Magnetic Field Model Equations and a Load of Legendre Polynomials

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1 Legendre Polynomials

This is the form of a Legendre polynomial (Rodrigues' formula):

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \tag{1}$$

where n is the degree of the polynomial. The first 5 degrees (0-4) are shown below:

$$P_0(x) = \frac{\mathrm{d}^0}{\mathrm{d}x^0}(x^2-1)^0 = 1, \tag{2}$$

$$P_1(x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1) = x, \tag{3}$$

$$P_2(x) = \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2-1)^2 = \frac{1}{2}(3x^2-1), \tag{4}$$

$$P_3(x) = \frac{1}{48}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^2-1)^3 = \frac{1}{2}(5x^3-3x), \tag{5}$$

$$P_4(x) = \frac{1}{384}\frac{\mathrm{d}^4}{\mathrm{d}x^4}(x^2-1)^4 = \frac{1}{8}(35x^4-30x^2+3), \tag{6}$$

and x can be substituted for $\cos\theta$:

$$P_0(\cos\theta) = 1, \tag{7}$$

$$P_1(\cos\theta) = \cos\theta, \tag{8}$$

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1), \tag{9}$$

$$P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta), \tag{10}$$

$$P_4(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30x^2 + 3). \tag{11}$$

1.1 Derivatives

These derivatives of the above equations 2–6 with respect to x will come in handy later on...

For equation 2:

$$\frac{\mathrm{d}}{\mathrm{d}x}P_0(x) = 0. \tag{12}$$

For equation 3:

$$\frac{\mathrm{d}}{\mathrm{d}x}P_1(x) = 1. \tag{13}$$

For equation 4:

$$\frac{\mathrm{d}}{\mathrm{d}x} P_2(x) = 3x, \tag{14}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} P_2(x) = 3. \tag{15}$$

For equation 5:

$$\frac{\mathrm{d}}{\mathrm{d}x} P_3(x) = \frac{1}{2}(15x^2 - 3), \tag{16}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} P_3(x) = 15x, \tag{17}$$
$$\frac{\mathrm{d}^3}{\mathrm{d}x^3} P_3(x) = 15. \tag{18}$$

For equation 6:

$$\frac{\mathrm{d}}{\mathrm{d}x} P_4(x) = \frac{5}{2}(7x^3 - 3x), \tag{19}$$
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} P_4(x) = \frac{15}{2}(7x^2 - 1), \tag{20}$$
$$\frac{\mathrm{d}^3}{\mathrm{d}x^3} P_4(x) = 105x, \tag{21}$$
$$\frac{\mathrm{d}^4}{\mathrm{d}x^4} P_4(x) = 105. \tag{22}$$

2 Ferrers Normalized Legendre Polynomials

The associated Legendre polynomials are defined by *Ferrers* (1877) as:

$$P_{n,m}(x) = (1 - x^2)^{\frac{m}{2}} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_n(x), \tag{23}$$

where m is the order.

Some equations and their derivatives...

$$P_{0,0}(x) = 1, \tag{24}$$
$$P_{0,0}(\cos\theta) = 1, \tag{25}$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta} P_{0,0} = 0. \tag{26}$$

$$P_{1,0}(x) = x, \tag{27}$$
$$P_{1,0}(\cos\theta) = \cos\theta, \tag{28}$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta} P_{1,0} = -\sin\theta. \tag{29}$$

$$P_{1,1}(x) = (1 - x^2)^{\frac{1}{2}}, \tag{30}$$
$$P_{1,1}(\cos\theta) = \sin\theta, \tag{31}$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta} P_{1,1} = \cos\theta. \tag{32}$$

$$P_{2,0}(x) = \frac{1}{2}(3x^2 - 1), \tag{33}$$

$$P_{2,0}(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1), \tag{34}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,0} = -3\cos\theta\sin\theta. \tag{35}$$

$$P_{2,1}(x) = 3x(1 - x^2)^{\frac{1}{2}}, \tag{36}$$

$$P_{2,1}(\cos\theta) = 3\cos\theta\sin\theta, \tag{37}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,1} = 3(2\cos^2\theta - 1). \tag{38}$$

$$P_{2,2}(x) = 3(1 - x^2), \tag{39}$$

$$P_{2,2}(\cos\theta) = 3\sin^2\theta, \tag{40}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,2} = 6\sin\theta\cos\theta. \tag{41}$$

$$P_{3,0}(x) = \frac{1}{2}(5x^3 - 3x), \tag{42}$$

$$P_{3,0}(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta), \tag{43}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,0} = \frac{3}{2}\sin\theta(1 - 5\cos^2\theta). \tag{44}$$

$$P_{3,1}(x) = \frac{1}{2}\sqrt{(1 - x^2)}(15x^2 - 3), \tag{45}$$

$$P_{3,1}(\cos\theta) = \frac{3}{2}\sin\theta(5\cos^2\theta - 1), \tag{46}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,1} = \frac{3}{2}\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}(5\cos^2\theta - 1) + (5\cos^2\theta - 1)\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\right], \tag{47}$$

$$= \frac{3}{2}(15\cos^3\theta - 11\cos\theta). \tag{48}$$

$$P_{3,2}(x) = 15x(1 - x^2), \tag{49}$$

$$P_{3,2}(\cos\theta) = 15\cos\theta\sin^2\theta, \tag{50}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,2} = 15\sin\theta(3\cos^2\theta - 1). \tag{51}$$

$$P_{3,3}(x) = 15(1 - x^2)^{\frac{3}{2}}, \tag{52}$$

$$P_{3,3}(\cos\theta) = 15\sin^3\theta, \tag{53}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,3} = 45\sin^2\theta\cos\theta. \tag{54}$$

$$P_{4,0}(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \tag{55}$$

$$P_{4,0}(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3), \tag{56}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,0} = \frac{5}{2}(7\cos^3\theta - 3\cos\theta). \tag{57}$$

$$P_{4,1}(x) = \frac{5}{2}(7x^3 - 3x)(1 - x^2)^{\frac{1}{2}}, \tag{58}$$

$$P_{4,1}(\cos\theta) = \frac{5}{2}\sin\theta(7\cos^3\theta - 3\cos\theta), \tag{59}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,1} = \frac{5}{2}\left[(7\cos^3\theta - 3\cos\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta + \sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}(7\cos^3\theta - 3\cos\theta)\right], \tag{60}$$

$$= \frac{5}{2}(28\cos^4\theta - 27\cos^3\theta + 3). \tag{61}$$

$$P_{4,2}(x) = \frac{15}{2}(7x^2 - 1)(1 - x^2), \tag{62}$$

$$P_{4,2}(\cos\theta) = \frac{15}{2}\sin^2\theta(7\cos^2\theta - 1), \tag{63}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,2} = \frac{15}{2}\left[(7\cos^2\theta - 1)\frac{\mathrm{d}}{\mathrm{d}\theta}\sin^2\theta + \sin^2\theta\frac{\mathrm{d}}{\mathrm{d}\theta}(7\cos^2\theta - 1)\right], \tag{64}$$

$$= 30\cos\theta\sin\theta(7\cos^2\theta - 4). \tag{65}$$

$$P_{4,3}(x) = 105x(1 - x^2)^{\frac{3}{2}}, \tag{66}$$

$$P_{4,3}(\cos\theta) = 105\cos\theta\sin^3\theta, \tag{67}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,3} = 105\left[\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\sin^3\theta + \sin^3\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\cos\theta\right], \tag{68}$$

$$= 105\sin^2\theta(4\cos^2\theta - 1). \tag{69}$$

$$P_{4,4}(x) = 105(1 - x^2)^2, \tag{70}$$

$$P_{4,4}(\cos\theta) = 105\sin^4\theta, \tag{71}$$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,4} = 420\sin^3\theta\cos\theta. \tag{72}$$

2.1 Recurrence Relations

There are three recurrence relations which are used to calculate the associate polynomials from lower order/degree polynomials.

A: $m < n - 1$

$$P_{n,m}(\cos\theta) = \frac{1}{n - m}\left[(2n - 1)\cos\theta P_{n-1,m} - (n + m - 1)P_{n-2,m}\right] \tag{73}$$

B: $m = n - 1$

$$P_{n,m}(\cos\theta) = (2n - 1)\sin\theta P_{n-1,m-1}, \tag{74}$$

C: $m = n$

$$P_{n,m}(\cos\theta) = (2n - 1)\sin\theta P_{n-1,m-1}. \tag{75}$$

They are based on code used elsewhere, but the difference being that a factor of $(-1)^m$ was removed from rule C (and also from their calculation of $S_{n,m}$ as the seem to cancel each other). Also rule B uses sin here as opposed to cos (I think the original code had a mistake in it). These changes effectively make rules B and C identical. The examples below should all match with those derived directly from equations 1 and 23 in the previous subsection.

Examples for case A:

$$
\begin{aligned}
P_{2,0} &= \frac{1}{2}\left[3\cos\theta P_{1,0} - P_{0,0}\right] && (76)\\
&= \frac{1}{2}(3\cos^2\theta - 1) && (77)\\
P_{3,0} &= \frac{1}{3}\left[5\cos\theta P_{2,0} - 2P_{1,0}\right] && (78)\\
&= \frac{1}{2}(5\cos^3\theta - 3\cos\theta) && (79)\\
P_{3,1} &= \frac{1}{2}\left[5\cos\theta P_{2,1} - 3P_{1,1}\right] && (80)\\
&= \frac{3}{2}\sin\theta\left[5\cos^2\theta - 1\right] && (81)\\
P_{4,0} &= \frac{1}{4}\left[7\cos\theta P_{3,0} - 3P_{2,0}\right] && (82)\\
&= \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3) && (83)\\
P_{4,1} &= \frac{1}{3}\left[7\cos\theta P_{3,1} - 4P_{2,1}\right] && (84)\\
&= \frac{5}{2}\sin\theta\cos\theta(7\cos^2\theta - 3) && (85)\\
P_{4,2} &= \frac{1}{2}\left[7\cos\theta P_{3,2} - 5P_{2,2}\right] && (86)\\
&= \frac{15}{2}\sin^2\theta(7\cos^2\theta - 1) && (87)
\end{aligned}
$$

Examples for case B:

$$
\begin{aligned}
P_{2,1} &= 3\sin\theta P_{1,0} && (88)\\
&= 3\sin\theta\cos\theta && (89)\\
P_{3,2} &= 5\sin\theta P_{2,1} && (90)\\
&= 15\sin^2\theta\cos\theta && (91)\\
P_{4,3} &= 7\sin\theta P_{3,2} && (92)\\
&= 105\sin^3\theta\cos\theta && (93)
\end{aligned}
$$

Examples for case C (I think this rule is basically the same as B):

$$
\begin{aligned}
P_{1,1} &= \sin\theta P_{0,0} && (94)\\
&= \sin\theta && (95)\\
P_{2,2} &= 3\sin\theta P_{1,1} && (96)\\
&= 3\sin^2\theta && (97)\\
P_{3,3} &= 5\sin\theta P_{2,2} && (98)\\
&= 15\sin^3\theta && (99)\\
P_{4,4} &= 7\sin\theta P_{3,3} && (100)\\
&= 105\sin^4\theta && (101)
\end{aligned}
$$

The derivatives can be calculated using similar rules:

A: $m < n-1$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{n,m} = \frac{1}{n-m}\left[(2n-1)\left(\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{n-1,m} - \sin\theta P_{n-1,m}\right) - (n+m-1)\frac{\mathrm{d}}{\mathrm{d}\theta}P_{n-2,m}\right] \tag{102}$$

B: $m = n-1$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{n,m} = (2n-1)\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{n-1,m-1} + \cos\theta P_{n-1,m-1}\right] \tag{103}$$

C: $m = n$

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{n,m} = (2n-1)\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{n-1,m-1} + \cos\theta P_{n-1,m-1}\right] \tag{104}$$

Examples for case A:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,0} &= \frac{1}{2}\left[3\left(\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{1,0} - \sin\theta P_{1,0}\right) - \frac{\mathrm{d}}{\mathrm{d}\theta}P_{0,0}\right] && (105)\\
&= \frac{1}{2}(3\cos^2\theta - 1) && (106)\\
\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,0} &= \frac{1}{3}\left[5\left(\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,0} - \sin\theta P_{2,0}\right) - 2\frac{\mathrm{d}}{\mathrm{d}\theta}P_{1,0}\right] && (107)\\
&= \frac{1}{2}(5\cos^3\theta - 3\cos\theta) && (108)\\
\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,1} &= \frac{1}{2}\left[5\left(\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,1} - \sin\theta P_{2,1}\right) - 3\frac{\mathrm{d}}{\mathrm{d}\theta}P_{1,1}\right] && (109)\\
&= \frac{3}{2}\sin\theta(5\cos^2\theta - 1) && (110)\\
\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,0} &= \frac{1}{4}\left[7\left(\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,0} - \sin\theta P_{3,0}\right) - 3\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,0}\right] && (111)\\
&= \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3) && (112)\\
\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,1} &= \frac{1}{3}\left[7\left(\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,1} - \sin\theta P_{3,1}\right) - 4\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,1}\right] && (113)\\
&= \frac{5}{2}\sin\theta\cos\theta(7\cos^2\theta - 3) && (114)\\
\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,2} &= \frac{1}{2}\left[7\left(\cos\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,2} - \sin\theta P_{3,2}\right) - 5\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,2}\right] && (115)\\
&= \frac{15}{2}\sin^2\theta(7\cos^2\theta - 1) && (116)
\end{aligned}$$

Examples for case B:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,1} &= 3\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{1,0} + \cos\theta P_{1,0}\right] && (117)\\
&= 6\cos^2\theta - 3 && (118)\\
\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,2} &= 5\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,1} + \cos\theta P_{2,1}\right] && (119)\\
&= 15\sin\theta(3\cos^2\theta - 1) && (120)\\
\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,3} &= 7\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,2} + \cos\theta P_{3,2}\right] && (121)\\
&= 105\sin^2\theta(4\cos^2\theta - 1) && (122)
\end{aligned}$$

Examples for case C:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,2} = 3\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{1,1} + \cos\theta P_{1,1}\right] \tag{123}$$
$$= 6\cos\theta\sin\theta \tag{124}$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,3} = 5\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{2,2} + \cos\theta P_{2,2}\right] \tag{125}$$
$$= 45\cos\theta\sin^2\theta \tag{126}$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta}P_{4,4} = 7\left[\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}P_{3,3} + \cos\theta P_{3,3}\right] \tag{127}$$
$$= 420\cos\theta\sin^3\theta \tag{128}$$

The recurrence relations listed above can be used to calculate any of the associated Legendre polynomials provided that $P_{0,0}$, $P_{1,0}$ and $P_{1,1}$ are defined initially. Here are some examples below, where the left polynomial is formed using the polynomial(s) to the right of the arrow and the letter in brackets corresponds to the three rules listed above:

$$P_{2,0} \leftarrow P_{1,0}, P_{0,0} \qquad (C) \tag{129}$$
$$P_{2,1} \leftarrow P_{1,0} \qquad (B) \tag{130}$$
$$P_{2,2} \leftarrow P_{1,1} \qquad (A) \tag{131}$$

$$P_{3,0} \leftarrow P_{2,0}, P_{1,0} \qquad (C) \tag{132}$$
$$P_{3,1} \leftarrow P_{2,1}, P_{1,1} \qquad (C) \tag{133}$$
$$P_{3,2} \leftarrow P_{2,1} \qquad (B) \tag{134}$$
$$P_{3,3} \leftarrow P_{2,2} \qquad (A) \tag{135}$$

$$P_{4,0} \leftarrow P_{3,0}, P_{2,0} \qquad (C) \tag{136}$$
$$P_{4,1} \leftarrow P_{3,1}, P_{2,1} \qquad (C) \tag{137}$$
$$P_{4,2} \leftarrow P_{3,2}, P_{2,2} \qquad (C) \tag{138}$$
$$P_{4,3} \leftarrow P_{3,2} \qquad (B) \tag{139}$$
$$P_{4,4} \leftarrow P_{3,3} \qquad (A) \tag{140}$$

Note that the same rules apply for the derivatives too.

3 Schmidt Normalized Legendre Polynomials

The Schmidt normalized Legendre polynomials, P_n^m, are defined by

$$P_n^m = S_n^m P_{n,m}, \tag{141}$$

where

$$S_{n,m} = \sqrt{(2-\delta_m^0)\frac{(n-m)!}{(n+m)!}}, \tag{142}$$

and $\delta_m^0 = 1$ when $m = 0$, and is $\delta_m^0 = 0$ otherwise.

4 Calculating the Magnetic Field Model

The magnetic field, $\mathbf{B}$, is calculated from a scalar potential,

$$\mathbf{B} = -\nabla V, \tag{143}$$

where $\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)$ in spherical polar coordinates, V is defined as (e.g. *Connerney et al.*, 1998; *Winch et al.*, 2005),

$$V = a \sum_{n=1}^{n_{max}} \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^{n} \left\{P_n^m(\cos\theta)\left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right]\right\}, \tag{144}$$

and a is the radius of Jupiter (71,398 km).
Using equation 143 on 144, each component of the magnetic field is defined by,

$$B_r = -\frac{\partial V}{\partial r} = \sum_{n=1}^{n_{max}} \left(\frac{a}{r}\right)^{n+2} (n+1) \sum_{m=0}^{n} \left\{P_n^m(\cos\theta)\left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right]\right\}, \tag{145}$$

$$B_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\sum_{n=1}^{n_{max}} \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} \left\{\frac{\mathrm{d}}{\mathrm{d}\theta}P_n^m(\cos\theta)\left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right]\right\}, \tag{146}$$

$$B_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = -\frac{1}{\sin\theta} \sum_{n=1}^{n_{max}} \left(\frac{a}{r}\right)^{n+2} \sum_{m=0}^{n} \left\{mP_n^m(\cos\theta)\left[h_n^m \cos(m\phi) - g_n^m \sin(m\phi)\right]\right\}. \tag{147}$$

In the code, the above equations use $a = 1$ and r in units of R_J.

4.1 Cartesian Solution

Possibly dodgy derivation of the Cartesian solution to equation 143, i.e.

$$\mathbf{B} = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) V. \tag{148}$$

This will be done by splitting equation 144 into several parts and combining them at the end...

4.1.1 Part 1

The derivatives of

$$a\left(\frac{a}{r}\right)^{n+1}, \tag{149}$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and we will define $u = a/r$.

$$\frac{\partial}{\partial x}\left[a\left(\frac{a}{r}\right)^{n+1}\right] = a\frac{\mathrm{d}u}{\mathrm{d}u}u^{n+1} \cdot \frac{\mathrm{d}u}{\mathrm{d}r} \cdot \frac{\mathrm{d}r}{\mathrm{d}x}, \tag{150}$$

$$= a\left((n+1)u^n\right) \cdot \left(-\frac{u^2}{a}\right) \cdot \left(\frac{a}{r}\right), \tag{151}$$

$$= -(n+1)\left(\frac{x}{r}\right)\left(\frac{a}{r}\right)^{n+2}. \tag{152}$$

All three components have similar solutions:

$$\frac{\partial}{\partial x}\left[a\left(\frac{a}{r}\right)^{n+1}\right] = -(n+1)\left(\frac{x}{r}\right)\left(\frac{a}{r}\right)^{n+2}, \tag{153}$$

$$\frac{\partial}{\partial y}\left[a\left(\frac{a}{r}\right)^{n+1}\right] = -(n+1)\left(\frac{y}{r}\right)\left(\frac{a}{r}\right)^{n+2}, \tag{154}$$

$$\frac{\partial}{\partial z}\left[a\left(\frac{a}{r}\right)^{n+1}\right] = -(n+1)\left(\frac{z}{r}\right)\left(\frac{a}{r}\right)^{n+2}. \tag{155}$$

4.1.2 Part 2

The derivatives of $P_n^m(\cos\theta)$ for each component can be done using the chain rule, e.g.:

$$\frac{\partial}{\partial x}P_n^m(\cos\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}P_n^m(\cos\theta)\frac{\partial\theta}{\partial x}. \tag{156}$$

Using

$$\theta = \arccos \frac{z}{r} = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \tag{157}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}u} \arccos u = \frac{-1}{\sqrt{1 - u^2}}, \tag{158}$$

the derivatives of θ are:

$$\frac{\partial \theta}{\partial x} = \frac{xz}{\rho r^2}, \tag{159}$$

$$\frac{\partial \theta}{\partial y} = \frac{yz}{\rho r^2}, \tag{160}$$

$$\frac{\partial \theta}{\partial z} = -\frac{\rho}{r^2}, \tag{161}$$

where $\rho = \sqrt{x^2 + y^2}$.

Legendre polynomials are therefore,

$$\frac{\partial}{\partial x} P_n^m(\cos\theta) = \left(\frac{xz}{\rho r^2}\right) \frac{\mathrm{d}}{\mathrm{d}\theta} P_n^m(\cos\theta), \tag{162}$$

$$\frac{\partial}{\partial y} P_n^m(\cos\theta) = \left(\frac{yz}{\rho r^2}\right) \frac{\mathrm{d}}{\mathrm{d}\theta} P_n^m(\cos\theta), \tag{163}$$

$$\frac{\partial}{\partial z} P_n^m(\cos\theta) = \left(-\frac{\rho}{r^2}\right) \frac{\mathrm{d}}{\mathrm{d}\theta} P_n^m(\cos\theta), \tag{164}$$

where $\frac{\mathrm{d}}{\mathrm{d}\theta} P_n^m(\cos\theta)$ is already calculated using the spherical polar version of the code (see section 2.1).

4.1.3 Part 3

This section deals with the derivative of

$$g_n^m \cos(m\phi) + h_n^m \sin(m\phi), \tag{165}$$

which will be treated as a chain rule, e.g.

$$\frac{\partial}{\partial x} \left(g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right) = \frac{\mathrm{d}}{\mathrm{d}\phi} \left(g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right) \cdot \frac{\partial \phi}{\partial x}. \tag{166}$$

The first step in that chain rule is as follows (and used in the spherical polar version),

$$\frac{\mathrm{d}}{\mathrm{d}\phi} \left(g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right) = m \left(h_n^m \cos(m\phi) - g_n^m \sin(m\phi)\right). \tag{167}$$

Using $\phi = \arctan(y/x)$ and the fact that

$$\frac{\mathrm{d}}{\mathrm{d}u} \arctan u = \frac{1}{1 + u^2}, \tag{168}$$

the three derivatives of ϕ are,

$$\frac{\partial \phi}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2} = \frac{-y}{\rho^2}, \tag{169}$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{\rho^2}, \tag{170}$$

$$\frac{\partial \phi}{\partial z} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot 0 = 0. \tag{171}$$

Combining equation 167 with equations 169-171 gives,

$$\frac{\partial}{\partial x} \left(g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right) = -m\frac{y}{\rho^2} \left(h_n^m \cos(m\phi) - g_n^m \sin(m\phi)\right), \tag{172}$$

$$\frac{\partial}{\partial y} \left(g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right) = m\frac{x}{\rho^2} \left(h_n^m \cos(m\phi) - g_n^m \sin(m\phi)\right), \tag{173}$$

$$\frac{\partial}{\partial z} \left(g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right) = 0 \tag{174}$$

4.1.4 Combined solution

Using the three parts above to solve equation 148:

$$
\begin{aligned}
B_x = -\frac{\partial V}{\partial x} = &\sum_{n=1}^{n_{max}} (n+1) \left(\frac{x}{r}\right) \left(\frac{a}{r}\right)^{(n+2)} \sum_{m=0}^{n} \left\{P_n^m(\cos\theta) \left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right]\right\} \\
&- a \sum_{n=1}^{n_{max}} \left(\frac{a}{r}\right)^{(n+1)} \sum_{m=0}^{n} \left\{ \left(\frac{xz}{\rho r^2}\right) \frac{\mathrm{d}}{\mathrm{d}\theta} P_n^m(\cos\theta) \left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right] \right. \\
&\left. + m \frac{y}{\rho^2} P_n^m(\cos\theta) \left[g_n^m \sin(m\phi) - h_n^m \cos(m\phi)\right] \right\}
\end{aligned} \tag{175}
$$

$$
\begin{aligned}
B_y = -\frac{\partial V}{\partial y} = &\sum_{n=1}^{n_{max}} (n+1) \left(\frac{y}{r}\right) \left(\frac{a}{r}\right)^{(n+2)} \sum_{m=0}^{n} \left\{P_n^m(\cos\theta) \left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right]\right\} \\
&- a \sum_{n=1}^{n_{max}} \left(\frac{a}{r}\right)^{(n+1)} \sum_{m=0}^{n} \left\{ \left(\frac{yz}{\rho r^2}\right) \frac{\mathrm{d}}{\mathrm{d}\theta} P_n^m(\cos\theta) \left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right] \right. \\
&\left. + m \frac{x}{\rho^2} P_n^m(\cos\theta) \left[h_n^m \cos(m\phi) - g_n^m \sin(m\phi)\right] \right\}
\end{aligned} \tag{176}
$$

$$
\begin{aligned}
B_z = -\frac{\partial V}{\partial z} = &\sum_{n=1}^{n_{max}} (n+1) \left(\frac{z}{r}\right) \left(\frac{a}{r}\right)^{(n+2)} \sum_{m=0}^{n} \left\{P_n^m(\cos\theta) \left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right]\right\} \\
&- a \sum_{n=1}^{n_{max}} \left(\frac{a}{r}\right)^{(n+1)} \sum_{m=0}^{n} \left\{ \left(\frac{\rho}{r^2}\right) \frac{\mathrm{d}}{\mathrm{d}\theta} P_n^m(\cos\theta) \left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right] \right\}.
\end{aligned} \tag{177}
$$

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