Legendre Polynomials

Matt James

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1 Legendre Polynomials

This is the form of a Legendre polynomial (Rodrigues' formula):

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (v^2 - 1)^n \tag{1}$$

where n is the degree of the polynomial. The first 5 degrees (0-4) are shown below:

$$P_0(x) = \frac{\mathrm{d}^0}{\mathrm{d}x^0}(x^2-1)^0 \qquad = 1, \tag{2}$$

$$P_1(x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1) \qquad = x, \tag{3}$$

$$P_2(x) = \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2-1)^2 \qquad = \frac{1}{2}(3x^2-1), \tag{4}$$

$$P_3(x) = \frac{1}{48}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^2-1)^3 \qquad = \frac{1}{2}(5x^3-3x), \tag{5}$$

$$P_4(x) = \frac{1}{384}\frac{\mathrm{d}^4}{\mathrm{d}x^4}(x^2-1)^4 \qquad = \frac{1}{8}(35x^4-30x^2+3), \tag{6}$$

where x can be substituted for $\cos\theta$:

$$P_0(\cos\theta) = 1, \tag{7}$$

$$P_1(\cos\theta) = \cos\theta, \tag{8}$$

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1), \tag{9}$$

$$P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta), \tag{10}$$

$$P_4(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30x^2 + 3). \tag{11}$$

1.1 Derivatives

These derivatives of the above equations 2–6 with respect to x will come in handy later on...

For equation 2:

$$\frac{\mathrm{d}}{\mathrm{d}x}P_0(x) = 0. \tag{12}$$

For equation 3:

$$\frac{\mathrm{d}}{\mathrm{d}x}P_1(x) = 1. \tag{13}$$

For equation 4:

$$\frac{\mathrm{d}}{\mathrm{d}x}P_2(x) = 3x, \tag{14}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_2(x) = 3. \tag{15}$$

For equation 5:

$$\frac{\mathrm{d}}{\mathrm{d}x}P_3(x) = \frac{1}{2}(15x^2 - 3), \tag{16}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_3(x) = 15x, \tag{17}$$

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}P_3(x) = 15. \tag{18}$$

For equation 6:

$$\frac{\mathrm{d}}{\mathrm{d}x}P_4(x) = \frac{5}{2}(7x^3 - 3x), \tag{19}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_4(x) = \frac{15}{2}(7x^2 - 1), \tag{20}$$

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3}P_4(x) = 105x, \tag{21}$$

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4}P_4(x) = 105. \tag{22}$$

2 Ferrers Normalized Legendre Polynomials

The Ferrers normalized Legendre polynomials are defined by *Ferrers* (1877) as:

$$P_{n,m}(x) = (1 - x^2)^{\frac{m}{2}} \frac{\mathrm{d}^m}{\mathrm{d}x^m}P_n(x), \tag{23}$$

where m is the order.

Some equations and their derivatives...

$$P_{0,0}(x) = 1, \tag{24}$$

$$P_{0,0}(\cos\theta) = 1, \tag{25}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{0,0} = 0. \tag{26}$$

$$P_{1,0}(x) = x, \tag{27}$$

$$P_{1,0}(\cos\theta) = \cos\theta, \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{1,0} = -\sin\theta. \tag{29}$$

$$P_{1,1}(x) = (1 - x^2)^{\frac{1}{2}}, \tag{30}$$

$$P_{1,1}(\cos\theta) = \sin\theta, \tag{31}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{1,1} = \cos\theta. \tag{32}$$

$$P_{2,0}(x) = \frac{1}{2}(3x^2 - 1), \tag{33}$$
$$P_{2,0}(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1), \tag{34}$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,0} = -3\cos\theta\sin\theta. \tag{35}$$

$$P_{2,1}(x) = 3x(1 - x^2)^{\frac{1}{2}}, \tag{36}$$
$$P_{2,1}(\cos\theta) = 3\cos\theta\sin\theta, \tag{37}$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,1} = 3(2\cos^2\theta - 1). \tag{38}$$

$$P_{2,2}(x) = 3(1 - x^2), \tag{39}$$
$$P_{2,2}(\cos\theta) = 3\sin^2\theta, \tag{40}$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,2} = 6\sin\theta\cos\theta. \tag{41}$$

$$P_{3,0}(x) = \frac{1}{2}(5x^3 - 3x), \tag{42}$$
$$P_{3,0}(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta), \tag{43}$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,0} = \frac{3}{2}\sin\theta(1 - 5\cos^2\theta). \tag{44}$$

$$P_{3,1}(x) = \frac{1}{2}\sqrt{(1 - x^2)}(15x^2 - 3), \tag{45}$$
$$P_{3,1}(\cos\theta) = \frac{3}{2}\sin\theta(5\cos^2\theta - 1), \tag{46}$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,1} = \frac{3}{2}\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}(5\cos^2\theta - 1) + (5\cos^2\theta - 1)\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\right], \tag{47}$$
$$= \frac{3}{2}(15\cos^3\theta - 11\cos\theta). \tag{48}$$

$$P_{3,2}(x) = 15x(1 - x^2), \tag{49}$$
$$P_{3,2}(\cos\theta) = 15\cos\theta\sin^2\theta, \tag{50}$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,2} = 15\sin\theta(3\cos^2\theta - 1). \tag{51}$$

$$P_{3,3}(x) = 15(1 - x^2)^{\frac{3}{2}}, \tag{52}$$
$$P_{3,3}(\cos\theta) = 15\sin^3\theta, \tag{53}$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,3} = 45\sin^2\theta\cos\theta. \tag{54}$$

$$P_{4,0}(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \tag{55}$$
$$P_{4,0}(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3), \tag{56}$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,0} = \frac{5}{2}(7\cos^3\theta - 3\cos\theta). \tag{57}$$

$$P_{4,1}(x) = \frac{5}{2}(7x^3 - 3x)(1-x^2)^{\frac{1}{2}}, \tag{58}$$

$$P_{4,1}(\cos\theta) = \frac{5}{2}\sin\theta(7\cos^3\theta - 3\cos\theta), \tag{59}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,1} = \frac{5}{2}\left[(7\cos^3\theta - 3\cos\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta + \sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}(7\cos^3\theta - 3\cos\theta)\right], \tag{60}$$

$$= \frac{5}{2}(28\cos^4\theta - 27\cos^3\theta + 3). \tag{61}$$

$$P_{4,2}(x) = \frac{15}{2}(7x^2 - 1)(1-x^2), \tag{62}$$

$$P_{4,2}(\cos\theta) = \frac{15}{2}\sin^2\theta(7\cos^2\theta - 1), \tag{63}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,2} = \frac{15}{2}\left[(7\cos^2\theta - 1)\frac{\mathrm{d}}{\mathrm{d}\theta}\sin^2\theta + \sin^2\theta\frac{\mathrm{d}}{\mathrm{d}\theta}(7\cos^2\theta - 1)\right], \tag{64}$$

$$= 30\cos\theta\sin\theta(7\cos^2\theta - 4). \tag{65}$$

$$P_{4,3}(x) = 105x(1-x^2)^{\frac{3}{2}}, \tag{66}$$

$$P_{4,3}(\cos\theta) = 105\cos\theta\sin^3\theta, \tag{67}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,3} = 105\left[\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\sin^3\theta + \sin^3\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\cos\theta\right], \tag{68}$$

$$= 105\sin^2\theta(4\cos^2\theta - 1). \tag{69}$$

$$P_{4,4}(x) = 105(1-x^2)^2, \tag{70}$$

$$P_{4,4}(\cos\theta) = 105\sin^4\theta, \tag{71}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,4} = 420\sin^3\theta\cos\theta. \tag{72}$$

2.1 Recurrence Relations

There are three recurrence relations which can be used to calculate the associate polynomials from lower order/degree polynomials.

A: $m < n-1$

$$P_{n,m}(\cos\theta) = \frac{1}{n-m}\left[(2n-1)\cos\theta P_{n-1,m} - (n+m-1)P_{n-2,m}\right] \tag{73}$$

B: $m = n-1$

$$P_{n,m}(\cos\theta) = (2n-1)\sin\theta P_{n-1,m-1}, \tag{74}$$

C: $m = n$

$$P_{n,m}(\cos\theta) = (2n-1)\sin\theta P_{n-1,m-1}. \tag{75}$$

Examples for case A:

$$
\begin{aligned}
P_{2,0} &= \frac{1}{2}\left[3\cos\theta P_{1,0} - P_{0,0}\right] && (76)\\
&= \frac{1}{2}(3\cos^2\theta - 1) && (77)\\
P_{3,0} &= \frac{1}{3}\left[5\cos\theta P_{2,0} - 2P_{1,0}\right] && (78)\\
&= \frac{1}{2}(5\cos^3\theta - 3\cos\theta) && (79)\\
P_{3,1} &= \frac{1}{2}\left[5\cos\theta P_{2,1} - 3P_{1,1}\right] && (80)\\
&= \frac{3}{2}\sin\theta\left[5\cos^2\theta - 1\right] && (81)\\
P_{4,0} &= \frac{1}{4}\left[7\cos\theta P_{3,0} - 3P_{2,0}\right] && (82)\\
&= \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3) && (83)\\
P_{4,1} &= \frac{1}{3}\left[7\cos\theta P_{3,1} - 4P_{2,1}\right] && (84)\\
&= \frac{5}{2}\sin\theta\cos\theta(7\cos^2\theta - 3) && (85)\\
P_{4,2} &= \frac{1}{2}\left[7\cos\theta P_{3,2} - 5P_{2,2}\right] && (86)\\
&= \frac{15}{2}\sin^2\theta(7\cos^2\theta - 1) && (87)
\end{aligned}
$$

Examples for case B:

$$
\begin{aligned}
P_{2,1} &= 3\sin\theta P_{1,0} && (88)\\
&= 3\sin\theta\cos\theta && (89)\\
P_{3,2} &= 5\sin\theta P_{2,1} && (90)\\
&= 15\sin^2\theta\cos\theta && (91)\\
P_{4,3} &= 7\sin\theta P_{3,2} && (92)\\
&= 105\sin^3\theta\cos\theta && (93)
\end{aligned}
$$

Examples for case C (I think this rule is basically the same as B):

$$
\begin{aligned}
P_{1,1} &= \sin\theta P_{0,0} && (94)\\
&= \sin\theta && (95)\\
P_{2,2} &= 3\sin\theta P_{1,1} && (96)\\
&= 3\sin^2\theta && (97)\\
P_{3,3} &= 5\sin\theta P_{2,2} && (98)\\
&= 15\sin^3\theta && (99)\\
P_{4,4} &= 7\sin\theta P_{3,3} && (100)\\
&= 105\sin^4\theta && (101)
\end{aligned}
$$

The derivatives can be calculated using similar rules:

A: $m < n-1$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{n,m} = \frac{1}{n-m}\left[(2n-1)\left(\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{n-1,m} - \sin\theta P_{n-1,m}\right) - (n+m-1)\frac{\mathrm{d}}{\mathrm{d}\theta}P_{n-2,m}\right] \tag{102}$$

B: $m = n-1$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{n,m} = (2n-1)\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{n-1,m-1} + \cos\theta P_{n-1,m-1}\right] \tag{103}$$

C: $m = n$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{n,m} = (2n-1)\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{n-1,m-1} + \cos\theta P_{n-1,m-1}\right] \tag{104}$$

Examples for case A:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,0} = \frac{1}{2}\left[3\left(\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{1,0} - \sin\theta P_{1,0}\right) - \frac{\mathrm{d}}{\mathrm{d}\theta}P_{0,0}\right] \tag{105}$$

$$= \frac{1}{2}(3\cos^2\theta - 1) \tag{106}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,0} = \frac{1}{3}\left[5\left(\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,0} - \sin\theta P_{2,0}\right) - 2\frac{\mathrm{d}}{\mathrm{d}\theta}P_{1,0}\right] \tag{107}$$

$$= \frac{1}{2}(5\cos^3\theta - 3\cos\theta) \tag{108}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,1} = \frac{1}{2}\left[5\left(\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,1} - \sin\theta P_{2,1}\right) - 3\frac{\mathrm{d}}{\mathrm{d}\theta}P_{1,1}\right] \tag{109}$$

$$= \frac{3}{2}\sin\theta(5\cos^2\theta - 1) \tag{110}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,0} = \frac{1}{4}\left[7\left(\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,0} - \sin\theta P_{3,0}\right) - 3\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,0}\right] \tag{111}$$

$$= \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3) \tag{112}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,1} = \frac{1}{3}\left[7\left(\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,1} - \sin\theta P_{3,1}\right) - 4\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,1}\right] \tag{113}$$

$$= \frac{5}{2}\sin\theta\cos\theta(7\cos^2\theta - 3) \tag{114}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,2} = \frac{1}{2}\left[7\left(\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,2} - \sin\theta P_{3,2}\right) - 5\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,2}\right] \tag{115}$$

$$= \frac{15}{2}\sin^2\theta(7\cos^2\theta - 1) \tag{116}$$

Examples for case B:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,1} = 3\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{1,0} + \cos\theta P_{1,0}\right] \tag{117}$$

$$= 6\cos^2\theta - 3 \tag{118}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,2} = 5\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,1} + \cos\theta P_{2,1}\right] \tag{119}$$

$$= 15\sin\theta(3\cos^2\theta - 1) \tag{120}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,3} = 7\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,2} + \cos\theta P_{3,2}\right] \tag{121}$$

$$= 105\sin^2\theta(4\cos^2\theta - 1) \tag{122}$$

Examples for case C:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,2} = 3\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{1,1} + \cos\theta P_{1,1}\right] \tag{123}$$

$$= 6\cos\theta\sin\theta \tag{124}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,3} = 5\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,2} + \cos\theta P_{2,2}\right] \tag{125}$$

$$= 45\cos\theta\sin^2\theta \tag{126}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,4} = 7\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,3} + \cos\theta P_{3,3}\right] \tag{127}$$

$$= 420\cos\theta\sin^3\theta \tag{128}$$

The recurrence relations listed above can be used to calculate any of the associated Legendre polynomials provided that $P_{0,0}$, $P_{1,0}$ and $P_{1,1}$ are defined initially. Here are some examples below, where the left polynomial is formed using the polynomial(s) to the right of the arrow and the letter in brackets corresponds to the three rules listed above:

$$P_{2,0} \leftarrow P_{1,0}, P_{0,0} \qquad (C) \tag{129}$$

$$P_{2,1} \leftarrow P_{1,0} \qquad (B) \tag{130}$$

$$P_{2,2} \leftarrow P_{1,1} \qquad (A) \tag{131}$$

$$P_{3,0} \leftarrow P_{2,0}, P_{1,0} \qquad (C) \tag{132}$$

$$P_{3,1} \leftarrow P_{2,1}, P_{1,1} \qquad (C) \tag{133}$$

$$P_{3,2} \leftarrow P_{2,1} \qquad (B) \tag{134}$$

$$P_{3,3} \leftarrow P_{2,2} \qquad (A) \tag{135}$$

$$P_{4,0} \leftarrow P_{3,0}, P_{2,0} \qquad (C) \tag{136}$$

$$P_{4,1} \leftarrow P_{3,1}, P_{2,1} \qquad (C) \tag{137}$$

$$P_{4,2} \leftarrow P_{3,2}, P_{2,2} \qquad (C) \tag{138}$$

$$P_{4,3} \leftarrow P_{3,2} \qquad (B) \tag{139}$$

$$P_{4,4} \leftarrow P_{3,3} \qquad (A) \tag{140}$$

Note that the same rules apply for the derivatives too.

3 Schmidt Normalized Legendre Polynomials

The Schmidt normalized Legendre polynomials, P_n^m, are defined by

$$P_n^m = S_n^m P_{n,m}, \tag{141}$$

where

$$S_{n,m} = \sqrt{(2-\delta_m^0)\frac{(n-m)!}{(n+m)!}}, \tag{142}$$

and $\delta_m^0 = 1$ when $m = 0$, and is $\delta_m^0 = 0$ otherwise.

References

Ferrers, N. (1877), *An elementary treatise on spherical harmonics and subjects connected with them.*