`vsmodel` Derivation

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This document describes the derivation of the model equations for the `vsmodel` package.

1 Derivation of the Model

In this section we derive the components of the Volland-Stern electric field model (*Volland*, 1973; *Stern*, 1975) in polar coordinates (section 1.1) and cylindrical coordinates (section 1.2).

As in *Zhao et al.* (2017), we start with an electrostatic potential:

$$U(r, \phi) = -\frac{a}{r} - br^{\gamma} \sin\phi, \tag{1}$$

where r is the radial distance in units of R_E; ϕ is the azimuth; a is the corotation constant, a=92.4 kV R_E; γ is the shielding exponent ($\gamma = 2$); and b is related to the convection electric field using

$$b = \frac{0.045}{(1.0 - 0.159Kp + 0.0093K_p^2)^3} \text{ (kV } R_E^{-2}) \tag{2}$$

from *Maynard and Chen* (1975).

The electric field is then obtained using $\mathbf{E} = -\nabla U$.

The following sections derive the radial, E_r, and azimuthal, E_ϕ, components of the electric field using two slightly different methods, ultimately with the same result (thankfully) - this is likely due to the fact that we are only defining the two component in the equatorial plane.

1.1 Polar Coordinates

In the case where we use polar coordinates, the electric field is calculated using:

$$\mathbf{E} = -\nabla U = -\frac{\partial U}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\phi}, \tag{3}$$

where

$$\frac{\partial U}{\partial r} = ar^{-2} - b\gamma r^{\gamma-1}\sin\phi, \tag{4}$$
$$\frac{\partial U}{\partial \theta} = 0, \tag{5}$$
$$\frac{\partial U}{\partial \phi} = -br^{\gamma}\cos\phi, \tag{6}$$

and because we are looking at the equatorial plane, $\theta = \pi$.
Which means that the electric field components become

$$E_r = -ar^{-2} + b\gamma r^{\gamma-1}\sin\phi, \tag{7}$$
$$E_\theta = 0, \tag{8}$$
$$E_\phi = br^{\gamma-1}\cos\phi. \tag{9}$$

1.2 Cylindrical Coordinates

Alternatively, we can try the same derivation using the cylindrical coordinate system to achieve effectively the same result. *In all honestly - I'm not 100% sure which I should be using, but both give the same result.*

The cylindrical equivalent to equation **??**:

$$\mathbf{E} = -\nabla U = -\frac{\partial U}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\phi} - \frac{\partial U}{\partial z}\hat{\mathbf{z}}, \tag{10}$$

where

$$\frac{\partial U}{\partial r} = ar^{-2} - b\gamma r^{\gamma-1}\sin\phi, \tag{11}$$
$$\frac{\partial U}{\partial \phi} = -br^{\gamma}\cos\phi, \tag{12}$$
$$\frac{\partial U}{\partial z} = 0. \tag{13}$$

From equations 10, 11, 12 and 13:

$$E_r = -ar^{-2} + b\gamma r^{\gamma-1}\sin\phi, \tag{14}$$
$$E_\phi = br^{\gamma-1}\cos\phi, \tag{15}$$
$$E_z = 0. \tag{16}$$

2 SM Model Field

The previous section describes the model in polar/cylindrical coordinates (they are equivalent at $z = 0$), here we convert the model to SM coordinates by rotating about the z-axis.

Considering an Electric field vector $\mathbf{E}(r, \phi)$ with components E_r and E_ϕ – we need to rotate this vector by ϕ to transform into SM coordinates.

Start by expressing the components of $\mathbf{E}$ in terms of some polar coordinates ρ and α:

$$E_r = \rho \cos \alpha, \tag{17}$$
$$E_\phi = \rho \sin \alpha, \tag{18}$$

then rotate by ϕ,

$$E_x = \rho \cos (\alpha + \phi), \tag{19}$$
$$E_y = \rho \sin (\alpha + \phi). \tag{20}$$

Using the trigonometric identities,

$$\sin (\alpha \pm \phi) = \sin \alpha \cos \phi \pm \cos \alpha \sin \phi, \tag{21}$$
$$\cos (\alpha \pm \phi) = \cos \alpha \cos \phi \mp \sin \alpha \sin \phi, \tag{22}$$

equations 19 and 20 become

$$E_x = \rho \cos \alpha \cos \phi - \rho \sin \alpha \sin \phi, \tag{23}$$
$$E_y = \rho \sin \alpha \cos \phi + \rho \cos \alpha \sin \phi. \tag{24}$$

Then, substituting in equations 17 and 18, gives

$$E_x = E_r \cos \phi - E_\phi \sin \phi, \tag{25}$$
$$E_y = E_r \sin \phi + E_\phi \cos \phi. \tag{26}$$

References

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