Matt Kinsinger, MAT 372 Final

[4] Suppose that $w \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution to the given partial differential equation. Since $\Omega \subseteq \mathbb{R}^3$ is bounded, $\overline{\Omega}$ is both closed and bounded in $\mathbb{R}^3$, hence $\overline{\Omega}$ is compact. w is continuous on a compact set, thus w attains a maximum and a minimun on $\overline{\Omega}$.

- Case 1: Suppose that both the max and min of w on $\overline{\Omega}$ occur on $\partial\Omega$. Then

  $$\max w = \min w = 0.$$

  So $w = 0$ on $\overline{\Omega}$.

- Case 2: Suppose that max w occurs at $x_M \in \Omega$, and min w occurs anywhere in $\overline{\Omega}$. Since Ω is open, x_M is an interior point of Ω. Since $w \in C^2(\Omega)$, both $Dw(x_M)$ and $D^2w(x_M)$ exist. Moreover, $Dw(x_M)y = 0$ and $D^2w(x_M)y^2 \leq 0$ for all $y \in \mathbb{R}^3$. This implies that

  $$c \cdot (\nabla w)_{x_M} = c \cdot \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right)_{x_M} = c \cdot (0,0,0) = 0.$$

  hence

  $$\begin{aligned} -\nabla^2 w + c \cdot \nabla w + w &= 0 \\ w &= \nabla^2 w \end{aligned}$$

  But,

  $$\begin{aligned} 0 &\geq D^2w(x_M)e_1^2 = D_{11}w(x_M) \\ 0 &\geq D^2w(x_M)e_2^2 = D_{22}w(x_M) \\ 0 &\geq D^2w(x_M)e_3^2 = D_{33}w(x_M) \end{aligned}$$

  It follows that

  $$\max_{\Omega}\ w = w(x_M) = \nabla^2 w(x_M) = D_{11}w(x_M) + D_{22}w(x_M) + D_{33}w(x_M) \leq 0$$

  Moreover, because $\Omega \subseteq \overline{\Omega}$,

  $$\min_{\overline{\Omega}}\ w \leq \min_{\Omega}\ w \leq \max_{\Omega}\ w \leq \max_{\overline{\Omega}}\ w \leq 0. \tag{1}$$

- Case 3: Suppose that min w occurs at $x_m \in \Omega$, and max w occurs anywhere in $\overline{\Omega}$. We can follow that same arguement in Case 1, but with

  $$\begin{aligned} 0 &\leq D^2w(x_m)e_1^2 = D_{11}w(x_m) \\ 0 &\leq D^2w(x_m)e_2^2 = D_{22}w(x_m) \\ 0 &\leq D^2w(x_m)e_3^2 = D_{33}w(x_m) \end{aligned}$$

Thus,

$$\min_{\Omega} \; w = w(x_m) = \nabla^2 w(x_m) = D_{11}w(x_m) + D_{22}w(x_m) + D_{33}w(x_m) \geq 0$$

Therefore, because $\Omega \subseteq \overline{\Omega}$,

$$\max_{\overline{\Omega}} \; w \geq \max_{\Omega} \; w \geq \min_{\Omega} \; w \geq \min_{\overline{\Omega}} \geq 0. \tag{2}$$

(1) and (2) together imply that max $w = 0$ and min $w = 0$ on $\overline{\Omega}$. Thus, $w = 0$ on $\overline{\Omega}$. Since w was an arbitrary solution, $w = 0$ is the only solution. $\square$