

Matt Kinsinger, MAT 372 Final

[1] For any set $E \subseteq \mathbb{R}^2$ define the projection operator $P : E \rightarrow \mathbb{R}$ by $P(x, y) = x$.

(a) Suppose that E is bounded. Then for all $u \in E$, there exists $M \geq 0$ such that $\|u\| \leq M$. Let $u = (u_1, u_2) \in E$. Then,

$$|P(u)| = |P(u_1, u_2)| = |u_1| \leq \|u\| \leq M.$$

Thus, $P(E)$ is bounded. □

(b) Let $E = \{(x, \frac{1}{x}) : x > 0\} \subseteq \mathbb{R}^2$, which is closed because any $u \in \mathbb{R}^2 \cap E^c$ can be enclosed in an open ball $B(u) \subseteq E^c$, i.e. E^c is open. But, $P(E) = \{x : x > 0\}$ is open. □

(c) Suppose the E is compact. Let $\bigcup_{\alpha=1}^{\infty} G_{\alpha}$ be a union of open sets such that $P(E) \subseteq \bigcup_{\alpha=1}^{\infty} G_{\alpha}$. It follows that

$$E \subseteq P(E) \times \left(\bigcup_{k \in \mathbb{N}} (-k, k) \right) \subseteq \left(\bigcup_{\alpha=1}^{\infty} G_{\alpha} \right) \times \left(\bigcup_{k \in \mathbb{N}} (-k, k) \right)$$

Since E is compact, there exists numbers N, M such that

$$E \subseteq \left(\bigcup_{\alpha=1}^N G_{\alpha} \right) \times \left(\bigcup_{k=1}^M (-k, k) \right)$$

Thus, $P(E) \subseteq \bigcup_{\alpha=1}^N G_{\alpha}$. So $P(E)$ is compact. □

More succinctly (but less fun) you could use continuity of P and preservation of compact sets by continuous functions. □

[2] Let $B : \mathbb{R}^p \times \mathbb{R}^p \rightarrow \mathbb{R}^q$ be a bounded bilinear function. Set $g(x) = B(x, x)$. For $x, u \in \mathbb{R}^p$, prove:

- (i) $Dg(x)(u) = B(x, u) + B(u, x) = Dg(u)(x)$
- (ii) $g(x + u) = g(x) + g(u) + Dg(x)(u)$.

Since B is a bounded bilinear function, there exists $M > 0$ such that $\|B(x, y)\| \leq M\|x\|\|y\|$ for all $x, y \in \mathbb{R}^p$.

Let $x, u \in \mathbb{R}^p$.

(i) To check that $F_x(u) = B(x, u) + B(u, x)$ is linear in u , let $u = cv + z$ for scalar c and $v, z \in \mathbb{R}^p$.

$$\begin{aligned}
F_x(cv + z) &= B(x, cv + z) + B(cv + z, x) \\
&= cB(x, v) + B(x, z) + cB(v, x) + B(z, x) \\
&= c[B(x, v) + B(v, x)] + B(x, z) + B(z, x) \\
&= cF_x(v) + F_x(z).
\end{aligned}$$

Thus F_x is linear in u .

Let $\epsilon > 0$. Choose $t \in \mathbb{R}$ such that $\|tu\| < \frac{\epsilon}{M}$. Note that

$$\begin{aligned}
g(x + tu) &= B(x + tu, x + tu) \\
&= B(x, x + tu) + tB(u, x + tu) \\
&= B(x, x) + tB(x, u) + tB(u, x) + t^2B(u, u)
\end{aligned} \tag{1}$$

It follows that

$$\begin{aligned}
\|g(x + tu) - g(x) - F_x(tu)\| &= \|g(x + tu) - g(x) - [B(x, tu) + B(tu, x)]\| \\
&= \|g(x + tu) - g(x) - [tB(x, u) + tB(u, x)]\| \\
&= \|B(x, x) + t^2B(u, u) - g(x)\| \\
&= \|t^2B(u, u)\| \\
&\leq t^2M\|u\|^2 \\
&= M\|tu\|\|tu\| \\
&< \epsilon\|tu\|.
\end{aligned}$$

Thus, $Dg(x)(u) = F_x(u) = B(x, u) + B(u, x)$. Using symmetry, this same argument shows that $Dg(u)(x) = B(x, u) + B(u, x)$ as well. \square

(ii)

$$\begin{aligned}
g(x + u) &= B(x + u, x + u) \\
&= B(x, x + u) + B(u, x + u) \\
&= B(x, x) + B(x, u) + B(u, x) + B(u, u) \\
&= B(x, x) + B(u, u) + B(x, u) + B(u, x) \\
&= g(x) + g(u) + Dg(x)(u).
\end{aligned}$$

\square

[4] Suppose that $w \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution to the given partial differential equation. Since $\Omega \subseteq \mathbb{R}^3$ is bounded, $\overline{\Omega}$ is both closed and bounded in \mathbb{R}^3 , hence $\overline{\Omega}$ is compact. w is continuous on a compact set, thus w attains a maximum and a minimum on $\overline{\Omega}$.

- Case 1: Suppose that both the max and min of w on $\overline{\Omega}$ occur on $\partial\Omega$. Then

$$\max w = \min w = 0.$$

So $w = 0$ on $\overline{\Omega}$.

- Case 2: Suppose that $\max w$ occurs at $x_M \in \Omega$, and $\min w$ occurs anywhere in $\overline{\Omega}$. Since Ω is open, x_M is an interior point of Ω . Since $w \in C^2(\Omega)$, both $Dw(x_M)$ and $D^2w(x_M)$ exist. Moreover, $Dw(x_M)y = 0$ and $D^2w(x_M)y^2 \leq 0$ for all $y \in \mathbb{R}^3$. This implies that

$$c \cdot (\nabla w)_{x_M} = c \cdot \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z} \right)_{x_M} = c \cdot (0, 0, 0) = 0.$$

hence

$$\begin{aligned} -\nabla^2 w + c \cdot \nabla w + w &= 0 \\ w &= \nabla^2 w \end{aligned}$$

But,

$$\begin{aligned} 0 &\geq D^2w(x_M)e_1^2 = D_{11}w(x_M) \\ 0 &\geq D^2w(x_M)e_2^2 = D_{22}w(x_M) \\ 0 &\geq D^2w(x_M)e_3^2 = D_{33}w(x_M) \end{aligned}$$

It follows that

$$\max_{\Omega} w = w(x_M) = \nabla^2 w(x_M) = D_{11}w(x_M) + D_{22}w(x_M) + D_{33}w(x_M) \leq 0$$

Moreover, because $\Omega \subseteq \overline{\Omega}$,

$$\min_{\overline{\Omega}} w \leq \min_{\Omega} w \leq \max_{\Omega} w \leq \max_{\overline{\Omega}} w \leq 0. \quad (1)$$

- Case 3: Suppose that $\min w$ occurs at $x_m \in \Omega$, and $\max w$ occurs anywhere in $\overline{\Omega}$. We can follow that same argument in Case 1, but with

$$\begin{aligned} 0 &\leq D^2w(x_m)e_1^2 = D_{11}w(x_m) \\ 0 &\leq D^2w(x_m)e_2^2 = D_{22}w(x_m) \\ 0 &\leq D^2w(x_m)e_3^2 = D_{33}w(x_m) \end{aligned}$$

Thus,

$$\min_{\Omega} w = w(x_m) = \nabla^2 w(x_m) = D_{11}w(x_m) + D_{22}w(x_m) + D_{33}w(x_m) \geq 0$$

Therefore, because $\Omega \subseteq \overline{\Omega}$,

$$\max_{\overline{\Omega}} w \geq \max_{\Omega} w \geq \min_{\Omega} w \geq \min_{\overline{\Omega}} w \geq 0. \quad (2)$$

(1) and (2) together imply that $\max w = 0$ and $\min w = 0$ on $\overline{\Omega}$. Thus, $w = 0$ on $\overline{\Omega}$. Since w was an arbitrary solution, $w = 0$ is the only solution. \square

[5] Let $\Omega = [0, 1] \times [0, 1]$. Suppose $f : \Omega \rightarrow \mathbb{R}$ is given by $f(x, y) = xy$.

(a) Use the definition of the Darboux integral to show that f is Darboux integrable, and find the integral.

Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $\frac{2}{N} < \epsilon$. Let P_ϵ be a partition such that

$$P_1 = \{x_0, x_1, \dots, x_N\} = \left\{0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N}{N} = 1\right\}$$

$$P_2 = \{y_0, y_1, \dots, y_N\} = \left\{0, \frac{1}{N}, \frac{2}{N}, \dots, \frac{N}{N} = 1\right\}.$$

Note that on each square $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $1 \leq i \leq N$, $1 \leq j \leq N$:

$$\begin{aligned} \sup_{R_{ij}} f(x, y) &= x_i y_j \\ \inf_{R_{ij}} f(x, y) &= x_{i-1} y_{j-1}. \end{aligned}$$

It follows that

$$\begin{aligned} U(f, P_\epsilon) - L(f, P_\epsilon) &= \sum_{i,j=1}^N \left[\sup_{R_{ij}} f(x, y) - \inf_{R_{ij}} f(x, y) \right] (x_i - x_{i-1}) (y_j - y_{j-1}) \\ &= \sum_{i,j=1}^N [x_i y_j - x_{i-1} y_{j-1}] (x_i - x_{i-1}) (y_j - y_{j-1}) \\ &= \sum_{i,j=1}^N [x_i y_j - (x_i y_{j-1}) + (x_i y_{j-1}) - x_{i-1} y_{j-1}] (x_i - x_{i-1}) (y_j - y_{j-1}) \\ &= \sum_{i,j=1}^N \left[x_i (y_j - y_{j-1}) + y_{j-1} (x_i - x_{i-1}) \right] (x_i - x_{i-1}) (y_j - y_{j-1}) \\ &\leq \sum_{i,j=1}^N \left[(1) (y_j - y_{j-1}) + (1) (x_i - x_{i-1}) \right] (x_i - x_{i-1}) (y_j - y_{j-1}) \\ &= \sum_{i,j=1}^N \left[\frac{1}{N} + \frac{1}{N} \right] \frac{1}{N} \frac{1}{N} \\ &= \frac{2}{N^3} \sum_{i=1}^N \sum_{j=1}^N 1 \\ &= \frac{2}{N^3} N^2 \\ &= \frac{2}{N} \\ &< \epsilon. \end{aligned}$$

Thus f is darbox integrable on Ω . □

Let P_n , $n \in \mathbb{N}$, be a partition of Ω consisting of squares of side length $\frac{1}{n}$. Then

$$\begin{aligned}
 U(f, P_n) &= \sum_{i,j=1}^n \left[\sup_{R_{ij}} f(x, y) \right] (x_i - x_{i-1}) (y_j - y_{j-1}) \\
 &= \sum_{i,j=1}^n \left[\frac{i}{n} \frac{j}{n} \right] \frac{1}{n} \frac{1}{n} \\
 &= \frac{1}{n^4} \left[\sum_{i=1}^n i \left[\sum_{j=1}^n j \right] \right] \\
 &= \\
 &= \frac{1}{n^4} \left[\frac{n(n+1)}{2} \right] \sum_{i=1}^n i \\
 &= \left[\frac{(n+1)}{2n^3} \right] \left[\frac{n(n+1)}{2} \right] \\
 &= \frac{(n+1)^2}{4n^2} \\
 &= \frac{n^2 + 2n + 1}{4n^2} \\
 &> \frac{1}{4}
 \end{aligned}$$

for all n.

Thus, $\inf_P U(f, P) \geq \frac{1}{4}$.

Similarly,

$$\begin{aligned}
L(f, P_n) &= \sum_{i,j=1}^n \left[\inf_{R_{ij}} f(x, y) \right] (x_i - x_{i-1}) (y_j - y_{j-1}) \\
&= \sum_{i,j=1}^n \left[\frac{i-1}{n} \frac{j-1}{n} \right] \frac{1}{n} \frac{1}{n} \\
&= \frac{1}{n^4} \left[\sum_{i=1}^n (i-1) \left[\sum_{j=1}^n (j-1) \right] \right] \\
&= \\
&= \frac{1}{n^4} \left[\frac{n(n+1)}{2} - n \right] \sum_{i=1}^n (i-1) \\
&= \frac{1}{n^3} \left[\frac{(n+1)}{2} - 1 \right] \left[\frac{n(n+1)}{2} - n \right] \\
&= \frac{1}{n^2} \frac{n-1}{2} \frac{n-1}{2} \\
&= \frac{n^2 - 2n + 1}{4n^2} \\
&< \frac{1}{4} \quad \text{for all } n.
\end{aligned}$$

Thus, $\sup_P L(f, P) \leq \frac{1}{4}$.

But, since f is darbox integrable on Ω , it must be that

$$\sup_P L(f, P) = \inf_P U(f, P) = \frac{1}{4} = \int_{\Omega} f.$$

□

(b) Repeat (a) using theorems.

Since the projection functions $p_x(x, y) = x$ and $p_y(x, y) = y$ are both continuous, by the algebra of continuous functions, their product $f(x, y) = xy$ is continuous. Thus, $f(x, y)$ is darbox integrable on Ω . Let $x' \in [0, 1]$. Clearly $\int_0^1 f(x', y) dy = \int_0^1 x' y dy$ exists. Hence, by the Jones iterated integrals theorem,

$$g(x) = \int_0^1 f(x, y) dy$$

is integrable on $[0, 1]$, and $\int_0^1 g(x) \, dx = \iint_{\Omega} f(x, y) \, dA$. Therefore,

$$\begin{aligned} \iint_{\Omega} f(x, y) \, dA &= \int_0^1 g(x) \, dx \\ &= \int_0^1 \left(\int_0^1 f(x, y) \, dy \right) dx \\ &= \int_0^1 \left(\int_0^1 xy \, dy \right) dx \\ &= \int_0^1 \frac{x}{2} \, dx \\ &= \frac{1}{4}. \end{aligned}$$

□