Physics 220 Homework 3

Problem 1.18

Consider $|\phi\rangle$ to be an eigenstate/ket of $\hat{A}\hat{B}$,

$$\hat{A}\hat{B}|\phi\rangle = \lambda|\phi\rangle$$

This means $\left\{\hat{A}, \hat{B}\right\}|\phi\rangle = \hat{A}\hat{B}|\phi\rangle - \hat{B}\hat{A}|\phi\rangle = 0$. Rearranging the equation,

$$\hat{A}\hat{B}|\phi\rangle = -\hat{B}\hat{A}|\phi\rangle = \lambda|\phi\rangle \implies \hat{B}\hat{A}|\phi\rangle = -\lambda|\phi\rangle$$

This suggests that switching the order of the operators yields the same eigenstate and eigenvalue magnitude but with the opposite sign.

Problem 1.21

1.21

$|\psi\rangle = |+\rangle_z = \begin{pmatrix}1\\0\end{pmatrix}$

$\langle S_x^2\rangle = (1\ 0)\left[\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]^2\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar^2}{4}(1\ 0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar^2}{4}(1\ 0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar^2}{4}(1\ 0)\begin{pmatrix}1\\0\end{pmatrix}$

$= \frac{\hbar^2}{4}$

$(\langle S_x\rangle)^2 = \left[(1\ 0)\frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]^2 = \frac{\hbar^2}{4}\left[(1\ 0)\begin{pmatrix}0\\1\end{pmatrix}\right]^2 = 0$

$\langle(\Delta S_x)^2\rangle = \frac{\hbar^2}{4} - 0 = \frac{\hbar^2}{2}$

$\langle(\Delta S_y)^2\rangle = \langle S_y^2\rangle - \langle S_y\rangle^2$

$\langle S_y^2\rangle = \left[(1\ 0)\frac{\hbar^2}{4}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right] = \frac{\hbar^2}{4}(1\ 0)\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}i\\0\end{pmatrix} = \frac{\hbar^2}{4}(1\ 0)\begin{pmatrix}0\\-1\end{pmatrix} = 0$

$\langle S_y\rangle^2 = \left[(1\ 0)\frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]^2 = \frac{\hbar^2}{4}\left[(1\ 0)\begin{pmatrix}i\\0\end{pmatrix}\right]^2 = -\frac{\hbar^2}{4}$

$\langle S_y^2\rangle - \langle S_y\rangle^2 = 0 - \left(-\frac{\hbar^2}{4}\right) = \frac{\hbar^2}{4}$

$[S_x, S_y] = \frac{\hbar^2}{4}\begin{pmatrix}2i&0\\0&-2i\end{pmatrix}$

$\frac{1}{4}\left|\left\langle\frac{\hbar^2}{4}\begin{pmatrix}2i&0\\0&-2i\end{pmatrix}\right\rangle\right|^2 = \frac{1}{4}\frac{\hbar^4}{16}\left|(1\ 0)\begin{pmatrix}2i&0\\0&-2i\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right|^2 = \frac{\hbar^4}{64}|2i|^2 = \frac{\hbar^4}{16}$

$\frac{\hbar^2}{4}\cdot\frac{\hbar^2}{4} \geq \frac{\hbar^4}{16}$

with $|\psi\rangle = |+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$\langle(\Delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2 = 0$

$\langle(\Delta S_y)^2\rangle = \frac{\hbar^2}{4}$

$\frac{1}{4}|\langle[S_x, S_y]\rangle|^2 = \frac{1}{4}\left|\frac{1}{2}(1\ 1)\begin{pmatrix}2i&0\\0&-2i\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right|^2 = 0$

$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = 0 \geq \frac{1}{4}|\langle[S_x, S_y]\rangle|^2 = 0$

Problem 1.23

The dispersion of the position wave function is

$$\begin{aligned}\left\langle (\Delta x)^2 \right\rangle = \left\langle x^2 \right\rangle - \langle x \rangle^2 &= \int_0^a \psi^\star(x) x^2 \psi(x)\, dx - \left(\int_0^a \psi^\star(x) x \psi(x)\, dx \right)^2 \\ &= \frac{2}{a} \int_0^a x^2 \sin^2\left(\frac{n\pi x}{a}\right) dx - \left(\frac{2}{a} \int_0^a x \sin^2\left(\frac{n\pi x}{a}\right) dx \right)^2 \\ &= \frac{4\pi^2 n^2 a^2 - 6a^2}{12\pi^2 n^2} - \frac{a^2}{4}\end{aligned}$$

The dispersion of the momentum wave function is

$$\begin{aligned}\left\langle (\Delta p)^2 \right\rangle &= \left\langle p^2 \right\rangle - \langle p \rangle^2 \\ &= \hbar^2 \frac{2}{a} \left(\frac{n\pi}{a}\right)^2 \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx - 0^2 = \left(\frac{n\pi\hbar}{a}\right)^2\end{aligned}$$

Taking the product of the dispersions,

$$\begin{aligned}\left\langle (\Delta x)^2 \right\rangle \left\langle (\Delta p)^2 \right\rangle &= \left(\frac{4\pi^2 n^2 - 6}{12} - \frac{n^2\pi^2}{4} \right) \hbar^2 \\ &= \left(\frac{\pi^2 n^2 - 6}{12} \right) \geq \frac{1}{4} \implies n^2 \geq \frac{9}{\pi^2} \implies n \geq \frac{3}{\pi}\end{aligned}$$

n must be an integer 1 or greater, so the inequality holds.

Problem 1.25

(a) Operator $\hat{A}$ exhibits a degenerate spectrum as, out of the three eigenvalues it has, two of them are $-a$. Determining the eigenvalues of $\hat{B}$,

$$\det\left(\hat{B} - \lambda \hat{I}\right) = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -b-\lambda & -ib \\ 0 & ib & -b-\lambda \end{vmatrix} = (b - \lambda) \begin{vmatrix} -\lambda & -ib \\ ib & -\lambda \end{vmatrix} = -(\lambda - b)(\lambda + b)(\lambda - b) = 0$$

From the characteristic polynomial, it is clear that the polynomial has degenerate roots, and therefore $\hat{B}$ has a degenerate spectrum.

(b) $\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$, which computes to

$$\begin{aligned}&\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} - \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \\ &= \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} - \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} = \vec{0}\end{aligned}$$

(c) The eigenvalue of $\hat{A}\hat{B}$ are,

$$\det\left(\hat{A}\hat{B}-\lambda\hat{I}\right)=0 \implies \begin{vmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{vmatrix} = -(\lambda-ab)(\lambda+ab)(\lambda-ab)=0$$

The characteristic polynomial has three roots, two of which are degenerate (i.e. ab). This means that each eigenstate cannot be distinguished solely by its corresponding eigenvalue, as the same eigenvalue may correspond to more than one eigenstate.

Problem 1.30

(a) The classical Poisson bracket expression for

$$\{x, F(p_x)\}_{x,p_x} = \frac{\partial x}{\partial x}\frac{\partial F(p_x)}{\partial p_x} - \frac{\partial F(p_x)}{\partial x}\frac{\partial x}{\partial p_x} = \frac{\partial F(p_x)}{\partial p_x}$$

(b) The commutator of the operators $\hat{x}, \exp\left(i\hat{p}_x a/\hbar\right)$, where

$$\exp\left(\frac{i\hat{p}_x a}{\hbar}\right) = \sum_{n=0}^{\infty}\frac{a^n}{n!}\frac{\partial^n}{\partial x^n}$$

The commutator evaluates to

$$x\sum_{n=0}^{\infty}\frac{a^n}{n!}\frac{\partial^n}{\partial x^n}f(x) - \sum_{n=0}^{\infty}\frac{a^n}{n!}\frac{\partial^n}{\partial x^n}[xf(x)]$$

Applying Leibniz's product rule, where

$$\frac{d^n}{dx^n}[f(x)g(x)] = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{\partial^k}{\partial x^k}(x)\frac{\partial^{(n-k)}f(x)}{\partial x^{(n-k)}}$$

The commutator expression further evaluates to

$$x\sum_{n=0}^{\infty}\frac{a^n}{n!}\frac{\partial^n}{\partial x^n}f(x) - \sum_{n=0}^{\infty}a^n\left(\frac{x}{n!}f^{(n)}(x) + \frac{f^{(n-1)}(x)}{(n-1)!}\right) = -a\sum_{n'=0}^{\infty}\frac{a^{n'}}{n'!}f^{(n)}(x) = -a\exp\left(\frac{ip_x a}{\hbar}\right)$$

(c) Applying the commutator to $|x'\rangle$ as an operator evaluates to

$$\left[\hat{x}, \exp\left(\frac{i\hat{p}_x a}{\hbar}\right)\right]|x'\rangle = \left(\hat{x}\exp\left(\frac{i\hat{p}_x a}{\hbar}\right) - \exp\left(\frac{i\hat{p}_x a}{\hbar}\right)\hat{x}\right)|x'\rangle = -a\exp\left(\frac{ip_x a}{\hbar}\right)|x'\rangle$$

Rearranging the second term on the left-hand side of the equation,

$$\hat{x}\exp\left(\frac{i\hat{p}_x a}{\hbar}\right)|x'\rangle = (x-a)\exp\left(\frac{i\hat{p}_x a}{\hbar}\right)|x'\rangle$$

The eigenvalue corresponding to the eigenstate is $x - a$.