Thu Oct 24th)

1) Consider a dynamical flow which is defined on R^n with $\boldsymbol{x} = (x_1, \dots x_n)$ and a vector field ***v(x)*** as follows

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{v}(\boldsymbol{x}(t))$$

   a) Solving the differential equations and finding **x**(t) defines a map from $R^n \to R^n$. Show that this map is volume preserving if the vector field ***v*** is divergence free, i.e.

$$\sum_i \frac{\partial v_i}{\partial x_i} = 0$$

   b) Show that by setting $\boldsymbol{x} = (q_1, \dots q_n, p_1, \dots p_n)$ and

$$v_i = \sum J_{ij} \frac{\partial H}{\partial x_j}$$

   where *J* is the symplectic matrix, the vector field ***v*** is divergence-free if Hamilton equations hold. Hence you have proved Liouville's theorem.

2) This problem concerns the Runge-Lenz vector and its algebra under Poisson brackets. The Runge-Lenz vector is defined as

$$\vec{K} = \frac{\vec{p} \times \vec{L}}{m} - \frac{\vec{r}}{r}$$

   a) Show that the following Poisson bracket is given by

$$\{L_i, K_j\} = \epsilon_{ijk} K_k$$

   b) Show that

$$\{K_i, K_j\} = -\frac{2}{m}\left(\frac{p^2}{2m} - \frac{1}{r}\right)\epsilon_{ijk} L_k$$

   c) Show that for a Hamiltonian of the form

$$H = \frac{p^2}{2m} + V(r)$$

   one has

$$\{\vec{K}, H\} = \frac{\vec{r} \times (\vec{p} \times \vec{r})}{m}\left(\frac{-1 + r^2 V'(r)}{r^3}\right)$$

3) A particle with mass m and charge e moves in a magnetic field **B.**

a) Show that the following Poisson brackets hold

$$\{m\dot{x}_i, m\dot{x}_j\} = \epsilon_{ijk} B_k, \qquad \{m\dot{x}_i, x_j\} = -\delta_{ij}$$

b) Consider the magnetic field of magnetic monopole with

$$\boldsymbol{B} = g_m \frac{\widehat{\boldsymbol{x}}}{|x|^2}$$

where $\hat{x}$ is the unit vector in the **x** direction. Show that the following generalization of the angular momentum

$$\boldsymbol{J} = m\boldsymbol{x} \times \dot{\boldsymbol{x}} - e g_m \widehat{\boldsymbol{x}}$$

commutes with the Hamiltonian H

c) Can you interpret the fact that the usual angular momentum is not conserved while the **J** defined above is?

4) Show that the following coordinate transformations are canonical

a) $P = \frac{1}{2}(p^2 + q^2), \quad Q = \arctan\left(\frac{q}{p}\right)$

b) $P = 1/q, \quad Q = pq^2$

c) $P = 2\sqrt{q}(1 + \sqrt{q}\cos p)\sin p, \quad Q = \log(1 + \sqrt{q}\cos p)$