Physics 220 Fall 2024

Problem Set # 5 (due on Thu Oct 31$^{\text{th}}$)

1) Consider the following Hamiltonian

$$H = \frac{p^2 q^4}{2m} + \frac{k}{2q^2}$$

and a canonical transformation given by the generating function

$$F_1(q, Q) = -\sqrt{mk}\,\frac{Q}{p}$$

a) Find the canonical transformation $Q = Q(q, p), P = P(q, p)$
b) Find the new Hamiltonian $H(P, Q)$
c) Find the general solution $Q(t), P(t)$
d) Solve for q(t),p(t) with the initial conditions $q(0) = q_0, p(0) = p_0,$

2) Use Hamilton-Jacobi theory to solve for the dynamics of a particle with Cartesian coordinates *(x,y,z)* moving in the gravitational potential of two fixed masses m_+, m_- located at the points *(±c,0,0)*. Restrict motion to the plane *z=0* for the sake of simplicity. This system has Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{\mu_+}{r_+} + \frac{\mu_-}{r_-}$$

where for brevity $\mu_\pm = G m_\pm m > 0$ and $r_\pm = \sqrt{(x \mp c)^2 + y^2}$. [Hint: change variables to elliptical coordinates $x = c\cosh q_1 \cos q_2$ and $y = c \sinh q_1 \sin q_2$

3) Consider an infinitesimal canonical transformation with generating function

$$F_2(q, P) = Pq + \epsilon S(P, q)$$

a) Show that to first order in ϵ one has

$$p = P + \epsilon \frac{\partial S}{\partial q}, \qquad Q = q + \epsilon \frac{\partial S}{\partial P}$$

b) From this, show that the infinitesimal canonical transformation satisfies Hamiltonian equations with ϵ viewed as “time”

$$\left.\frac{dP}{d\epsilon}\right|_{\epsilon=0} = -\frac{\partial H}{\partial q}, \qquad \left.\frac{dQ}{d\epsilon}\right|_{\epsilon=0} = \frac{\partial H}{\partial p}$$

where $H(p, q) = S(p, q)$