`vsmodel` Derivation

Matt James

January 18, 2021

This document describes the derivation of the model equations for the `vsmodel` package.

1 Derivation of the Model

In this section we derive the components of the Volland-Stern electric field model (*Volland*, 1973; *Stern*, 1975) in cylindrical coordinates. The Volland-Stern model starts with an electrostatic potential, $U(r, \phi)$, from which the electric field is obtained using

$$\mathbf{E} = -\nabla U, \tag{1}$$

where

$$\nabla U = \frac{\partial U}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial U}{\partial \phi}\hat{\phi} + \frac{\partial U}{\partial z}\hat{\mathbf{z}}. \tag{2}$$

The coordinate system is such that z lies along the dipole axis of the planet; x points approximately sunward where the magnetic equatorial plane intersects the plane containing both the dipole axis and the Earth-Sun line; the y axis points approximately duskward. In cylindrical coordinates: r is the radial distance from the z axis (i.e. $r = \sqrt{x^2 + y^2}$); ϕ is the azimuth, equal to 0 at noon where $\phi = \arctan y, x$.

The potential, U, is made up of a corotational component, U_{cor}, and a convection component, U_{cnv}. The corotation component used here is defined by

$$U_{cor} = -\frac{a}{r}, \tag{3}$$

where $a = 92.4$ keV is the corotation constant used in *Zhao et al.* (2017).

The electric field due to corotation, $\mathbf{E_{cor}}$, is given by:

$$E_r = -\frac{a}{r^2}, \tag{4}$$

$$E_\phi = 0, \tag{5}$$

$$E_z = 0, \tag{6}$$

where $\mathbf{E_{cor}} = [E_r, E_\phi, E_z]$.

There are two options for the convection component of this model - the simpler option is from *Maynard and Chen* (1975) and is derived in section 2; the more complicated option described by *Goldstein et al.* (2005) separates the convection component into solar wind electric field (section 3) and SAPS (section 4) parts. The overall electric field is simply obtained by adding the separate components of the model, i.e. $\mathbf{E} = \mathbf{E_{cor}} + \mathbf{E_{cnv}}$.

2 The *Maynard and Chen* (1975) E-field

This section describes the electric field due to convection as described by *Maynard and Chen* (1975). The potential used is

$$U_{mc} = -A_{mc} r^{\gamma} \sin\phi, \tag{7}$$

where

$$A_{mc} = \frac{0.045}{(1.0 - 0.159Kp + 0.0093K_p^2)^3} \text{ (kV } R_E^{-2}) \tag{8}$$

and $\gamma = 2$ is the shielding parameter.

The electric field components become

$$E_r = \gamma A_{mc} r^{(\gamma-1)} \sin\phi, \tag{9}$$

$$E_\phi = A_{mc} r^{(\gamma-1)} \cos\phi, \tag{10}$$

$$E_z = 0. \tag{11}$$

3 Solar Wind Electric Field

This section uses the electric field due to the solar wind propagation past the Earth as described in *Goldstein et al.* (2005). The electric field due to the solar wind is given by,

$$E_{sw} = -V_{sw} B_z, \tag{12}$$

where V_{sw} is the x component of the solar wind velocity (negative Sunward), and B_z is the north-south component of the interplanetary magnetic field (IMF). E_{sw} has a minimum value of 0.1 mV m^{-1}, so when the IMF is northward there is still a little bit of a viscous interaction with the magnetosphere.

The potential is given by

$$U_{sw} = -A_{sw} r^2 \sin\phi, \tag{13}$$

where

$$A_{sw} = 0.12 E_{sw} (6.6)^{(1-\gamma)}, \text{ (kV } R_E^{-2}) \tag{14}$$

and E_{sw} in this case should be converted from mV m^{-1} to kV R_E^{-1}.

The electric field components become

$$E_r = \gamma A_{sw} r^{(\gamma-1)} \sin\phi, \tag{15}$$

$$E_\phi = A_{sw} r^{(\gamma-1)} \cos\phi, \tag{16}$$

$$E_z = 0. \tag{17}$$

4 SAPS Electric Field

This section uses the electrostatic potential due to SAPS as described in *Goldstein et al.* (2005),

$$U_{saps}(r,\phi,K_p) = -V_s(K_p)F(r,\phi,K_p)G(\phi), \tag{18}$$

where

$$V_s(Kp) = 0.75K_p^2 \text{ (in kV)}, \tag{19}$$

$$F(r,\phi,K_p) = \frac{1}{2} + \frac{1}{\pi}\arctan\left[\frac{2}{\alpha(\phi,K_p)}\{r - R_s(\phi,K_p)\}\right], \tag{20}$$

$$G(\phi) = \sum_{m=0}^{2}\{A_m\cos[m(\phi-\phi_0)] + B_m\sin[m(\phi-\phi_0)]\}, \tag{21}$$

and

$$\alpha(\phi,K_p) = 0.15 + (2.55 - 0.27K_p)\left[1 + \cos\left(\phi - \frac{7\pi}{12}\right)\right], \tag{22}$$

$$R_s(\phi,K_p) = R_0(K_p)\left(\frac{1+\beta}{1+\beta\cos(\phi-\pi)}\right)^{\kappa}, \tag{23}$$

$$R_0(K_p) = 4.4 - 0.6(K_p - 5) \text{ (in } R_E), \tag{24}$$

$$A_m = [0.53, 0.37, 0.1], \tag{25}$$

$$B_m = [0.0, 0.21, -0.1], \tag{26}$$

$$\phi_0 = \frac{\pi}{2}, \tag{27}$$

$$\beta = 0.97, \tag{28}$$

$$\kappa = 0.14. \tag{29}$$

The cylindrical components of the electric field are given by,

$$E_r = V_s(K_p)\frac{\partial F(r,\phi,K_p)}{\partial r}G(\phi), \tag{30}$$

$$E_\phi = \frac{1}{r}V_s\left[F(r,\phi,K_p)\frac{\mathrm{d}G(\phi)}{\mathrm{d}\phi} + \frac{\partial F(r,\phi,K_p)}{\partial\phi}G(\phi)\right], \tag{31}$$

$$E_z = 0. \tag{32}$$

4.1 Derivatives: $\frac{\partial F}{\partial r}$

$$\begin{aligned}
\frac{\partial F}{\partial r} &= \frac{\partial}{\partial r}\left(\frac{1}{2}+\frac{1}{\pi}\arctan\left[\frac{2}{\alpha}\left\{r-R_s\right\}\right]\right) &(33)\\
&= \frac{\partial}{\partial r}\left(\frac{1}{2}+\frac{1}{\pi}\arctan\left[f\right]\right) &(34)\\
&= 0+\frac{1}{\pi}\left(\frac{\partial}{\partial f}\arctan f\right)\frac{\partial f}{\partial r}, &(35)
\end{aligned}$$

where

$$\frac{\partial}{\partial f}\arctan f = \frac{1}{1+f^2}, \tag{36}$$

and

$$\frac{\partial f}{\partial r} = \frac{2}{\alpha}. \tag{37}$$

So,

$$\boxed{\frac{\partial F}{\partial r} = \frac{2}{\alpha\pi}\left[\frac{1}{1+f^2}\right].} \tag{38}$$

4.2 Derivatives: $\frac{\partial F}{\partial \phi}$

$$\begin{aligned}
\frac{\partial F}{\partial \phi} &= \frac{\partial}{\partial \phi}\left(\frac{1}{2}+\frac{1}{\pi}\arctan\left[\frac{2}{\alpha}\left\{r-R_s\right\}\right]\right) &(39)\\
&= \frac{\partial}{\partial \phi}\left(\frac{1}{2}+\frac{1}{\pi}\arctan\left[f\right]\right) &(40)\\
&= 0+\frac{1}{\pi}\left(\frac{\partial}{\partial f}\arctan f\right)\frac{\partial f}{\partial \phi}, &(41)
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial}{\partial f}\arctan f &= \frac{1}{1+f^2}, &(42)\\
\frac{\partial f}{\partial \phi} &= \frac{\partial}{\partial \phi}(g(\phi,K_p)h(\phi,K_p)), &(43)\\
&= \frac{\partial g}{\partial \phi}h + g\frac{\partial h}{\partial \phi}, &(44)\\
\frac{\partial g}{\partial \phi} &= \frac{\partial}{\partial \phi}\left(\frac{2}{\alpha}\right) = \frac{\partial g}{\partial \alpha}\frac{\partial \alpha}{\partial \phi} = -\frac{2}{\alpha^2}\frac{\partial \alpha}{\partial \phi}, &(45)\\
\frac{\partial h}{\partial \phi} &= -\frac{\partial R_s}{\partial \phi}. &(46)
\end{aligned}$$

$$\boxed{\therefore \frac{\partial F}{\partial \phi} = \frac{1}{\pi}\left(\frac{1}{1+f^2}\right)\left[\frac{-2}{\alpha^2}\frac{\partial \alpha}{\partial \phi}h - g\frac{\partial R_s}{\partial \phi}\right].} \tag{47}$$

4.3 Derivatives: $\frac{\partial R_s}{\partial \phi}$

$$\frac{\partial R_s}{\partial \phi} = R_0\frac{\mathrm{d}}{\mathrm{d}\phi}\left\{\left(\frac{1+\beta}{1+\beta\cos(\phi-\pi)}\right)^{\kappa}\right\} = R_0\frac{\mathrm{d}}{\mathrm{d}S}\left\{S^{\kappa}\right\}\frac{\mathrm{d}S}{\mathrm{d}\phi}, \tag{48}$$

$$S = \frac{1+\beta}{1+\beta\cos(\phi-\pi)} = \frac{p}{q}, \tag{49}$$

$$\frac{\mathrm{d}}{\mathrm{d}S}\left\{S^{\kappa}\right\} = \kappa S^{(\kappa-1)} = \kappa\left(\frac{1+\beta}{1+\beta\cos(\phi-\pi)}\right)^{\kappa-1}, \tag{50}$$

$$\frac{\mathrm{d}S}{\mathrm{d}\phi} = \frac{\frac{\mathrm{d}p}{\mathrm{d}\phi}q - p\frac{\mathrm{d}q}{\mathrm{d}\phi}}{q^2} = -\frac{p}{q^2}\frac{\mathrm{d}q}{\mathrm{d}\phi} = \frac{1+\beta}{(1+\beta\cos(\phi-\pi))^2}\cdot\beta\sin(\phi-\pi), \tag{51}$$

$$\boxed{\therefore \frac{\partial R_s}{\partial \phi} = -R_0\frac{1}{q}\frac{\mathrm{d}q}{\mathrm{d}\phi}\kappa\left(\frac{p}{q}\right)^{\kappa} = R_0\frac{(1+\beta)\beta\sin(\phi-\pi)}{(1+\beta\cos(\phi-\pi))^2}\kappa\left(\frac{1+\beta}{1+\beta\cos(\phi-\pi)}\right)^{\kappa-1}.} \tag{52}$$

4.4 Derivatives: $\frac{\partial \alpha}{\partial \phi}$

$$\boxed{\frac{\mathrm{d}\alpha}{\mathrm{d}\phi} = -(2.55-0.27K_p)\sin\left(\phi-\frac{7\pi}{12}\right)} \tag{53}$$

4.5 Derivatives: $\frac{\mathrm{d}G}{\mathrm{d}\phi}$

$$\boxed{\frac{\mathrm{d}G}{\mathrm{d}\phi} = \sum_{m=0}^{2}\left\{-mA_m\sin\left[m(\phi-\phi_0)\right] + mB_m\cos\left[m(\phi-\phi_0)\right]\right\}.} \tag{54}$$

5 SM Model Field

The previous section describes the model in cylindrical coordinates, here we convert the model to SM coordinates by rotating about the z-axis.

Considering an Electric field vector $\mathbf{E}(r,\phi)$ with components E_r and E_ϕ – we need to rotate this vector by ϕ to transform into SM coordinates.

Start by expressing the components of $\mathbf{E}$ in terms of some polar coordinates ρ and α:

$$E_r = \rho \cos \alpha, \tag{55}$$
$$E_\phi = \rho \sin \alpha, \tag{56}$$

then rotate by ϕ,

$$E_x = \rho \cos(\alpha + \phi), \tag{57}$$
$$E_y = \rho \sin(\alpha + \phi). \tag{58}$$

Using the trigonometric identities,

$$\sin(\alpha \pm \phi) = \sin \alpha \cos \phi \pm \cos \alpha \sin \phi, \tag{59}$$
$$\cos(\alpha \pm \phi) = \cos \alpha \cos \phi \mp \sin \alpha \sin \phi, \tag{60}$$

equations 57 and 58 become

$$E_x = \rho \cos \alpha \cos \phi - \rho \sin \alpha \sin \phi, \tag{61}$$
$$E_y = \rho \sin \alpha \cos \phi + \rho \cos \alpha \sin \phi. \tag{62}$$

Then, substituting in equations 55 and 56, gives

$$E_x = E_r \cos \phi - E_\phi \sin \phi, \tag{63}$$
$$E_y = E_r \sin \phi + E_\phi \cos \phi. \tag{64}$$

References

Goldstein, J., et al. (2005), Magnetospheric model of subauroral polarization stream, *Journal of Geophysical Research: Space Physics*, *110*(A9), doi: https://doi.org/10.1029/2005JA011135.

Maynard, N. C., et al. (1975), Isolated cold plasma regions: Observations and their relation to possible production mechanisms, *Journal of Geophysical Research (1896-1977)*, *80*(7), 1009–1013, doi: https://doi.org/10.1029/JA080i007p01009.

Stern, D. P. (1975), The motion of a proton in the equatorial magnetosphere, *Journal of Geophysical Research (1896-1977)*, *80*(4), 595–599, doi: https://doi.org/10.1029/JA080i004p00595.

Volland, H. (1973), A semiempirical model of large-scale magnetospheric electric fields, *Journal of Geophysical Research (1896-1977)*, *78*(1), 171–180, doi: https://doi.org/10.1029/JA078i001p00171.

Zhao, H., et al. (2017), Van allen probes measurements of energetic particle deep penetration into the low l region (l ¡ 4) during the storm on 8 april 2016, *Journal of Geophysical Research: Space Physics*, *122*(12), 12,140–12,152, doi: https://doi.org/10.1002/2017JA024558.