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Intermediate nonlinear Schrödinger equation for internal waves in a fluid of finite depth

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Abstract

A new evolution equation is derived by means of an asymptotic multi-scale technique for quasi-harmonic internal waves in a fluid of finite depth. This equation is shown to generalize the nonlinear Schrödinger equation which appears in the small-depth limit. Soliton solutions to the equation are found in an explicit form and describe the localized dips propagating along a modulationally stable wave background.

1. Self-modulation of quasi-harmonic wave packets propagating in nonlinear dispersive media is a classical problem which has been investigated thoroughly enough (see, e.g., Ref. [1]). It is well known that the nonlinear Schrödinger (NLS) equation which can be integrated by means of the inverse scattering transform [2,3] is used as a general evolution equation for a slowly varying amplitude of one-dimensional wave packets. The NLS equation appears in the greatest majority of physical problems as a result of asymptotic multi-scale reduction of the original equations [4]. However, in some cases the coefficient of the nonlinear (or dispersive) term in the NLS equation is vanishing and it is necessary to calculate higher-order nonlinear-dispersive terms for a correct description of wave packet evolution. For example, the so-called derivative nonlinear Schrödinger (DNLS) equation was found for magneto-hydrodynamic waves propagating along the magnetic field [5] and was shown to be integrable [6].

Another situation occurs for internal waves (interfacial waves) in a two-layer stratified fluid when one layer (for instance, the upper one) is thin and the other (the lower one) is deep [7]. The analysis which was developed for this case in the framework of the Benjamin–Ono (BO) model reveals an evolution equation which is a modification of the DNLS equation [8]. However, some consequences of the derived equation seem to be paradoxical. First of all, as it follows from results of Ref. [9] this equation is not integrable although it was derived from the integrable BO equation [10]. Besides, the single-soliton solution which was also found in Ref. [8] is localized in space exponentially, while the soliton solution of the BO equation is localized algebraically [10]. These facts contradict the conventional opinion [4] that the equation obtained by means of an asymptotic multi-scale expansion inherits both the properties of integrability of the original equation and the construction of its explicit solutions.

This paper is devoted to solution of these paradoxes and to revision of Ref. [8]. Using a modification of the asymptotic multi-scale technique we shall derive a new evolution equation which we call the interme-

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diate nonlinear Schrödinger (INLS) equation. This equation generalizes the NLS equation and differs from that found in Ref. [8]. The N-soliton solution of the INLS equation allows us to suppose that it belongs to the class of integrable equations.

2. In order to describe the waves propagating along the interface between two fluids of different densities so that the wave scales greatly exceed the characteristic depth of the upper layer and are comparable with the depth of the lower layer, we can use the intermediate long-wave (ILW) equation (see, e.g., Refs. [11,12]) which may be written in the following form,

$$u_t + \delta^{-1}u_x + 2uu_x + \boldsymbol{T}(u_{xx}) = 0, \tag{1}$$

where

$$\boldsymbol{T}(u) = (2\delta)^{-1}\text{p.v.} \int_{-\infty}^{+\infty} \coth[\pi(z-x)/2\delta]u(z)\,\mathrm{d}z$$

and the dimensionless parameter δ stands for total fluid depth.

The same equation was found for shear-flow waves of vorticity which are long compared to the depth of the boundary layer (where the velocity of the main flow changes essentially) [13]. For shallow water ($\delta \to 0$), Eq. (1) transforms to the Korteweg–de Vries (KdV) equation and for deep water ($\delta \to \infty$), it transforms to the BO equation also containing a nonlocal Hilbert operator $\boldsymbol{H}(u) = \pi^{-1}\text{p.v.}\int_{-\infty}^{+\infty} u(z)\,\mathrm{d}z/(z-x)$ [11,12].

As is well known [14], the ILW equation can be solved by means of a modification of the inverse scattering transform, namely the nonlocal Riemann–Hilbert problem. Moreover, this technique generalized to the matrix case (but for a local problem) allowed one to find a broad class of integrable nonlocal nonlinear equations with singular kernels including an exotic generalization of the NLS equation [15,16]. Nevertheless, physical applications of the constructed nonlocal equations remain scanty. Here we derive a new nonlocal generalization of the NLS equation which is an asymptotic reduction of the ILW equation and describes a wave packet dynamics in a fluid of finite depth.

Let us consider the packet of quasi-harmonic waves which are short-wave compared to the fluid depth δ,

$$u = \epsilon^{-1}[A(x,t,\tau)\exp(\mathrm{i}\theta) + \text{c.c.}] + \sum_{n=2}^{\infty} \epsilon^{n-2}u_n(x,t,\tau,X,T), \tag{2}$$

where $\epsilon \ll 1$ is a formal small parameter, $\tau = t/\epsilon^2, X = x/\epsilon^2, T = t/\epsilon^4, \theta = KX - \epsilon^4\omega(K/\epsilon^2)T$, and the dispersion dependence $\omega(k)$ for linear perturbations of Eq. (1) has the form

$$\omega(k) = \delta^{-1}k - k^2\coth(\delta k). \tag{3}$$

The introduction of different scales of wave packet variations implies that the high-frequency terms, which are proportional to $\exp(\mathrm{i}\theta), \exp(2\mathrm{i}\theta)$ and so on, lie in the region of BO dispersion, where $\omega(k) \approx -k|k|$. At the same time, the low-frequency modulation of the wave packet as well as the mean flow induced by nonlinear effects lie in the intermediate region, where $\omega(k)$ is expressed by Eq. (3).

We would like to emphasize that such a modification of an asymptotic multi-scale expansion is different from a conventional one [1]. The standard multi-scale expansion, when the high-frequency terms lie in the intermediate region and the low-frequency terms lie in the region of the KdV dispersion (where $\omega(k) \approx -\frac{1}{3}\delta k^3$), leads to the ordinary NLS equation [7]. The difference between these expansions is depicted in Fig. 1.

Direct substitution of expansion (2) into Eq. (1) enables us to find the first terms of the asymptotic series

$$u_2 = n - \frac{|A|^2}{|k|} + \left(\frac{A^2}{|k|}\exp(2\mathrm{i}\theta) + \text{c.c.}\right), \tag{4a}$$

$$u_3 = \left(\frac{A^3}{k^2}\exp(3\mathrm{i}\theta) + \text{c.c.}\right), \tag{4b}$$

$$u_4 = \tilde{n} + \left(\frac{\mathrm{i}(A^2)_x}{2k|k|}\exp(2\mathrm{i}\theta) + \text{c.c.}\right) + \left(\frac{A^4}{k^2|k|}\exp(4\mathrm{i}\theta) + \text{c.c.}\right), \tag{4c}$$

where we introduce a long-wave field $n(x,t)$ which obeys the original equation (1) and an induced long-wave flow $\tilde{n}(x,t,\tau)$ related to $A(x,t,\tau)$. Because the secular terms appear at the first and zero harmonics when we are looking for higher-order terms of the series (2) [1] our asymptotic expansion becomes di-

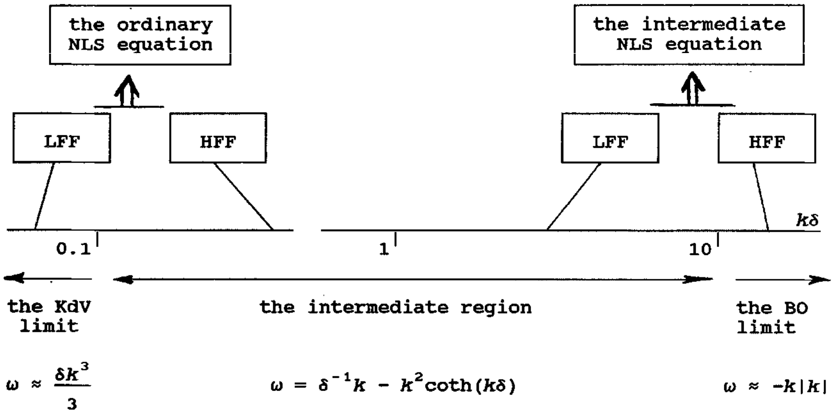

Fig. 1. Scheme of the standard and novel asymptotic expansions leading to the ordinary and the intermediate NLS equations (LFF: low-frequency field; HFF: high-frequency field).

verging. In order to avoid such divergences we need to remove the secular terms specifying the induced mean flow

$$\tilde{n} = \frac{\mathrm{i}}{2k|k|}(AA_x^* - A_xA^*) - \frac{|A|^4}{k^2|k|} + \frac{1}{2k^2}\boldsymbol{T}(|A|_x^2) \quad (5)$$

and evolution equations for the amplitude $A(x,t,\tau)$,

$$\mathrm{i}(A_\tau - 2|k|A_x) - 2kA[n + (2\delta)^{-1}] = 0, \quad (6a)$$

$$\mathrm{i}[A_t + \delta^{-1}A_x + 2(nA)_x] - \sigma A_{xx} - \frac{2\mathrm{i}}{|k|}A\boldsymbol{P}_\sigma(|A|_x^2) = 0, \quad (6b)$$

where $\sigma = \mathrm{sign}(k)$ and $\boldsymbol{P}_\sigma = \frac{1}{2}(1 - \mathrm{i}\sigma\boldsymbol{T})$ is a nonlocal operator determining the analytical function in a strip of the complex extension of x (which we designate through z) between $\mathrm{Im}(z) = 0$ and $\mathrm{Im}(z) = 2\sigma\delta$ [14].

Eq. (6a) is easily integrated,

$$A = a(x + 2|k|\tau, t)\exp\left[\mathrm{i}\sigma\left(\int n\,\mathrm{d}x + \frac{x}{2\delta} - \frac{t}{4\delta^2}\right)\right], \quad (7)$$

and, finally, we find the nonlinear equation for the amplitude $a(x,t)$ of the high-frequency wave packet interacting with the given low-frequency field $n(x,t)$,

$$\mathrm{i}a_t - \sigma a_{xx} + 2\mathrm{i}a\boldsymbol{P}_\sigma\left(n - \frac{|a|^2}{|k|}\right)_x = 0. \quad (8)$$

In the limit $\delta \to 0$, the operator $\boldsymbol{P}_\sigma$ has the form $\boldsymbol{P}_\sigma = \mathrm{i}\sigma\int(\quad)\,\mathrm{d}x/2\delta + \mathrm{O}(1)$ so that Eq. (8) becomes an ordinary NLS equation with an inhomogeneous, nonstationary potential $n(x,t)$ obeying the KdV equation. It should be noted that the limit $\delta \to 0$ needs rescaling the variables n, x, t according to Eq. (7).

In the other limit $\delta \to \infty$, Eq. (8) has the same form but the operator $\boldsymbol{P}_\sigma = \frac{1}{2}(1 - \mathrm{i}\sigma\boldsymbol{H})$ is a projective operator determining a function which is analytical in the upper (for $\sigma = +1$) and in the lower (for $\sigma = -1$) half-planes of the complex extension of x. Here the field $n(x,t)$ obeys the BO equation.

Thus, by analogy with Eq. (1), Eq. (8) is referred to as the intermediate nonlinear Schrödinger (INLS) equation. Note that the nonlocal term was missing in Ref. [8] where an equation of the DNLS-type was derived for $\delta = \infty$. Therefore, the properties of the found evolution equation change essentially.

3. We now consider a problem of wave packet self-modulation which is described by Eq. (8) for $n = 0$. The linearization against the wave background of constant amplitude $a = \rho + (u + \mathrm{i}w)\exp[\mathrm{i}(\kappa x - \Omega t)]$, where $\rho = \mathrm{const}$ and $u, w \ll \rho$, shows that the wave background is modulationally stable and the dispersion dependence $\Omega(\kappa)$ for linear perturbations has the form

$$\Omega(\Omega + 2c\kappa) = \kappa^4 + 2c\kappa^3\coth(\kappa\delta), \quad (9)$$

where $c = \rho^2/|k|$.

The dispersion dependence (9) describes two branches of acoustic type for the wave disturbances propagating along a modulationally stable background. In the dispersionless limit ($\kappa \to 0$ but finite $\kappa\delta$), one branch corresponds to the perturbations propagating to the left with velocity $2c$ and the other branch corresponds to standing perturbations. In the shallow water limit $\delta \to 0$, both the branches become symmetric so that the waves move to the left and to the right with equal velocities $\pm(2c/\delta)^{1/2} + O(1)$.

It is known [1] that localized wave packets diverge due to dispersive effects in a modulationally stable case. However, in this case there may exist solitary waves on a wave background and solutions describing dynamics of such solitons were found for the NLS equation in Ref. [3]. In order to understand the features of wave processes in the framework of the INLS equation we consider evolution of small-amplitude, smoothly modulated wave perturbations propagating along the background of constant amplitude $a = \rho =$ const. It follows from Eq. (9) in the limit $\kappa \to 0$ that smoothly modulated perturbations decay into a superposition of weakly interacting waves of both types. Their evolution occurring due to nonlinear and dispersive effects can be described by means of two asymptotic expansions,

$$a^+ = [\rho + \mu R^+(X,T) + O(\mu^2)] \times \exp\{i[S^+(X,T) + O(\mu)]\}, \tag{10a}$$

$$a^- = [\rho + \mu R^-(\xi,T) + O(\mu^2)] \times \exp\{i[\mu S^-(\xi,T) + O(\mu^2)]\}, \tag{10b}$$

where $\xi = \mu(x + 2ct), X = \mu x, T = \mu^2 t$, and $\mu \ll 1$.

The substitution of the expansions (10a), (10b) into Eq. (8) gives relations for parameters $S^\pm$ and $R^\pm$,

$$S^+ = -\frac{2\rho}{k}\int R^+\,dX, \quad S^- = -\frac{\sigma}{\rho}T(R^-), \tag{11}$$

and restricts their variation by the equations

$$\pm R_T^\pm + \frac{4\rho}{|k|}R^\pm R_X^\pm - T(R_{XX}^\pm) = 0. \tag{12}$$

The found equations (12) coincide with the ILW model but are slightly different from the original equation (1) by the form of their coefficients. Thus, dynamics of wave perturbations against a modulationally stable background in the framework of the INLS equation generalizes the dynamics of long waves described by the ILW equation and reduces to it in the small-amplitude limit. At $\delta \to 0$, the same correspondence between the NLS and KdV equations follows from Eqs. (8), (12) and is well known [4].

4. The relationships between the INLS and ILW equations which we discussed above imply a similarity between their soliton solutions. N-soliton solutions to the ILW equation were found in Refs. [17,18]. Here we consider N-soliton solutions to Eq. (8) using the Hirota bilinear method [19]. For this we replace the dependent variables as follows,

$$a = \rho G/F, \tag{13a}$$

$$a^* = \rho \bar{G}/\bar{F}, \tag{13b}$$

$$|a|^2 = \rho^2 - ik[\log(F/\bar{F})]_x. \tag{13c}$$

Besides, we choose a reference frame propagating to the left with velocity c by replacing the independent variables $(x,t) \to (x + ct, t)$.

The conditions of the transition from Eq. (8) to the Hirota bilinear equations are the following restrictions imposed on the functions $F, \bar{F}$,

$$F = f(x - i\sigma\delta), \quad \bar{F} = f(x + i\sigma\delta), \tag{14}$$

where $f(x)$ is a real function with the zero lying in the complex plane z outside the strip $-\delta \leqslant \mathrm{Im}(z) \leqslant \delta$. In this case, $P_\sigma(-|a|^2/|k|)_x = i\sigma[\log(F)]_{xx}$ [11] and Eq. (8) at $n = 0$ transforms to the system of bilinear equations

$$(iD_t + icD_x + \sigma D_{xx})(F \cdot G) = 0, \tag{15a}$$

$$(iD_t + icD_x + \sigma D_{xx})(\bar{G} \cdot \bar{F}) = 0, \tag{15b}$$

$$iD_x(F \cdot \bar{F}) + c\sigma(G \cdot \bar{G} - F \cdot \bar{F}) = 0. \tag{15c}$$

Eqs. (15a), (15b) are referred to as the bilinear Bäcklund transformation (BBT) between solutions of the Kadomtsev–Petviashvili (KP) and related equations [19,20]. On the other hand, it can readily be shown [21] that Eq. (15c) is satisfied if the functions $(\bar{F}, G)$ and $(\bar{G}, F)$ are also related by the BBT equations

$$(iD_t - icD_x + \sigma D_{xx})(\bar{F} \cdot G) = 0, \tag{15d}$$

$$(iD_t - icD_x + \sigma D_{xx})(\bar{G} \cdot F) = 0. \tag{15e}$$

Therefore, Eq. (15c) can be replaced by Eqs. (15d), (15e). As a result, soliton solutions to the INLS equation, similarly to the solutions to the ILW equation, are reductions of the well-known soliton solutions to the KP equation [20]. Moreover, the functions $F, \bar{F}, G, \bar{G}$ have identical structures in the form of exponential polynomials,

$$F = \sum_{\nu=(0,1)} \exp\left(\sum_{n=1}^{N} \nu_n(\eta_n + \mathrm{i}\phi_n) + \sum_{1\leqslant n<m\leqslant N} \nu_n\nu_m A_{nm}\right) \equiv F(\phi), \tag{16a}$$

$$\bar{F} = F(\bar{\phi}), \tag{16b}$$

$$G = F(\psi), \tag{16c}$$

$$\bar{G} = F(\bar{\psi}), \tag{16d}$$

where

$$\eta_n = \kappa_n(x - v_n t - x_{0n}),$$

$$\exp(A_{nm}) = \frac{(\kappa_n - \kappa_m)^2 + (v_n - v_m)^2}{(\kappa_n + \kappa_m)^2 + (v_n - v_m)^2}.$$

The action of bilinear equations (15a), (15b), (15d), (15e) is merely a shift of the phase constants according to the equations

$$\cot[\tfrac{1}{2}(\psi_n - \phi_n)] = \cot[\tfrac{1}{2}(\bar{\phi}_n - \bar{\psi}_n)] = \frac{v_n - c}{\kappa_n},$$

$$\cot[\tfrac{1}{2}(\psi_n - \bar{\phi}_n)] = \cot[\tfrac{1}{2}(\phi_n - \bar{\psi}_n)] = \frac{v_n + c}{\kappa_n}. \tag{17}$$

Obviously, Eqs. (17) give only three relations between four constants $\phi_n, \bar{\phi}_n, \psi_n, \bar{\psi}_n$. However, in order to satisfy condition (14) we should specify the parameters ϕ_n and $\bar{\phi}_n$ to be $\phi_n = -\sigma\delta\kappa_n$, $\bar{\phi}_n = +\sigma\delta\kappa_n$. Hence the velocity of the nth soliton v_n depends on the parameter κ_n. Such a dependence can be found from Eqs. (17) and has the form

$$v_n^2 + \kappa_n^2 = c^2 + 2c\kappa_n \cot(\kappa_n\delta). \tag{18}$$

Besides, it should be noted that the parameters κ_n belong to the interval $0 \leqslant \kappa_n\delta \leqslant \pi$ [11].

Simple analysis shows that the N-soliton solution (16) describes the scattering of N solitons moving with velocities v_n, which results in a sum phase shift of each soliton caused by a pair collision with the other solitons (see, e.g., Ref. [1]). Such a dynamics is well known for one-dimensional exponentially localized solitons including the solitons of the NLS and ILW equations [3,14].

As follows from Eq. (13c), each nth soliton is a dip in the distribution $|a|^2$. The profile of such a dip coincides with the profile of the soliton in the ILW equation. However, its velocity v_n is related to the parameter κ_n by Eq. (18) which is different from that for solitons in the ILW equation (except the limiting case $\kappa_n \to 0$). Obviously, the velocity of the solitons decreases with the growth of their amplitudes symmetrically for both the branches of the dispersion dependence (18) so that for $\kappa_n = \kappa_c$, where $\kappa_c \tan(\frac{1}{2}\kappa_c\delta) = c$, the branches merge and there appears an immobile soliton with zero amplitude at the minimum and the phase jump by π.

The structure and properties of solitons in the INLS equation coincide qualitatively with those of dark solitons in the NLS equation [3]. However, their quantitative features are determined by the parameter δ. In the limit $\delta \to \infty$, $\kappa_n\delta \to \pi$ we can obtain from Eqs. (16) the polynomial functions (see, e.g., Ref. [11]) which describe algebraic solitons of Eq. (8) propagating along the wave background of amplitude ρ.

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