Matt Kinsinger, MAT 372 Final

[1] For any set $E \subseteq \mathbb{R}^2$ define the projection operator $P : E \to \mathbb{R}$ by $P(x, y) = x$.

(a) Suppose that E is bounded. Then for all $u \in E$, there exists $M \geq 0$ such that $\|u\| \leq M$. Let $u = (u_1, u_2) \in E$. Then,

$$|P(u)| = |P(u_1, u_2)| = |u_1| \leq \|u\| \leq M.$$

Thus, $P(E)$ is bounded. $\square$

(b) Let $E = \left\{(x, \frac{1}{x}) : x > 0\right\} \subseteq \mathbb{R}^2$, which is closed because any $u \in \mathbb{R}^2 \cap E^c$ can be enclosed in an open ball $B(u) \subseteq E^c$, i.e. E^c is open. But, $P(E) = \{x : x > 0\}$ is open. $\square$

(c) Suppose the E is compact. Let $\bigcup\limits_{\alpha=1}^{\infty} G_\alpha$ be a union of open sets such that $P(E) \subseteq \bigcup\limits_{\alpha=1}^{\infty} G_\alpha$. It follows that

$$E \subseteq P(E) \times \left(\bigcup_{k \in \mathbb{N}} (-k, k)\right) \subseteq \left(\bigcup_{\alpha=1}^{\infty} G_\alpha\right) \times \left(\bigcup_{k \in \mathbb{N}} (-k, k)\right)$$

Since E is compact, there exists numbers N,M such that

$$E \subseteq \left(\bigcup_{\alpha=1}^{N} G_\alpha\right) \times \left(\bigcup_{k=1}^{M} (-k, k)\right)$$

Thus, $P(E) \subseteq \bigcup\limits_{\alpha=1}^{N} G_\alpha$. So $P(E)$ is compact. $\square$

More succinctly (but less fun) you could use continuity of P and preservation of compact sets by continuous functions. $\square$

[2] Let $B : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^q$ be a bounded bilinear function. Set $g(x) = B(x, x)$. For $x, u \in \mathbb{R}^p$, prove:

$$\text{(i) } Dg(x)(u) = B(x, u) + B(u, x) = Dg(u)(x)$$
$$\text{(ii) } g(x + u) = g(x) + g(u) + Dg(x)(u).$$

Since B is a bounded bilinear function, there exists $M > 0$ such that $\|B(x, y)\| \leq M\|x\|\|y\|$ for all $x, y \in \mathbb{R}^p$.

Let $x, u \in \mathbb{R}^p$.

(i) To check that $F_x(u) = B(x,u) + B(u,x)$ is linear in u, let $u = cv + z$ for scaler c and $v, z \in \mathbb{R}^p$.

$$\begin{aligned}
F_x(cv+z) &= B(x, cv+z) + B(cv+z, x) \\
&= cB(x,v) + B(x,z) + cB(v,x) + B(z,x) \\
&= c\,[B(x,v) + B(v,x)] + B(x,z) + B(z,x) \\
&= cF_x(v) + F_x(z).
\end{aligned}$$

Thus F_x is linear in u.

Let $\epsilon > 0$. Choose $t \in \mathbb{R}$ such that $\|tu\| < \frac{\epsilon}{M}$. Note that

$$\begin{aligned}
g(x+tu) &= B(x+tu, x+tu) \\
&= B(x, x+tu) + tB(u, x+tu) \\
&= B(x,x) + tB(x,u) + tB(u,x) + t^2 B(u,u) \qquad (1)
\end{aligned}$$

It follows that

$$\begin{aligned}
\|g(x+tu) - g(x) - F_x(tu)\| &= \|g(x+tu) - g(x) - [B(x,tu) + B(tu,x)]\,\| \\
&= \|g(x+tu) - g(x) - [tB(x,u) + tB(u,x)]\,\| \\
&= \|B(x,x) + t^2 B(u,u) - g(x)\| \\
&= \|t^2 B(u,u)\| \\
&\leq t^2 M \|u\|^2 \\
&= M\|tu\|\|tu\| \\
&< \epsilon\|tu\|.
\end{aligned}$$

Thus, $Dg(x)(u) = F_x(u) = B(x,u) + B(u,x)$. Using symmetry, this same argument shows that $Dg(u)(x) = B(x,u) + B(u,x)$ as well. □

(ii)

$$\begin{aligned}
g(x+u) &= B(x+u, x+u) \\
&= B(x, x+u) + B(u, x+u) \\
&= B(x,x) + B(x,u) + B(u,x) + B(u,u) \\
&= B(x,x) + B(u,u) + B(x,u) + B(u,x) \\
&= g(x) + g(u) + Dg(x)(u).
\end{aligned}$$

□

[4] Suppose that $w \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution to the given partial differential equation. Since $\Omega \subseteq \mathbb{R}^3$ is bounded, $\overline{\Omega}$ is both closed and bounded in $\mathbb{R}^3$, hence $\overline{\Omega}$ is compact. w is continuous on a compact set, thus w attains a maximum and a minimun on $\overline{\Omega}$.

- Case 1: Suppose that both the max and min of w on $\overline{\Omega}$ occur on $\partial\Omega$. Then

$$\max w = \min w = 0.$$

So $w = 0$ on $\overline{\Omega}$.

- Case 2: Suppose that max w occurs at $x_M \in \Omega$, and min w occurs anywhere in $\overline{\Omega}$. Since Ω is open, x_M is an interior point of Ω. Since $w \in C^2(\Omega)$, both $Dw(x_M)$ and $D^2w(x_M)$ exist. Moreover, $Dw(x_M)y = 0$ and $D^2w(x_M)y^2 \leq 0$ for all $y \in \mathbb{R}^3$. This implies that

$$c \cdot (\nabla w)_{x_M} = c \cdot \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right)_{x_M} = c \cdot (0,0,0) = 0.$$

hence

$$\begin{aligned} -\nabla^2 w + c \cdot \nabla w + w = 0 \\ w = \nabla^2 w \end{aligned}$$

But,

$$\begin{aligned} 0 \geq D^2 w(x_M)e_1^2 = D_{11}w(x_M) \\ 0 \geq D^2 w(x_M)e_2^2 = D_{22}w(x_M) \\ 0 \geq D^2 w(x_M)e_3^2 = D_{33}w(x_M) \end{aligned}$$

It follows that

$$\max_{\Omega}\ w = w(x_M) = \nabla^2 w(x_M) = D_{11}w(x_M) + D_{22}w(x_M) + D_{33}w(x_M) \leq 0$$

Moreover, because $\Omega \subseteq \overline{\Omega}$,

$$\min_{\overline{\Omega}}\ w \leq \min_{\Omega}\ w \leq \max_{\Omega}\ w \leq \max_{\overline{\Omega}}\ w \leq 0. \qquad (1)$$

- Case 3: Suppose that min w occurs at $x_m \in \Omega$, and max w occurs anywhere in $\overline{\Omega}$. We can follow that same arguement in Case 1, but with

$$\begin{aligned} 0 \leq D^2 w(x_m)e_1^2 = D_{11}w(x_m) \\ 0 \leq D^2 w(x_m)e_2^2 = D_{22}w(x_m) \\ 0 \leq D^2 w(x_m)e_3^2 = D_{33}w(x_m) \end{aligned}$$

Thus,

$$\min_{\Omega}\ w = w(x_m) = \nabla^2 w(x_m) = D_{11}w(x_m) + D_{22}w(x_m) + D_{33}w(x_m) \geq 0$$

Therefore, because $\Omega \subseteq \overline{\Omega}$,

$$\max_{\overline{\Omega}}\ w \geq \max_{\Omega}\ w \geq \min_{\Omega}\ w \geq \min_{\overline{\Omega}} \geq 0. \qquad (2)$$

(1) and (2) together imply that max $w = 0$ and min $w = 0$ on $\overline{\Omega}$. Thus, $w = 0$ on $\overline{\Omega}$. Since w was an arbitrary solution, $w = 0$ is the only solution. $\square$