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PHY 151 Final Project
RC-circuits and Magnetic Forces

Kirchoff's loop law for circuits

Ohm's Law : $V = IR$

ε = constant voltage source

v_R = voltage across the resistor R

v_C = voltage across the capacitor C

$q(t)$ = charge on the capacitor at time t

By using Kirchoff's loop law on our circuit we attain the following differential equation:

$$\begin{aligned}
 \varepsilon - v_R - v_C &= 0 \\
 \varepsilon - v_C &= v_R \\
 \varepsilon - \frac{q(t)}{C} &= i(t)R && \text{since } v_C(t) = \frac{q(t)}{C}, \text{ and } v_R(t) = i(t)R \\
 \varepsilon - \frac{q(t)}{C} &= q'(t)R && \text{since } i(t) = q'(t) \\
 \left(\frac{1}{RC}\right) [\varepsilon C - q(t)] &= q'(t) \\
 \frac{1}{RC} &= \frac{q'(t)}{\varepsilon C - q(t)} \\
 \frac{1}{RC} &= -[\ln(\varepsilon C - q(t))]' \\
 \int \frac{1}{RC} dt &= \int -[\ln(\varepsilon C - q(t))]' dt \\
 \frac{t}{RC} + K &= -\ln(\varepsilon C - q(t)) \\
 K - \frac{t}{RC} &= \ln(\varepsilon C - q(t)) \\
 e^{K - \frac{t}{RC}} &= e^{\ln(\varepsilon C - q(t))} \\
 e^K e^{-\frac{t}{RC}} &= \varepsilon C - q(t) \\
 K e^{-\frac{t}{RC}} &= \varepsilon C - q(t) \tag{1}
 \end{aligned}$$

$$\begin{aligned}
 K &= \varepsilon C - 0 && \text{since } q(0) = 0 \\
 K &= \varepsilon C \tag{2}
 \end{aligned}$$

$$\begin{aligned}
 \varepsilon C e^{-\frac{t}{RC}} &= \varepsilon C - q(t) && \text{by (1) and (2)} \\
 q(t) &= \varepsilon C - \varepsilon C e^{-\frac{t}{RC}} \\
 q(t) &= \varepsilon C \left[1 - e^{-\frac{t}{RC}} \right] \tag{3}
 \end{aligned}$$

(3) is an expression for the charge on the capacitor at time t . Since the charges are deposited on the capacitor by the current we can differentiate this expression to find an expression for the current in the circuit at time t :

$$\begin{aligned} i(t) &= q'(t) \\ &= \varepsilon C \left[\frac{1}{RC} e^{\frac{-t}{RC}} \right] \\ &= \frac{\varepsilon}{R} \left[e^{\frac{-t}{RC}} \right] \end{aligned} \tag{4}$$

The distance of the proton from the wire of the circuit is small in comparison with the length of the wire. I will approximate the wire as an infinitely long current carrying wire. We can use the Biot-Savart law to find the magnetic field at any distance r away from the wire.

$$|\vec{B}| = \frac{\mu_0 I}{4\pi r} \tag{5}$$

In this project I am trying to keep the proton within the area between the wires (see figure A). The direction of the current in the wires make this happen. Using the right-hand rule we see that the \vec{B} -fields are directed in opposite directions:

$$\begin{aligned} \vec{B}_L &= 0 \hat{x} + 0 \hat{y} + \frac{\mu_0 I}{4\pi r} \hat{z} && \text{(the magnetic field due to the wire on the left side)} \\ \vec{B}_R &= 0 \hat{x} + 0 \hat{y} - \frac{\mu_0 I}{4\pi r} \hat{z} && \text{(the magnetic field due to the wire on the right side)} \end{aligned}$$

Due to the set up of my system the proton will only have velocity components in the x and y directions. The force on our moving proton in the presence of a magnetic field is the superposition of the force due to the left and right side wires:

$$\begin{aligned} \vec{F}_m &= \vec{F}_{m,L} + \vec{F}_{m,R} \\ &= \left[q\vec{v} \times \vec{B}_L \right] + \left[q\vec{v} \times \vec{B}_R \right] \end{aligned} \tag{6}$$

I will consider the momentum of the proton in order to calculate the affect of the net magnetic force on the velocity of the proton over small time intervals:

$$\vec{p} = m\vec{v}$$

$$\Delta\vec{p} = m\Delta\vec{v}$$

We can relate the change in momentum of the proton to the force F_m over time Δt :

$$\begin{aligned}\Delta\vec{p} &= m\Delta\vec{v} \\ &= \vec{F}_m\Delta t\end{aligned}$$

Solving for $|\Delta\vec{v}|$ and using (6) for $|\vec{F}_m|$ we have:

$$\begin{aligned}\Delta\vec{v} &= \frac{1}{m}\vec{F}_m\Delta t \\ &= \frac{1}{m}\left[\vec{F}_{m,L} + \vec{F}_{m,R}\right]\Delta t\end{aligned}\tag{7}$$

$$\begin{aligned}&= \frac{1}{m}\left[q\vec{v} \times \vec{B}_L\right]\Delta t + \frac{1}{m}\left[q\vec{v} \times \vec{B}_R\right]\Delta t \\ &= \frac{1}{m}\left[q\vec{v} \times \left(0\hat{x} + 0\hat{y} + \frac{\mu_0 I}{4\pi r}\hat{z}\right)\right]\Delta t + \frac{1}{m}\left[q\vec{v} \times \left(0\hat{x} + 0\hat{y} - \frac{\mu_0 I}{4\pi r}\hat{z}\right)\right]\Delta t\end{aligned}\tag{8}$$

We can visualize an approximation of the motion of the proton in the magnetic field created by the wire of charge using VPython (see final page for the source code). I will use time intervals of $\Delta t = .001$. At each point in time the program will use equation (8) to approximate the change in the proton's velocity over that time interval. A true calculation would require integration, but summing over $\Delta t = .001$ will give us a useful approximation for the purposes of this project. With t_i representing the points in time at which we are calculating the velocity of the proton, we can approximate the velocity of the proton over the i^{th} time interval Δt by:

$$v_{avg,i} = (v_{i-1}) + \frac{\Delta v}{2}$$

We can then calculate the change in position of the proton over the i^{th} time interval by:

$$\Delta s_i = (v_{avg,i})\Delta t\tag{9}$$

By looking at equation (9) for the Δv of the proton we can decide on some parameters to vary while holding all others fixed and test to see how they affect the path of the proton. In particular we can test for a range of values of:

Resistance (R), Capacitance (C), Input voltage (ε), proton initial velocity (v_0),
proton charge (q), proton mass (m)

This table summarizes the relationships between the system variables of:

$$R, C, \varepsilon, v_0, q, \text{ and } m,$$

and the magnitudes of the time dependent variables of:

$$\begin{aligned} &\text{current } [i(t_i)], \text{ magnetic field } [B(t_i)], \text{ magnetic force } [F_m(t_i)], \\ &\text{proton's change in velocity } [\Delta v(t_i)], \text{ and proton's change in position } [\Delta s(t_i)]. \end{aligned}$$

Where t_i represents the time at a given time step in our VPython program.

	$i(t_i)$	$B(t_i)$	$F_m(t_i)$	$\Delta v(t_i)$	$\Delta s(t_i)$
$\uparrow R$	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
$\uparrow C$	\uparrow	\uparrow	\uparrow	\uparrow	\uparrow
$\uparrow \varepsilon$	\uparrow	\uparrow	\uparrow	\uparrow	\uparrow
$\uparrow v_0$	$-$	$-$	\uparrow	\uparrow	\uparrow
$\uparrow q$	$-$	$-$	\uparrow	\uparrow	\uparrow
$\uparrow m$	$-$	$-$	$-$	\downarrow	\downarrow

The remaining of the paper elaborates on the relationships shown in this table.

Changing the Resistance

$$R_1 = 1000 \, \Omega, R_2 = 2000 \, \Omega, R_3 = 4000 \, \Omega$$

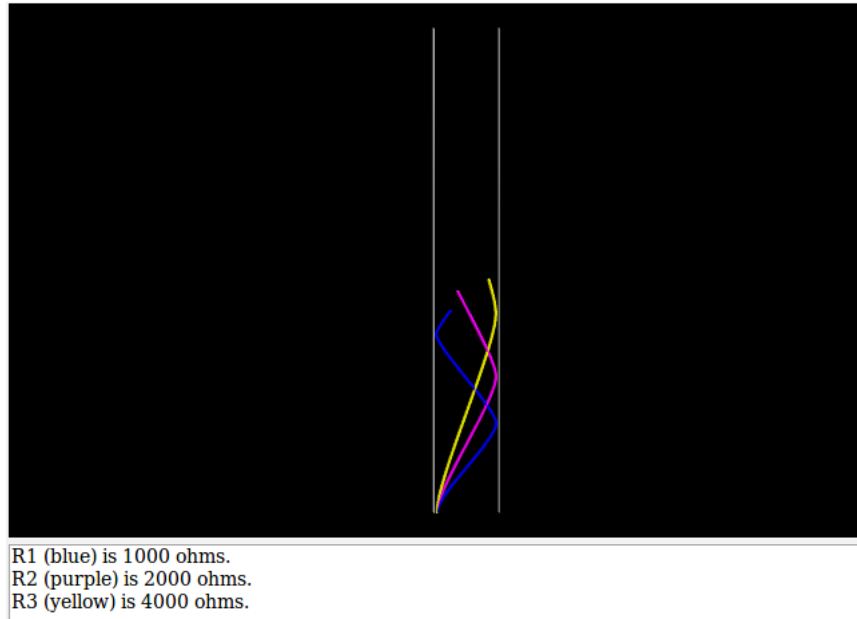
We can see in (4)

$$i(t) = \frac{\varepsilon}{R} \left[e^{\frac{-t}{RC}} \right]$$

that the relationship between current and resistance is somewhat complicated with R appearing in two different locations having opposing effects:

(*)	$\frac{\varepsilon}{R}$	↑ R , ↓ $i(t)$
(**)	$e^{\frac{t}{RC}}$	↑ R , ↑ $i(t)$

We see in our VPython simulation that (*) has a stronger effect. Hence, increasing R has the affects shown in the previous table.



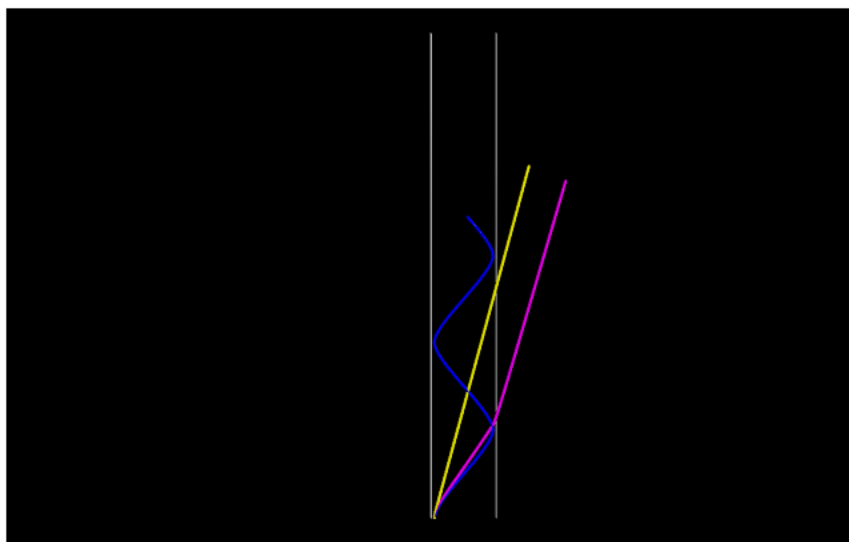
Changing the Capacitance

$$C_1 = .2 \text{ Farads}, C_2 = .0002 \text{ Farads}, C_3 = .00002 \text{ Farads}$$

We can see from (4)

$$i(t) = \frac{\varepsilon}{R} \left[e^{\frac{-t}{RC}} \right]$$

that increasing C will decrease the current in the wires at time t . This is shown in our simulation. In the second and third cases the current decreases so rapidly that the magnetic field from the wire on the right is not even strong enough to change the direction of the proton before it passes the wire.



C1 (blue) is 0.2 Farads.
C2 (purple) is 0.0002 Farads.
C3 (yellow) is 0.00002 Farads.

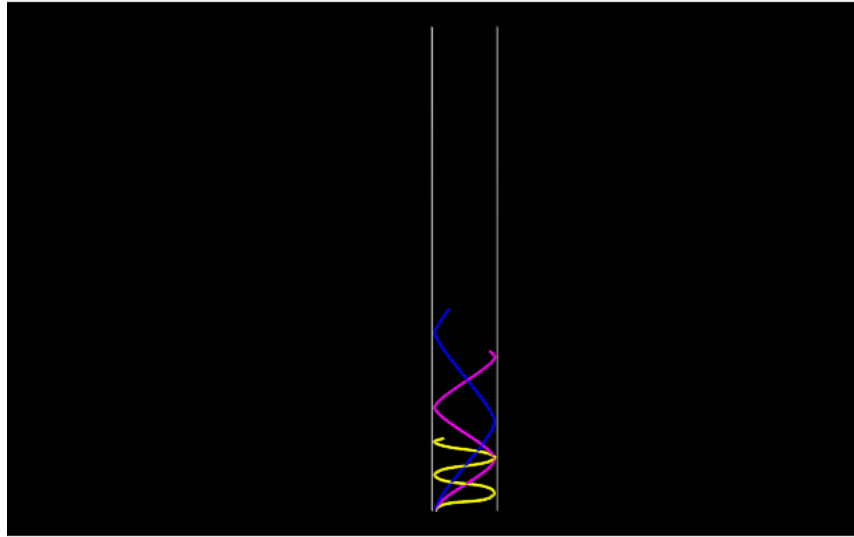
Changing the Voltage source

$\varepsilon_1 = 100$ volts, $\varepsilon_2 = 200$ volts, $\varepsilon_3 = 400$ volts

We can see from (4)

$$i(t) = \frac{\varepsilon}{R} \left[e^{\frac{-t}{RC}} \right]$$

that increasing ε will increase the current at any time t . This is shown in our simulation where the cases with higher ε show the proton changing direction more rapidly.



Vb1 (blue) is 100 volts.
Vb2 (purple) is 200 volts.
Vb3 (yellow) is 400 volts.

Changing the initial velocity

$$v_{0,1} = 10 \text{ m/s } \hat{y}, v_{0,2} = 20 \text{ m/s } \hat{y}, v_{0,3} = 30 \text{ m/s } \hat{y}$$

We can see from (6)

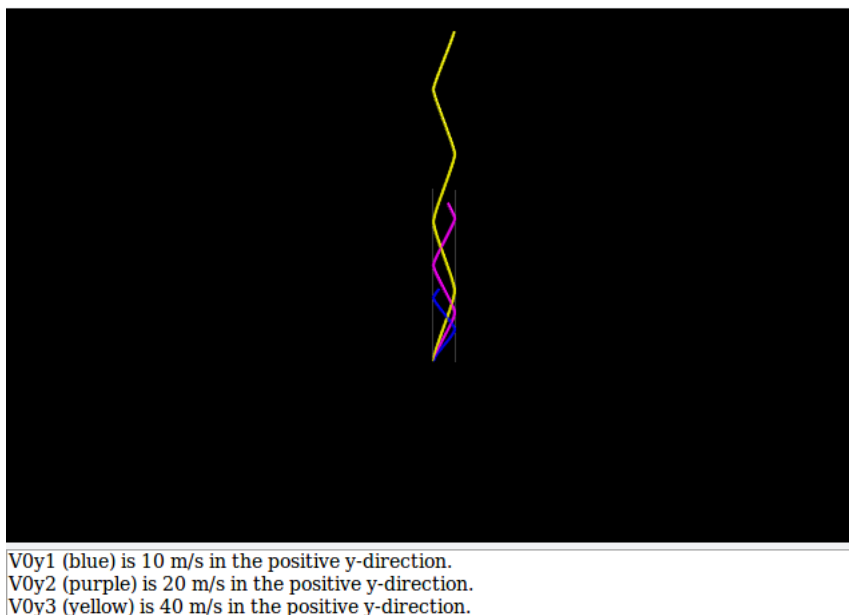
$$\vec{F}_m = \vec{F}_{m,L} + \vec{F}_{m,R} = \left[q\vec{v} \times \vec{B}_L \right] + \left[q\vec{v} \times \vec{B}_R \right]$$

that an increase in the velocity of the proton will result in an increase in the magnetic force felt by the proton. Also, we see in (9)

$$\Delta s_i = (v_{avg,i}) \Delta t$$

that a higher velocity also increases the change in position over each time interval.

The question is how will this affect the path of the proton? Will the increase in the magnetic force be strong enough to make the proton's change in position in the x-direction greater than the change in position in the y-direction? The simulation shows us that it is not. As the initial velocity (in the positive y-direction) increases, the change in the overall path of the proton is larger in the y-direction than in the x-direction.



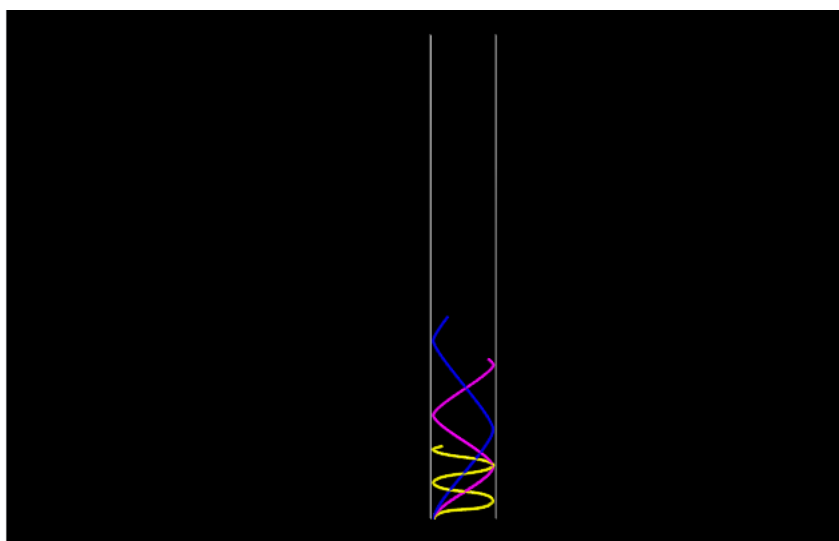
Changing the Charge of the particle

$$q_1 = 1.6 \times 10^{-19} \text{ Coulombs}, q_2 = 2q_1, q_3 = 4q_1$$

We can see from (6)

$$\vec{F}_m = \vec{F}_{m,L} + \vec{F}_{m,R} = \left[q\vec{v} \times \vec{B}_L \right] + \left[q\vec{v} \times \vec{B}_R \right]$$

that doubling the charge of the particle will double the magnetic force felt from each wire. Thus, our simulation should show that the particle changes direction more quickly as we increase the charge of the particle.



q1 (blue) is 1.6000000000000002e-19 coulombs.
q2 (purple) is 3.2000000000000003e-19 coulombs.
q3 (yellow) is 6.400000000000001e-19 coulombs.

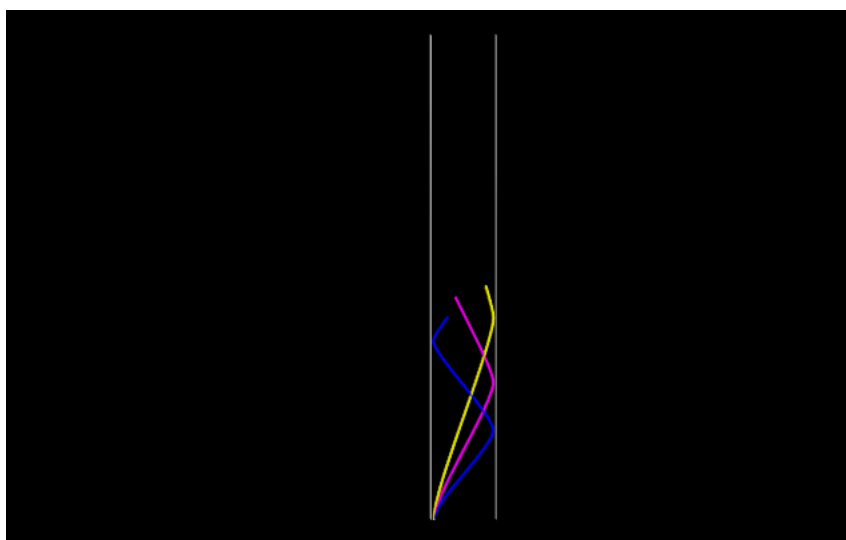
Changing the Mass of the particle

$$m_1 = 1.7 \times 10^{-27} \text{ kilograms}, m_2 = 2m_1, m_3 = 4m_1$$

We can see from (7)

$$\Delta \vec{v} = \frac{1}{m} \left[\vec{F}_{m,L} + \vec{F}_{m,R} \right] \Delta t$$

that increasing the mass of the particle will decrease Δv of the particle. We see this in our simulation by the path of the particle changing less in the x-direction relative to the y-direction as the mass of the particle increases.



m1 (blue) is 1.7e-27 kilograms.
m2 (purple) is 3.4e-27 kilograms.
m3 (yellow) is 6.8e-27 kilograms.

Constant Mass:Charge ratio for the particle

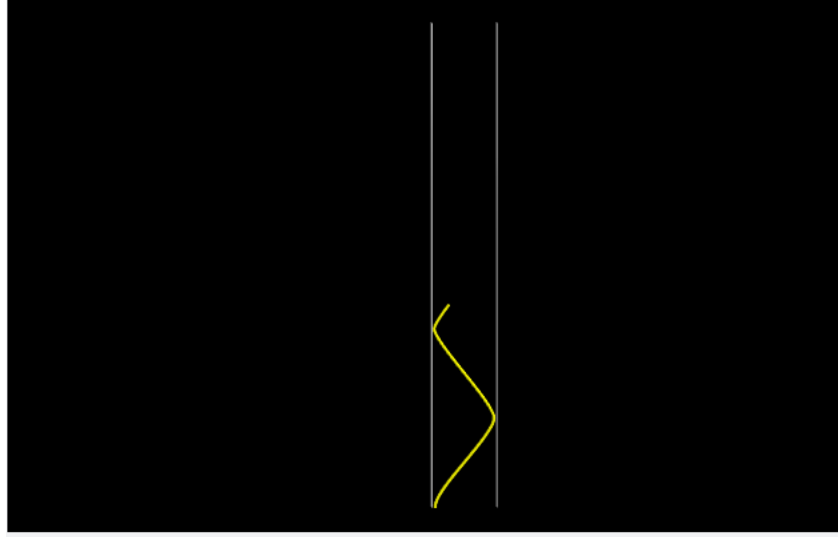
$$\begin{array}{lll} m_1 = 1.7 \times 10^{-27} \text{ kilograms} & m_2 = 2m_1 & m_3 = 4m_1 \\ q_1 = 1.6 \times 10^{-19} \text{ coulombs} & q_2 = 2q_1 & q_3 = 4q_1 \end{array}$$

$$\frac{m_1}{q_1} = \frac{m_2}{q_2} = \frac{m_3}{q_3}$$

We see from (8)

$$\Delta \vec{v} = \frac{1}{m} \left[q \vec{v} \times \left(0 \hat{x} + 0 \hat{y} + \frac{\mu_0 I}{4\pi r} \hat{z} \right) \right] \Delta t + \frac{1}{m} \left[q \vec{v} \times \left(0 \hat{x} + 0 \hat{y} - \frac{\mu_0 I}{4\pi r} \hat{z} \right) \right] \Delta t$$

that if the ratio of $\frac{q}{m}$ remains the same, then $\Delta \vec{v}$ over each time interval does not change. This is shown in the simulation where each of the three cases have traced the same line.



```
m1 = 1.7e-27 kilograms.
m2 = 3.4e-27 kilograms.
m3 = 6.8e-27 kilograms.
q1 = 1.6000000000000002e-19 coulombs.
q2 = 3.2000000000000003e-19 coulombs.
q3 = 6.400000000000001e-19 coulombs.
```

VPython source code (with R_1 , R_2 , and R_3)

```

1 GlowScript 2.4 VPython
2 from visual import *
3
4 #variable definitions
5 q=1.6*10**-19 #charge
6 m=1.7*10**-27 #mass
7 mu0=(4*pi)*10**-7
8 Vb=100 #voltage of battery source
9 R=20000 #reistance of resistor
10 R1=1000
11 R2=2000
12 R3=4000
13 print("R1 (blue) is " +R1+ " ohms.")
14 print("R2 (purple) is " +R2+ " ohms.")
15 print("R3 (yellow) is " +R3+ " ohms.")
16 C=.003 #Capacitance
17 It=0 #current at time t
18 r0x=.1 #initial distance between proton and wire
19 r0y=-20
20 v0x=0
21 v0y=10
22
23 myboxL=box(pos=vector(-.2,0,0), length=.10, height=40, width=.01)
24 myboxR=box(pos=vector(5.2,0,0), length=.10, height=40, width=.01)
25 proton1=sphere(pos=vector(r0x,r0y,0),radius=0.03,color=vector(0,1,1),charge=q,mass=m,v=vector(v0x,v0y,0),trail=curve(color=vector(0,0,1)))
26 proton2=sphere(pos=vector(r0x,r0y,0),radius=0.03,color=vector(0,1,1),charge=q,mass=m,v=vector(v0x,v0y,0),trail=curve(color=vector(1,0,1)))
27 proton3=sphere(pos=vector(r0x,r0y,0),radius=0.03,color=vector(0,1,1),charge=q,mass=m,v=vector(v0x,v0y,0),trail=curve(color=vector(1,1,0)))
28
29 #Particle dynamics
30 t=0
31 dt=.001
32 #It=(Vb/R)*(exp(-t/(R*C)))
33 #print (It)
34 k=0
35 while t<=2:
36     rate(800)
37     t=t+dt
38     It=(Vb/R1)*(exp(-t/(R1*C))) #current at time t
39     BL=vector(0,0,(mu0*It)/(4*pi*abs(proton1.pos.x))) #magnitude of the magnetic field produced by the wire at the proton's position
40     BR=vector(0,0,-(mu0*It)/(4*pi*abs(5-proton1.pos.x)))
41     B1=BL+BR
42     Fm=q*cross(proton1.v,B1) #magnetic force on the proton
43     deltav=(1/proton1.mass)*Fm*dt #change in velocity of the proton, using change in momentum
44     vxavg=proton1.v+(deltav/2) #average velocity of the proton over the time frame of dt
45     proton1.pos.x=proton1.pos.x+(vxavg.x*dt)
46     proton1.pos.y=proton1.pos.y+(vxavg.y*dt)
47     proton1.v=proton1.v+deltav
48     proton1.trail.append(proton1.pos)
49
50 t=0
51 It=0 #zero out the current
52 while t<=2:
53     rate(800)
54     t=t+dt
55     It=(Vb/R2)*(exp(-t/(R2*C))) #current at time t
56     BL=vector(0,0,(mu0*It)/(4*pi*abs(proton2.pos.x))) #magnitude of the magnetic field produced by the wire at the proton's position
57     BR=vector(0,0,-(mu0*It)/(4*pi*abs(5-proton2.pos.x)))
58     B2=BL+BR
59     Fm=q*cross(proton2.v,B2) #magnetic force on the proton
60     deltav=(1/proton2.mass)*Fm*dt #change in velocity of the proton, using change in momentum
61     vxavg=proton2.v+(deltav/2) #average velocity of the proton over the time frame of dt
62     proton2.pos.x=proton2.pos.x+(vxavg.x*dt)
63     proton2.pos.y=proton2.pos.y+(vxavg.y*dt)
64     proton2.v=proton2.v+deltav
65     proton2.trail.append(proton2.pos)
66
67 t=0
68 It=0 #zero out the current
69 while t<=2:
70     rate(800)
71     t=t+dt
72     It=(Vb/R3)*(exp(-t/(R3*C))) #current at time t
73     BL=vector(0,0,(mu0*It)/(4*pi*abs(proton3.pos.x))) #magnitude of the magnetic field produced by the wire at the proton's position
74     BR=vector(0,0,-(mu0*It)/(4*pi*abs(5-proton3.pos.x)))
75     B3=BR+BL
76     Fm=q*cross(proton3.v,B3) #magnetic force on the proton
77     deltav=(1/proton3.mass)*Fm*dt #change in velocity of the proton, using change in momentum
78     vxavg=proton3.v+(deltav/2) #average velocity of the proton over the time frame of dt
79     proton3.pos.x=proton3.pos.x+(vxavg.x*dt)
80     proton3.pos.y=proton3.pos.y+(vxavg.y*dt)
81     proton3.v=proton3.v+deltav
82     proton3.trail.append(proton3.pos)

```