# Sample BEAMER Presentation

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# Introduction

- Heat as a function of 1-dimensional space and time: $u(x, t)$
- Notation for $u$ evaluated at discretized points in space and time

$$u(x_i, t_n) = u_i^n$$

- The heat equation, a second order partial differential equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

- Three approximation strategies
  - Euler's method
  - Improved Euler's method
  - Implicit Time Stepping

## Euler's Method

$$u_i^{n+1} \approx u_i^n + \frac{k\Delta t}{\Delta x^2}\left[u_{i+1}^n - 2u_i^n + u_{i-1}^n\right]$$

- This can be derived using Taylor series expansions in each variable

## Improved Euler's method

- Approximate the slopes at the endpoints of each time interval
  - $\widetilde{K}_{i+1}$ is found using an approximation for $u_i^{n+1}$
- Use the average of these slopes to linearly approximate the solution over the interval

$$u_i^{n+1} = u_i^n + \Delta t \left[ \frac{K_i + \widetilde{K}_{i+1}}{2} \right]$$

- $K_i \approx$ slope at $u_i^n$
- $\widetilde{K}_{i+1} \approx$ slope at $u_i^{n+1}$

- Pieces to the approximation

  - $K_i = \frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{k}{\Delta x^2}\left[u_{i-1}^n - 2u_i^n + u_{i+1}^n\right]$
  - $\tilde{u}_i^{n+1} = u_i^n + \frac{k\Delta t}{\Delta x^2}\left[u_{i-1}^n - 2u_i^n + u_{i+1}^n\right]$
  - $\widetilde{K}_{i+1} = \frac{u_i^{n+2} - u_i^{n+1}}{\Delta t} = \frac{k}{\Delta x^2}\left[\tilde{u}_{i-1}^{n+1} - 2\tilde{u}_i^{n+1} + \tilde{u}_{i+1}^{n+1}\right]$

## Implicit Time Stepping

- Benefits
  - Increased stabilty of approximation
  - Able to handle larger diffusivity constants
- Drawbacks
  - Need to solve a linear system $Ax = b$

- Setting up the approximation
  - Use the Taylor's series expansion, but the RHS of our final approximation expression is in terms of $u_x^{t_{n+1}}$ rather than $u_x^{t_n}$.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{k}{\Delta x^2}\left[u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right]$$

$$u_i^n = u_i^{n+1} - \frac{k\Delta t}{\Delta x^2}\left[u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right]$$

$$u_i^n = u_i^{n+1}\left[1 + 2r\right] - r\left[u_{i-1}^{n+1} + u_{i+1}^{n+1}\right] \qquad , r = \frac{k\Delta t}{\Delta x^2}$$

  - We are using the value at the *next* time step

- Fix $n$ (time) and let $i$ (space) float over all of the interior points of our rod
    - Boundary conditions $u(x_1, t)$ and $u(x_N, t)$ are known for all $t$
    - Interior points

$$
\begin{aligned}
u_2^n &= u_2^{n+1}\Big[1+2r\Big] - r\Big[u_1^{n+1} + u_3^{n+1}\Big] \\
u_3^n &= u_3^{n+1}\Big[1+2r\Big] - r\Big[u_2^{n+1} + u_4^{n+1}\Big] \\
u_4^n &= u_4^{n+1}\Big[1+2r\Big] - r\Big[u_3^{n+1} + u_5^{n+1}\Big] \\
&\;\vdots \\
u_{N-2}^n &= u_{N-2}^{n+1}\Big[1+2r\Big] - r\Big[u_{N-3}^{n+1} + u_5^{N-1}\Big] \\
u_{N-1}^n &= u_{N-1}^{n+1}\Big[1+2r\Big] - r\Big[u_{N-2}^{n+1} + u_N^{n+1}\Big]
\end{aligned}
$$

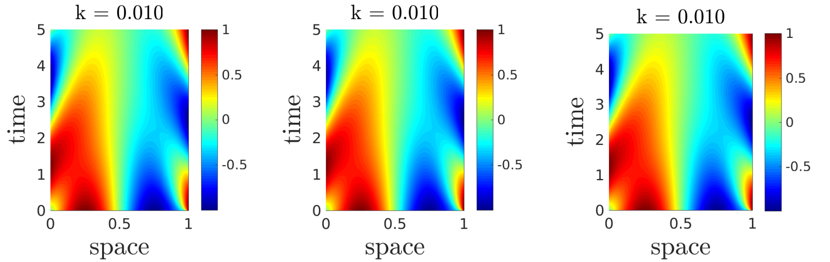
We move our known terms $u_1^{n+1}$ and $u_N^{n+1}$ to the RHS.

Recognizing the pattern we build the linear system:

$$\begin{pmatrix} 1+2r & -r & 0 & 0 & . & . & 0 \\ -r & 1+2r & -r & 0 & 0 & . & 0 \\ 0 & -r & 1+2r & -r & 0 & . & 0 \\ . & & . & . & . & & . \\ . & & . & . & . & & . \\ . & & . & . & . & & . \\ 0 & 0 & . & . & -r & 1+2r & -r \\ 0 & 0 & . & . & 0 & -r & 1+2r \end{pmatrix} \begin{pmatrix} u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ . \\ . \\ . \\ u_{N-2}^{n+1} \\ u_{N-1}^{n+1} \end{pmatrix} = \begin{pmatrix} u_2^n + ru_1^{n+1} \\ u_3^n \\ u_4^n \\ . \\ . \\ . \\ u_{N-2}^n \\ u_{N-1}^n + ru_N^{n+1} \end{pmatrix}$$

## Comparing plots

- We need the initial temperature profile of the rod and the temperatures for all points in time of the endpoints of the rod
    - We will give these values in our MatLab code
- Example 1
    - Length of rod, L: 1 meter
    - Number of points: 50
    - Total time, $t_f$: 5 seconds
    - Number of time points: 2000
    - Diffusivity constant: k
    - $r = k\frac{\Delta t}{\Delta x^2}$ $r < 0.50$ required
    - $u(1,:) = \sin\left(\frac{2\pi}{t_f}\ t\right)$ Left endpoint boundary conditions
    - $u(N,:) = \cos\left(\frac{2\pi}{t_f}\ t\right)$ Right endpoint boundary conditions
    - $u(:,1) = \sin\left(\frac{2\pi}{L}\right)\ x$ Initial conditions




Euler's Improved Euler's Implicit Time Stepping

## Exploring Error

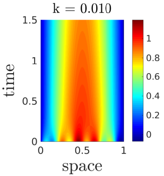
- Simplify IC and Boundary conditions so the we can solve for an explicit solution
  - IC
  $$u(x,0) = \sin(\pi x) + 0.2\sin(10\pi x)$$
  - Boundary conditions
  $$u(0,t) = u(1,t) = 0$$
  - Guess a solution to $u_t = ku_{xx}$

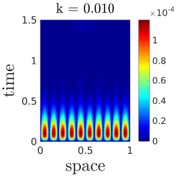
$$u(t,x) = e^{-\pi^2 kt}\sin(\pi x) + 0.2e^{-(10\pi)^2 kt}\sin(10\pi x)$$

$$
\begin{aligned}
u_t &= \frac{\partial}{\partial t}\Big[u(x,t)\Big] \\
&= \frac{\partial}{\partial t}\Big[e^{-\pi^2 kt}\sin(\pi x) + 0.2e^{-(10\pi)^2 kt}\sin(10\pi x)\Big] \\
&= -\pi^2 k e^{-\pi^2 kt}\sin(\pi x) + 0.2\left[-(10\pi)^2 k\right] e^{-(10\pi)^2 kt}\sin(10\pi x) \\
&= k\Big[-\pi^2 e^{-\pi^2 kt}\sin(\pi x) + 0.2\left[-(10\pi)^2 k\right] e^{-(10\pi)^2 t}\sin(10\pi x)\Big] \\
&= k\frac{\partial^2 u}{\partial x^2}\Big[e^{-\pi^2 kt}\sin(\pi x) + 0.2e^{-(10\pi)^2 kt}\sin(10\pi x)\Big] \\
&= k\frac{\partial^2 u}{\partial x^2}\Big[u(x,t)\Big] \\
&= k u_{xx}
\end{aligned}
$$

It is a solution! We can use this to measure the error in our approximations!

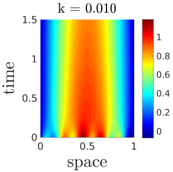
# Plots with $M = 2000, N = 200, k = 0.01 \rightarrow r = 0.2972$

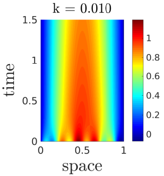


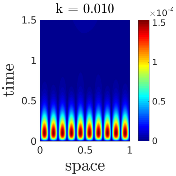




Exact solution

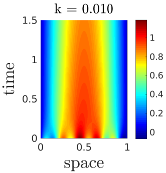
Euler's

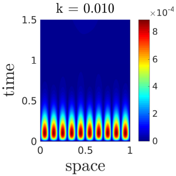



Improved Euler's

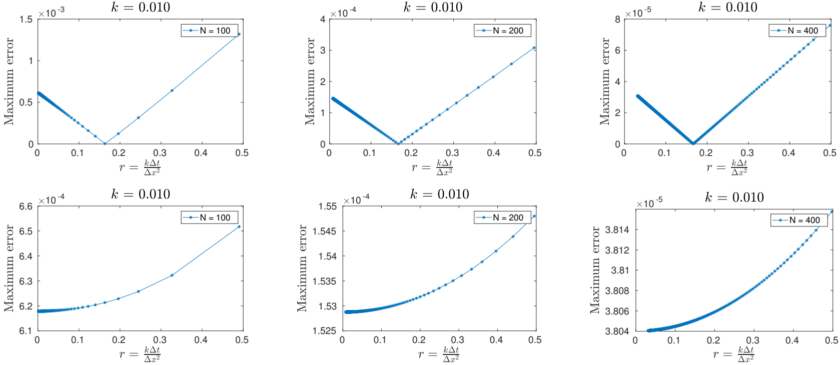



Implicit

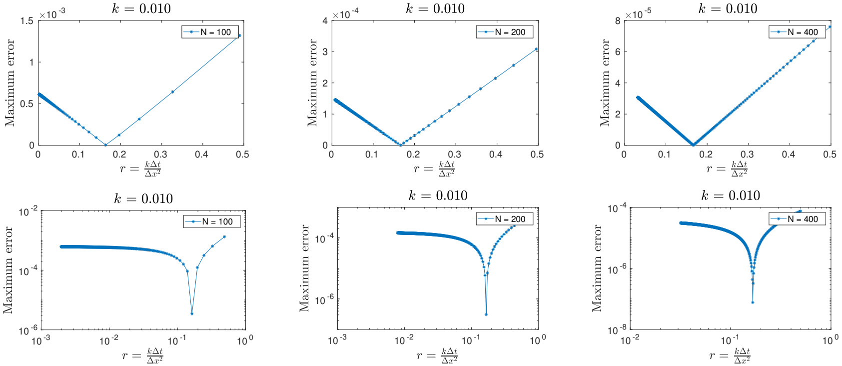
At which point in time should we sample the error across the rod?

- $t$ large and $t \approx 0$ the error is nearly zero
- Error appears non-trivial at $t \approx 0.10$ seconds
- Experiment:
  - A given number of sample points in space
  - Range through different numbers of sample points in time
    - $r = \frac{k\Delta t}{\Delta x^2} \downarrow \quad \text{as } \Delta t \downarrow$




Implicit time stepping: The code ran forever

# Log plot to get a closer look at Euler's method approximation




At some point just above $r = 0.10$ we see the error becoming very small!

Intersting...Why?

$$U_t = kU_{xx}$$

Approximation: $$\frac{U^{n+1} - U^n}{\Delta t} = k\frac{U_{j+1} - 2U_j - U_{j-1}}{\Delta x^2}$$

Recall the Taylor series expansion in one variable

$$u(x_j \pm \Delta x) = \sum_{k=0}^{N} (\pm \Delta x)^k f^{(k)}(x_j) + error\ term.$$

$$u(t^n \pm \Delta t) = \sum_{k=0}^{N} (\pm \Delta t)^k f^{(k)}(t^n) + error\ term.$$

x's first...

$$U_{j+1} = u(x_j + \Delta x)$$
$$= u(x_j) + \Delta x U_x(x_j) + \frac{\Delta x^2}{2!} U_{xx}(x_j) + \frac{\Delta x^3}{3!} U_x xx(x_j) + \frac{\Delta x^4}{4!} U_{4x}(x_j) + \ldots.$$

$$U_{j-1} = u(x_j - \Delta x)$$
$$= u(x_j) - \Delta x U_x(x_j) + \frac{\Delta x^2}{2!} U_{xx}(x_j) - \frac{\Delta x^3}{3!} U_x xx(x_j) + \frac{\Delta x^4}{4!} U_{4x}(x_j) + \ldots.$$

$$\frac{U_{j+1} - 2U_j + U_{j-1}}{\Delta x^2} = U_{xx}(x_j) + 2\frac{\Delta x^2}{4!} U_{4x}(x_j) + \ldots.\Delta x^{large} \quad (1)$$

Now t's...

$$U^{n+1} = u(t^n + \Delta t) = u(t^n) + \Delta t U_t(t^n) + \frac{\Delta t^2}{2!} U_{tt}(t^n) + \frac{\Delta t^3}{3!} U_{ttt}(t^n) + \ldots . \Delta t^{large}$$

$$\frac{U^{n+1} - U^n}{\Delta t} = U_t(t^n) + \frac{\Delta t}{2!} U_{tt}(t^n) + \frac{\Delta t^2}{3!} U_{ttt}(t^n) + \ldots . \Delta t^{large} \qquad (2)$$

$$(2) = k(1)$$

$$\frac{U^{n+1} - U^n}{\Delta t} = k\left[\frac{U_{j+1} - 2U_j + U_{j-1}}{\Delta x^2}\right]$$

$$U_t(t^n) + \frac{\Delta t}{2!}U_{tt}(t^n) + \frac{\Delta t^2}{3!}U_{ttt}(t^n) + \ldots. = k\left[U_{xx}(x_j) + 2\frac{\Delta x^2}{4!}U_{4x}(x_j) + \ldots.\right]$$

$$U_t = kU_{xx} - \frac{\Delta t}{2}U_{tt} - \frac{\Delta t^2}{6}U_{3t} + k\frac{\Delta x^2}{12}U_{4x} + \ldots. \tag{3}$$

# Remarks

# Acknowledgments