UCLA

Physics 220 – Midterm

Instructor: Prof. Pietro Musumeci

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Name: ______________________________
Student Number: ______________________________

This exam contains 6 pages (including this cover page) and 3 questions. Total of points is 60.
Good luck and Good skill!

Distribution of Marks

Question	Points	Score
1	20	
2	20	
3	20	
Total:	60	

1. Consider a three-dimensional mechanical system in cylindrical coordinates (r, φ, z) with a potential of the form $V(r, k\varphi + z)$ with some length scale $k \in \mathbb{R}$. The Lagrangian is given by:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2\right) - V(r, k\varphi + z) \tag{1}$$

   (a) (8 points) Show that this system has two continuous symmetries and write down the corresponding coordinate transformations.

   (b) (12 points) Use Noether's Theorem to compute the two conserved quantities.

Solutions:

(a) 1. First observe that the Lagrangian is time-independent. We know that:

$$\begin{aligned} &t \mapsto t + \epsilon \\ &L(q^i(t+\epsilon), \dot{q}^i(t+\epsilon)) = L(q^i(t), \dot{q}^i(t)) + \epsilon\frac{d}{dt}L + \mathcal{O}(\epsilon^2) \end{aligned} \tag{2}$$

The Lagrangian changes by a total derivative. Thus, it is a symmetry.

2. The potential only depends on $k\varphi + z$. If we shift both z and φ, such that this stays constant, we do not change the Lagrangian:

$$\begin{aligned} &z \mapsto z + \epsilon \;\; \varphi \mapsto \varphi - \frac{\epsilon}{k} \\ &L(r, \varphi', z') = L(r, \varphi, z) \end{aligned} \tag{3}$$

(b) Noether's Theorem states a symmetry is associated to a conserved quantity:

$$Q = \frac{\partial L}{\partial \dot{q}^i}\delta q^i - \Lambda \tag{4}$$

where Λ is the function, whose total derivative the Lagrangian changes by and δq^i is the variation of the coordinate q^i under the symmetry. Using this, we find:

$$\begin{aligned} &Q_1 = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2\right) + V(r, k\varphi + z) \\ &Q_2 = m\dot{z} - \frac{m}{k}r^2\dot{\varphi} \end{aligned} \tag{5}$$

2. Consider a particle of mass m which is constrained to move without friction on a parabola which is of the form $z = cy^2$ at $t = 0$. $c \in \mathbb{R}$ is some constant inverse length scale. The parabola rotates with angular velocity ω around the z-axis and the mass feels a constant gravitational acceleration g in negative z-direction.
   (a) (4 points) Write down the constraint equations of this system.
   (b) (8 points) Write down the Lagrangian L and the Euler-Lagrange equations. You are free to eliminate the constraints and do not need to write everything using Lagrange-multipliers.
   (c) (8 points) Compute the Hamiltonian H of the system. Is it a conserved quantity? Compare it to the definition of energy, you learned in the Newtonian formulation of mechanics. Is it the same?

Solutions:

(a) I am going to solve this problem in cylindrical coordinates (r, φ, z) and reduce it to a 1D-problem in r. There are multiple other options.

1. The first constraint is $\dot{\varphi} = \omega =$ const., due to the constant angular velocity. Indeed, with initial condition $\varphi = \frac{\pi}{2}$, we have $\varphi(t) = \frac{\pi}{2} + \omega t$
2. The second constraint is found by observing the motion of the parabola. At $t = 0$, it is on the y-axis, but at time t, it will be on the axis given by $(r\cos(\varphi(t)), r\sin(\varphi(t)))^T$. Thus, the actual constrained is just:

$$z = c\left(x(t)^2 + y(t)^2\right) = cr^2 \tag{6}$$

such that $\dot{z} = 2cr\dot{r}$.

(b) The Lagrangian in cylindrical coordinates is:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2\right) - mgz + \lambda_1(\dot{\varphi} - \omega) + \lambda_2(z - cr^2) \tag{7}$$

where λ_i are Lagrange multipliers. By computing the equations of motion of the Lagrange multipliers, we can constrain the motion to:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\omega^2 + 4c^2r^2\dot{r}^2\right) - mgcr^2 \tag{8}$$

and compute the Euler-Lagrange equation for r:

$$m\ddot{r} + 4mc^2r^2\ddot{r} + 8mc^2r\dot{r}^2 = mr\omega^2 - 2mgcr \tag{9}$$

(c) The Hamiltonian is conserved, but not the standard Newtonian energy $E = T + U$:

$$\begin{aligned} p_r &= m\dot{r} + 4mc^2r^2\dot{r} \\ H(r, p_r) &= p_r\dot{r} - L = \frac{p_r^2}{2m\left(1 + 4c^2r^2\right)} - \frac{mr^2\omega^2}{2} + mgcr^2 \end{aligned} \tag{10}$$

3. Consider the Hamiltonian system defined by:

$$H(x, p_x, y, p_y) = p_x^2 + x^2 p_y^2 + x^2 y^2 \tag{11}$$

   (a) (4 points) Write down the Hamilton-Jacobi equation for this system.

   (b) (8 points) Solve the Hamilton-Jacobi equation using a separation *ansatz*.

   (c) (8 points) Use the solution to the Hamilton-Jacobi equation to solve for $x(t), y(t), p_x(t), p_y(t)$.

Solutions:

(a) We assume a generating function of Type 2 $F_2(q^i, P_i)$, such that:

$$\begin{aligned} p_i &= \frac{\partial F_2}{\partial q^i} \\ Q^i &= \frac{\partial F_2}{\partial P_i} \end{aligned} \tag{12}$$

Then the Hamilton-Jacobi Equation is simply:

$$\left(\frac{\partial F_2}{\partial x}\right)^2 + x^2 \left(\left(\frac{\partial F_2}{\partial y}\right)^2 + y^2\right) + \frac{\partial F_2}{\partial t} = 0 \tag{13}$$

(b) Nest, we would like to solve it, using a separation ansatz. For this, first observe that:

$$\begin{aligned} \dot{p_y} &= -2yx^2 \;\; \dot{y} = 2p_y x^2 \\ p_y \dot{p_y} + y\dot{y} &= 0 = \frac{1}{2}\frac{d}{dt}(y^2 + p_y^2) \end{aligned} \tag{14}$$

is a conserved quantity. Thus, a convenient choice for our new canonical momenta is:

$$P_1 = H \;\; P_2 = y^2 + p_y^2 \tag{15}$$

Next, we want to apply our separation ansatz and write:

$$F_2(q, P, t) = W_x(x, P) + W_y(y, P) - P_1 t \tag{16}$$

which simplifies the equation to:

$$\left(\frac{\partial W_x}{\partial x}\right)^2 + x^2 \left(\left(\frac{\partial W_y}{\partial y}\right)^2 + y^2\right) - P_1 = 0 \tag{17}$$

Now we can shuffle everything that doesn't depend on y over to the right-hand side to get:

$$\left(\frac{\partial W_y}{\partial y}\right)^2 + y^2 = P_2^2 \tag{18}$$

Solving for the derivative, we find:

$$W_y = \int dy' \sqrt{P_2^2 - y'^2} + \tilde{G}(P) \tag{19}$$

Observe, that I can choose my sign arbitrarily here, since it will not bother the Hamiltonian and otherwise will just change the sign of the Q's. These are however constant, so it should not what value exactly they take on. Here, I added some arbitrary function of P. This will in general only shift the Q's. We might as well set it to zero. This is a standard integral, which can be found by substituting $y/P_2 = \sin(\theta)$:

$$W_y = \frac{P_2^2}{2}\left(\frac{y}{P_2}\sqrt{1-\frac{y}{P_2}^2} + \sin^{-1}\left(\frac{y}{P_2}\right)\right) \tag{20}$$

Next, we can repeat the procedure for W_x:

$$\begin{aligned} \left(\frac{\partial W_x}{\partial x}\right)^2 &= P_1 - P_2^2 x^2 \\ W_x &= \int dx' \sqrt{P_1 - P_2^2 x'^2} + \tilde{H}(P) \end{aligned} \tag{21}$$

which is the same integral up to a rescaling of the integration variable:

$$W_x = \frac{P_1}{2P_2}\left(\frac{P_2 x}{\sqrt{P_1}}\sqrt{1-\frac{P_2 x}{\sqrt{P_1}}^2} + \sin^{-1}\left(\frac{yP_2}{\sqrt{P_1}}\right)\right) \tag{22}$$

Putting everything together, we find our overall solution.

(c) For the final part of this problem, we need to find our now canonical coordinates and invert them. For this, it will prove useful to keep the integral form of the W_i's. Let us first compute Q^1:

$$Q^1 = \frac{\partial F_2}{\partial P_1} = \frac{1}{P_2}\sin^{-1}\left(\frac{P_2 x}{\sqrt{P_1}}\right) - t \tag{23}$$

which we can solve as:

$$x(t) = \frac{\sqrt{P_1}}{P_2}\sin(2P_2(Q^1 - t)) \tag{24}$$

Next, we can do the same part for the other canonical coordinate:

$$Q^2 = \frac{\partial F_2}{\partial P_2} = \frac{\partial W_x}{\partial P_2} + \frac{\partial W_y}{\partial P_2} - t = P_2 \sin^{-1}\left(\frac{y}{P_2}\right) + \frac{P_2^2}{2P_1}\left(\frac{P_2 x}{\sqrt{P_1}}\sqrt{1-\left(\frac{P_2 x}{\sqrt{P_1}}\right)^2} - \sin^{-1}\left(\frac{P_2 x}{\sqrt{P_1}}\right)\right) \tag{25}$$

which you can solve for $y(t)$ and then use the solution for $x(t)$ to find:

$$y(t) = P_2 \sin\left(\frac{1}{P_2}\left(Q^2 - \frac{P_2^2}{P_1}\left(\frac{1}{2}\sin(2P_2(Q^1 - t))\cos(2P_2(Q^1 - t)) - P_2(Q^1 - t)\right)\right)\right) \tag{26}$$

Finally, you can solve for $p_y(t)$ using P_2 and $p_x(t)$ using P_1.

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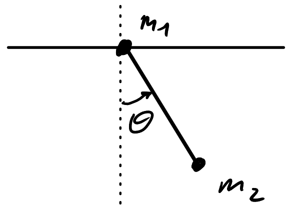
Problem 4

a) We use the x coordinate of m_1 and the angle θ (see drawing)



Then:

$$x_2 = x + l\sin\theta$$

$$y_2 = -l\cos\theta$$

So:

$$\dot{x}_2 = \dot{x} + l\cos\theta\,\dot{\theta}$$

$$\dot{y}_2 = l\sin\theta\,\dot{\theta}$$

So:

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) - mg\,y_2 =$$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + l^2(\cos^2\theta + \sin^2\theta)\dot{\theta}^2 + 2l\dot{x}\cos\theta\dot{\theta}\right) +$$

$$+ mgl\cos\theta$$

$$= \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2 l^2\dot{\theta}^2 + m_2 l\cos\theta\,\dot{\theta}\dot{x} + mgl\cos\theta$$

b) x is cyclic, hence:

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2 l\cos\theta\dot{\theta} =$$

$$= m_1\dot{x} + m_2(\dot{x} + l\cos\theta\dot{\theta})$$

$$= m_1\dot{x} + m_2\dot{x}_2 \qquad \text{is conserved}$$