Matt Kinsinger, MAT 372 Final

[1] For any set $E \subseteq \mathbb{R}^2$ define the projection operator $P : E \to \mathbb{R}$ by $P(x, y) = x$.

(a) Suppose that E is bounded. Then for all $u \in E$, there exists $M \geq 0$ such that $\|u\| \leq M$. Let $u = (u_1, u_2) \in E$. Then,

$$|P(u)| = |P(u_1, u_2)| = |u_1| \leq \|u\| \leq M.$$

Thus, $P(E)$ is bounded. $\square$

(b) Let $E = \left\{(x, \frac{1}{x}) : x > 0\right\} \subseteq \mathbb{R}^2$, which is closed because any $u \in \mathbb{R}^2 \cap E^c$ can be enclosed in an open ball $B(u) \subseteq E^c$, i.e. E^c is open. But, $P(E) = \{x : x > 0\}$ is open. $\square$

(c) Suppose the E is compact. Let $\bigcup_{\alpha=1}^{\infty} G_\alpha$ be a union of open sets such that $P(E) \subseteq \bigcup_{\alpha=1}^{\infty} G_\alpha$. It follows that

$$E \subseteq P(E) \times \left(\bigcup_{k \in \mathbb{N}} (-k, k)\right) \subseteq \left(\bigcup_{\alpha=1}^{\infty} G_\alpha\right) \times \left(\bigcup_{k \in \mathbb{N}} (-k, k)\right)$$

Since E is compact, there exists numbers N,M such that

$$E \subseteq \left(\bigcup_{\alpha=1}^{N} G_\alpha\right) \times \left(\bigcup_{k=1}^{M} (-k, k)\right)$$

Thus, $P(E) \subseteq \bigcup_{\alpha=1}^{N} G_\alpha$. So $P(E)$ is compact. $\square$

More succinctly (but less fun) you could use continuity of P and preservation of compact sets by continuous functions. $\square$

[2] Let $B : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^q$ be a bounded bilinear function. Set $g(x) = B(x, x)$. For $x, u \in \mathbb{R}^p$, prove:

$$\text{(i) } Dg(x)(u) = B(x, u) + B(u, x) = Dg(u)(x)$$
$$\text{(ii) } g(x + u) = g(x) + g(u) + Dg(x)(u).$$

[proof]

Since B is a bounded bilinear function, there exists $M > 0$ such that $\|B(x, y)\| \leq M\|x\|\|y\|$ for all $x, y \in \mathbb{R}^p$.

Let $x, u \in \mathbb{R}^p$.

(i) To check that $F_x(u) = B(x, u) + B(u, x)$ is linear in u, let $u = cv + z$ for scaler c and $v, z \in \mathbb{R}^p$.

$$\begin{aligned}
F_x(cv + z) &= B(x, cv + z) + B(cv + z, x) \\
&= cB(x, v) + B(x, z) + cB(v, x) + B(z, x) \\
&= c\left[B(x, v) + B(v, x)\right] + B(x, z) + B(z, x) \\
&= cF_x(v) + F_x(z).
\end{aligned}$$

Thus F_x is linear in u.

Let $\epsilon > 0$. Choose $t \in \mathbb{R}$ such that $\|tu\| < \frac{\epsilon}{M}$. Note that

$$\begin{aligned}
g(x + tu) &= B(x + tu, x + tu) \\
&= B(x, x + tu) + tB(u, x + tu) \\
&= B(x, x) + tB(x, u) + tB(u, x) + t^2 B(u, u) \qquad (1)
\end{aligned}$$

It follows that

$$\begin{aligned}
\|g(x + tu) - g(x) - F_x(tu)\| &= \|g(x + tu) - g(x) - \left[B(x, tu) + B(tu, x)\right]\| \\
&= \|g(x + tu) - g(x) - \left[tB(x, u) + tB(u, x)\right]\| \\
&= \|B(x, x) + t^2 B(u, u) - g(x)\| \\
&= \|t^2 B(u, u)\| \\
&\leq t^2 M\|u\|^2 \\
&= M\|tu\|\|tu\| \\
&< \epsilon\|tu\|.
\end{aligned}$$

Thus, $Dg(x)(u) = F_x(u) = B(x, u) + B(u, x)$. Using symmetry, this same argument shows that $Dg(u)(x) = B(x, u) + B(u, x)$ as well. $\square$

(ii)

$$\begin{aligned}
g(x + u) &= B(x + u, x + u) \\
&= B(x, x + u) + B(u, x + u) \\
&= B(x, x) + B(x, u) + B(u, x) + B(u, u) \\
&= B(x, x) + B(u, u) + B(x, u) + B(u, x) \\
&= g(x) + g(u) + Dg(x)(u).
\end{aligned}$$

$\square$

[3] Find sufficient conditions on f and g so that the equations

$$f(xy) + g(xz) = 0, \qquad g(xy) + f(yz) = 0$$

have a solution near $x = y = z = 1$. Assume that $f(1) = g(1) = 0$.

[proof]

Suppose that for any $0 < \epsilon < 1$ we have $\Omega = (1-\epsilon, 1+\epsilon)$ and $f, g \in C^1(\Omega)$ with $f'(1) \neq 0$ and $g'(1) \neq 0$.

It follows that the mappings $x \mapsto f'(1)x$ and $x \mapsto g'(1)x$ are both bijections from $\mathbb{R}$ to $\mathbb{R}$. By the inversion theorem there exists and open nbhd $U_f \subseteq \Omega$ and $U_g \subseteq \Omega$ such that $V_f = f(U_f)$ and $V_g = g(U_g)$ are both open nbhds of $f(1) = g(1) = 0$, and both $f : U_f \to V_f$ and $g : U_g \to V_g$ are bijections. Let $U = U_f \cap U_g$ and $V = V_f \cap V_g$. Then further restrict V (and the corresponding U) so that V is an open nbhd with center 0. Hence $z \in V \iff -z \in V$.

Choose $r \in V$. By the surjectivity of f there exists $w \in U$ such that $f(w) = r$. By the bijectivity of g, $g(w) = q$ for some $q \in V$. Moreover, both $-r \in V$ and $-q \in V$. It follows from the bijectivity of f and g that there exists $s, t \in U$ such that $f(s) = -q$ and $g(t) = -r$. Note that by how we restricted Ω, each of w, s, and t are positive numbers, and both

$$f(w) + g(t) = 0$$

$$g(w) + f(s) = 0.$$

Set

$$xy = w, \qquad xz = t, \qquad \text{and} \quad yz = s.$$

Since w, s, t are positive, x, y, z are also positive. We just need to solve these three equations to find values for x, y, z.

$$xy = w \to y = \frac{w}{x}. \qquad xz = t \to z = \frac{t}{x}. \qquad s = yz = \frac{w}{x}\frac{t}{x} = \frac{wt}{x^2} \to x = \sqrt{\frac{wt}{s}}.$$

Thus,

$$y = \frac{w}{\sqrt{\frac{wt}{s}}} = \sqrt{\frac{ws}{t}} \qquad \text{and} \qquad z = \frac{t}{\sqrt{\frac{wt}{s}}} = \sqrt{\frac{ts}{w}}.$$

Since $w, s, t \in U \subseteq \Omega$ and $\Omega = (1-\epsilon, 1+\epsilon)$ for an arbitrary $\epsilon > 0$, we can make each of x, y, z as near to 1 as we like. It follows that

$$f(xy) + g(xz) = 0$$

and

$$g(xy) + f(yz) = 0$$

has a solution near $x = y = z = 1$ $\square$

[4] Let **c** be a constant vector in $\mathbb{R}^3$ and $\Omega \subseteq \mathbb{R}^3$ be a bounded open set. Consider the partial differential equation, where $w : \Omega \to \mathbb{R}$ is the unknown,

$$-\nabla^2 w + c \cdot \nabla w + w = 0,$$

$$w|_{\partial\Omega} = 0.$$

Suppose a solution exists such that $w \in C^2(\Omega) \cap C(\overline{\Omega})$. Prove, with details, $w = 0$ is the only solution.

[proof]

Suppose that $w \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution to the given partial differential equation. Since $\Omega \subseteq \mathbb{R}^3$ is bounded, $\overline{\Omega}$ is both closed and bounded in $\mathbb{R}^3$, hence $\overline{\Omega}$ is compact. w is continuous on a compact set, thus w attains a maximum and a minimun on $\overline{\Omega}$.

- Case 1: Suppose that both the max and min of w on $\overline{\Omega}$ occur on $\partial\Omega$. Then

  $$\max w = \min w = 0.$$

  So $w = 0$ on $\overline{\Omega}$.

- Case 2: Suppose that max w occurs at $x_M \in \Omega$. Since Ω is open, x_M is an interior point of Ω. Since $w \in C^2(\Omega)$, both $Dw(x_M)$ and $D^2w(x_M)$ exist. Moreover, $Dw(x_M)y = 0$ and $D^2w(x_M)y^2 \leq 0$ for all $y \in \mathbb{R}^3$. This implies that

  $$c \cdot (\nabla w)_{x_M} = c \cdot \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right)_{x_M} = c \cdot (0,0,0) = 0.$$

  hence

  $$\begin{aligned} -\nabla^2 w + c \cdot \nabla w + w &= 0 \\ w &= \nabla^2 w \end{aligned}$$

  But,

  $$\begin{aligned} 0 &\geq D^2w(x_M)e_1^2 = D_{11}w(x_M) \\ 0 &\geq D^2w(x_M)e_2^2 = D_{22}w(x_M) \\ 0 &\geq D^2w(x_M)e_3^2 = D_{33}w(x_M) \end{aligned}$$

  It follows that

  $$\max_{\Omega}\, w = w(x_M) = \nabla^2 w(x_M) = D_{11}w(x_M) + D_{22}w(x_M) + D_{33}w(x_M) \leq 0$$

  Moreover, because $\Omega \subseteq \overline{\Omega}$,

  $$\min_{\overline{\Omega}}\, w \leq \min_{\Omega}\, w \leq \max_{\Omega}\, w \leq \max_{\overline{\Omega}}\, w \leq 0. \qquad (1)$$

  Thus, if min w occurs on $\partial\Omega$, then $\min w = \max w = 0$ on $\overline{\Omega}$.

Suppose that min w occurs at $x_m \in \Omega$. We can follow that same arguement in Case 1, but with

$$\begin{aligned}
0 \leq D^2 w(x_m) e_1^2 &= D_{11} w(x_m) \\
0 \leq D^2 w(x_m) e_2^2 &= D_{22} w(x_m) \\
0 \leq D^2 w(x_m) e_3^2 &= D_{33} w(x_m)
\end{aligned}$$

Thus,

$$\min_{\Omega}\ w = w(x_m) = \nabla^2 w(x_m) = D_{11} w(x_m) + D_{22} w(x_m) + D_{33} w(x_m) \geq 0$$

Therefore, because $\Omega \subseteq \overline{\Omega}$,

$$\max_{\overline{\Omega}}\ w \geq \max_{\Omega}\ w \geq \min_{\Omega}\ w \geq \min_{\overline{\Omega}} \geq 0. \tag{2}$$

(1) and (2) along with $w|_{\partial\Omega} = 0$ imply that max $w = 0$ and min $w = 0$ on $\overline{\Omega}$. Thus, $w = 0$ on $\overline{\Omega}$. Since w was an arbitrary solution, $w = 0$ is the only solution. $\square$

[5] Let $\Omega = [0,1] \times [0,1]$. Suppose $f : \Omega \to \mathbb{R}$ is given by $f(x,y) = xy$.

(a) Use the definition of the Darboux integral to show that f is Darboux integrable, and find the integral.

Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $\frac{2}{N} < \epsilon$. Let P_ϵ be a partition such that

$$P_1 = \{x_0, x_1, ..., x_N\} = \left\{0, \frac{1}{N}, \frac{2}{N}, ..., \frac{N}{N} = 1\right\}$$

$$P_2 = \{y_0, y_1, ..., y_N\} = \left\{0, \frac{1}{N}, \frac{2}{N}, ..., \frac{N}{N} = 1\right\}.$$

Note that on each square $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $1 \leq i \leq N$, $1 \leq j \leq N$:

$$\sup_{R_{ij}} f(x,y) = x_i y_j, \quad \text{and} \quad \inf_{R_{ij}} f(x,y) = x_{i-1} y_{j-1}.$$

It follows that

$$\begin{aligned}
U(f, P_\epsilon) - L(f, P\epsilon) &= \sum_{i,j=1}^{N} \left[\sup_{R_{ij}} f(x,y) - \inf_{R_{ij}} f(x,y)\right] (x_i - x_{i-1})(y_j - y_{j-1}) \\
&= \sum_{i,j=1}^{N} [x_i y_j - x_{i-1} y_{j-1}] (x_i - x_{i-1})(y_j - y_{j-1}) \\
&= \sum_{i,j=1}^{N} [x_i y_j - (x_i y_{j-1}) + (x_i y_{j-1}) - x_{i-1} y_{j-1}] (x_i - x_{i-1})(y_j - y_{j-1}) \\
&= \sum_{i,j=1}^{N} \Big[x_i (y_j - y_{j-1}) + y_{j-1} (x_i - x_{i-1})\Big] (x_i - x_{i-1})(y_j - y_{j-1}) \\
&\leq \sum_{i,j=1}^{N} \Big[(1)(y_j - y_{j-1}) + (1)(x_i - x_{i-1})\Big] (x_i - x_{i-1})(y_j - y_{j-1}) \\
&= \sum_{i,j=1}^{N} \left[\frac{1}{N} + \frac{1}{N}\right] \frac{1}{N} \frac{1}{N} \\
&= \frac{2}{N^3} \sum_{i=1}^{N} \sum_{j=1}^{N} 1 \\
&= \frac{2}{N^3} N^2 \\
&= \frac{2}{N} \\
&< \epsilon.
\end{aligned}$$

Thus f is darboux integrable on Ω. $\square$

Let P_n, $n \in \mathbb{N}$, be a partition of Ω consisting of squares of side length $\frac{1}{n}$. Then

$$\begin{aligned}
U(f, P_n) &= \sum_{i,j=1}^{n} \left[\sup_{R_{ij}} f(x,y)\right] (x_i - x_{i-1})\,(y_j - y_{j-1}) \\
&= \sum_{i,j=1}^{n} \left[\frac{i}{n}\frac{j}{n}\right] \frac{1}{n}\frac{1}{n} \\
&= \frac{1}{n^4}\left[\sum_{i=1}^{n} i \left[\sum_{j=1}^{n} j\right]\right] \\
&= \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right] \sum_{i=1}^{n} i \\
&= \left[\frac{(n+1)}{2n^3}\right]\left[\frac{n(n+1)}{2}\right] \\
&= \frac{(n+1)^2}{4n^2} \\
&= \frac{n^2+2n+1}{4n^2} \\
&> \frac{1}{4}, \qquad\qquad \text{for all n. Thus, } \inf_P U(f,P) \geq \frac{1}{4}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
L(f, P_n) &= \sum_{i,j=1}^{n} \left[\inf_{R_{ij}} f(x,y)\right] (x_i - x_{i-1})\,(y_j - y_{j-1}) \\
&= \sum_{i,j=1}^{n} \left[\frac{i-1}{n}\frac{j-1}{n}\right] \frac{1}{n}\frac{1}{n} \\
&= \frac{1}{n^4}\left[\sum_{i=1}^{n}(i-1) \left[\sum_{j=1}^{n}(j-1)\right]\right] \\
&= \frac{1}{n^4}\left[\frac{n(n+1)}{2} - n\right] \sum_{i=1}^{n}(i-1) \\
&= \frac{1}{n^3}\left[\frac{(n+1)}{2} - 1\right]\left[\frac{n(n+1)}{2} - n\right] \\
&= \frac{1}{n^2}\frac{n-1}{2}\frac{n-1}{2} \\
&= \frac{n^2-2n+1}{4n^2} \\
&< \frac{1}{4} \qquad\qquad \text{for all n. Thus, } \sup_P L(f,P) \leq \frac{1}{4}.
\end{aligned}$$

But, since f is darboux integrable on Ω, it must be that

$$\sup_P L(f,P) = \inf_P U(f,P) = \frac{1}{4} = \int_\Omega f.$$

$\square$

(b) Repeat (a) using theorems.

Since the projection functions $p_x(x,y) = x$ and $p_y(x,y) = y$ are both continuous, by the algebra of continuous functions, their product $f(x,y) = xy$ is continuous. Thus, $f(x,y)$ is darboux integrable on Ω. Let $x' \in [0,1]$. Clearly $\int_0^1 f(x',y)\,dy = \int_0^1 x'y\,dy$ exists. Hence, by the Jones iterated integrals theorem,

$$g(x) = \int_0^1 f(x,y)\,dy$$

is integrable on $[0,1]$, and $\int_0^1 g(x)\,dx = \iint_\Omega f(x,y)\,dA$. Therefore,

$$\begin{aligned}
\iint_\Omega f(x,y)\,dA &= \int_0^1 g(x)\,dx \\
&= \int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx \\
&= \int_0^1 \left(\int_0^1 xy\,dy \right) dx \\
&= \int_0^1 \frac{x}{2}\,dx \\
&= \frac{1}{4}.
\end{aligned}$$

$\square$