1.2

What is $\{\cdots\}$ operator → Sakurai 26 Anticommutator

$$\{A,B\} = AB + BA$$
$$[A,B] = AB - BA$$

$$[AB, CD] = -AC\{D,B\} + A\{C,B\}D - C\{D,A\}B + \{C,A\}DB$$

$$[AB, CD] = ABCD - CDAB$$

$$-AC(DB+BD) + A(CB+BC)D - C(DA+AD)B + (CA+AC)DB = -ACDB - ACBD + A(CBD+BCD) - C(DAB+ADB) + (CA+AC)DB = \cancel{-ACDB} \cancel{-ACBD} \cancel{+ACBD} + ABCD \cancel{-CDAB} - CADB \cancel{+CADB} \cancel{+ACDB}$$

$$= ABCD - CADB$$

$$\therefore [AB, CD] = -AC\{D,B\} + A\{C,B\}D - C\{D,A\}B + \{C,A\}DB$$

1.6

use Einstein notation to solve

1.6 Using the rules of bra-ket algebra, prove or evaluate the following:

a. $\mathrm{tr}(XY) = \mathrm{tr}(YX)$, where X and Y are operators;

b. $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators;

c. $\exp[if(A)] = ?$ in ket-bra form, where A is a Hermitian operator whose eigenvalues are known;

d. $\sum_{a'} \psi^*_{a'}(\mathbf{x}')\psi_{a'}(\mathbf{x}'')$, where $\psi_{a'}(\mathbf{x}') = \langle \mathbf{x}'|a'\rangle$.

$X_{ij} = X^*_{ji}$ $\quad Y_{kl} = Y^*_{lk}$

$\rightarrow A_{ik} = X_{ij} Y_{jk}$ $\quad \mathrm{tr}(XY) = \sum_i A_{ii} = \sum_i X_{ii} Y_{ii}$

$\mathrm{tr}(YX) = \sum_i Y_{ii} X_{ii} = \sum X_{ii} Y_{ii} = \mathrm{tr}(XY)$

$(X_{ij} Y_{jk})^\dagger = (A_{ik})^\dagger = A^*_{ki}$ $\quad \longrightarrow (XY)^\dagger = Y^\dagger X^\dagger$

$Y^\dagger_{ij} X^\dagger_{jk} = Y^*_{ji} X^*_{kj} = X^*_{kj} Y^*_{ji} = A^*_{ki}$

1.7

use something like $\int dx |x\rangle\langle x|$

$$|a\rangle = \int dx\, |x\rangle\langle x|a\rangle$$ basis expansion

$$|\alpha\rangle = \sum_i |a^{(i)}\rangle\langle a^{(i)}|\alpha\rangle$$
$$|\beta\rangle = \sum_i |a^{(i)}\rangle\langle a^{(i)}|\beta\rangle$$

$$\langle\beta| = \langle\beta|\left(\sum_i |a^{(i)}\rangle\langle a^{(i)}|\right)$$

$$\rightarrow |\alpha\rangle\langle\beta| = \left(\sum_i |a^{(i)}\rangle\langle a^{(i)}|\alpha\rangle\right)\left(\langle\beta|\sum_j |a^{(j)}\rangle\langle a^{(j)}|\right)$$

$$= \sum_{ij} \langle a^{(i)}|\alpha\rangle\langle\beta|a^{(j)}\rangle\, |a^{(i)}\rangle\langle a^{(j)}|$$

let $\alpha_i = \langle a^{(i)}|\alpha\rangle$ $\beta_j^* = \langle\beta|a^{(j)}\rangle$

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} \alpha_0\beta_0^* & \alpha_0\beta_1^* & \alpha_0\beta_2 & \cdots & \alpha_0\beta_j \\ \alpha_1\beta_1^* & & \ddots & & \\ \vdots & & & & \\ \alpha_i\beta_0^* & & & & \alpha_i\beta_j^* \end{pmatrix}$$

$$|\alpha\rangle = |S_z;+\rangle = \begin{pmatrix}1\\0\end{pmatrix} = |+\rangle_z$$
$$|\beta\rangle = |S_x;+\rangle = \frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt2}\left(|+\rangle_z + |-\rangle_z\right)$$

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ 0 & 0 \end{pmatrix}$$

1.11

Construct $|\hat{S}\cdot\hat{n};+\rangle$ s.t $S\cdot\hat{n}\,|S\cdot\hat{n};+\rangle = \left(\frac{\hbar}{2}\right)|S\cdot\hat{n};+\rangle$

$$S_x = \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right) = \frac{\hbar}{2}\left(\begin{pmatrix}1\\0\end{pmatrix}(0\ 1) + \begin{pmatrix}0\\1\end{pmatrix}(1\ 0)\right) = \frac{\hbar}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$$S_y = \frac{i\hbar}{2}\left(-|+\rangle\langle-| + |-\rangle\langle+|\right) = \frac{i\hbar}{2}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix}$$

$$S_z = \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)$$

$$= \frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\hat{n} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$$

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \left(\frac{\hbar}{2}\right)^2\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} + \begin{pmatrix}0&-i\\i&0\end{pmatrix}\begin{pmatrix}0&-i\\i&0\end{pmatrix} + \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]$$

$$= \frac{\hbar^2}{4}\begin{pmatrix}3&0\\0&3\end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$|\hat{n}\rangle = \sin\theta\cos\varphi\,|+\rangle_x + \sin\theta\sin\varphi\,|+\rangle_y + \cos\theta\,|+_z\rangle$$

$$S = \frac{\hbar}{2}\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}|+\rangle_x + \begin{pmatrix}0&-i\\i&0\end{pmatrix}|+\rangle_y + \begin{pmatrix}1&0\\0&-1\end{pmatrix}|+\rangle_z\right]$$

$$S\cdot\hat{n} = \frac{\hbar}{2}\left[\begin{pmatrix}0&\sin\theta\cos\varphi\\ \sin\theta\cos\varphi&0\end{pmatrix} + \begin{pmatrix}0&-i\sin\theta\sin\varphi\\ i\sin\theta\sin\varphi&0\end{pmatrix} + \begin{pmatrix}\cos\theta&0\\0&-\cos\theta\end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\begin{bmatrix}\cos\theta & e^{-i\varphi}\sin\theta\\ \sin\theta e^{i\varphi} & -\cos\theta\end{bmatrix}$$

$$\begin{pmatrix}c-1 & se^{-i\varphi}\\ se^{i\varphi} & -c-1\end{pmatrix} \sim \begin{pmatrix}c-1 & se^{-i\varphi}\\ 0 & -c-1-\frac{se^{i\varphi}}{c-1}\cdot(se^{-i\varphi})\end{pmatrix}$$

$$-c-1-\frac{s^2}{c-1} = \frac{-(c+1)(c-1)-s^2}{c-1} = \frac{-(c^2-1)-s^2}{c-1}$$

$$(c-1)x + (se^{-i\varphi})y = (\cos\theta-1)x = -(se^{-i\varphi})y$$

let $\quad x = \cos\frac{\theta}{2} \quad y = \sin\frac{\theta}{2}e^{i\varphi}$

$$\cos\theta\cos\frac{\theta}{2} - \cos\frac{\theta}{2} = -\sin\theta\sin\frac{\theta}{2}$$

$$\cos\frac{\theta}{2} = \cos\theta\cos\frac{\theta}{2} - \sin\theta\sin\frac{\theta}{2} = \cos\left(\theta - \frac{\theta}{2}\right)$$

$\therefore$ eigenstate of $S\cdot\hat{n}$: $\cos\left(\frac{\theta}{2}\right)|+\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|-\rangle$

$$= \cos\left(\frac{\alpha}{2}\right)|+\rangle + \sin\left(\frac{\alpha}{2}\right)e^{i\varphi}|-\rangle$$

1.12

$$\hat{H} = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|) \doteq a\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\text{eig}\,\hat{H} = a\begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = a[-(1-\lambda)(1+\lambda) - 1] = a[-(1-\lambda^2) - 1] = a(-2+\lambda^2)$$

$$= 0 \rightarrow a(\lambda^2 - 2) = 0 \rightarrow \lambda = \pm\sqrt{2}$$

$$\lambda = \sqrt{2} \longrightarrow |\lambda_1\rangle = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} \sim \begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \qquad E_{\lambda_1} = \sqrt{2}\,a$$

$$\lambda = -\sqrt{2} \qquad |\lambda_2\rangle = \begin{pmatrix} -\sqrt{2}+1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} \frac{-\sqrt{2}+1}{\sqrt{4+2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} \end{pmatrix} \qquad E_{\lambda_2} = -\sqrt{2}\,a$$

$$(1+\sqrt{2})^2 + 1 = 1 + 2\sqrt{2} + 2 + 1 \rightarrow \sqrt{4+2\sqrt{2}} = C$$