ON THE INTEGRABILITY OF THE BENJAMIN-ONO EQUATION ON THE TORUS

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ABSTRACT. In this paper we prove that the Benjamin-Ono equation, when considered on the torus, is an integrable (pseudo)differential equation in the strongest possible sense: it admits global Birkhoff coordinates on the space $L^2(\mathbb{T})$. These are coordinates which allow to integrate it by quadrature and hence are also referred to as nonlinear Fourier coefficients. As a consequence, all the $L^2(\mathbb{T})$ solutions of the Benjamin–Ono equation are almost periodic functions of the time variable. The construction of such coordinates relies on the spectral study of the Lax operator in the Lax pair formulation of the Benjamin–Ono equation and on the use of a generating functional, which encodes the entire Benjamin–Ono hierarchy.

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1. Introduction

In this paper we consider the Benjamin-Ono (BO) equation on the torus,

(1.1)
$$\partial_t u = H \partial_x^2 u - \partial_x (u^2), \quad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}, \ t \in \mathbb{R},$$

where $u \equiv u(t,x)$ is real valued and H denotes the Hilbert transform, defined for $f = \sum_{n \in \mathbb{Z}} \widehat{f}(n)e^{inx}$, $\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx}dx$, by

$$Hf(x) := \sum_{n \in \mathbb{Z}} -i \operatorname{sign}(n) \widehat{f}(n) e^{inx}$$

with $\operatorname{sign}(\pm n) := \pm 1$ for any $n \geq 1$, whereas $\operatorname{sign}(0) := 0$. This pseudodifferential equation (ΨDE) in one space dimension has been introduced by Benjamin [3] and Ono [26] to model long, one-way internal gravity waves in a two-layer fluid. It has been extensively studied, in particular the wellposedness problem on the torus as well as on the real line. On appropriate Sobolev spaces, the BO equation (1.1) can be written in Hamiltonian form

$$\partial_t u = \partial_x (\nabla \mathcal{H}(u)), \qquad \mathcal{H}(u) := \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{1}{2} (|\partial_x|^{1/2} u)^2 - \frac{1}{3} u^3 \right] dx$$

where $|\partial_x|^{1/2}$ is the square root of the Fourier multiplier operator $|\partial_x|$ given by

$$|\partial_x|f(x) = \sum_{n\in\mathbb{Z}} |n|\widehat{f}(n)e^{inx}.$$

Note that the L^2 -gradient $\nabla \mathcal{H}$ of \mathcal{H} can be computed to be $|\partial_x|u-u^2$ and that $\partial_x \nabla \mathcal{H}$ is the Hamiltonian vector field corresponding to the Gardner bracket, defined for any two functionals $F, G: L^2 \to \mathbb{C}$ with sufficiently regular L^2 -gradients by

$$\{F,G\} := \frac{1}{2\pi} \int_{0}^{2\pi} (\partial_x \nabla F) \nabla G dx$$
.

The main result of this paper says that the BO equation (1.1) is an integrable Ψ DE. In fact, we show that it admits global Birkhoff coordinates and hence is an integrable Ψ DE in the strongest possible sense. To state our results more precisely, we first need to introduce some notation. Denote by L^2 the \mathbb{C} -Hilbert space $L^2(\mathbb{T},\mathbb{C})$ of L^2 -integrable, complex valued functions with the standard inner product

(1.2)
$$\langle f|g\rangle := \frac{1}{2\pi} \int_{0}^{2\pi} f(x)\overline{g(x)} \, dx$$

and the corresponding norm $||f|| := \langle f|f\rangle^{1/2}$. Furthermore, we denote by L_r^2 the \mathbb{R} -Hilbert space $L^2(\mathbb{T},\mathbb{R})$, consisting of elements $u \in L^2$

which are real valued and let

$$L_{r,0}^2 := \{ u \in L_r^2 \mid \langle u | 1 \rangle = 0 \}$$
.

For any subset $J \subset \mathbb{Z}_{\geq 0}$ and any $s \in \mathbb{R}$, $h^s(J) \equiv h^s(J, \mathbb{C})$ denotes the weighted ℓ^2 -sequence space

$$h^{s}(J) = \{(z_{n})_{n \in J} \subset \mathbb{C} : ||(z_{n})_{n \in J}||_{s} < \infty\}$$

where

$$\|(z_n)_{n\in J}\|_s := \left(\sum_{n\in J} \langle n \rangle^{2s} |z_n|^2\right)^{1/2}, \quad \langle n \rangle := \max\{1, |n|\}.$$

In case where $J = \mathbb{N} := \{n \in \mathbb{Z} : n \geq 1\}$ we write h_+^s instead of $h^s(\mathbb{N})$. In the sequel, we view h_+^s as the \mathbb{R} -Hilbert space $h^s(\mathbb{N}, \mathbb{R}) \oplus h^s(\mathbb{N}, \mathbb{R})$ by identifying a sequence $(z_n)_{n \in \mathbb{N}} \in h_+^s$ with the pair of sequences $((\operatorname{Re} z_n)_{n \in \mathbb{N}}, (\operatorname{Im} z_n)_{n \in \mathbb{N}})$ in $h^s(\mathbb{N}, \mathbb{R}) \oplus h^s(\mathbb{N}, \mathbb{R})$.

The main result of this paper then reads as follows.

Theorem 1. There exists a homeomorphism

$$\Phi: L_{r,0}^2 \to h_+^{1/2}, \ u \mapsto (\zeta_n(u))_{n \ge 1}$$

so that the following holds:

- (B1) For any $n \geq 1$, $\zeta_n : L^2_{r,0} \to \mathbb{C}$ is real analytic.
- (B2) The Poisson brackets between the coordinate functions ζ_n are well defined and for any $n, k \geq 1$,

$$\{\zeta_n, \overline{\zeta_k}\} = -i\delta_{nk}, \qquad \{\zeta_n, \zeta_k\} = 0.$$

(B3) On its domain of definition, $\mathcal{H} \circ \Phi^{-1}$ is a (real analytic) function, which only depends on the actions $|\zeta_n|^2$, $n \geq 1$.

The coordinates ζ_n are referred to as complex Birkhoff coordinates.

Remark 1.1. (i) The Birkhoff map Φ is bounded, meaning that for any bounded subset B of $L_{r,0}^2$, the image $\Phi(B)$ is bounded. Indeed, this is a direct consequence of the trace formula of Proposition 3.1, saying that for any $u \in L_{r,0}^2$,

$$||u||^2 = 2\sum_{n=1}^{\infty} n|\zeta_n|^2$$
.

In analogy to Parseval's identity in Fourier analysis, we refer to this trace formula as Parseval's identity for the nonlinear map Φ .

(ii) When restricted to submanifolds of finite gap potentials (cf. Definition 2.2), the map Φ is a canonical, real analytic diffeomorphism onto corresponding Euclidean spaces – see Theorem 3 in Section 7 for details.

In subsequent work [11] we plan to further study the regularity of the Birkhoff map of Theorem 1 and its restrictions to the scale of Sobolev spaces $H_{r,0}^s$, $s \geq 0$, where

$$H^s_{r,0} := \left\{ u \in L^2_{r,0} \, | \, \|u\|_s < \infty \right\}, \qquad \|u\|_s := \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |\widehat{u}(n)|^2 \right)^{1/2}$$

and to make a detailed analysis of the solution map of the Benjamin–Ono equation, including qualitative properties of solutions of (1.1) such as their long time behaviour. As an immediate application of Theorem 1 in this direction we mention the following result on the solutions of the BO equation for initial data in $L_{r,0}^2$ obtained by quadrature, when equation (1.1) is expressed in Birkhoff coordinates.

Theorem 2. For every initial data u(0) in $L_{r,0}^2$, the solution of the BO equation with initial data u(0),

$$t \in \mathbb{R} \mapsto u(t) \in L^2_{r,0}$$
,

is almost periodic. Its orbit is relatively compact in $L_{r,0}^2$.

Remark 1.2. (i) The solutions of the BO equation of Theorem 2 coincide with the ones obtained by Molinet in [20] (cf. also [21]). Since the average is conserved by (1.1), the results of Theorem 2 easily extend to solutions with initial data in L_r^2 .

- (ii) Expressing a given solution $t \in \mathbb{R} \mapsto u(t) \in L^2_{r,0}$ of (1.1) in Birkhoff coordinates one immediately sees that Corollary 1.3 in [31] on the recurrence of solutions extends in the sense that for any initial datum $u(0) \in L^2_{r,0}$ there exists a sequence $0 < t_1 < t_2 < \cdots$ of times with $t_n \to \infty$ so that $\lim_{n \to \infty} ||u(t_n) u(0)|| = 0$.
- (iii) As another immediate application of Theorem 1, we present a new proof of the result due to Amick& Toland [2], characterizing the traveling wave solutions of the BO equation as solutions with initial data given by one gap potentials see Proposition B.1 in Appendix B for details.

Outline of the proof of Theorem 1: At the heart of the proof is the Lax pair formulation $\partial_t L_u = [B_u, L_u]$ of the BO equation, discovered by Nakamura [24] and Bock&Kruskal [4]. It is reviewed at the beginning of Section 2 – see also Appendix A. For any $u \in L_r^2$, L_u is a selfadjoint pseudodifferential operator of order one whose spectrum is conserved along the flow of the BO equation. In contrast to integrable PDEs with a Lax pair formulation such as the Korteweg-de Vries or the nonlinear Schrödinger equation, the Lax operator L_u for the BO equation is not a differential operator. When considered on the Hardy space $L_+^2 := \{h \in L^2 : \hat{h}(n) = 0 \ \forall n < 0\}$, the operator L_u is given by $L_u = -i\partial_x - T_u$ where T_u is the Toeplitz operator $T_u h = \Pi(uh)$ and Π denotes the Szegő projector $\Pi: L_r^2 \to L_+^2$. Its spectrum consists of real eigenvalues

which when listed in increasing order and with their multiplicities take the form $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ with $\lambda_n \to \infty$ as $n \to \infty$. In our analysis of the spectrum of L_u , the shift operator $S: L^2_+ \to L^2_+$, $h \mapsto e^{ix}h$ plays an important role. We point out that this operator played also a major role in the study of the Szegő equation (cf. [8], [9]). Using commutator relations between L_u and S (cf. Proposition 2.1) we infer that for any $u \in L^2_r$

$$\gamma_n(u) := \lambda_n(u) - \lambda_{n-1}(u) - 1 \ge 0, \quad \forall n \ge 1.$$

Since all eigenvalues are simple, $\gamma_n \equiv \gamma_n(u)$, $n \geq 1$, can be shown to depend real analytically on u. Note that the eigenvalues λ_n , $n \geq 1$, can then be expressed in terms of the γ_k 's by $\lambda_n = \lambda_0 + n + \sum_{k=1}^n \gamma_k$. Furthermore, we provide an orthonormal basis of eigenfunctions f_n , where each f_n depends analytically on u (Definition 2.1).

A key ingredient of the proof of Theorem 1 is the generating function $\mathcal{H}_{\lambda}(u)$, which we study in Section 3. For any $u \in L_r^2$, $\lambda \mapsto \mathcal{H}_{\lambda}(u)$ is the meromorphic function, defined by

$$\mathcal{H}_{\lambda}(u) := \langle (L_u + \lambda Id)^{-1} 1 | 1 \rangle$$
.

Note that $\mathcal{H}_{\lambda}(u)$ is holomorphic on $\mathbb{C} \setminus \{-\lambda_n(u) : n \geq 0\}$ and might have simple poles at $\lambda = -\lambda_n(u), n \geq 0$. By Proposition 3.1, $\mathcal{H}_{\lambda}(u)$ can be expressed as a function of λ_0 and the γ_n 's, and hence is a family of integrals of the BO equation, parametrized by the spectral parameter λ . For our study of the BO equation it plays a role comparable to the one of the discriminant in the normal form theory of the KdV equation. In particular, \mathcal{H}_{λ} admits a product representation (Proposition 3.1) and an expansion at $\lambda = \infty$ whose coefficients constitute the BO hierarchy. The functionals $\langle u|1\rangle$, $||u||^2$ as well as the BO Hamiltonian \mathcal{H} are elements in this hierarchy. As an application we derive a trace formula for $||u||^2$ (Proposition 3.1), implying that for any $u \in L^2_{r,0}$ the sequence $(\sqrt{\gamma_n})_{n\geq 1}$ is in $h^{1/2}_+$. Furthermore, we show that for any $n\geq 0$, the absolute value $|\langle 1|f_n\rangle|$ is a function of λ_0 and the γ_n 's alone (Corollary 3.1).

Our candidates for the Birkhoff coordinates $(\zeta_n)_{n\geq 1}$ are then obtained by normalizing $(\langle 1|f_n\rangle)_{n\geq 1}$ so that $|\zeta_n|=\sqrt{\gamma_n}$ for any $n\geq 1$ (cf. Section 4) and the corresponding action angle coordinates are $(\gamma_n)_{n\geq 1}$, $(\arg\langle 1|f_n\rangle)_{n\geq 1}$. These variables are very reminiscent of the canonical variables constructed for the Toda system with Dirichlet boundary condition in [23]. In fact, the Toda system admits a Lax pair formulation where the Lax operator L is a Jacobi matrix. In the case of Dirichlet boundary conditions, all eigenvalues of L are simple. Canonical variables are then given by these eigenvalues and the first components of appropriately normalized eigenvectors – see [23] for more details.

In the sequel, the map $\Phi: u \mapsto (\zeta_n(u))_{n\geq 1}$, defined on $L^2_{r,0}$ and with values in $h^{1/2}_+$, is studied in detail. A key observation is that any

 $u \in L^2_{r,0}$ can be reconstructed from $\langle 1|f_n\rangle$, $n \geq 0$, allowing to prove that Φ is one to one (formula (4.5), Proposition 4.2). The generating function also plays a major role for proving that $(\zeta_n)_{n\geq 1}$, $(\overline{\zeta}_n)_{n\geq 1}$ satisfy the canonical relations stated in Theorem 1. Since for any $n\geq 1$, the L^2 -gradient of $\nabla \zeta_n$ is in H^1 (Proposition 5.1), it follows that the Poisson brackets among these functionals are well defined. A key ingredient for proving that they satisfy canonical relations, is the observation that, for λ sufficiently large, the ΨDE $\partial_t u = \partial_x \nabla \mathcal{H}_{\lambda}(u)$ admits the Lax pair formulation $\frac{dL_u}{dt} = [B_u^{\lambda}, L_u]$ where B_u^{λ} is a certain skew adjoint operator (Proposition 6.1). As a consequence, the eigenvalues λ_p commute with \mathcal{H}_{λ} , implying that $\{\gamma_p, \gamma_n\}$ vanishes on $L^2_{r,0}$ for any $p, n \geq 1$ (Corollary 6.1). In a similar fashion one computes $\{\mathcal{H}_{\lambda}, \langle 1, f_n \rangle\}$ and then derives that for any $p \geq 1, n \geq 0$, $\{\gamma_p, \langle 1, f_n \rangle\} = i \langle 1, f_n \rangle \delta_{pn}$ on $L^2_{r,0}$ (Proposition 6.2).

As a last key ingredient into the proof of the claimed canonical relations for $(\zeta_n)_{n\geq 1}$, $(\overline{\zeta}_n)_{n\geq 1}$ we consider finite gap potentials: we say that $u\in L^2_r$ is a *finite gap potential* if the set $\{n\geq 1: \gamma_n(u)>0\}$ is finite. We then show that for any $N\geq 1$, the restriction of Φ to the submanifold \mathcal{U}_N of special finite gap potentials, defined by

$$\mathcal{U}_N := \{ u \in L_{r,0}^2 : \gamma_N(u) > 0, \gamma_n(u) = 0 \ \forall n > N \} ,$$

is a symplectic diffeomorphism onto $\mathbb{C}^{N-1} \times \mathbb{C}^*$ (cf. Theorem 3 in Section 7).

In the course of the proof we derive a formula for an arbitrary element in \mathcal{U}_N , showing in particular that Πu is of the form $\Pi u(x) = R(e^{ix})$ where R is a rational function, $R(z) = -z \frac{Q'(z)}{Q(z)}$, and Q(z) a polynomial of degree N with roots outside the unit disc – see (7.9), Remark 7.1. We note that $\bigcup_{N\geq 0} \mathcal{U}_N$ is the set of all finite gap potentials in $L^2_{r,0}$ and is dense in $L^2_{r,0}$. In Section 8 we make a synopsis of the proof of Theorem 1 and as an application of Theorem 1 show Theorem 2.

Related work: The Benjamin-Ono equation has been extensively studied. For an excellent account we refer to the recent survey by J.C. Saut [30]. Besides the foundational work of Benjamin [3] and Ono [26], let us point out a few highlights, relevant in the context of the present paper. The first results about (1.1) by PDE methods can be found in [29], [1] and the most recent ones in [20], [21], [13] – we refer to [21] and [30] for numerous other references.

Concerning the investigation of the integrability of (1.1), besides the discovery of the Lax pair formulation of this equation already referred to above (cf. [24], [4]), we mention the pioneering work of Fokas&Ablowitz [7], Coifman&Wickerhauser [5], Kaup&Matsuno [16], and the more recent contributions by Wu [32], [33], on the scattering transform for the BO equation, which concerns the integrability of the BO equation

when considered on the real line. Surprisingly, the integrability of the BO equation on the torus has not been studied in detail so far. We point out the work by Satsuma&Ishimori [28] on the construction of multi-phase solutions of (1.1) by Hirota's bilinear method, the one by Amick&Toland [2] on the characterisation of the solitary wave solutions as well as the work by Dobrokhotov&Krichever [6] where multi-phase solutions are constructed by the method of finite zone integration. We refer to these solutions as finite gap solutions and they are treated in detail in Section 7. In their work on the quantum BO equation, Nazarov&Sklyanin [25] introduced a generating function for the classical BO hierarchy of the type introduced in Section 3, which admits a quantum analogue. The approach of Nazarov&Sklyanin has been further developed by Moll [22] in the context of the classical BO equation. Finally, we mention the work by Tzvetkov&Visciglia on the construction of invariant Gaussian measures and applications to the long time behaviour of solutions of (1.1) (cf. [31] and references therein). Our analysis of the integrability of (1.1) borrows from the one carried out for the Korteweg-de Vries and the defocusing nonlinear Schrödinger equations as well as the Szegő equation. As the BO equation, the Korteweg-de Vries and the defocusing nonlinear Schrödinger equations are integrable in the sense that they admit global Birkhoff coordinates - see [14], [12] and references therein. Furthermore, it turned out that the Szegő equation, an integrable ΨDE introduced and studied in detail by Gérard and Grellier, is closely related to the BO equation and tools developed for the study of the integrability of this equation were of great use in the present paper – cf. [8], [9], [10] and references therein.

Notation: We have already introduced the Hilbert spaces L^2 , $L^2_r, L^2_{r,0}$, and the Hardy space L^2_+ as well as the Szegő projector $\Pi: L^2 \to L^2_+$. More generally, for any $1 \leq p \leq \infty$, $L^p \equiv L^p(\mathbb{T},\mathbb{C})$ denotes the standard complex L^p —space and the L^p —norm of an element $f \in L^p$ is denoted by $\|f\|_{L^p}$. The subspace of real valued functions in L^p is denoted by $L^p_r \equiv L^p(\mathbb{T},\mathbb{R})$. Furthermore, we have already introduced the Sobolev spaces $H^s_{r,0}$. More generally, we denote by H^s the standard Sobolev space $H^s(\mathbb{T},\mathbb{C})$ and by H^s_r the corresponding real subspace. In order to define the L^2 —gradient of a differentiable functional $\mu: f \mapsto \mu(f) = \mathrm{Re}\mu(f) + i\mathrm{Im}\mu(f) \in \mathbb{C}$, defined on L^2_r , introduce the complex bilinear form,

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{0}^{2\pi} f(x)g(x) dx$$
, $\forall f, g \in L^2$.

Note that the function g in the latter integral is not complex conjugated. The L^2 -gradient $\nabla \mu(f)$ of μ at $f \in L^2_r$ is then defined as the

element in L^2 , uniquely determined by

$$\langle \nabla \mu(f), g \rangle = d\mu(f)[g] , \quad \forall g \in L_r^2 .$$

The L^2 -gradient of a functional, defined on a subspace of L_r^2 such as a Sobolev space, is defined in a similar fashion by extending the bilinear form $\langle f, g \rangle$ as a dual paring between the subspace and its dual. In case the functional μ is defined on a complex neighborhood of L_r^2 in L^2 and analytic there, the L^2 -gradient of μ is defined in the same way.

Finally, we recall that we have already introduced the sequence spaces $h^s(J)$ where $J \subset \mathbb{Z}$. In case s = 0, we write $\ell^2(J)$ for $h^0(J)$ and endow it with the standard inner product

$$\langle (z_n)_{n\in J}|(w_n)_{n\in J}\rangle_{\ell^2}=\sum_{n\in J}z_n\overline{w_n}.$$

More generally, for any $1 \leq p < \infty$ and $s \in \mathbb{R}$, denote by $\ell^{p,s}(J) \equiv \ell^{p,s}(J,\mathbb{C})$ the weighted ℓ^p -sequence space, consisting of sequences $\xi = (\xi_n)_{n \in J} \subset \mathbb{C}$ so that

$$\|\xi\|_{\ell^{p,s}} := \left(\sum_{n \in J} \langle n \rangle^{ps} |\xi_n|^p\right)^{1/p} < \infty.$$

Correspondingly we define the sequence spaces $\ell^{p,s}(J,\mathbb{R})$. The spaces $\ell^{\infty,s}(J) \equiv \ell^{\infty,s}(J,\mathbb{C})$ and $\ell^{\infty,s}(J,\mathbb{R})$ are defined in a similar fashion. In the sequel we will often use the notation ℓ^p_n to denote the n-th element of a sequence in ℓ^p .

2. The Lax operator

Nakamura [24] and Bock&Kruskal [4] discovered that the BO equation admits a Lax pair. In the periodic setup it can be described as follows. For any $u \in L_r^2$, introduce the unbounded operators L_u and B_u on L_+^2 .

(2.1)
$$L_u = D - T_u$$
, $B_u := iD^2 + 2iT_{D(\Pi u)} - 2iDT_u$.

where D is the unbounded operator on L_+^2 , defined by $D := -i\partial_x$, T_u the Toeplitz operator on L_+^2 with symbol u,

$$T_u: L_+^2 \to L_+^2, h \mapsto T_u h = \Pi(uh)$$

and $\Pi:L^2\to L^2_+$ the Szegő projector. For any $u\in L^2_r,\ T_u$ is an unbounded operator, satisfying the estimate

$$||T_u(h)|| \le ||u|| \, ||h||_{L^{\infty}}.$$

Hence $L_u = D - T_u$ is an unbounded operator with domain $H^1_+ := H^1 \cap L^2_+$ and B_u is such an operator with domain $H^2_+ := H^2 \cap L^2_+$. Here $H^m \equiv H^m(\mathbb{T}, \mathbb{C})$ denotes the standard Sobolev space of order $m \geq 1$. Furthermore, it follows that T_u is D-compact. Arguing as in Appendix C (cf. Lemma C.1) one verifies that L_u is selfadjoint and B_u skew-adjoint. The Lax pair formulation of the BO equation then reads

(2.2)
$$\frac{d}{dt}L_u = [B_u, L_u].$$

Remark 2.1. In order to make the paper self-contained, we verify the Lax pair formulation of the BO equation in Appendix A. Clearly, the operator B_u is not uniquely determined. Instead of B_u one might take any operator of the form $B_u + P_u$ where P_u is a skew adjoint operator commuting with L_u . In particular, one might choose instead of B_u the operator $\tilde{B}_u := B_u - iL_u^2$. By a straightforward computation one gets

$$\tilde{B}_u = i(T_{|\partial_x|u} - T_u^2) ,$$

which is a pseudodifferential operator of order 0. This Lax pair formulation will be also a consequence of Proposition 6.1.

The Lax pair formulation of the BO equation implies that at least formally, the spectrum of the operator L_u is left invariant by the solution map of the BO equation. For this reason, we want to analyze it in more detail. In addition of being selfadjoint, L_u has a compact resolvent. Furthermore note that for u=0, $L_0=D$ is a nonnegative operator and its spectrum consists of simple eigenvalues, which we list in increasing order, $\lambda_n=n$, $n\geq 0$. Since T_u is D-compact it then follows that for any $u\in L_r^2$, L_u is bounded from below and its spectrum consists of eigenvalues which are bounded from below and have finite multiplicities. We list these eigenvalues in increasing order and with their multiplicities so that

$$\lambda_n(u) \le \lambda_{n+1}(u), \quad \forall n \ge 0.$$

Our first result says that all eigenvalues of L_u are simple. More precisely, we have the following

Proposition 2.1. For any $u \in L_r^2$ and $n \geq 0$, the eigenvalues $\lambda_n \equiv \lambda_n(u)$ satisfy

$$\lambda_{n+1} \geq \lambda_n + 1$$
.

As a consequence, all eigenvalues of L_n are simple.

Remark 2.2. Since for any $u \in L_r^2$, $\langle L_u 1 | 1 \rangle = -\langle u | 1 \rangle$, it follows from the variational characterization of λ_0 that $\lambda_0 \leq -\langle u | 1 \rangle$. Furthermore, if $\lambda_0 = -\langle u | 1 \rangle$, then $f_0 = 1$ is a normalized eigenfunction corresponding to λ_0 . Since $L_u 1 = -\Pi u$, one has $\Pi u = \langle u | 1 \rangle 1$, implying that $u = \langle u | 1 \rangle$.

The proof of Proposition 2.1 is based on properties of the shift operator S and its commutator relations with L_u . More precisely, S is the operator on L_+^2 , defined by

$$S: L^2_+ \to L^2_+, h \mapsto Sh(x) = e^{ix}h(x)$$
.

Its adjoint with respect to the L^2 -inner product (1.2) is given by

$$S^* = T_{e^{-ix}} ,$$

implying that the following identities hold

$$(2.3) S^*S = Id , SS^* = Id - \langle \cdot | 1 \rangle 1 .$$

Furthermore, since for any $f \in L^2$, $\Pi(e^{ix}f) = S\Pi(f) + \langle e^{ix}f|1\rangle$ one has for any $h \in H^1_+$,

$$(2.4) T_u(Sh) = ST_uh + \langle uSh|1\rangle 1.$$

Using that $S^*1 = 0$, one then gets

$$(2.5) S^*T_uS = T_u$$

and consequently, since DS = SD + S and hence $S^*DS = D + Id$,

$$(2.6) S^*L_uS = L_u + Id.$$

Moreover, combining DS = SD + S with (2.4), one sees that

$$(2.7) L_u S = SL_u + S - \langle uS \cdot | 1 \rangle 1.$$

With these preparations we are now ready to prove Proposition 2.1.

Proof. Apply the max-min formula for λ_n ,

$$\lambda_n = \max_{\dim F = n} m(F) , m(F) := \min\{\langle L_u h | h \rangle : h \in H^1_+ \cap F^\perp, \|h\| = 1\} ,$$

where, in the above maximum, F describes all \mathbb{C} -subspaces of L^2_+ of dimension n. If F is such a vector subspace, observe that $G(F) := \mathbb{C}1 \oplus S(F)$ is a vector subspace of dimension n+1, and that $G(F)^{\perp} \cap H^1_+$ is precisely $S(F^{\perp} \cap H^1_+)$. Therefore, we have

$$m(G(F)) = \min\{\langle L_u Sh | Sh \rangle : h \in H^1_+ \cap F^\perp, ||h|| = 1\}.$$

From (2.6), we infer $\langle L_u Sh, Sh \rangle = \langle L_u h|h \rangle + ||h||^2$ and consequently

$$m(G(F)) = m(F) + 1.$$

Substituting this identity into the above formula for λ_{n+1} , we conclude that $\lambda_{n+1} \ge \max_{\dim F = n} m(G(F)) = \lambda_n + 1$.

We shall prove in Section 5 that the eigenvalues λ_n are real analytic. At this stage, we establish that they are Lipschitz continuous.

Proposition 2.2. For every $n \geq 0$, the functional λ_n is uniformly Lipschitz continuous on bounded subsets of $L_r^2(\mathbb{T})$.

Proof. For any $h \in H^1_+$,

$$(2.8) |\langle L_u h, h \rangle - \langle L_v h, h \rangle| = |\langle (u - v), |h|^2 \rangle| \le ||u - v|| ||h||_{L^4}^2$$

and if in addition ||h|| = 1, one has by the Sobolev embedding theorem

$$(2.9) ||h||_{L^4}^2 < C\langle (1+D)^{1/2}h|h\rangle < C\langle (1+D)h|h\rangle^{1/2}.$$

Applying the max-min formula and (2.8) for v = 0, we obtain

$$\lambda_n(u) \leq \max_{\dim F = n} \min \{ \langle Dh + C \| u \| \langle (1+D)^{1/2} h | h \rangle : h \in F^{\perp} \cap H^1_+, \| h \| = 1 \}$$

$$\leq n + C \| u \| (1+n)^{1/2} .$$

Given a subspace $F \subset L^2_+$ with dim F = n, select $h \in H^1_+ \cap \bigoplus_{k=0}^n \ker(L_u - \lambda_k(u)Id) \cap F^\perp$ with ||h|| = 1. From (2.8) and (2.9) one then infers that

$$\langle L_v h | h \rangle \le \langle L_u h | h \rangle + C \| u - v \| \langle (1+D)h | h \rangle^{1/2}$$
.

On the other hand, $\langle L_u h | h \rangle \leq \lambda_n(u)$, implying that

$$\langle Dh|h\rangle = \langle uh|h\rangle + \lambda_n(u) \le C||u||\langle (1+D)h|h\rangle^{1/2} + n + C||u||(1+n)^{1/2},$$
 yielding

$$\langle Dh|h\rangle \leq \tilde{C}(n+1+||u||^2)$$
.

We then conclude by the max-min formula that

$$\lambda_n(v) \le \lambda_n(u) + K \|u - v\| (n+1 + \|u\|^2)^{1/2}$$

where K > 0 is an absolute constant.

In view of Proposition 2.1 we introduce for any $n \geq 1$ and $u \in L_r^2$,

(2.10)
$$\gamma_n(u) := \lambda_n(u) - \lambda_{n-1}(u) - 1 ,$$

and refer to it as the n-th gap length – see Proposition C.1 in Appendix C for an explanation of this terminology. It then follows that for any $n \ge 1$

(2.11)
$$\lambda_n(u) = n + \lambda_0(u) + \sum_{k=1}^n \gamma_k(u) \ge n + \lambda_0(u) .$$

Next we obtain a useful characterisation for the vanishing of γ_n . For any $u \in L^2_r$ and $n \geq 0$, we denote by f_n a L^2 -normalized eigenfunction of L_u for the eigenvalue λ_n . Hence $f_n \in H^1_+$ and $||f_n|| = 1$. In view of Proposition 2.1, f_n is determined uniquely up to a phase factor for any $n \geq 0$. Later we will show that this indeterminacy can be removed in such a way that the eigenfunctions f_n depend real analytically on u.

Lemma 2.1. For any $u \in L_r^2$, $\langle f_0 | 1 \rangle \neq 0$. Furthermore, for any $n \geq 1$ and $u \in L_r^2$, the following statements are equivalent:

$$(1)\langle f_n|1\rangle = 0$$
, $(2)L_u(Sf_{n-1}) = \lambda_n Sf_{n-1}$, $(3)\langle Sf_{n-1}|u\rangle = 0$, $(4)\gamma_n = 0$.

Proof. Assume that, for some $n \ge 0$, we have $\langle f_n | 1 \rangle = 0$. Then $f_n = Sg_n$ for some g_n , and, by (2.7),

$$\lambda_n S g_n = L_u S g_n = S(L_u g_n + g_n) - \langle S g_n | u \rangle 1.$$

Applying S^* to both sides, we infer, in view of (2.3), that

$$L_u g_n = (\lambda_n - 1) g_n .$$

In the case n = 0, the latter identity contradicts the fact that λ_0 is the smallest eigenvalue of L_u and hence $\langle f_0|1\rangle \neq 0$. In the case $n \geq 1$, the

identity implies that $\lambda_n - 1 = \lambda_{n-1}$ and that g_n is collinear to f_{n-1} , showing that (1) implies (2).

Applying (2.7) to f_{n-1} immediately yields the equivalence of (2) and (3), and that (3) implies (4).

Finally, assume that $\gamma_n = 0$ for some $n \ge 1$. Then, again by applying (2.7) to f_{n-1} ,

$$L_u S f_{n-1} = (\lambda_{n-1} + 1) S f_{n-1} - \langle S f_{n-1} | u \rangle 1 = \lambda_n S f_{n-1} - \langle S f_{n-1} | u \rangle 1$$
.

Taking the inner product with f_n of both sides of the latter identity, one gets

$$\langle uSf_{n-1}|1\rangle\langle 1|f_n\rangle=0$$
,

which means that either (1) holds, or (3) holds, hence (2) holds, so that Sf_{n-1} is collinear to f_n , implying (1).

In a next step we show that the eigenfunctions f_n can be normalized in such a way that they depend continuously in u. First we need to make some preliminary considerations.

Lemma 2.2. For any $u \in L_r^2$ and $k, p \ge 0$

$$(\lambda_p - \lambda_k - 1)\langle f_p | Sf_k \rangle = -\langle f_p | 1 \rangle \langle u | Sf_k \rangle.$$

Proof. Recall from (2.7) that for any $k \geq 0$,

$$(2.12) L_u(Sf_k) = (\lambda_k + 1)Sf_k - \langle Sf_k | u \rangle 1.$$

Computing the inner product of f_p with both sides of (2.12) leads to the claimed formula.

Lemma 2.3. For any $u \in L_r^2$ and $n \ge 1$, $\langle f_n | Sf_{n-1} \rangle \ne 0$.

Proof. Assume that for some $u \in L_r^2$ and $n \ge 1$, $\langle f_n | S f_{n-1} \rangle = 0$. From Lemma 2.2, we infer

$$\langle f_n|1\rangle\langle u|Sf_{n-1}\rangle=0.$$

By Lemma 2.1, this implies that Sf_{n-1} and f_n are collinear, contradicting the assumption $\langle f_n | Sf_{n-1} \rangle = 0$.

Lemma 2.1(i) and Lemma 2.3 allow to define an orthonormal basis of eigenfunctions of L_u as follows.

Definition 2.1. For any $u \in L_r^2$, f_0 is defined to be the L^2 -normalized eigenfunction of L_u corresponding to the eigenvalue λ_0 , uniquely determined by the condition

$$\langle 1|f_0\rangle > 0$$
.

The L^2 -normalized eigenfunction f_1 of L_u , corresponding to the eigenvalue λ_1 , is then defined to be the eigenfunction uniquely determined by the condition

$$\langle f_1|Sf_0\rangle > 0$$
.

Assume that for any $0 \le k \le n$, the eigenfunctions f_k have been defined. Then the L^2 -normalized eigenfunction of L_u , corresponding to the eigenvalue λ_{n+1} , is defined by the condition

$$\langle f_{n+1}|Sf_n\rangle > 0$$
.

In the remaining part of the paper, for any $u \in L_r^2$, $f_n(\cdot, u)$, $n \ge 0$, will always denote the orthonormal basis of L_+^2 introduced in Definition 2.1.

Remark 2.3. It is easy to see that the map $u \mapsto f_n(\cdot, u)$ is continuous from L_r^2 to H_+^1 . Indeed, if $u^{(k)} \to u$ in L_r^2 , $f_n^{(k)} := f_n(\cdot, u^{(k)})$ satisfies

$$||Df_n^{(k)}||^2 = \lambda_n(u^{(k)})\langle f_n^{(k)} | Df_n^{(k)} \rangle + \langle u^{(k)}f_n^{(k)} | Df_n^{(k)} \rangle$$

which, by Proposition 2.2 and Sobolev inequalities, implies that $f_n^{(k)}$ is bounded in H_+^1 as $k \to \infty$. By Rellich's theorem and Proposition 2.2, we infer that any weak * limit point g_n of $(f_n^{(k)})_{k\geq 1}$ in H_+^1 satisfies $L_u g_n = \lambda_n(u) g_n$, $||g_n|| = 1$ and $||Dg_n||^2 = \lim_{k\to\infty} ||Df_n^{(k)}||^2$. Furthermore, the conditions $\langle 1, f_0 \rangle \geq 0$, $\langle f_{n+1} | Sf_n \rangle \geq 0$ are also stable by weak * convergence in H_+^1 , implying that $g_0 = f_0$ (Lemma 2.1) and $g_n = f_n$ for any $n \geq 1$ (Lemma 2.3). In Section 5, we will prove that $u \mapsto f_n(\cdot, u)$ is a real analytic map from L_r^2 to H_+^1 .

The complex numbers $\langle 1|f_n\rangle$, $n \geq 1$, appropriately rescaled, are our candidates for the complex Birkhoff coordinates. Note that for any $u \in L_r^2$, one can express Πu in terms of $\langle 1|f_n\rangle$, $n \geq 0$, as follows. Since f_n , $n \geq 0$, is an orthonormal basis of L_+^2 , one has $\Pi u = \sum_{n=0}^{\infty} \langle \Pi u|f_n\rangle f_n$. When combined with the identity

(2.13)
$$\lambda_n \langle 1|f_n \rangle = \langle 1|L_u f_n \rangle = \langle L_u 1|f_n \rangle = -\langle \Pi u|f_n \rangle = -\langle u|f_n \rangle$$
 one is led to the following trace formula

(2.14)
$$\Pi u = -\sum_{n=0}^{\infty} \lambda_n \langle 1|f_n \rangle f_n$$

and hence

(2.15)
$$\|\Pi u\|^2 = \sum_{n=0}^{\infty} \lambda_n^2 |\langle 1|f_n\rangle|^2$$

and

(2.16)
$$\langle u | 1 \rangle = -\sum_{n=0}^{\infty} \lambda_n |\langle 1 | f_n \rangle|^2.$$

Similarly, the expansion $1 = \sum_{n=0}^{\infty} \langle 1|f_n \rangle f_n$ leads to $\sum_{n=0}^{\infty} |\langle 1|f_n \rangle|^2 = 1$. We record the following property of the eigenfunctions f_n .

Lemma 2.4. For any $u \in L_r^2$ with $\gamma_n = 0$ for a given $n \geq 1$, $f_n = Sf_{n-1}$.

Proof. By Lemma 2.1(iii), for any $n \geq 1$ with $\gamma_n = 0$, the function Sf_{n-1} is an L^2 -normalized eigenfunction of L_u , corresponding to the eigenvalue λ_n and satisfies the condition $\langle f_n | Sf_{n-1} \rangle > 0$.

In analogy with the notion of a finite gap potential, introduced in the context of the Korteweg-de Vries equation and the nonlinear Schrödinger equation, we define the corresponding notion in the context of the BO equation.

Definition 2.2. A potential $u \in L_r^2$ is said to be a finite gap potential if $J(u) = \{n \geq 1 : \gamma_n(u) > 0\} \subset \mathbb{N}$ is finite. In case J(u) consists of one element only, u is referred to as a one gap potential.

Note that for any finite gap potential u, the expansion (2.14), is a finite sum. Finite gap potentials play a similar role as trigonometric polynomials in harmonic analysis. For any given subset $J \subset \mathbb{N}$ introduce

$$G_J := \{ u \in L_r^2 : \gamma_j(u) > 0 \ \forall j \in J; \ \gamma_j(u) = 0 \ \forall j \in \mathbb{N} \setminus J \} .$$

Elements in the set G_J are referred to as J-gap potentials. We refer to Section 7 for a detailed studied of J-gap potentials with $J = \{1, \dots, N\}$ and to Appendix B for a description of one gap potentials.

Remark 2.4. One might ask if there exists a potential $u \in L_r^2$ with the property that $\gamma_n(u) > 0$ for every $n \ge 1$. We claim that $u(x) = 2\alpha \cos x$ is such a potential for every $\alpha \in \mathbb{R} \setminus \{0\}$. Indeed, we have for any $n \ge 1$,

$$\langle Sf_{n-1}|u\rangle = \alpha \langle Sf_{n-1}|e^{ix}\rangle = \alpha \langle f_{n-1}|1\rangle.$$

Hence, by Lemma 2.1, $\langle f_n|1\rangle = 0$ if and only if $\langle f_{n-1}|u\rangle = 0$. Since $\langle f_0|1\rangle \neq 0$, we conclude that $\langle f_n|1\rangle \neq 0$ for every n.

We finish this section with a brief discussion of the symmetry induced by translation. For any $\tau \in \mathbb{R}$, denote by Q_{τ} the linear isometry on L_r^2 , given by the translation by τ ,

$$Q_{\tau}: L_r^2 \to L_r^2, \ , \ Q_{\tau}u(x) := u(x+\tau).$$

Lemma 2.5. For any $\tau \in \mathbb{R}$, $u \in L_r^2$, and $n \geq 0$, one has

$$\lambda_n(Q_{\tau}u) = \lambda_n(u), \qquad f_n(x, Q_{\tau}u) = e^{-in\tau} f_n(x + \tau, u).$$

As a consequence,

(2.17)
$$\langle 1 | f_n(\cdot, Q_{\tau}u) \rangle = e^{in\tau} \langle 1 | f_n(\cdot, u) \rangle, \quad \forall n \ge 0.$$

Proof. For any $\tau \in \mathbb{R}$, $u \in L_r^2$, and $n \geq 0$, note that $f_n \equiv f_n(\cdot, u)$ satisfies

$$L_{Q_{\tau}u}(Q_{\tau}f_n) = Q_{\tau}(L_uf_n) = \lambda_n(u)Q_{\tau}f_n .$$

It implies that $\lambda_n(Q_{\tau}u) = \lambda_n(u)$ and that $Q_{\tau}f_n(\cdot,u)$ is an L^2 -normalized eigenfunction of $L_{Q_{\tau}u}$, corresponding to the eigenvalue λ_n . Furthermore,

$$\langle 1 | f_0(\cdot + \tau, u) \rangle = \langle 1 | f_0(\cdot, u) \rangle > 0$$

implying that $f_0(\cdot, Q_{\tau}u) = f_0(\cdot + \tau, u)$. Similarly, for any $n \ge 1$,

$$\langle e^{-in\tau} f_n(\cdot + \tau, u) \mid Se^{-i(n-1)\tau} f_{n-1}(\cdot + \tau, u) \rangle = \langle f_n(\cdot + \tau, u) \mid e^{i\tau} Sf_{n-1}(\cdot + \tau, u) \rangle$$

which equals $\langle f_n(\cdot, u) | Sf_{n-1}(\cdot, u) \rangle > 0$. Arguing by induction one then concludes that $f_n(\cdot, Q_{\tau}u) = e^{-in\tau}f_n(\cdot + \tau, u)$ as claimed.

3. Generating function and trace formulae

One of the main results of this section are trace formulas for $|\langle 1|f_n\rangle|^2$, stated in Corollary 3.1. They will be used to define our candidates of (complex) Birkhoff coordinates by appropriately scaling $\langle 1|f_n\rangle$. A key ingredient for the proof of these trace formulas is the generating function \mathcal{H}_{λ} . We remark that \mathcal{H}_{λ} plays a role for the normal form theory of the BO equation comparable to the one of the discriminant in the normal form theory of the KdV equation.

For any $u \in L_r^2$, the generating function is the meromorphic function $\lambda \mapsto \mathcal{H}_{\lambda}(u)$, defined by

(3.1)
$$\mathcal{H}_{\lambda}(u) := \langle (L_u + \lambda Id)^{-1} 1 | 1 \rangle.$$

Note that $\mathcal{H}_{\lambda}(u)$ is holomorphic on $\mathbb{C} \setminus \{-\lambda_n(u) : n \geq 0\}$ and might have simple poles at $\lambda = -\lambda_n(u), n \geq 0$. Substituting the expansion $1 = \sum_{n=0}^{\infty} \langle 1 | f_n \rangle f_n$ into the expression for $\mathcal{H}_{\lambda}(u)$ one obtains

(3.2)
$$\mathcal{H}_{\lambda}(u) = \sum_{n=0}^{\infty} \frac{|\langle 1|f_n\rangle|^2}{\lambda_n + \lambda} .$$

Proposition 3.1. For any $u \in L_r^2$, the following identities hold:

(i)
$$\mathcal{H}_{\lambda}(u) = \frac{1}{\lambda_0 + \lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\gamma_n}{\lambda_n + \lambda}\right),$$

(ii)
$$||u||^2 - \langle u|1\rangle^2 = 2\sum_{n=1}^{\infty} n\gamma_n$$
, $\langle u|1\rangle = -\lambda_0 - \sum_{n=1}^{\infty} \gamma_n$.

Hence $\gamma_n(u) = \frac{1}{n} \ell_n^1$ and both $\langle u|1\rangle$ and $||u||^2$ are spectral invariants.

Proof. (i) The proof makes use of the shift operator S and in particular identity (2.6). Assume that λ is real and $\lambda \geq -\lambda_0 + 1$, so that $L_u + \lambda Id$: $H^1_+ \to L^2_+$ is invertible. By (2.6), one has

$$(3.3) S^*(L_u + \lambda Id)S = L_u + (\lambda + 1)Id,$$

implying that the operator $S^*(L_u + \lambda Id)S : H^1_+ \to L^2_+$ is invertible. To compute its inverse we need the following

Lemma 3.1. Assume that $A: H^1_+ \to L^2_+$ is positive and selfadjoint so that A and S^*AS are invertible and A^{-1} is positive on L^2_+ . Then

$$(S^*AS)^{-1} = S^*A^{-1}S - \frac{\langle \cdot \mid S^*A^{-1}1 \rangle}{\langle A^{-1}1 \mid 1 \rangle} S^*A^{-1}1.$$

Proof of Lemma 3.1. Consider $f \in L^2_+$ and let $h := (S^*AS)^{-1}f$. Applying S to both sides of the equation

$$S^*ASh = f$$
,

we infer from (2.3) that

$$ASh = Sf + \langle ASh|1\rangle 1$$
,

hence

$$(3.4) Sh = A^{-1}Sf + \langle ASh|1\rangle A^{-1}1.$$

In particular, we have

$$\langle A^{-1}Sf|1\rangle = -\langle ASh|1\rangle\langle A^{-1}1|1\rangle$$
.

Since A^{-1} is positive, $\langle A^{-1}1|1\rangle > 0$ and hence we obtain

$$\langle ASh|1\rangle = -\frac{\langle A^{-1}Sf|1\rangle}{\langle A^{-1}1|1\rangle} = -\frac{\langle f|S^*A^{-1}1\rangle}{\langle A^{-1}1,1\rangle} \ .$$

Substituting this identity into the equation (3.4) for Sh and applying S^* to both sides, we obtain the claimed statement.

Let us now go back to the proof of item(i). Since $L_u + \lambda Id$ is selfadjoint and positive we can apply Lemma 3.1 to $A := L_u + \lambda Id$ to get, in view of (3.3),

$$(3.5) \qquad (L_u + (\lambda + 1)Id)^{-1} - S^*(L_u + \lambda Id)^{-1}S$$

$$= -\frac{\langle \cdot | S^*(L_u + \lambda Id)^{-1} 1 \rangle}{\langle (L_u + \lambda Id)^{-1} 1 | 1 \rangle} S^*(L_u + \lambda Id)^{-1}1.$$

Since the latter operator is of rank 1, it is of trace class and hence so is the operator $(L_u + (\lambda + 1)Id)^{-1} - S^*(L_u + \lambda Id)^{-1}S$. To compute its trace we write it as a sum of two operators,

$$(L_u + (\lambda + 1)Id)^{-1} - S^*(L_u + \lambda Id)^{-1}S = A_1 + A_2$$

where

$$A_1 := (L_u + (\lambda + 1)Id)^{-1} - (L_u + \lambda Id)^{-1},$$

$$A_2 := (L_u + \lambda Id)^{-1} - S^*(L_u + \lambda Id)^{-1}S.$$

In view of Proposition 2.1, A_1 is of trace class and

$$\operatorname{Tr} A_1 = \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n + \lambda + 1} - \frac{1}{\lambda_n + \lambda} \right).$$

Since by (3.5), $A_1 + A_2$ has rank 1 and hence is of trace class, the operator A_2 is also of trace class. Computing its trace with respect to the orthonormal basis $(e^{inx})_{n\geq 0}$ of L^2_+ , we obtain

$$\operatorname{Tr} A_2 = \operatorname{Tr} ((L_u + \lambda Id)^{-1} - S^* (L_u + \lambda Id)^{-1} S) = \langle (L_u + \lambda Id)^{-1} 1 | 1 \rangle$$
.

On the other hand, by (3.5), $Tr(A_1 + A_2)$ equals

$$\operatorname{Tr}\left(-\frac{\langle \cdot | S^*(L_u + \lambda Id)^{-1} 1 \rangle}{\langle (L_u + \lambda Id)^{-1} 1 | 1 \rangle} S^*(L_u + \lambda Id)^{-1} 1\right) = -\frac{\|S^*(L_u + \lambda Id)^{-1} 1\|^2}{\langle L_u + \lambda Id \rangle^{-1} 1 | 1 \rangle}$$

and hence by (2.3)

$$\operatorname{Tr}(A_1 + A_2) = -\frac{\|(L_u + \lambda Id)^{-1}1\|^2}{\langle (L_u + \lambda Id)^{-1}1|1\rangle} + \langle (L_u + \lambda Id)^{-1}1|1\rangle .$$

Altogether we have proved that

$$\sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n + \lambda + 1} - \frac{1}{\lambda_n + \lambda} \right) = -\frac{\|(L_u + \lambda Id)^{-1}1\|^2}{\langle (L_u + \lambda Id)^{-1}1|1 \rangle}$$

or (3.6)

$$\sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n + \lambda + 1} - \frac{1}{\lambda_n + \lambda} \right) = -\frac{\sum_{n=0}^{\infty} \frac{|\langle 1|f_n \rangle|^2}{(\lambda_n + \lambda)^2}}{\sum_{n=0}^{\infty} \frac{|\langle 1|f_n \rangle|^2}{\lambda_n + \lambda}} = \frac{d}{d\lambda} \log \mathcal{H}_{\lambda}(u) .$$

Isolating the term $-(\lambda_0 + \lambda)^{-1}$ on the left hand side of the latter identity and then rewriting the remaining sum as a telescopic series yields

(3.7)
$$\frac{d}{d\lambda}\log \mathcal{H}_{\lambda}(u) = -\frac{1}{\lambda_0 + \lambda} + \sum_{n=1}^{\infty} \frac{\gamma_n}{(\lambda + \lambda_{n-1} + 1)(\lambda + \lambda_n)}$$

or, written in a slightly more convenient form,

$$\frac{d}{d\lambda}\log\left((\lambda_0+\lambda)\mathcal{H}_{\lambda}(u)\right) = \sum_{n=1}^{\infty} \frac{\gamma_n}{(\lambda+\lambda_{n-1}+1)(\lambda+\lambda_n)}.$$

Integrating both sides of the latter identity from λ to $+\infty$, we infer

$$-\log\left((\lambda_0 + \lambda)\mathcal{H}_{\lambda}(u)\right) = -\sum_{n=1}^{\infty}\log\left(1 - \frac{\gamma_n}{\lambda + \lambda_n}\right),\,$$

implying that the formula of item (i) holds for any $\lambda > |-\lambda_0 + 1|$. By analyticity, it is then valid for any $\lambda \in \mathbb{C} \setminus \{-\lambda_n : n \geq 0\}$.

(ii) The claimed trace formulas are obtained by expanding \mathcal{H}_{λ} at $\lambda = \infty$. For $\varepsilon := \frac{1}{\lambda} > 0$ with $\lambda > |-\lambda_0 + 1|$, define

(3.8)
$$\tilde{\mathcal{H}}_{\varepsilon} := \frac{1}{\varepsilon} \mathcal{H}_{\frac{1}{\varepsilon}} = \sum_{n=0}^{\infty} \frac{|\langle 1|f_n \rangle|^2}{1 + \varepsilon \lambda_n} ,$$

so that in view of (2.15) - (2.16), $\tilde{\mathcal{H}}_0 = 1$, $\frac{d}{d\varepsilon}|_{\varepsilon=0}\tilde{\mathcal{H}}_{\varepsilon} = \langle u|1\rangle$, and

(3.9)
$$\frac{d^2}{d\varepsilon^2}|_{\varepsilon=0}\tilde{\mathcal{H}}_{\varepsilon} = 2\|\Pi u\|^2 = \|u\|^2 + \langle u|1\rangle^2$$

and hence, using these identities,

$$\frac{d}{d\varepsilon}|_{\varepsilon=0}\log\tilde{\mathcal{H}}_{\varepsilon} = \frac{d}{d\varepsilon}|_{\varepsilon=0}\tilde{\mathcal{H}}_{\varepsilon} = \langle u|1\rangle$$

and

$$(3.10) \qquad \frac{d^2}{d\varepsilon^2}|_{\varepsilon=0}\log\tilde{\mathcal{H}}_{\varepsilon} = -\left(\frac{d}{d\varepsilon}|_{\varepsilon=0}\tilde{\mathcal{H}}_{\varepsilon}\right)^2 + \frac{d^2}{d\varepsilon^2}|_{\varepsilon=0}\tilde{\mathcal{H}}_{\varepsilon} = ||u||^2.$$

Furthermore, identity (3.7) becomes

$$(3.11) \qquad -\frac{d}{d\varepsilon}\log\tilde{\mathcal{H}}_{\varepsilon} = \frac{\lambda_0}{1+\varepsilon\lambda_0} + \sum_{n=1}^{\infty} \frac{\gamma_n}{(1+\varepsilon(\lambda_{n-1}+1))(1+\varepsilon\lambda_n)} \ .$$

Passing to the limit as $\varepsilon \to 0$ in both sides, we obtain that the series $\sum_{n=1}^{\infty} \gamma_n$ is summable — notice that $\gamma_n \geq 0$ and $\lambda_n \geq 0$ for n large enough — and that

$$(3.12) -\langle u|1\rangle = \lambda_0 + \sum_{n=1}^{\infty} \gamma_n .$$

Taking the derivatives of both sides of (3.11) with respect to ε , one concludes that $\frac{d^2}{d\varepsilon^2}\log \tilde{\mathcal{H}}_{\varepsilon}$ equals

$$\frac{\lambda_0^2}{(1+\varepsilon\lambda_0)^2} + \sum_{n=1}^{\infty} \frac{\gamma_n}{(1+\varepsilon(\lambda_{n-1}+1))(1+\varepsilon\lambda_n)} \left(\frac{\lambda_{n-1}+1}{1+\varepsilon(\lambda_{n-1}+1)} + \frac{\lambda_n}{1+\varepsilon\lambda_n}\right).$$

Passing to the limit as $\varepsilon \to 0$ and using (3.10), one concludes that

$$||u||^2 = \lambda_0^2 + \sum_{n=1}^{\infty} \gamma_n (\lambda_n + \lambda_{n-1} + 1)$$
.

From identity (3.12) and the definition of γ_n we have for every $n \geq 0$,

(3.13)
$$\lambda_n = n - \langle u|1\rangle - \sum_{k=n+1}^{\infty} \gamma_k .$$

As a consequence,

$$\lambda_n + \lambda_{n-1} + 1 = 2n - 2\langle u|1\rangle - \gamma_n - 2\sum_{k=n+1}^{\infty} \gamma_k ,$$

so that

$$||u||^2 = \lambda_0^2 + 2\sum_{n=1}^{\infty} n\gamma_n - 2\langle u|1\rangle \sum_{n=1}^{\infty} \gamma_n - \left(\sum_{n=1}^{\infty} \gamma_n\right)^2,$$

which, in view of (3.12), leads to

$$||u||^2 = \langle u|1\rangle^2 + 2\sum_{n=1}^{\infty} n\gamma_n ,$$

whence the claimed formula for $||u||^2$.

Remark 3.1. (i) It follows from the proof of Proposition 3.1 that $\mathcal{H}_{\lambda}(u)$ equals the trace of the rank one operator $(L_u + \lambda Id)^{-1} - S^*(L_u + \lambda Id)^{-1}S$. (ii) Writing $u \in L_r^2$ as v + c with $c = \langle u|1 \rangle$ and hence $v \in L_{r,0}^2$, one computes

$$\mathcal{H}(u) = \mathcal{H}(v) - c \frac{1}{2\pi} \int_{0}^{2\pi} v^2 dx - \frac{1}{3}c^3.$$

By the computations in the proof of Proposition 3.1 one has (cf. also (8.1))

$$\mathcal{H}(v) = -\frac{1}{6} \frac{d^3}{d\varepsilon^3} |_{\varepsilon=0} \tilde{\mathcal{H}}_{\varepsilon}(v) , \quad ||v||^2 = \frac{d^2}{d\varepsilon^2} |_{\varepsilon=0} \tilde{\mathcal{H}}_{\varepsilon}(v) ,$$

and
$$c = \langle u|1\rangle = \frac{d}{d\varepsilon}|_{\varepsilon=0}\tilde{\mathcal{H}}_{\varepsilon}(u)$$
.

The identity of Proposition 3.1(i) can be used to obtain product representations for $|\langle 1|f_n\rangle|^2$, which will be used later to appropriately scale $\langle 1|f_n\rangle$ in order to obtain our candidates for (complex) Birkhoff coordinates.

Corollary 3.1. For any $u \in L_r^2$ and $n \ge 1$,

$$|\langle 1|f_0\rangle|^2 = \kappa_0 , \qquad \kappa_0 := \prod_{p=1}^{\infty} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_0}\right) ,$$
$$|\langle 1|f_n\rangle|^2 = \gamma_n \kappa_n , \quad \kappa_n := \frac{1}{\lambda_n - \lambda_0} \prod_{1 \le p \ne n} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}\right) .$$

In particular, $|\langle 1, f_n \rangle|^2$, $n \geq 0$, are spectral invariants.

Remark 3.2. The formulas of Corollary 3.1 provide a new proof of the fact that $\langle 1|f_0\rangle \neq 0$ and that, for $n \geq 1$, $\langle 1|f_n\rangle = 0$ if and only if $\gamma_n = 0$.

Proof. The claimed product representations are obtained by computing with the help of the identity of Proposition 3.1(i) the residue of $\mathcal{H}_{\lambda}(u)$ at $\lambda = -\lambda_n$ for any $n \geq 0$.

As an application of Proposition 3.1 and Corollary 3.1 we analyze the isospectral set Iso(u) of an arbitrary potential $u \in L_r^2$, defined as

$$Iso(u) := \{ v \in L_r^2 : \lambda_n(v) = \lambda_n(u) \ \forall n \ge 0 \}.$$

Note that Iso(u) is closed by Proposition 2.2 and that

Iso(u) = {
$$v \in L_r^2 : \mathcal{H}_{\lambda}(v) = \mathcal{H}_{\lambda}(u) \ \forall \lambda \in \mathbb{C} \setminus {\{\lambda_n(u) : n \ge 0\}}$$
.

First we need to make some preliminary considerations. Recall that by Proposition 3.1, $(\gamma_n(u))_{n\geq 1}$ is in the weighted ℓ^1 -sequence space $\ell^{1,1}(\mathbb{N},\mathbb{R})$ and $\gamma_n(u)\geq 0$ for any $n\geq 1$. Hence the map Γ

(3.14)
$$\Gamma: L_{r,0}^2 \to \mathcal{C}_+^{1,1}, u \mapsto (\gamma_n(u))_{n \ge 1}$$

is well defined. Here $\mathcal{C}^{1,1}_+$ denotes the positive cone in $\ell^{1,1}(\mathbb{N},\mathbb{R})$,

$$\mathcal{C}_{+}^{1,1} := \{ (r_n)_{n \ge 1} \in \ell^{1,1}(\mathbb{N}, \mathbb{R}) : r_n \ge 0 \ \forall n \ge 1 \}.$$

Proposition 3.2. The map Γ is proper.

Proof. Let $(u^{(k)})_{k\geq 1}$ be a sequence in $L^2_{r,0}$ so that

$$\Gamma(u^{(k)}) \to \gamma = (\gamma_n)_{n \ge 1} \in \mathcal{C}_+^{1,1} \quad \text{as } k \to \infty.$$

By Proposition 3.1(ii) and the assumption,

$$||u^{(k)}||^2 = 2\sum_{n=1}^{\infty} n\gamma_n(u^{(k)}) \rightarrow 2\sum_{n=1}^{\infty} n\gamma_n \text{ as } k \to \infty,$$

and thus we may assume, after extracting a subsequence if needed, that $u^{(k)} \rightharpoonup u$ weakly in $L_{r,0}^2$. The proof will be complete if we establish that this convergence is strong, or equivalently that

$$||u||^2 = 2\sum_{n=1}^{\infty} n\gamma_n .$$

Since (again by Proposition 3.1(ii))

$$L_{u^{(k)}} \ge \lambda_0(u^{(k)}) = -\sum_{n=1}^{\infty} \gamma_n(u^{(k)}) \to -\sum_{n=1}^{\infty} \gamma_n$$
,

we infer that there exists $c > |-\lambda_0 + 1|$ so that for any $k \ge 1$ and $\lambda \ge c$,

$$L_{u^{(k)}} + \lambda Id : H^1_+ \longrightarrow L^2_+$$

is an isomorphism whose inverse is bounded uniformly in k. Therefore

$$w_{\lambda}^{(k)} := (L_{u^{(k)}} + \lambda Id)^{-1}[1],$$

is a well defined, bounded sequence of H^1_+ . Let us choose an arbitrary countable subset Λ of $[c,+\infty)$ with a cluster point. By a diagonal procedure, we extract a subsequence of $w^{(k)}_{\lambda}$, again denoted by $w^{(k)}_{\lambda}$, so that for every $\lambda \in \Lambda$, $w^{(k)}_{\lambda}$ converges weakly in H^1_+ to some element $w_{\lambda} \in H^1_+$ as $k \to \infty$. By Rellich's theorem we infer that, weakly in L^2_+ ,

$$(L_{u^{(k)}} + \lambda Id)w_{\lambda}^{(k)} \rightharpoonup (L_u + \lambda Id)w_{\lambda} \text{ as } k \to \infty.$$

But since $(L_{u^{(k)}} + \lambda Id)w_{\lambda}^{(k)} = 1$ for any $k \geq 1$ it then follows that for every $\lambda \in \Lambda$, $(L_u + \lambda Id)w_{\lambda} = 1$ and hence by the definition of $\mathcal{H}_{\lambda}(u)$,

$$\mathcal{H}_{\lambda}(u^{(k)}) = \langle w_{\lambda}^{(k)} | 1 \rangle \rightarrow \langle w_{\lambda} | 1 \rangle = \mathcal{H}_{\lambda}(u) \quad \forall \lambda \in \Lambda.$$

On the other hand, for every $n \geq 0$,

$$\lambda_n(u^{(k)}) = n - \sum_{j=n+1}^{\infty} \gamma_j(u^{(k)}) \rightarrow n - \sum_{j=n+1}^{\infty} \gamma_j \text{ as } k \to \infty$$

uniformly with respect to $n \geq 0$. Hence, setting $\lambda_n := n - \sum_{j=n+1}^{\infty} \gamma_j$ for any $n \geq 0$ one infers that for any $\lambda \in [c, \infty)$,

$$\frac{1}{\lambda_0(u^{(k)}) + \lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\gamma_n(u^{(k)})}{\lambda_n(u^{(k)}) + \lambda} \right) \rightarrow \frac{1}{\lambda_0 + \lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\gamma_n}{\lambda_n + \lambda} \right).$$

By Proposition 3.1(i), it then follows that

$$\mathcal{H}_{\lambda}(u) = \frac{1}{\lambda_0 + \lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\gamma_n}{\lambda_n + \lambda} \right)$$

for any $\lambda \in \Lambda$ and hence by analyticity, for any $\lambda \in [c, +\infty)$. Thus (3.9) applies and we conclude that $||u||^2 = 2 \sum_{n=1}^{\infty} n \gamma_n$.

Our result on Iso(u) then reads as follows.

Proposition 3.3. For every $u \in L_r^2$, Iso(u) is a compact subset of L_r^2 . In fact, $Iso(u) - \langle u|1 \rangle$ is a compact subset of the sphere in $L_{r,0}^2$ of radius $||u - \langle u|1 \rangle||$, centered at 0. In particular, for any $c \in \mathbb{R}$, Iso(c) consists of the constant potential c only.

Proof. Let $v \in \text{Iso}(u)$. Since by Proposition 3.1(ii)

$$\langle v|1\rangle = -\lambda_0(u) - \sum_{n=1}^{\infty} \gamma_n(u) = \langle u|1\rangle$$

and $\gamma_n(u - \langle u|1\rangle) = \gamma_n(u)$ by the definition of γ_n , $n \geq 1$, one has

$$v - \langle u|1\rangle \in \Gamma^{-1}\{(\gamma_n(u))_{n\geq 1}\}.$$

Using that Γ is proper by Proposition 3.2 and that $\|u - \langle u|1\rangle\|$ is a spectral invariant, it follows that $\mathrm{Iso}(u - \langle u|1\rangle) = \mathrm{Iso}(u) - \langle u|1\rangle$ is contained in a compact subset of the sphere in $L^2_{r,0}$ of radius $\|u - \langle u|1\rangle\|$, centered at 0 . Since $\mathrm{Iso}(u)$ is closed one then concludes that $\mathrm{Iso}(u) - \langle u|1\rangle$ and in turn $\mathrm{Iso}(u)$ are compact. \square

4. Complex Birkhoff Coordinates

In this section we introduce our candidates of complex Birkhoff coordinates, define the corresponding Birkhoff map Φ , and discuss first properties of Φ .

For any $u \in L^2_{r,0}$ and $n \geq 0$, define

(4.1)
$$\zeta_n(u) := \frac{\langle 1|f_n\rangle}{\sqrt{\kappa_n}}$$

where we recall that $\kappa_0 = \prod_{p \geq 1} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_0}\right)$ and, for any $n \geq 1$,

$$\kappa_n = \frac{1}{\lambda_n - \lambda_0} \prod_{1 \le p \ne n} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n} \right) .$$

The functionals ζ_n , $n \geq 1$, are our candidates for complex Birkhoff coordinates of the BO equation. By Corollary 3.1, $\zeta_0 = 1$ and for any $n \geq 1$, $|\zeta_n|^2 = \gamma_n$.

Proposition 4.1. The map

(4.2)
$$\Phi: L_{r_0}^2 \to h_+^{1/2}, \ u \mapsto (\zeta_n(u))_{n \ge 1}$$

is continuous and proper.

Proof. Since by Proposition 3.1, $\gamma_n = \frac{1}{n} \ell_n^1$ and in addition $|\zeta_n| = |\gamma_n|^{1/2}$, it follows that $\Phi(u) \in h_+^{1/2}$ for any $u \in L_{r,0}^2$. Furthermore, since for any $n \geq 1$, we know from Proposition 2.2, Remark 2.3 and Proposition 3.1 (ii), that $\langle 1, f_n \rangle$ and κ_n depend continuously on u, we infer that ζ_n depends continuously on u. Since by Proposition 3.1 (ii), Φ is a bounded map, it then follows that whenever $u_k \to u$ in $L_{r,0}^2$, $\Phi(u_k)$ converges to $\Phi(u)$ weakly in the Hilbert space $h_+^{1/2}$. Since, again by Proposition 3.1,

$$\|\Phi(v)\|_{1/2}^2 = 2\|v\|^2$$
, $v \in L_{r,0}^2$

we infer that $\|\Phi(u_k)\|_{1/2} \to \|\Phi(u)\|_{1/2}$ and hence that $\Phi(u_k) \to \Phi(u)$ in $h_+^{1/2}$. This shows that Φ is continuous. Finally, by Proposition 3.2, the map Γ is proper and so is Φ .

In a next step we want to show that the map Φ is one-to-one. We prove this fact by showing that any potential $u \in L^2_{r,0}$ can be expressed in terms of the complex numbers $\zeta_n(u)$, $n \geq 1$. First note that the function Πu extends as a holomorphic function to the unit disc |z| < 1, which by a slight abuse of notation, we denote by $\Pi u(z)$. It is given by its Taylor expansion at z = 0,

$$\Pi u(z) = \sum_{k=0}^{\infty} \widehat{u}(k) z^k = \sum_{k=0}^{\infty} \langle \Pi u | S^k 1 \rangle z^k = \sum_{k=0}^{\infty} \langle (S^*)^k \Pi u | 1 \rangle z^k .$$

Denote by $||S^*||$ the norm of the operator $S^*: L_+^2 \to L_+^2$. Since $|z|||S^*|| = |z| < 1$, one has $\sum_{k=0}^{\infty} (zS^*)^k = (Id - zS^*)^{-1}$ and hence

(4.3)
$$\Pi u(z) = \langle (Id - zS^*)^{-1} \Pi u | 1 \rangle , \qquad \forall |z| < 1.$$

Denote by $M \equiv M(u)$ the matrix representation of the operator S^* : $L^2_+ \to L^2_+$ in the basis $(f_n)_{n \geq 0}$,

$$(4.4) \quad M = (M_{np})_{n,p \ge 0}, \qquad M_{np} \equiv M_{np}(u) = \langle S^* f_p | f_n \rangle = \langle f_p | S f_n \rangle .$$

Then (4.3) together with the formula (2.14) for Πu yields the following

Lemma 4.1. For any $u \in L^2_r$ and any $z \in \mathbb{C}$ with |z| < 1,

(4.5)
$$\Pi u(z) = \langle (Id - zM(u))^{-1}X(u)|Y(u)\rangle_{\ell^2},$$

with X(u) and Y(u) being the column vectors

(4.6)
$$X(u) := -(\lambda_p \langle 1|f_p \rangle)_{p \geq 0} , Y(u) := (\langle 1|f_n \rangle)_{n \geq 0} .$$

Recall that by (2.3), $MM^* = Id$ and hence Id - zM is invertible for any |z| < 1.

Proposition 4.2. The map Φ is one-to-one.

Proof. First recall that that for any $u \in L^2_{r,0}$, $\langle 1|f_0\rangle = \sqrt{\kappa_0}$ with $\kappa_0 = \prod_{p\geq 1} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_0}\right)$ and for any $n\geq 1$

$$\langle 1|f_n\rangle = \sqrt{\kappa_n}\zeta_n \;, \qquad \kappa_n = \frac{1}{\lambda_n - \lambda_0} \prod_{1 \le n \ne n} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}\right)$$

where $\gamma_p = |\zeta_p|^2$ and $\lambda_n = n - \sum_{k=n+1}^{\infty} \gamma_k$. Therefore the components $\langle 1|f_n\rangle$ of Y(u) and the components $-\lambda_p\langle 1|f_p\rangle$ of X(u) can be expressed in terms of the ζ_k , $k\geq 1$. We claim that the coefficients M_{np} can also be expressed in this way. Indeed, for any given $n\geq 0$ one argues as follows: if $\zeta_{n+1}=0$, then $\gamma_{n+1}=0$ and hence $f_{n+1}=Sf_n$, implying that

$$M_{np} = \delta_{p,n+1}.$$

If $\zeta_{n+1} \neq 0$, then $\gamma_{n+1} \neq 0$ and $\langle 1|f_{n+1}\rangle \neq 0$. We then use that by Lemma 2.2 for any $p \geq 0$,

$$(\lambda_p - \lambda_n - 1)M_{np} = -\langle u|Sf_n\rangle\langle f_p|1\rangle$$

and hence in particular for p = n+1, $\gamma_{n+1}\langle f_{n+1}|Sf_n\rangle = -\langle u|Sf_n\rangle\langle f_{n+1}|1\rangle$. Combining these two identities then yields

(4.7)
$$M_{np} = \frac{\gamma_{n+1} \langle f_{n+1} | Sf_n \rangle}{\langle f_{n+1} | 1 \rangle} \frac{\langle f_p | 1 \rangle}{\lambda_p - \lambda_n - 1}.$$

Since $\langle f_{n+1}|Sf_n\rangle>0$ the identity $1=\|Sf_n\|^2=\sum_{p=0}^\infty |M_{np}|^2$ then reads

(4.8)
$$1 = \langle f_{n+1} | Sf_n \rangle^2 \frac{\gamma_{n+1}^2}{|\langle f_{n+1} | 1 \rangle|^2} \sum_{n=0}^{\infty} \frac{|\langle f_p | 1 \rangle|^2}{(\lambda_p - \lambda_n - 1)^2}$$

Altogether we thus have shown that all terms in the formula (4.7) for M_{np} can be expressed in terms of ζ_k , $k \geq 1$, and hence also $\Pi u(z)$ for |z| < 1 can be expressed in this way. Since $\Pi u(z)$, |z| < 1, completely determines u we have shown that Φ is one-to-one.

Remark 4.1. By the definition (3.2) of \mathcal{H}_{λ} and formula (4.8),

$$1 = \langle f_{n+1} | S f_n \rangle^2 \frac{\gamma_{n+1}^2}{|\langle f_{n+1} | 1 \rangle|^2} (-\mathcal{H}'_{-\lambda_n - 1}).$$

In view of Proposition 3.1(i) this leads to $\langle f_{n+1}|Sf_n\rangle = \sqrt{\mu_{n+1}}$ with

(4.9)
$$\mu_{n+1} := \left(1 - \frac{\gamma_{n+1}}{\lambda_{n+1} - \lambda_0}\right) \prod_{1 \le p \ne n+1} \frac{\left(1 - \frac{\gamma_p}{\lambda_p - \lambda_{n+1}}\right)}{\left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n - 1}\right)}.$$

We finish this section with discussing two symmetry properties of the map Φ . For any $u \in L_r^2$, denote by u_* the element in L_r^2 , given by $u_*(x) := u(-x)$. Note that the fixed points of the involution $u \mapsto u_*$ are the even functions in L_r^2 .

Proposition 4.3. For any $u \in L_r^2$,

$$\lambda_n(u_*) = \lambda_n(u), \quad f_n(x, u_*) = \overline{f_n(-x, u)}, \quad \forall n \ge 0.$$

As a consequence, for any $u \in L^2_{r,0}$,

$$\zeta_n(u_*) = \overline{\zeta_n(u)} \qquad \forall n \ge 1.$$

Hence u is even if and only if for any $n \geq 1$, $\zeta_n(u) \in \mathbb{R}$.

Proof. Given any $u \in L^2_{r,0}$ and $n \geq 0$, let $g_n(x) := \overline{f_n(-x,u)}$ and, to shorten notation, write λ_n for $\lambda_n(u)$ and $f_n(x)$ instead of $f_n(x,u)$. Taking the complex conjugate of both sides of the identity $L_u f_n = \lambda_n f_n$ and then evaluate them at -x one obtains $L_{u_*} g_n = \lambda_n g_n$. Hence $\lambda_n = \lambda_n(u_*)$ and g_n is an eigenfunction of L_{u_*} , corresponding to the eigenvalue $\lambda_n(u_*)$. One verifies inductively that the eigenfunctions g_n , $n \geq 0$, satisfy the normalisation conditions of Definition 2.1, implying that $g_n(x,u) = f_n(x,u_*)$.

Since $\kappa_n(u)$, $n \geq 0$, are spectral invariants of L_u one has $\kappa_n(u_*) = \kappa_n(u)$ and in turn for any $n \geq 1$,

$$\zeta_n(u_*) = \frac{1}{\sqrt{\kappa_n(u)}} \langle 1|f_n(\cdot, u_*)\rangle = \frac{1}{\sqrt{\kappa_n(u)}} \langle 1|g_n\rangle = \overline{\zeta_n(u)}.$$

It then follows that for $u \in L_{r,0}$ even, $\zeta_n(u) \in \mathbb{R}$ for any $n \geq 1$. Conversely, if for a given $u \in L_{r,0}$, $\zeta_n(u) \in \mathbb{R}$ for any $n \geq 1$, then $\Phi(u) = \Phi(u_*)$. Since Φ is one-to-one we conclude that $u = u_*$.

The second result concerns potentials in $L_{r,0}^2$ which are $2\pi/K$ periodic for some integer $K \geq 2$. Let $L_{r,0}^{2,K}$ denote the subspace of $L_{r,0}^2$, consisting of such elements and by $L^{2,K}$, $L_{+}^{2,K}$ the corresponding subspaces of L^2 , L_{+}^2 . For any $u \in L_{r,0}^{2,K}$, let $L_u^{(K)}$ be the Lax operator $-i\partial_x - T_u^{(K)}$, acting on $L_{+}^{2,K}$ where $T_u^{(K)}$ denotes the Toeplitz operator given by $T_u^{(K)}(f) = \Pi^{(K)}(uf)$ and $\Pi^{(K)}$ is the Szegő projector $L^{2,K} \to L_{+}^{2,K}$. The spectrum of $L_u^{(K)}$ is given by a sequence of eigenvalues which we list in increasing order, $(\lambda_n^{(K)}(u))_{n>0}$. Following the

arguments of the proof of Proposition 2.1 with the operator S replaced by

$$S^{(K)}: L^{2,K}_+ \to L^{2,K}_+, f \mapsto e^{iKx}f$$

one verifies that $\lambda_n^{(K)}(u) \geq \lambda_{n-1}^{(K)}(u) + K$ for any $n \geq 1$. Since all the eigenvalues of L_u are simple and any eigenvalue $\lambda_n^{(K)}$ of $L_u^{(K)}$ is also an eigenvalue of L_u it follows by a simple homotopy argument applied to the path $t \mapsto tu$, $0 \leq t \leq 1$, that $\lambda_n^{(K)}(u) = \lambda_{nK}(u)$ for any $n \geq 0$.

Proposition 4.4. Assume that $u \in L^2_{r,0}$ is $2\pi/K$ -periodic for some integer $K \geq 2$. Then for any (n,k) with $n \geq 0$ and $1 \leq k \leq K-1$,

$$\lambda_{nK+k}(u) = \lambda_{nK}(u) + k, \qquad f_{nK+k}(x,u) = e^{ikx} f_{nK}(x,u).$$

Hence for such pairs (n, k), $\gamma_{nK+k}(u) = 0$ and thus $\zeta_{nK+k}(u) = 0$.

Proof. Let $u \in L^2_{r,0}$ be $2\pi/K$ – periodic for some integer $K \geq 2$. Since for any $n \geq 0$, $\lambda_n^{(K)}(u) = \lambda_{nK}(u)$, and in view of Lemma 2.4 it suffices to prove that for any (n,k) with $n \geq 0$, $1 \leq k \leq K-1$ one has

(4.10)
$$\lambda_{nK+k}(u) = \lambda_{nK}(u) + k .$$

To verify the latter identity, denote by $f_n^{(K)}$ the eigenfunction of $L_u^{(K)}$ corresponding to the eigenvalue $\lambda_n^{(K)}$.

Since u and $f_n^{(K)}$ are both $2\pi/K$ periodic one has $\Pi(uf_n^{(K)}) = \Pi^{(K)}(uf_n^{(K)})$ and hence for any $1 \le k \le K - 1$, $e^{ikx}\Pi(uf_n^{(K)}) = \Pi(e^{ikx}uf_n^{(K)})$ implying that

$$-i\partial_x(e^{ikx}f_n^{(K)}) - \Pi^{(K)}(e^{ikx}uf_n^{(K)}) = (\lambda_n^{(K)} + k)e^{ikx}f_n^{(K)}$$

By the homotopy argument mentioned above, (4.10) then follows. \Box

Remark 4.2. Conversely, any element $u \in L^2_{r,0}$ with $\zeta_{nK+k}(u) = 0$ for any (n,k) with $n \geq 0$ and $1 \leq k \leq K-1$, is $2\pi/K$ -periodic. Indeed, denote by $\Phi^{(K)}: L^{2,K}_{r,0} \to h^{1/2}_+, u \mapsto (\zeta^{(K)}_n(u))_{n\geq 1}$ the Birkhoff map on $L^{2,K}_{r,0}$, where $(\zeta^{(K)}_n(u))_{n\geq 1}$ are the Birkhoff coordinates of u, viewed as an element in $L^{2,K}_{r,0}$. Arguing as in the proof of Theorem 1 concerning the case K=1 (cf Section 8) one infers that $\Phi^{(K)}$ is bijective. Hence there exists $v \in L^{2,K}_{r,0}$ so that $\Phi^{(K)}(v) = (\zeta_{nK}(u))_{n\geq 1}$. By Proposition 4.4 it then follows that $\Phi(v) = (\zeta_n(u))_{n\geq 1}$ and hence by the uniqueness of Φ one concludes that v=u.

5. Analyticity and Gradients

In this section, we establish that the Birkhoff coordinates ζ_n are real analytic functions, and that their gradients $\nabla \zeta_n$ are real analytic maps with values in H^1 . Hence the Gardner brackets $\{\zeta_n, F\}$ are well defined real analytic functions for every real analytic functional F on $L_{r_0}^2$.

As a first result we establish that the eigenvalues λ_n are real analytic. Note that for any $v \in L^2$ ($\equiv L^2(\mathbb{T}, \mathbb{C})$), the Lax operator $L_v = -i\partial_x - T_v$ on L^2_+ with domain H^1_+ is no longer selfadjoint, but it is still a closed unbounded operator with compact resolvent. Hence its spectrum consists of a sequence of complex eigenvalues, each with finite multiplicity. For any r > 0, $\lambda \in \mathbb{C}$ and for any $\varepsilon > 0$, $u \in L^2_r$, let

$$D_r(\lambda) = \{ z \in \mathbb{C} : |z - \lambda| < r \}, \quad B_{\varepsilon}(u) = \{ v \in L^2 : ||v - u|| < \varepsilon \}$$

and for $N \geq 1$ and $n \geq 0$, denote by $\operatorname{Box}_{N,n}(u)$ the closed rectangle in \mathbb{C} given by the set of complex numbers λ satisfying

$$-N + \lambda_0(u) \le \operatorname{Re}(\lambda) \le \lambda_n(u) + 1/2$$
, $|\operatorname{Im}(\lambda)| \le N$.

For $u \in L_r^2$, we define $r_0(u) := 1/4$, $\tau_0(u) := \lambda_0(u)$, and, for $p \ge 1$,

$$r_p(u) := \frac{1}{4} + \frac{\gamma_p(u)}{2} , \ \tau_p(u) := \frac{\lambda_p(u) + \lambda_{p-1}(u) + 1}{2} .$$

Lemma 5.1. For any $u \in L_r^2$, $N \ge 1$, and $n \ge 0$, there exists $\varepsilon_n > 0$ so that for any $v \in B_{\varepsilon_n}(u)$ and $0 \le k \le n$, $\#(\operatorname{spec}(L_v) \cap D_{r_k}(\tau_k(u))) = 1$ and

$$spec(L_v) \cap Box_{N,n}(u) \subset \bigcup_{0 \le p \le n} D_{r_p(u)}(\tau_p(u))$$
.

The unique eigenvalue of L_v in $D_{r_k(u)}(\tau_k(u))$, denoted by $\lambda_k(v)$, is real analytic on $B_{\varepsilon_n}(u)$. It then follows that for any $1 \leq k \leq n$, γ_k is an analytic functional on $B_{\varepsilon_n}(u)$ as well.

Proof. For simplicity, we do not indicate the dependence of r_k and τ_k on u. In a first step we prove that there exists $\delta_n > 0$ so that for any $v \in B_{\delta_n}(u)$

$$spec(L_v) \cap Box_{N,n}(u) \subset \bigcup_{0 \le k \le n} D_{r_k}(\tau_k)$$
.

Assume to the contrary that such a $\delta_n > 0$ does not exist. Then there exists a sequence $(v_\ell)_{\ell \geq 1} \subset L^2$ with $\|v_\ell - u\| < 1/\ell$ and an eigenvalue $\mu_\ell \in spec(L_{v_\ell})$ in $\operatorname{Box}_{N,n}(u)$ so that $|\mu_\ell - \tau_k| \geq r_k$ for any $0 \leq k \leq n$. Let g_ℓ be an eigenfunction of L_{v_ℓ} in H^1_+ corresponding to μ_ℓ , $L_{v_\ell}g_\ell = \mu_\ell g_\ell$, with $\|g_\ell\| = 1$. Since $(\mu_\ell)_{\ell \geq 1}$ is bounded, $(g_\ell)_{\ell \geq 1}$ is a bounded sequence in H^1_+ . Hence by choosing subsequences if needed, we can assume without loss of generality that $(\mu_\ell)_{\ell \geq 1}$ converges to a complex number μ and $(g_\ell)_{\ell \geq 1}$ converges weakly in H^1_+ to an element $g \in H^1_+$. Then $\mu \in \operatorname{Box}_{N,n}(u) \setminus \bigcup_{0 \leq k \leq n} D_{r_k}(\tau_k)$, and $\lim_{\ell \to \infty} \mu_\ell g_\ell = \mu g$ as well as $\lim_{\ell \to \infty} L_{v_\ell} g_\ell = L_u g$ weakly in H^1_+ . It then follows that $L_u g = \mu g$ and hence that μ is an eigenvalue of L_u , contradicting the fact that $specL_u \cap \operatorname{Box}_{N,n}(u) \subset \bigcup_{0 \leq k \leq n} D_{r_k}(\tau_k)$. Hence there exists $\varepsilon_n > 0$ so that for any $v \in B_{\varepsilon_n}(u)$

$$specL_v \cap Box_{N,n}(u) \subset \bigcup_{0 \le k \le n} D_{r_k}(\tau_k)$$
.

For any $0 \le k \le n$, denote by C_k the circle of radius $1/3 + \gamma_k(u)/2$ with counterclockwise orientation, centered at τ_k . Then for any $\lambda \in C_k$ and $v \in B_{\varepsilon_n}(u)$, $L_v - \lambda$ is invertible and

$$C_k \times B_{\varepsilon_n}(u) \to \mathcal{L}(L_+^2, H_+^1), (\lambda, v) \mapsto (\lambda - L_v)^{-1}$$

is analytic and so is the Riesz projector

$$P_k: B_{\varepsilon_n}(u) \to \mathcal{L}(L_+^2, H_+^1), \ v \mapsto \frac{1}{2\pi i} \int_{C_k} (\lambda - L_v)^{-1} d\lambda$$
.

Since for any given $v \in B_{\varepsilon_n}(u)$, L_v has a compact resolvent, $P_k(v)$ is an operator of finite rank. Hence its trace, $TrP_k(v)$, is finite. Actually, $TrP_k(v) = 1$ since $TrP_k(v)$ is constant and $TrP_k(u) = 1$. Hence for any $v \in B_{\varepsilon_n}(u)$ and $0 \le k \le n$, L_v has precisely one eigenvalue in $D_{r_k}(\tau_k)$ and this eigenvalue is simple. We denote it by $\lambda_k(v)$. By functional calculus one has $\lambda_k(v) = Tr\frac{1}{2\pi i} \int_{C_k} \lambda(\lambda - L_v)^{-1} d\lambda$ and hence it follows that $\lambda_k : B_{\varepsilon_n}(u) \to \mathbb{C}, v \mapsto \lambda_k(v)$ is analytic.

As a second step, we prove that the eigenfunctions f_n defined by Definition 2.1 are real analytic maps with values in H^1_+ . To state this result in more detail, we first need to make some preliminary considerations. Denote by $L^2_- \equiv L^2_-(\mathbb{T}, \mathbb{C})$ the Hardy space

$$L_{-}^{2} := \{ h \in L^{2} : h(x) = \sum_{k=-\infty}^{0} \widehat{h}(k)e^{ikx} \}$$

and the corresponding Szegő projector $\Pi^-:L^2\to L^2_-$. For any $u\in L^2_r$, denote by L^-_u the operator

$$L_u^- = i\partial_x - T_u^- : H_-^1 \to L_-^2 , \qquad H_-^1 := H^1 \cap L_-^2$$

where T_u^- denotes the Toeplitz operator

$$T_u^-: H_-^1 \to L_-^2 \ , \ h \mapsto \Pi^-(uh) \ .$$

Using that u is real valued, one verifies that the spectrum of L_u^- coincides with the one of L_u and that for any $n \geq 0$, $L_u^- f_n^- = \lambda_n f_n^-$ where $f_n^- = \overline{f_n}$. Hence $(f_n^-)_{n\geq 0}$ is an orthonormal basis of L_n^2 , normalized in such a way that $\langle 1|f_0^-\rangle > 0$ and $\langle f_{n+1}^-|S^-f_n^-\rangle > 0$ for any $n\geq 0$. Here $S^-:L_n^2\to L_n^2$, $h\mapsto e^{-ix}h$ denotes the shift operator to the left. Furthermore, the Riesz projector $P_n^-(u)$ onto $span(f_n^-)$ is given by

$$P_n^-(u) = \frac{1}{2\pi i} \int_{C_n} (\lambda - L_u^-)^{-1} d\lambda$$

where C_n is the same circle appearing in the definition of the Riesz projector $P_n(u)$, defined in the proof of Lemma 5.1. Finally we introduce for any $n \geq 0$ the functions

(5.1)
$$f_{n,1}(\cdot, u) := (f_n(\cdot, u) + f_n^-(\cdot, u))/2$$

and

(5.2)
$$f_{n,2}(\cdot, u) := (f_n(\cdot, u) - f_n^-(\cdot, u))/2i$$

implying that $f_n(\cdot, u) = f_{n,1}(\cdot, u) + i f_{n,2}(\cdot, u)$. Since for u real valued, $f_{n,1}(x, u)$ is the real part of $f_n(x, u)$ and $f_{n,2}(x, u)$ its imaginary part one has

$$\langle 1|f_n(\cdot,u)\rangle = \langle 1, f_{n,1}(\cdot,u) - if_{n,2}(\cdot,u)\rangle = \langle 1, f_n^-(\cdot,u)\rangle$$

and in turn

$$P_n(u)1 = \langle 1, f_n^-(\cdot, u) \rangle f_n(\cdot, u)$$
.

Lemma 5.2. For any $u \in L_r^2$ and $n \geq 0$ there exists $0 < \delta_n \leq \varepsilon_n$, with ε_n as in Lemma 5.1, so that for any $0 \leq k \leq n$, f_k and f_k^- admit analytic extensions

$$f_k: B_{\delta_n}(u) \to H^1_+ , \qquad f_k^-: B_{\delta_n}(u) \to H^1_- .$$

As a consequence, $f_{k,1}$ and $f_{k,2}$, defined by (5.1) - (5.2), admit analytic extensions, $f_{k,j}: B_{\delta_n}(u) \to H^1$ and so does $\langle 1, f_k^- \rangle : B_{\delta_n}(u) \to \mathbb{C}$ and $P_k 1$, meaning that

$$P_k(\cdot)1: B_{\delta_n}(u) \to H^1_+, v \mapsto P_k(v)1 = \langle 1, f_k^-(\cdot, v) \rangle f_k(\cdot, v)$$

is analytic. Furthermore, f_k satisfies the normalisation condition

$$\langle f_k, f_k^- \rangle = \langle f_{k,1}, f_{k,1} \rangle^2 + \langle f_{k,2}, f_{k,2} \rangle^2 = 1$$
.

Proof. We argue inductively and begin with f_0 . By the normalisation of f_0 , for any $v \in B_{\varepsilon_n}(u) \cap L^2_r$, $\langle 1|f_0(\cdot,v)\rangle > 0$. Since for such a v the operator L_v is selfadjoint, $P_0(v) = \frac{1}{2\pi i} \int_{C_0} (\lambda - L_v)^{-1} d\lambda$ is the L^2 -orthogonal projector onto $span(f_0(\cdot,v))$ and hence $P_0(v)1 = \langle 1|f_0(\cdot,v)\rangle f_0(\cdot,v)$. It implies that $\langle P_0(v)1|1\rangle = \langle 1|f_0(\cdot,v)\rangle^2 > 0$ and thus

$$\langle 1|f_0(\cdot,v)\rangle = \sqrt{\langle P_0(v)1|1\rangle} = \sqrt{\langle P_0(v)1,1\rangle}$$
.

Since $P_0: B_{\varepsilon_n}(u) \to \mathcal{L}(L_+^2, H_+^1)$ is analytic we can shrink ε_n if needed so that $Re\langle P_0(v)1, 1 \rangle > 0$ is uniformly bounded away from 0 on $B_{\varepsilon_n}(u)$. As a consequence, the principal branch of the root $\sqrt{\langle P_0(v)1, 1 \rangle}$ is analytic on $B_{\varepsilon_n}(u)$. The analytic extension of f_0 is then defined by

$$f_0(\cdot, v) := \frac{1}{\sqrt{\langle P_0(v)1, 1 \rangle}} P_0(v) 1 \in H^1_+.$$

By functional calculus,

$$L_{v}P_{0}(v)1 = \frac{1}{2\pi i} \int_{C_{0}} \lambda(\lambda - L_{v})^{-1}1d\lambda = \lambda_{0}(v)P_{0}(v)1$$

and hence $L_v f_0(\cdot, v) = \lambda_0(v) f_0(\cdot, v)$. Using the Riesz projector $P_0^-(v)$ instead of $P_0(v)$ one sees by the same arguments that the analytic extension of f_0^- is defined for $v \in B_{\varepsilon_n}(u)$ by

$$f_0^-(\cdot, v) := \frac{1}{\sqrt{\langle P_0^-(v)1, 1 \rangle}} P_0^-(v) 1 \in H_-^1,$$

and that $L_v^-(v)f_0^-(\cdot, v) = \lambda_0(v)f_0^-(\cdot, v)$.

Now assume that after shrinking ε_n if necessary, f_ℓ and f_ℓ^- , $0 \le \ell \le k$, have been extended analytically to $B_{\varepsilon_n}(u)$ for a given $k \le n-1$. More precisely, for any $0 \le \ell \le k$

$$f_{\ell}: B_{\varepsilon_n}(u) \to H^1_+, \qquad f_{\ell}^-: B_{\varepsilon_n}(u) \to H^1_-$$

are analytic maps and for any $v \in B_{\varepsilon_n}(u)$,

$$L_v f_{\ell}(\cdot, v) = \lambda_{\ell}(v) f_{\ell}(\cdot, v)$$
, $L_v^- f_{\ell}^-(\cdot, v) = \lambda_{\ell}(v) f_{\ell}^-(\cdot, v)$.

In case $v \in B_{\varepsilon_n} \cap L_r^2$, one has $\langle f_{k+1}(\cdot,v)|Sf_k(\cdot,v)\rangle > 0$. Since in such a case L_v is selfadjoint, the Riesz projector $P_{k+1}(v) = \frac{1}{2\pi i} \int_{C_{k+1}} (\lambda - L_v)^{-1} d\lambda$ is the L^2 -orthogonal projector onto $span(f_{k+1}(\cdot,v))$ and hence

$$P_{k+1}(v)Sf_k(\cdot, v) = \langle Sf_k(\cdot, v) \mid f_{k+1}(\cdot, v) \rangle f_{k+1}(\cdot, v).$$

It implies that

$$\langle Sf_k(\cdot, v) | f_{k+1}(\cdot, v) \rangle = \sqrt{\langle P_{k+1}(v) Sf_k(\cdot, v) | Sf_k(\cdot, v) \rangle}$$
.

Since $P_{k+1}: B_{\varepsilon_n}(u) \to \mathcal{L}(L_+^2, H_+^1)$ is analytic we can once more shrink ε_n if needed so that Re $\alpha_k > 0$ is uniformly bounded away from 0 on $B_{\varepsilon_n}(u)$ where $\alpha_k \equiv \alpha_k(v)$ is defined by

$$\alpha_k(v) := \langle P_{k+1}(v) S f_k(\cdot, v), S^- f_k^-(\cdot, v) \rangle$$
.

As a consequence, the principal branch of the root $\sqrt{\alpha_k}$ is analytic on $B_{\varepsilon_n}(u)$. The analytic extension of f_{k+1} to $B_{\varepsilon_n}(u)$ is then given by

$$f_{k+1}(\cdot, v) = \frac{1}{\sqrt{\alpha_k(v)}} P_{k+1}(v) S f_k(\cdot, v) \in H^1_+$$

and by functional calculus one has $L_v f_{k+1}(\cdot, v) = \lambda_{k+1}(v) f_{k+1}(\cdot, v)$. Using the Riesz projector $P_{k+1}^-(v)$ instead of $P_{k+1}(v)$ one sees by the arguments used above that the analytic extension of f_{k+1}^- is given for $v \in B_{\varepsilon_n}(u)$ by

$$f_{k+1}^-(\cdot,v) = \frac{1}{\sqrt{\alpha_k^-(v)}} P_{k+1}^-(v) S^- f_k^-(\cdot,v) \in H^1_-$$

and that $L_v^-(v)f_{k+1}^-(\cdot,v)=\lambda_{k+1}(v)f_{k+1}^-(\cdot,v)$. Here $\alpha_k^-(v)$ is defined by

$$\alpha_k^-(v) := \langle P_{k+1}^-(v) S^- f_k^-(\cdot\ ,v)\ ,\ Sf_k(\cdot\ ,v)\rangle\ .$$

We then denote by $0 < \delta_n \le \varepsilon_n$ the number obtained after shrinking ε_n successively so that for any $0 \le k \le n$, f_k extends to an analytic map, $f_k : B_{\delta_n}(u) \to H^1_+$ and similarly, f_k^- extends to one on $B_{\delta_n}(u)$ with values in H^1_- .

We now present a formula for the L^2 -gradient of the eigenvalues λ_n , $n \geq 0$. By Lemma 5.1, for any $n \geq 0$, $\lambda_n : u \in L^2_r \mapsto \lambda_n(u) \in \mathbb{R}$ is real analytic and hence its L^2 - gradient $\nabla \lambda_n$ is well defined. More precisely, we have the following

Corollary 5.1. For any $u \in L_r^2$, $n \ge 0$, and $0 \le k \le n$, the L^2 -gradient $\nabla \lambda_k(u)$ of λ_k at u is given by $\nabla \lambda_k(u) = -|f_k|^2(\cdot, u)$ and extends to an analytic function on $B_{\delta_n}(u)$ with $\delta_n > 0$ given as in Lemma 5.2. More precisely,

$$\nabla \lambda_k : B_{\delta_n}(u) \to H^1, \ v \mapsto -f_{k,1}(\cdot, v)^2 - f_{k,2}(\cdot, v)^2$$

is analytic where $f_{k,1}$ and $f_{k,2}$ are given as in Lemma 5.2.

Proof. Let $u \in L_r^2$ and $0 \le k \le n$ be given. To compute the gradient $\nabla \lambda_k(u)$, consider for any $v \in L_r^2$ the one parameter family $u_{\varepsilon} := u + \varepsilon v$. It is convenient to introduce the notation $f_k := f_k(\cdot, u)$, $\lambda_k := \lambda_k(u)$, and

$$\delta f_k := \frac{d}{d\varepsilon}|_{\varepsilon=0} f_k(\cdot, u_{\varepsilon}), \qquad \delta \lambda_k := \frac{d}{d\varepsilon}|_{\varepsilon=0} \lambda_k(u_{\varepsilon}).$$

Taking the derivative with respect to ε of both sides of the identity $(L_{u_{\varepsilon}} - \lambda_k(u_{\varepsilon})) f_k(\cdot, u_{\varepsilon}) = 0$ at $\varepsilon = 0$ one obtains

$$(L_u - \lambda_k)\delta f_k - T_v f_k - \delta \lambda_k f_k = 0.$$

Taking the inner product with f_k of both sides of the latter identity and using that $L_v - \lambda_k$ is selfadjoint one then concludes

$$-\langle T_v f_k | f_k \rangle - \delta \lambda_k \langle f_k | f_k \rangle = 0.$$

Since $||f_k|| = 1$ and $\Pi f_k = f_k$, the definition of the L^2 -gradient then implies that

$$\langle \nabla \lambda_k, v \rangle = \delta \lambda_k = -\langle v f_k | f_k \rangle = \langle -|f_k|^2 | v \rangle,$$

yielding the claimed formula for $\nabla \lambda_k(u)$. Since by Lemma 5.2, the real and imaginary parts $f_{k,1}$, $f_{k,2}$ of f_k , extend to analytic functions on $B_{\delta_n}(u)$ with values in H^1 and by Lemma 5.1, and λ_k also extends to an analytic functional on $B_{\delta_n}(u)$, it follows that the gradient $\nabla \lambda_k$ and its formula extend analytically,

$$\nabla \lambda_k : B_{\delta_n}(u) \to H^1, \ v \mapsto -f_{k,1}(\cdot, v)^2 - f_{k,2}(\cdot, v)^2$$

proving the lemma.

As a consequence of Corollary 5.1, we obtain a formula for the gradient of $\gamma_n = \lambda_n - \lambda_{n-1} - 1$,

(5.3)
$$\nabla \gamma_n = -|f_n|^2 + |f_{n-1}|^2.$$

As a third step, we show that for any $n \geq 0$, the L^2 -gradient $\nabla \langle 1|f_n\rangle$ is a real analytic map $L_r^2 \to H^1$, $u \mapsto \nabla \langle 1, f_n(\cdot, u)\rangle$. Recall that for any $u \in L_r^2$ and $n \geq 1$, the eigenfunctions f_k , $0 \leq k \leq n$, admit an analytic extension $f_k = f_{k,1} + i f_{k,2}$ to $B_{\delta_n}(u)$, i.e.

$$f_{k,j}: B_{\delta_n}(u) \to H^1, \ v \mapsto f_{k,j}(\cdot, v), \quad j = 1, 2,$$

are analytic maps (cf. Lemma 5.2). For any $v \in B_{\delta_n}(u)$, we denote by $df_{k,j}(\cdot,v)[w]$ the derivative of $f_{k,j}$ at v in direction $w \in L^2$.

Lemma 5.3. For any $n \geq 0$ and $v \in B_{\delta_n}(u)$ there exists a constant $c_n > 0$ so that for any $0 \leq k \leq n$, j = 1, 2, and $w \in L^2$,

$$||df_{k,j}(\cdot,v)[w]|| \le c_n ||w||_{H^{-1}}$$
.

The constant c_n can be chosen locally uniformly with respect to v. As a consequence, $\nabla \langle 1, f_{k,j} \rangle : B_{\delta_n} \to H^1$, j = 1, 2, are analytic.

Proof. The latter assertion easily follows from the first one, since for any $0 \le k \le n$, $j = 1, 2, v \in B_{\delta_n}(u)$, and $w \in L^2$,

$$|\langle \nabla \langle 1, f_{k,j}(\cdot, v) \rangle, w \rangle| = |\langle 1, df_{k,j}(\cdot, v)[w] \rangle|$$

$$\leq ||df_{k,j}(\cdot, v)[w]|| \leq c_n ||w||_{H^{-1}},$$

implying that $g_{k,j}(v) := \nabla \langle 1, f_{k,j}(\cdot, v) \rangle$ is bounded in H^1 uniformly with respect to $v \in B_{\delta_n}(u)$. Since $g_{k,j} : B_{\delta_n}(u) \to L^2$ is an analytic map (cf. Lemma 5.2), so is $\langle g_{k,j} | e^{inx} \rangle : B_{\delta_n}(u) \to \mathbb{C}$ for any $n \in \mathbb{Z}$. Together with the boundedness of $g_{k,j}$ in H^1 it then follows that $g_{k,j} : B_{\delta_n}(u) \to H^1$ is analytic (cf. e.g. [12, Appendix A]).

To prove the first assertion of the lemma, we proceed by induction. First we need to make some preliminary considerations. Without further reference, we will use the notation introduced in the derivation of Lemma 5.1 and Lemma 5.2. According to Lemma 5.2, $f_{k,1} = (f_k + f_k^-)/2$ and $f_{k,2} = (f_k - f_k^-)/2i$ where f_k^- , $k \ge 0$, are the eigenfunctions of $L_v^-: H_v^- \to L_v^-$, $h \mapsto i\partial_x h - \Pi^-(vh)$ and Π^- is the Szegő projector $L^2 \to L_v^2$. The first assertion of the lemma then follows by proving corresponding estimates for $df_k(\cdot, v)[w]$ and $df_k^-(\cdot, v)[w]$, $0 \le k \le n$. By the definition of r_n ,

$$(D_{r_n+\frac{1}{2}} \setminus D_{r_n+\frac{1}{4}}) \times B_{\delta_n} \to \mathcal{L}(L_+^2, H_+^1), \ (\lambda, v) \mapsto (\lambda - L_v)^{-1}$$

is analytic. Hence the adjoint of $(\lambda - L_v)^{-1}$ with respect to the dual pairing of H_+^{-1} and H_+^1 , also gives rise to an analytic map,

$$(D_{r_n+\frac{1}{2}} \setminus D_{r_n+\frac{1}{2}}) \times B_{\delta_n} \to \mathcal{L}(H_+^{-1}, L_+^2), \ (\lambda, v) \mapsto (\lambda - L_v)^{-1}.$$

Since the circles C_k , $0 \le k \le n$, are compact, it implies that for any $v \in B_{\delta_n}$ there exists $M_n > 0$ so that for any $\lambda \in C_k$, $0 \le k \le n$,

$$(5.4) \quad \|(\lambda - L_v)^{-1}\|_{L^2_+ \to H^1_+} \le M_n , \qquad \|(\lambda - L_v)^{-1}\|_{H^{-1}_+ \to L^2_+} \le M_n ,$$

where M_n can be chosen locally uniformly with respect to v. Furthermore, for any $0 \le k \le n$, the Riesz projector

$$P_k(v) = \frac{1}{2\pi i} \int_{C_k} (\lambda - L_v)^{-1} d\lambda$$

gives rise to an analytic map $P_k: B_{\delta_n} \to \mathcal{L}(L_+^2, H_+^1)$. For any $h \in L_+^2$, the derivative of $P_k(v)h$ with respect to v in direction $w \in L^2$ is given

by

$$(dP_k(v)[w])h = -\frac{1}{2\pi i} \int_{C_k} (\lambda - L_v)^{-1} \Pi(w(\lambda - L_v)^{-1}h) d\lambda .$$

We want to estimate the operator norm of $dP_k(v)[w]$, viewed as an operator on L^2_+ . Note that for any $g, h \in L^2_+$,

$$\langle (dP_k(v)[w])h, g \rangle = -\frac{1}{2\pi i} \int_{C_k} \langle w(\lambda - L_v)^{-1}h, (\lambda - L_v)^{-1}g \rangle d\lambda.$$

Taking into account (5.4) and that multiplication by $f \in H^1$ yields a bounded operator on H^{-1} , we infer that for any $v \in B_{\delta_n}$, there exists a constant $d_n > 0$ so that for any $0 \le k \le n$

where d_n can be chosen locally uniformly with respect to v.

We are now ready to present the induction argument with respect to $0 \le k \le n$ for proving the claimed estimates for $df_k(\cdot, v)[w]$ and $df_k^-(\cdot, v)[w]$. We start with k = 0. Recall that the formula for the analytic extension of f_0 to B_{δ_n} of Lemma (5.2) reads

$$f_0(\cdot, v) = \frac{1}{\sqrt{\langle P_0(v)1, 1 \rangle}} P_0(v)1,$$

implying that for any $w \in L^2$

$$df_0(\cdot, v)[w] = \frac{1}{\sqrt{\langle P_0(v)1, 1 \rangle}} (dP_0(v)[w]) 1 - \frac{1}{2} \frac{\langle (dP_0(v)[w])1, 1 \rangle}{\langle P_0(v)1, 1 \rangle^{\frac{3}{2}}} P_0(v) 1.$$

Applying (5.5) for k = 0, we immediately conclude that there exists $A_n > 0$ so that

$$||df_0(v)w||_{L^2} \le A_n ||w||_{H^{-1}}$$
.

The corresponding estimate for $df_0^-(v)[w]$ is proved in an analogous way by using the estimate for $dP_0^-(v)[w]$ corresponding to the one of $dP_0(v)[w]$.

Now assume that the claimed estimates for $df_{\ell}(\cdot, v)[w]$ and $df_{\ell}^{-}(\cdot, v)[w]$ are valid for any $0 \le \ell \le k$ for a given $0 \le k \le n-1$. We claim that they also hold for k+1. Let us first consider the one for $df_{k+1}(\cdot, v)[w]$ Again we use the formula for the analytic extension of f_{k+1} to B_{δ_n} derived in Lemma 5.2,

$$f_{k+1}(\cdot, v) = \frac{1}{\sqrt{\alpha_{k+1}(v)}} P_{k+1}(v) (Sf_k(\cdot, v))$$

where

$$\alpha_{k+1}(v) = \left\langle P_{k+1}(v)(Sf_k(\cdot, v)) | e^{-ix} f_k^-(\cdot, v) \right\rangle$$

The directional derivative $f_{k+1}(\cdot, v)[w]$ can then be computed as

$$df_{k+1}(\cdot, v)[w] = \frac{1}{\sqrt{\alpha_{k+1}(v)}} I - \frac{1}{2} \frac{d\alpha_{k+1}(v)[w]}{\alpha_{k+1}(v)^{3/2}} P_{k+1}(v) \left(Sf_k(\cdot, v) \right)$$

where

$$I := dP_{k+1}(v)[w](Sf_k(\cdot, v)) + P_{k+1}(v)(Sdf_k(\cdot, v)[w])$$
and $d\alpha_{k+1}(v)[w] = II_1 + II_2 + II_3$ with
$$II_1 := \langle dP_{k+1}(v)[w](Sf_k(\cdot, v)), \ e^{-ix}f_k^-(\cdot, v) \rangle$$

$$II_2 := \langle P_{k+1}(v)(Sdf_k(\cdot, v)[w]), \ e^{-ix}f_k^-(\cdot, v) \rangle$$

$$II_3 := \langle P_{k+1}(v)[Sf_k(\cdot, v)], \ e^{-ix}df_k^-(\cdot, v)[w] \rangle.$$

By increasing A_n if needed, the estimate (5.5) for k+1 together with the induction hypothesis then yield

$$||df_{k+1}(v)[w]||_{L^2} \le A_n ||w||_{H^{-1}},$$

The corresponding estimate for $df_{k+1}^-(v)[w]$, is proved in an analogous way by using the estimate for $dP_k^-(v)[w]$ corresponding to the one of $dP_k(v)[w]$. Altogether we have proved the induction step. Going through the arguments of the proof one verifies that the constant A_n can be chosen locally uniformly with respect to v.

Remark 5.1. At u = 0, one has $f_k(x) = e^{ikx}$ for any $k \ge 0$ and thus $\langle 1|f_n \rangle = 0$ for any $n \ge 1$. Let us compute the gradient of $\langle 1|f_n \rangle$ at u = 0 for $n \ge 1$. For any $v \in L^2_r$ and $\varepsilon \in \mathbb{R}$, define $\delta f_n := \frac{d}{d\varepsilon}|_{\varepsilon=0} f_n(\cdot, \varepsilon v)$ and $\delta \lambda_n := \frac{d}{d\varepsilon}|_{\varepsilon=0} \lambda_n(\cdot, \varepsilon v)$. Arguing as in the proof of Corollary 5.1 one has for any $n \ge 0$

$$(D-n)\delta f_n = T_v(e^{inx}) + \delta \lambda_n e^{inx} .$$

If $n \geq 1$, taking the inner product of both sides with 1, we obtain

$$-n\langle \delta f_n | 1 \rangle = \langle e^{inx}, v \rangle$$

which leads to

$$\nabla \langle 1|f_n \rangle = -\frac{1}{n} e^{-inx}$$
.

As a fourth step, we discuss the analytic extension of κ_n , $n \geq 0$, given by the formulae in Corollary 3.1. A difficulty here is that these formulae involve infinitely many eigenvalues λ_n , while the domain of analyticity of λ_n might shrink as $n \to \infty$. Therefore we need an extra argument. We shall appeal to the generating functional \mathcal{H}_{λ} . First we need to make some preliminary considerations. Let $u \in L_r^2$ be given. If needed, shrink the radius $\delta_n > 0$ of the ball $B_{\delta_n} \equiv B_{\delta_n}(u)$ so that for any $v \in B_{\delta_n}$ and $1 \leq k \leq n$, $\lambda_{k-1}(v) + 1$ is in the disk $D_{r_n}(\tau_n)$. For any $v \in B_{\delta_n}$ and $0 \leq k \leq n$, $\mathcal{H}_{\lambda}(v) = \langle (L_v + \lambda Id)^{-1}1, 1 \rangle$ is holomorphic for $-\lambda$ in $D_{r_k+1/4}(\tau_k) \setminus \{\lambda_k(v)\}$ and possibly has a simple pole at $-\lambda = \lambda_k(v)$. Let $P_k^{\perp}(v) = Id - P_k(v)$ where $P_k(v)$ denotes the Riesz projector, introduced in the proof of Lemma 5.1. Note that

$$\mathcal{H}_{\lambda,k}^{\perp}(v) := \langle (L_v + \lambda Id)^{-1} P_k^{\perp}(v) 1, 1 \rangle$$

is holomorphic on $-D_{r_k+1/4}(\tau_k)$. By Lemma 5.2,

$$P_k(v)1 = \langle 1, f_{k,1}(\cdot, v) - i f_{k,2}(\cdot, v) \rangle f_k(\cdot, v)$$

implying that $\mathcal{H}_{\lambda,k}(v) := \langle (L_v + \lambda Id)^{-1} P_k(v) 1, 1 \rangle$ equals

$$\left(\langle f_{k,1}(\cdot,v),1\rangle^2 + \langle f_{k,2}(\cdot,v),1\rangle^2\right) \frac{1}{\lambda_k(v)+\lambda}.$$

Hence the residue $res_k(v)$ of $\mathcal{H}_{\lambda}(v) = \mathcal{H}_{\lambda,k}(v) + \mathcal{H}_{\lambda,k}^{\perp}(v)$ at $\lambda = -\lambda_k(v)$ is given by

$$(5.6) res_k(v) = \langle f_{k,1}(\cdot, v), 1 \rangle^2 + \langle f_{k,2}(\cdot, v), 1 \rangle^2.$$

Furthermore, for any $v \in B_{\delta_n}$, $(\lambda_k(v) + \lambda)\mathcal{H}_{\lambda}(v)$ is holomorphic on $-D_{r_k+1/4}(\tau_k)$. Actually, the map

$$-D_{r_k+1/4}(\tau_k) \times B_{\delta_n} \to \mathbb{C}, \ (\lambda, v) \mapsto (\lambda_k(v) + \lambda)\mathcal{H}_{\lambda}(v)$$

is analytic. Let us now discuss this map in more detail. We begin with the case k = 0. For any (λ, v) in $-D_{r_0+1/4}(\tau_0) \times B_{\delta_n}$ set

$$\chi_0(\lambda, v) := (\lambda_0(v) + \lambda) \mathcal{H}_{\lambda}(v) , \quad \kappa_0(v) := \chi_0(-\lambda_0(v), v) .$$

Notice that Proposition 3.1 (i) implies that, for $v \in B_{\delta_n} \cap L_r^2$, the above definition of $\kappa_0(v)$ is consistent with the one in Corollary 3.1. With this notation we then get

$$res_0(v) = \langle f_{0,1}(\cdot, v), 1 \rangle^2 + \langle f_{0,2}(\cdot, v), 1 \rangle^2 = \kappa_0(v)$$
.

By Corollary 3.1, for any $v \in B_{\delta_n} \cap L_r^2$, one has $\kappa_0(v) > 0$. Hence by shrinking δ_n if necessary, we conclude that $\operatorname{Re} \kappa_0(v) > 0$ is uniformly bounded away from 0 on B_{δ_n} .

Now let us consider the case $1 \le k \le n$. Recall that our choice of δ_n mentioned above assures that $\lambda_{k-1}(v) + 1 \in D_{r_k}(\tau_k)$ for any $v \in B_{\delta_n}$. Hence it follows from Lemma 5.1 that the map

$$B_{\delta_n} \to \mathbb{C}, v \mapsto (\lambda_k(v) + \lambda)\mathcal{H}_{\lambda}(v)|_{\lambda = -\lambda_{k-1}(v) - 1}$$

is also analytic. Since by Proposition 3.1, for any $v \in B_{\delta_n} \cap L^2_r$,

$$(\lambda_k(v) + \lambda)\mathcal{H}_{\lambda}(v)|_{\lambda = -\lambda_{k-1}(v)-1} = 0$$
,

 $(\lambda_k(v) + \lambda)\mathcal{H}_{\lambda}(v)|_{\lambda = -\lambda_{k-1}(v)-1}$ vanishes identically on B_{δ_n} . Thus the function $(\lambda_k(v) + \lambda)\mathcal{H}_{\lambda}(v)$ is of the form $-(\lambda + \lambda_{k-1}(v) + 1)\chi_k(\lambda, v)$ where

$$\chi_k: -D_{r_k}(\tau_k) \times B_{\delta_n} \to \mathbb{C}$$

is analytic. Hence we have proved that for any $(\lambda, v) \in -D_{r_k}(\tau_k) \times B_{\delta_n}$,

(5.7)
$$\mathcal{H}_{\lambda}(v) = -\frac{\lambda_{k-1}(v) + 1 + \lambda}{\lambda_{k}(v) + \lambda} \chi_{k}(\lambda, v) .$$

Using that $\lambda_{k-1}(v) + 1 = \lambda_k(v) - \gamma_k(v)$ one then concludes from (5.6) that the residue $res_k(v)$ of $\mathcal{H}_{\lambda}(v)$ at $\lambda = -\lambda_k(v)$ is given by

$$\langle f_{k,1}(\cdot,v),1\rangle^2 + \langle f_{k,2}(\cdot,v),1\rangle^2 = \gamma_k(v)\kappa_k(v), \quad \kappa_k(v) := \chi_k(-\lambda_k(v),v).$$

Again Proposition 3.1 (i) implies that, for $v \in B_{\delta_n} \cap L_r^2$, the above definition of $\kappa_k(v)$ is consistent with the one in Corollary 3.1. Moreover, by Corollary 3.1, $\kappa_k(v) > 0$ for any $v \in B_{\delta_n} \cap L_r^2$. Hence by shrinking δ_n if necessary, we conclude that $\text{Re } \kappa_k(v) > 0$ is uniformly bounded away from 0 on B_{δ_n} . For later reference we record our findings as follows:

Lemma 5.4. After shrinking $\delta_n > 0$ if needed, the following holds: for any $u \in L_r^2$, $v \in B_{\delta_n}(u)$,

$$\langle f_{0,1}(\cdot, v), 1 \rangle^2 + \langle f_{0,2}(\cdot, v), 1 \rangle^2 = \kappa_0(v)$$

and for $1 \le k \le n$,

$$\langle f_{k,1}(\cdot, v), 1 \rangle^2 + \langle f_{k,2}(\cdot, v), 1 \rangle^2 = \gamma_k(v) \kappa_k(v)$$
.

For any $0 \le k \le n$, the functional $\kappa_k : B_{\delta_n}(u) \to \mathbb{C}$ is analytic with positive real part and $\text{Re } \kappa_k(v) > 0$ is uniformly bounded away from 0.

Finally we analyze the L^2 -gradient of κ_n .

Lemma 5.5. For any $n \geq 1$, $\nabla \kappa_n$ takes values in H_r^1 and $\nabla \kappa_n : L_r^2 \to H_r^1$ is real analytic.

Remark 5.2. (i) The map $\nabla \kappa_0 : L_r^2 \to H_r^1$ is real analytic. Indeed, by Lemma 5.4, $\kappa_0(u) = \langle 1, f_{0,1}(\cdot, u) \rangle^2 + \langle 1, f_{0,2}(\cdot, u) \rangle^2$. The claim then follows from Lemma 5.3.

(ii) For u=0 one infers from the product representation of κ_n and the facts that $f_n(x,0)=e^{inx},\ n\geq 0$, and both $\gamma_p(0)$ and $\nabla\gamma_p(0),\ p\geq 1$, vanish, that $\nabla\kappa_n(0)=0$ for any $n\geq 0$.

Proof. Let $n \geq 1$ and $u \in L_r^2$. For any $1 \leq k \leq n$, κ_n is an analytic map $B_{\delta_n}(u) \to \mathbb{C}$ (cf. Lemma 5.4), implying that $\nabla \kappa_n : B_{\delta_n}(u) \to L^2$ is such a map. To simplify notation, we write B_{δ_n} for $B_{\delta_n}(u)$. We now prove that $\nabla \kappa_n$ is in fact an analytic map $B_{\delta_n} \to H^1$. Let us start from the identity (5.7),

(5.8)
$$(\lambda_k + \lambda) \mathcal{H}_{\lambda}(v) = -(\lambda + \lambda_{k-1}(v) + 1) \chi_k(\lambda, v) ,$$

which holds for $(\lambda, v) \in -D_{r_k + \frac{1}{4}}(\tau_k) \times B_{\delta_n}$. Here χ_k is a holomorphic function on the latter domain and is related to κ_k by $\kappa_k(v) = \chi_k(-\lambda_k(v), v)$. By the chain rule we then have

$$\nabla \kappa_k(v) = -\frac{\partial \chi_k}{\partial \lambda}(-\lambda_k(v), v) \nabla \lambda_k(v) + \nabla \chi_k(-\lambda_k(v), v) .$$

Since $\nabla \lambda_k : B_{\delta_n} \to H^1$ is an analytic map (cf. Corollary 5.1, Lemma 5.2), it remains to prove that $v \mapsto \nabla \chi_k(-\lambda_k(v), v)$ is an analytic map $B_{\delta_n} \to H^1$. It suffices to prove that

$$\nabla \chi_k : -D_{r_k + \frac{1}{4}}(\tau_k) \times B_{\delta_n} \to H^1$$

is analytic. Indeed, let us take the gradient with respect to v of both sides of the identity (5.8). Writing

$$G_k(\lambda, v) := (\lambda_k(v) + \lambda) \mathcal{H}_{\lambda}(v) ,$$

we obtain the following identity,

$$(5.9) \nabla G_k(\lambda, v) = -(\lambda + \lambda_{k-1}(v) + 1) \nabla \chi_k(\lambda, v) - \chi_k(\lambda, v) \nabla \lambda_{k-1}(v) .$$

We claim that ∇G_k is H^1 valued and $\nabla G_k : -D_{r_k + \frac{1}{4}}(\tau_k) \times B_{\delta_n} \to H^1$ is analytic. Indeed, G_k satisfies the identity (cf. derivation of Lemma 5.4)

$$G_k(\lambda, v) = \langle f_{k,1}(\cdot, v), 1 \rangle^2 + \langle f_{k,2}(\cdot, v), 1 \rangle^2 + (\lambda_k + \lambda) \mathcal{H}_{\lambda,k}^{\perp}(v)$$

where

$$\mathcal{H}_{\lambda k}^{\perp}(v) = \langle (L_v + \lambda Id)^{-1} P_k^{\perp}(v) 1, 1 \rangle$$
, $P_k^{\perp}(v) = Id - P_k(v)$,

and $P_k(v)$ denotes the Riesz projector onto $span(f_k(\cdot, v))$. For any $(\lambda, v) \in -D_{r_k + \frac{1}{2}}(\tau_k) \times B_{\delta_n}$, we rewrite

$$(L_v + \lambda Id)^{-1} P_k^{\perp}(v) = (L_v + \lambda Id)^{-1} - (L_v + \lambda Id)^{-1} P_k(v)$$
,

using

$$(L_v + \lambda Id)^{-1} = \frac{1}{2\pi i} \int_{C_k} (L_v + \lambda Id)^{-1} \frac{1}{\mu + \lambda} d\mu$$
,

together with

$$(L_v + \lambda Id)^{-1} P_k(v) = \frac{1}{2\pi i} \int_{C_k} (L_v + \lambda Id)^{-1} (\mu Id - L_v)^{-1} d\mu ,$$

and the resolvent identity

$$(\mu + \lambda)(L_v + \lambda Id)^{-1}(\mu Id - L_v)^{-1} = (L_v + \lambda Id)^{-1} + (\mu Id - L_v)^{-1}.$$

It yields

$$(L_v + \lambda Id)^{-1} P_k^{\perp}(v) = -\frac{1}{2\pi i} \int_{C_l} \frac{1}{\mu + \lambda} (\mu Id - L_v)^{-1} d\mu$$
.

Consequently,

$$\nabla \mathcal{H}_{\lambda,k}^{\perp}(v) = \frac{1}{2\pi i} \int_{C_{t}} (\mu Id - L_{v})^{-1} [1] \overline{(\overline{\mu} Id - L_{\overline{v}})^{-1}} \overline{[1]} \frac{1}{\lambda + \mu} d\mu$$

implying that $\nabla \mathcal{H}_{\lambda,k}^{\perp}(v) \in H^1$ and $\nabla \mathcal{H}_{\cdot,k}^{\perp} : -D_{r_k+\frac{1}{4}}(\tau_k) \times B_{\delta_n} \to H^1$ is analytic. Finally, since $\nabla \langle f_{k,j}(\cdot,v), 1 \rangle : B_{\delta_n} \to H^1$, j=1,2, are analytic (cf. Lemma 5.3), it follows

$$\nabla G_k : -D_{r_k + \frac{1}{4}}(\tau_k) \times B_{\delta_n} \to H^1$$

is analytic. Coming back to (5.9), we infer that the holomorphic map

$$\nabla G_k + \chi_k \nabla \lambda_{k-1} : -D_{r_k + \frac{1}{4}}(\tau_k) \times B_{\delta_n} \to H^1$$

vanishes at $\lambda = -\lambda_{k-1}(v) - 1$ for any $v \in B_{\delta_n}$. Therefore there exists a holomorphic map

$$F_k: -D_{r_k+\frac{1}{4}}(\tau_k) \times B_{\delta_n} \to H^1$$

such that for any $(\lambda, v) \in -D_{r_k + \frac{1}{4}}(\tau_k) \times B_{\delta_n}$

$$\nabla G_k(\lambda, v) + \chi_k(\lambda, v) \nabla \lambda_{k-1}(v) = -(\lambda + \lambda_{k-1}(v) + 1) F_k(\lambda, v).$$

Comparing with (5.9), one deduces that

$$\nabla \chi_k(\lambda, v) = F_k(\lambda, v)$$

and thus has the claimed property.

With these preparations made, we now can study functionals ζ_n . Recall from the definition (4.1) that for any $n \geq 1$, $u \in L^2_{r,0}$, $\zeta_n(u) = \frac{\langle 1|f_n(\cdot,u)\rangle}{\sqrt{\kappa_n(u)}}$. The functionals $\langle 1|f_n\rangle:L^2_{r,0}\to\mathbb{C}$ (cf. Lemma 5.2), and $\kappa_n:L^2_{r,0}\to\mathbb{R}$ (Lemma 5.4) are real analytic. Hence we claim that $\zeta_n:L^2_{r,0}\to\mathbb{C}$ is real analytic. Indeed, let $u\in L^2_{r,0}$, $n\geq 1$, and $1\leq k\leq n$. By Lemma 5.2, $\langle 1,f_{k,1}-if_{k,2}\rangle$ is analytic on $B_{\delta_n}(u)\cap L^2_{r,0}$ and by Lemma 5.4, so is κ_k . Since by the same lemma, $\operatorname{Re}\kappa_k>0$ is uniformly bounded away from 0 on $B_{\delta_n}(u)\cap L^2_{r,0}$ it then follows that the principal branch of the root $\sqrt{\kappa_k}$ is a well defined, analytic functional which is also uniformly bounded away from 0 on this ball. We thus have proved that $\zeta_k=\frac{\langle 1|f_{k,1}\rangle}{\sqrt{\kappa_k}}-i\frac{\langle 1|f_{k,2}\rangle}{\sqrt{\kappa_k}}$ extends analytically to $B_{\delta_n}(u)\cap L^2_{r,0}$. Now we study the gradient of ζ_n . By the chain rule we have on $L^2_{r,0}$

(5.10)
$$\nabla \zeta_n = -\frac{1}{2} \frac{\nabla \kappa_n}{\kappa_n^{3/2}} \langle 1 | f_n \rangle + \frac{1}{\sqrt{\kappa_n}} \nabla \langle 1 | f_n \rangle.$$

Since $\nabla \langle 1|f_n\rangle: L^2_{r,0} \to H^1$ (cf. Lemma 5.3) and $\nabla \kappa_n: L^2_{r,0} \to H^1$ (cf. Lemma 5.5) are real analytic, the following result holds true.

Proposition 5.1. For any $n \ge 1$, the functional $\zeta_n : L_{r,0}^2 \to \mathbb{C}$ and the map $\nabla \zeta_n : L_{r,0}^2 \to H^1$ are real analytic.

Remark 5.3. The L^2 -gradient of ζ_n can be computed explicitly at u=0. Indeed, since $\kappa_n(0)=\frac{1}{n}$ and $\nabla\langle 1|f_n\rangle(0)=-\frac{1}{n}e^{-inx}$ (cf. Remark 5.1) one has

$$\nabla \zeta_n(0) = -\frac{1}{\sqrt{n}} e^{-inx} \,.$$

6. Poisson brackets

The main goal of this section is to compute the Poisson brackets $\{\gamma_p, \gamma_n\}$ and $\{\gamma_p, \langle 1|f_n\rangle\}$. As a first step we show that the flow, corresponding to the generating function \mathcal{H}_{λ} , leaves the eigenvalues of L_u invariant. Recall from (3.1) that for $u \in L_r^2$, the generating function is given by

$$\mathcal{H}_{\lambda}(u) = \langle (L_u + \lambda Id)^{-1} 1 | 1 \rangle.$$

Consider \mathcal{H}_{λ} with $\lambda \in \mathbb{R} \setminus \{-\lambda_n(u) : n \geq 0\}$. For any $v \in L_r^2$ one has

$$d\mathcal{H}_{\lambda}(u)[v] = \langle (L_u + \lambda Id)^{-1}T_v(L_u + \lambda Id)^{-1}1 | 1 \rangle = \langle v | |w_{\lambda}|^2 \rangle,$$

where $w_{\lambda} \equiv w_{\lambda}(\cdot, u)$ is given by

$$w_{\lambda} := (L_u + \lambda Id)^{-1} 1 \in H^1_+$$
.

Hence by the definition of the L^2 -gradient

$$\nabla \mathcal{H}_{\lambda}(u) = |w_{\lambda}(\cdot, u)|^2$$

and the Hamiltonian equation associated to \mathcal{H}_{λ} is

(6.1)
$$\partial_t u = \partial_x |w_\lambda(\cdot, u)|^2.$$

Since the map $u \in L_r^2 \mapsto w_\lambda(\cdot, u) \in H_+^1$ is Lipschitz on bounded subsets, so is the Hamiltonian vector field $u \in L_r^2 \mapsto \partial_x |w_\lambda(\cdot, u)|^2 \in L_r^2$ and hence the corresponding initial value problem admits a local in time solution for any initial data $u \in L_r^2$ and $\lambda \in \mathbb{C} \setminus \{\lambda_n(u) : n \geq 0\}$. Furthermore, the Toeplitz operators T_{w_λ} and $T_{\overline{w_\lambda}}$ are bounded operators on L_+^2 . The following result says that equation (6.1) has a Lax pair representation.

Proposition 6.1. For any $\lambda \in \mathbb{R} \setminus \{-\lambda_n(u) : n \geq 0\}$,

$$\frac{dL_u}{dt} = [B_u^{\lambda}, L_u] , \qquad B_u^{\lambda} := iT_{w_{\lambda}}T_{\overline{w_{\lambda}}}$$

along the evolution (6.1).

Remark 6.1. Recall that for $\varepsilon > 0$ we introduced $\tilde{\mathcal{H}}_{\varepsilon} = \frac{1}{\varepsilon}\mathcal{H}_{\frac{1}{\varepsilon}}$ and that $\frac{d^k}{d\varepsilon^k}\big|_{\varepsilon=0}\tilde{\mathcal{H}}_{\varepsilon}$, $k \geq 0$, are referred to as the Hamiltonians in the BO hierarchy. Introducing $\tilde{B}_{\varepsilon} := \frac{1}{\varepsilon}B_u^{\frac{1}{\varepsilon}}$, Proposition 6.1 then leads to a Lax pair formulation for the equations corresponding to the Hamiltonians in the BO hierarchy,

$$\frac{dL_u}{dt} = \left[\frac{d^k}{d\varepsilon^k}\Big|_{\varepsilon=0}\tilde{B}_{\varepsilon}, L_u\right],$$

where now u evolves according to the flow of $\frac{d^k}{d\varepsilon^k}\Big|_{\varepsilon=0} \tilde{\mathcal{H}}_{\varepsilon}$. In particular, in the case k=3, for any $u\in L^2_{r,0}$, $\mathcal{H}(u)=-\frac{1}{6}\frac{d^3}{d\varepsilon^3}\Big|_{\varepsilon=0} \tilde{\mathcal{H}}_{\varepsilon}(u)$ and one can check that the operator $-\frac{1}{6}\frac{d^3}{d\varepsilon^3}\Big|_{\varepsilon=0} \tilde{B}_{\varepsilon}$ coincides with $\tilde{B}_u-\frac{1}{2}i\|u\|^2Id$, where $\tilde{B}_u=i(T_{|\partial_x|u}-T_u^2)$ was introduced in Remark 2.1.

Recall that by Corollary 5.1, $\nabla \lambda_n = -|f_n|^2$ for any $n \geq 0$. Hence $\nabla \lambda_n \in H_r^1$ and the Poisson bracket $\{\lambda_p, \lambda_n\}$ is well-defined on L_r^2 for any $p \geq 0$. Clearly, the same is true for $\{\gamma_p, \gamma_n\}$.

Corollary 6.1. (i) For any $p, n \ge 0$ and $u \in L_r^2$, $\{\lambda_p, \lambda_n\}(u) = 0$. (ii) For any $p, n \ge 1$ and $u \in L_r^2$, $\{\gamma_p, \gamma_n\}(u) = 0$

Remark 6.2. By the arguments of the proof of Corollary 6.1,

$$\{\mathcal{H}_{\lambda}, \mathcal{H}_{\mu}\} = 0$$
, $\{\mathcal{H}_{\lambda}, \lambda_n\} = 0$

for any $\lambda, \mu \in \mathbb{C} \setminus \{-\lambda_k(u) : k \geq 0\}$ and $n \geq 0$.

Proof of Corollary 6.1. (i) In view of Proposition 6.1, for any $n \geq 0$ and λ in $\mathbb{R} \setminus \{-\lambda_k(u) : k \geq 0\}$, $\lambda_n(u)$ is constant along the evolution (6.1). Hence on L_r^2 , $\{\mathcal{H}_{\lambda}, \lambda_n\} = 0$. Using (3.2), we therefore have

$$\sum_{p=0}^{\infty} \left(-\frac{|\langle 1|f_p \rangle|^2}{(\lambda_p + \lambda)^2} \{\lambda_p, \lambda_n\} + \frac{\{|\langle 1|f_p \rangle|^2, \lambda_n\}}{\lambda_p + \lambda} \right) = 0.$$

By analytic continuation, the latter identity continues to hold for any λ in $\mathbb{C} \setminus \{-\lambda_k(u) : k \geq 0\}$. Since the left hand side of this identity is a meromorphic function in λ , the principal part of the expansion at any of its pole vanishes, implying that $\{\lambda_p, \lambda_n\}(u) = 0$ for any $p \geq 0$.

Let us come to the proof of Proposition 6.1.

Proof of Proposition 6.1. Assume that $u \equiv u(t)$ is a local in time solution of (6.1) with $\lambda \in \mathbb{R} \setminus \{\lambda_n(u(0)) : n \geq 0\}$. In the sequel of the proof, we denote also by D the operator $-i\partial_x$, when viewed as operator acting on H^1 . Since by the definition of the operator L_u one has

$$\frac{dL_u}{dt} = -T_{\partial_t u} = -iT_{D|w_\lambda|^2} ,$$

the claimed result is equivalent to the identity

$$[D - T_u, T_{w_{\lambda}} T_{\overline{w}_{\lambda}}] - T_{D|w_{\lambda}|^2} = 0.$$

We will use Hankel operators with symbols in L_+^{∞} to provide a formula for the commutators of Toeplitz operators. Given a symbol $v \in L_+^{\infty}$, the Hankel operator H_v is given by

$$H_v: L^2_+ \to L^2_+, h \mapsto H_v(h) = \Pi(v\overline{h}).$$

Notice that $H_v(h)$ is antilinear in h, but linear in v.

Lemma 6.1. For any $v, w \in L^{\infty}_{+}$

$$[T_v, T_{\overline{w}}] = \langle \cdot | w \rangle v - H_v H_w .$$

Furthermore, for any $u \in L_r^{\infty}$,

$$(6.4) H_{T_u w} = T_w H_{\Pi u} + H_w T_u - \langle \Pi u | \cdot \rangle w$$

(6.5)
$$H_{T_u w} = H_{\Pi u} T_{\overline{w}} + T_u H_w - \langle w | \cdot \rangle \Pi u .$$

Proof of Lemma 6.1. We shall make use of the following elementary identities, valid for any $f \in L^2$ and $h \in L^2_+$,

$$f = \Pi f + \overline{\Pi \overline{f}} - \langle f, 1 \rangle , \quad \Pi(f\overline{h}) = \Pi((\Pi f)\overline{h}) .$$

Then, for any $h \in L^2_+$,

$$[T_v, T_{\overline{w}}]h = \Pi(v\Pi(\overline{w}h)) - \Pi(\overline{w}\Pi(vh)) = \Pi(v\Pi(\overline{w}h)) - \Pi(\overline{w}vh).$$

Using that $\overline{w}h = \Pi(\overline{w}h) + \overline{\Pi(wh)} - \langle \overline{w}h|1 \rangle$ one then gets

$$[T_v, T_{\overline{w}}]h = -\Pi(v\overline{\Pi(w\overline{h})}) + \langle \overline{w}h|1\rangle v$$
,

which yields (6.3). As for (6.4), we have

$$H_{T_n w} h = \Pi(\Pi(uw)\overline{h}) = \Pi(uw\overline{h})$$
.

Since $u\overline{h} = \Pi(u\overline{h}) + \overline{\Pi(\overline{u}h)} - \langle u|h\rangle$ and u is real valued one obtains

$$H_{T_u w} h = \Pi(w \Pi(u \overline{h})) + \Pi(w \overline{\Pi(u h)}) - \langle u | h \rangle w$$

= $\Pi(w \Pi((\Pi u) \overline{h})) + \Pi(w \overline{\Pi(u h)}) - \langle \Pi u | h \rangle w$

which establishes (6.4). The identity (6.5) is proved in a similar way or alternatively, by taking adjoints with respect to the real inner product, induced by $\langle \cdot | \cdot \rangle$.

Let us now return to the proof of Proposition 6.1 and complete the verification of identity (6.2). We write w for w_{λ} for simplicity. Using (6.3), the Leibniz rule, and the antilinearity of H_w , we get

$$[D, T_w T_{\overline{w}}] = [D, T_{\overline{w}} T_w] + [D, \langle \cdot | w \rangle w - H_w^2]$$

$$= T_{D\overline{w}} T_w + T_{\overline{w}} T_{Dw} + \langle \cdot | w \rangle Dw - \langle \cdot | Dw \rangle w - H_{Dw} H_w + H_w H_{Dw}$$

$$= T_{D|w|^2} + \langle \cdot | w \rangle Dw - \langle \cdot | Dw \rangle w - H_{Dw} H_w + H_w H_{Dw}.$$

Regarding $[T_u, T_{\underline{w}}T_{\overline{w}}]$, note that $[T_u, T_wT_{\overline{w}}] = [T_u, T_w]T_{\overline{w}} + T_w[T_u, T_{\overline{w}}]$. Using $u = \Pi u + \overline{\Pi u} - \langle u|1\rangle$ and again (6.3), one gets

$$[T_u, T_w T_{\overline{w}}] = [T_{\overline{\Pi}u}, T_w] T_{\overline{w}} + T_w [T_{\Pi u}, T_{\overline{w}}]$$

$$= H_w H_{\Pi u} T_{\overline{w}} - T_w H_{\Pi u} H_w + \langle \cdot | w \rangle T_w (\Pi u) - \langle \cdot | T_w (\Pi u) \rangle w .$$

Combining these two identities, we get

$$[D - T_u, T_w T_{\overline{w}}] - T_{D|w|^2} = -(H_{Dw} - T_w H_{\Pi u}) H_w + H_w (H_{Dw} - H_{\Pi u} T_{\overline{w}}) + \langle \cdot | w \rangle (Dw - T_w (\Pi u)) - \langle \cdot , Dw - T_w (\Pi u) \rangle w.$$

We now have to analyze the terms $H_{Dw} - T_w H_{\Pi u}$ and $H_{Dw} - H_{\Pi u} T_{\overline{w}}$ in more detail. Since $w = (L_u + \lambda)^{-1} 1$, hence $Dw = T_u w + 1 - \lambda w$, and since by (6.4)

$$T_w H_{\Pi u} = H_{T_u w} - H_w T_u + \langle \Pi u | \cdot \rangle w$$

one sees that

$$H_{Dw} - T_w H_{\Pi u} = H_{T_u w + 1 - \lambda w} - H_{T_u w} + H_w T_u - \langle \Pi u | \cdot \rangle w$$

= $H_1 - \lambda H_w + H_w T_u - \langle \Pi u | \cdot \rangle w$.

Using that u is real valued and hence $\langle \Pi u | H_w h \rangle = \langle h | H_w(\Pi u) \rangle$ one is led to the following identity for the term $-(H_{Dw} - T_w H_{\Pi u}) H_w$,

$$-(H_{Dw}-T_wH_{\Pi u})H_w=-H_1H_w+\lambda H_wH_w-H_wT_uH_w+\langle\cdot|H_w(\Pi u)\rangle w.$$
 Similarly, by (6.5)

$$H_{\Pi u}T_{\overline{w}} = H_{T_u w} - T_u H_w + \langle w| \cdot \rangle \Pi u$$

one sees that

$$H_{Dw} - H_{\Pi u} T_{\overline{w}} = H_{T_u w + 1 - \lambda w} - H_{T_u w} + T_u H_w - \langle w | \cdot \rangle \Pi u$$

= $H_1 - \lambda H_w + T_u H_w - \langle w | \cdot \rangle \Pi u$.

Finally, since H_w is antilinear and λ is real one concludes that

$$H_w(H_{Dw} - H_{\Pi u}T_{\overline{w}}) = H_wH_1 - \lambda H_wH_w + H_wT_uH_w - \langle \cdot | w \rangle H_w(\Pi u).$$

Substituting these identities into the formula for $[D-T_u, T_w T_{\overline{w}}] - T_{D|w|^2}$, obtained above, one infers that

$$[D - T_u, T_w T_{\overline{w}}] - T_{D|w|^2} = -H_1 H_w + H_w H_1$$

+ $\langle \cdot | w \rangle (Dw - T_w(\Pi u) - H_w(\Pi u)) - \langle \cdot | Dw - T_w(\Pi u) - H_w(\Pi u) \rangle w$.
By (6.4),

$$T_{u}(w) = H_{T_{u}(w)}(1) = T_{w}H_{\Pi u}(1) + H_{w}T_{u}(1) - \langle \Pi u | 1 \rangle w$$

= $T_{w}(\Pi u) + H_{w}(\Pi u) - \langle u | 1 \rangle w$.

The identity $Dw = T_u w + 1 - \lambda w$ then reads

$$Dw - T_w(\Pi u) - H_w(\Pi u) = 1 - \lambda w - \langle u|1\rangle w.$$

Hence

$$[D - T_u, T_w T_{\overline{w}}] - T_{D|w|^2} = -H_1 H_w + H_w H_1 + \langle \cdot | w \rangle 1 - \langle \cdot | 1 \rangle w$$
 which vanishes since $\langle \cdot | w \rangle 1 = H_1 H_w$ and $\langle \cdot | 1 \rangle w = H_w H_1$.

The second main result of this section calculates the Poisson brackets of $\langle 1|f_n\rangle$ with the functionals γ_p . Recall that for any $p\geq 1$, $\nabla\gamma_p$ is in H^1 (cf. Corollary 5.1) and that for any $n\geq 0$, also $\nabla\langle 1|f_n\rangle$ is in H^1 (cf. Lemma 5.3). Hence the Poisson bracket $\{\gamma_p,\langle 1|f_n\rangle\}$ is well-defined on L_r^2 for any $p\geq 1$ and $n\geq 0$.

Proposition 6.2. For any $p \ge 1, n \ge 0$, and $u \in L^2_{r,0}$,

$$\{\gamma_p, \langle 1|f_n\rangle\}(u) = i \langle 1|f_n(\cdot, u)\rangle \delta_{pn}$$
.

Proof. Again we use the Hamiltonian flow of the generating function \mathcal{H}_{λ} and its Lax pair formulation provided by Proposition 6.1,

$$\frac{dL_u}{dt} = [B_u^{\lambda}, L_u] , \qquad B_u^{\lambda} = iT_{w_{\lambda}}T_{\overline{w}_{\lambda}} .$$

Given $u_0 \in L^2_{r,0}$, $n \geq 0$, and $\lambda \geq |-\lambda_0 + 1|$, denote by u = u(t) the trajectory issuing from u_0 at t = 0, and define $g_n^{\lambda} \equiv g_n^{\lambda}(t)$ by

$$\frac{dg_n^{\lambda}}{dt} = B_u^{\lambda} g_n^{\lambda} , \qquad g_n^{\lambda}(0) = f_n(u_0) .$$

Since B_u^λ is skew symmetric, $\|f_n(\cdot,u_0)\|=1$, $\|g_n^\lambda(t)\|=1$ and since

$$\frac{d}{dt}\Big((L_u - \lambda_n)g_n^{\lambda}\Big) = B_u^{\lambda}(L_u - \lambda_n)g_n^{\lambda}, \qquad \Big((L_u - \lambda_n)g_n^{\lambda}\Big)|_{t=0} = 0,$$

 $g_n^{\lambda}(t)$ is an eigenfunction of $L_{u(t)}$ corresponding to the eigenvalue λ_n , coinciding with the eigenfunction $f_n(\cdot, u(t))$ up to a phase factor. To

determine this phase factor we need to compute the evolution of $\langle g_0^{\lambda}|1\rangle$ and of $\langle g_n^{\lambda}|Sg_{n-1}^{\lambda}\rangle$, $n\geq 1$. Using that

$$T_{\overline{w}_{\lambda}} 1 = \overline{\langle w_{\lambda} | 1 \rangle} 1$$
, $\langle g_n^{\lambda} | w_{\lambda} \rangle = \langle (L_u + \lambda)^{-1} g_n^{\lambda} | 1 \rangle = \frac{1}{\lambda_n + \lambda} \langle g_n^{\lambda} | 1 \rangle$

one gets for any $n \geq 0$

$$\frac{d}{dt}\langle g_n^{\lambda}|1\rangle = \langle B_u^{\lambda}g_n^{\lambda}|1\rangle = i\langle g_n^{\lambda}|T_{w_{\lambda}}T_{\overline{w}_{\lambda}}1\rangle = i\langle w_{\lambda}|1\rangle\langle g_n^{\lambda}|w_{\lambda}\rangle$$

implying that

(6.6)
$$\frac{d}{dt}\langle g_n^{\lambda}|1\rangle = i\mathcal{H}_{\lambda}\frac{\langle g_n^{\lambda}|1\rangle}{\lambda_n + \lambda}.$$

Similarly, using in addition that B_u^{λ} is skew symmetric and that by (2.4)

$$ST_{w_{\lambda}} = T_{w_{\lambda}}S - \langle Sw_{\lambda} \cdot | 1 \rangle = T_{w_{\lambda}}S$$

one has for any $n \ge 1$

$$\frac{d}{dt}\langle g_n^{\lambda}|\,Sg_{n-1}^{\lambda}\rangle = \langle g_n^{\lambda}|\,[S,B_u^{\lambda}]g_{n-1}^{\lambda}\rangle = i\langle g_n^{\lambda}|\,T_{w_{\lambda}}[T_{\overline{w}_{\lambda}},S]g_{n-1}^{\lambda}\rangle.$$

Using (2.4) once more one gets

$$T_{w_{\lambda}}[T_{\overline{w}_{\lambda}}, S]g_{n-1}^{\lambda} = \langle S\overline{w}_{\lambda}g_{n-1}^{\lambda} | 1 \rangle T_{w_{\lambda}}1 = \overline{\langle w_{\lambda} | Sg_{n-1}^{\lambda} \rangle} w_{\lambda}$$

and in this way concludes that

$$\frac{d}{dt}\langle g_n^{\lambda}|Sg_{n-1}^{\lambda}\rangle = i\langle g_n^{\lambda}|w_{\lambda}\rangle\langle w_{\lambda}|Sg_{n-1}^{\lambda}\rangle = i\frac{\langle g_n^{\lambda}|1\rangle}{\lambda_n + \lambda}\langle w_{\lambda}|Sg_{n-1}^{\lambda}\rangle.$$

Note that by (2.7)

$$(\lambda_{n-1} + \lambda + 1)Sg_{n-1}^{\lambda} = (L_u + \lambda)Sg_{n-1}^{\lambda} + \langle Sg_{n-1}^{\lambda} | u \rangle 1 ,$$

and that

$$\langle w_{\lambda} | (L_u + \lambda) S g_{n-1}^{\lambda} \rangle = \langle 1 | S g_{n-1}^{\lambda} \rangle = 0$$
,

yielding

$$\langle w_{\lambda} | Sg_{n-1}^{\lambda} \rangle = \frac{\langle u | Sg_{n-1}^{\lambda} \rangle \mathcal{H}_{\lambda}}{\lambda_{n-1} + \lambda + 1}$$

Since by Lemma 2.2

$$-\langle g_n^{\lambda}|1\rangle\langle u|Sg_{n-1}^{\lambda}\rangle = (\lambda_n - \lambda_{n-1} - 1)\langle g_n^{\lambda}|Sg_{n-1}^{\lambda}\rangle = \gamma_n\langle g_n^{\lambda}|Sg_{n-1}^{\lambda}\rangle$$

we end up with the following identity

$$\frac{d}{dt}\langle g_n^{\lambda} | Sg_{n-1}^{\lambda} \rangle = -i\mathcal{H}_{\lambda} \cdot \frac{1}{\lambda_n + \lambda} \cdot \frac{\gamma_n}{\lambda_{n-1} + \lambda + 1} \cdot \langle g_n^{\lambda} | Sg_{n-1}^{\lambda} \rangle
(6.7) \qquad = i\mathcal{H}_{\lambda} \cdot \left(\frac{1}{\lambda_n + \lambda} - \frac{1}{\lambda_{n-1} + \lambda + 1} \right) \langle g_n^{\lambda} | Sg_{n-1}^{\lambda} \rangle$$

In view of (6.6) and (6.7) we set

$$\widetilde{g}_{p}^{\lambda}(t) = g_{p}^{\lambda}(t) e^{-it\alpha_{p}(\lambda)}, \ p \geq 0$$

where $\alpha_0(\lambda) := \frac{\mathcal{H}_{\lambda}}{\lambda_0 + \lambda}$ and for any $n \geq 1$,

$$\alpha_n(\lambda) := \mathcal{H}_{\lambda} \cdot \left(\frac{1}{\lambda_0 + \lambda} + \sum_{p=1}^n \left(\frac{1}{\lambda_p + \lambda} - \frac{1}{\lambda_{p-1} + \lambda + 1} \right) \right) ,$$

so that

$$\frac{d}{dt}\langle \widetilde{g}_0^{\lambda}|1\rangle = 0 , \qquad \frac{d}{dt}\langle \widetilde{g}_n^{\lambda}|S\widetilde{g}_{n-1}^{\lambda}\rangle = 0 , \ n \ge 1 ,$$

implying that for any $p \geq 0$, $\widetilde{g}_p(t) = f_p(u(t))$. Taking into account (6.6), one sees that for any $\lambda \geq |-\lambda_0+1|$ and $n \geq 1$, $\{\mathcal{H}_{\lambda}, \langle 1|f_n\rangle\} = \frac{d}{dt}\langle 1|f_n\rangle$, when evaluated at t=0, is given by

$$\{\mathcal{H}_{\lambda}, \langle 1|f_{n}\rangle\} = i\langle 1|f_{n}\rangle \left(\alpha_{n} - \frac{\mathcal{H}_{\lambda}}{\lambda_{n} + \lambda}\right)$$
$$= i\langle 1|f_{n}\rangle \cdot \mathcal{H}_{\lambda} \cdot \left(\frac{1}{\lambda_{0} + \lambda} - \frac{1}{\lambda_{n} + \lambda} + \sum_{p=1}^{n} \frac{1}{\lambda_{p} + \lambda} - \frac{1}{\lambda_{p-1} + \lambda + 1}\right)$$

or expressed in a shorter form,

$$\{\mathcal{H}_{\lambda}, \langle 1|f_n\rangle\} = i\langle 1|f_n\rangle \cdot \mathcal{H}_{\lambda} \cdot \sum_{p=0}^{n-1} \left(\frac{1}{\lambda_p + \lambda} - \frac{1}{\lambda_p + \lambda + 1}\right).$$

On the other hand, using formula (3.2) for \mathcal{H}_{λ} , one computes

$$\{\mathcal{H}_{\lambda}, \langle 1|f_n\rangle\} = \sum_{n=0}^{\infty} \left(-\frac{|\langle 1|f_p\rangle|^2}{(\lambda_p + \lambda)^2} \{\lambda_p, \langle 1|f_n\rangle\} + \frac{\{|\langle 1|f_p\rangle|^2, \langle 1|f_n\rangle\}}{\lambda_p + \lambda}\right)$$

Since the left and right hand side of the latter identity are meromorphic functions in λ , they have the same poles. In particular the principal part of the expansion at any of their poles of order two coincide, hence for any $p \geq 0$ with $\langle 1|f_p \rangle \neq 0$ one obtains

$$\{\lambda_p, \langle 1|f_n\rangle\} = \begin{cases} -i\langle 1|f_n\rangle & \text{if } 0 \le p \le n-1 \\ 0 & \text{if } p \ge n \end{cases},$$

Since, for $p \geq 0$, $\lambda_p = p - \sum_{k=p+1}^{\infty} \gamma_k$, it follows that for any $k \geq 1$,

(6.8)
$$\{\gamma_k, \langle 1|f_n\rangle\} = i\langle 1, f_n\rangle \delta_{kn}$$

at least on the set

$$\mathcal{O}_{\infty} = \{ u \in L^2_{r,0} : \gamma_j(u) > 0 \ \forall j \ge 1 \} .$$

Since for any $k \geq 1$, γ_k is a real analytic, not identically vanishing functional, \mathcal{O}_{∞} is a dense G_{δ} subset of $L^2_{r,0}$, and hence by continuity, identity (6.8) holds on all of $L^2_{r,0}$.

For any $n \geq 1$, on the dense open subset $L_{r,0}^2 \setminus Z_n$ of $L_{r,0}^2$, consisting of potentials u with $\gamma_n(u) > 0$, the argument φ_n of $\langle 1|f_n \rangle$ is well defined. Since $|\langle 1|f_n \rangle| = \gamma_n^{1/2} \kappa_n^{1/2}$, Corollary 6.1 and Proposition 6.2 imply that

(6.9)
$$\{\gamma_p, \varphi_n\} = \delta_{pn} , p \ge 1 , \qquad \forall u \in L_{r,0}^2 \setminus Z_n .$$

In order to show that the angle variables φ_n , $n \geq 1$, are conjugate to the action variables γ_p , $p \geq 1$, it remains to establish on $L^2_{r,0} \setminus (Z_p \cup Z_n)$ the complementary identities

$$\{\varphi_p, \varphi_n\} = 0.$$

In the subsequent section, we prove these identities for $1 \le p, n \le N$ with $N \ge 1$ arbitrary on the set

$$\mathcal{O}_N = \{ u \in L^2_{r_0} : \gamma_i(u) > 0 \ \forall 1 \le j \le N; \gamma_i(u) = 0 \ \forall j \ge N+1 \} .$$

In fact, we show that \mathcal{O}_N is a symplectic manifold of dimension 2N with $\gamma_1, \ldots, \gamma_N$ and $\varphi_1, \ldots, \varphi_N$ as action and angle variables in the classical sense.

7. FINITE GAP POTENTIALS

In this section we study the set G_J of J-gap potentials, introduced in Section 2, with $J \subset \mathbb{N}$ finite (cf. Definition 2.2). In fact, to simplify the exposition we will consider only the case $J = \{1, \ldots, N\}$ where N > 1 is an arbitrary integer. Then

$$G_{\{1,\dots,N\}} = \{u \in L_r^2 : \gamma_i(u) > 0 \ \forall 1 \le j \le N; \ \gamma_i(u) = 0 \ \forall j > N\}$$
.

Note that

$$G_{\{1,\dots,N\}} = \bigcup_{c \in \mathbb{R}} (c + \mathcal{O}_N) , \qquad \mathcal{O}_N = G_{\{1,\dots,N\}} \cap L^2_{r,0} ,$$

where $c + \mathcal{O}_N$ are the level sets of the restriction of the average functional $L_r^2 \to \mathbb{R}$, $u \mapsto \langle u | 1 \rangle$, to $G_{\{1,\ldots,N\}}$. Furthermore, $L_{r,0}^2$ is endowed with the symplectic form

(7.1)
$$\omega(u,v) = \langle u \,|\, \partial_x^{-1} v \rangle,$$

so that one has the identity $\{F,G\} = \omega(\partial_x(\nabla F), \partial_x(\nabla G))$. Note that the average functional is a Casimir for the Gardner bracket. The aim of this section is to characterize elements in \mathcal{O}_N and to show that the restriction of Φ to \mathcal{O}_N is a real analytic symplectic diffeomorphism onto its image where we recall that Φ denotes the Birkhoff map, introduced in (7.1). As an important application we will prove that Φ is onto (cf. Corollary 7.1). We refer to Appendix B for a study of the set of one gap potentials $G_{\{N\}}$ with $N \geq 1$ arbitrary.

Actually, we consider a slightly larger set than \mathcal{O}_N : for any $N \geq 1$, define

$$(7.2) \mathcal{U}_N := \{ u \in L^2_{r,0} : \gamma_N(u) > 0 , \gamma_j(u) = 0 \ \forall j > N \}.$$

It is convenient to define $\mathcal{U}_0 := \{0\}$. Note that for any $u \in \mathcal{U}_N$ and $n \geq N$ one has by (3.13) and Lemma 2.4

$$\lambda_n(u) = n$$
, $f_n(x, u) = e^{i(n-N)x} f_N(x, u)$.

Furthermore, denote by $\mathbb{C}_N[z]$ (respectively $\mathbb{C}_{\leq N}[z]$) the space of polynomials P in the complex variable z of degree N (of degree at most N) and by $\mathbb{C}_N^+[z]$ the open subset of polynomials $P \in \mathbb{C}_N[z]$ with the property that $\{P(z) = 0\} \subset \{|z| > 1\}$. We identify $\mathbb{C}_N[z]$ with the space $\mathbb{C}^N \times \mathbb{C}^*$ of coefficients of polynomials in $\mathbb{C}_N[z]$. The following theorem characterizes elements in \mathcal{U}_N .

Theorem 3. For any $N \ge 1$

(7.3)
$$\mathcal{U}_N = \left\{ u = h + \overline{h} : h(x) = -e^{ix} \frac{Q'(e^{ix})}{Q(e^{ix})}, Q \in \mathbb{C}_N^+[z] \right\}$$

where $Q'(z) := \partial_z Q(z)$. Furthermore, \mathcal{U}_N is a connected, real analytic, symplectic submanifold of $L^2_{r,0}$ of dimension 2N. The restriction Φ_N of Φ to \mathcal{U}_N ,

$$\Phi_N: \mathcal{U}_N \to \mathbb{C}^{N-1} \times \mathbb{C}^*, \ u \mapsto (\zeta_n(u))_{1 \le n \le N}$$

is a real analytic, symplectic diffeomorphism,

$$(\Phi_N)_*\omega = i\sum_{n=1}^N d\zeta_n \wedge d\overline{\zeta_n} .$$

In particular, \mathcal{O}_N is a dense open subset of \mathcal{U}_N with $\Phi_N(\mathcal{O}_N) = (\mathbb{C}^*)^N$.

Remark 7.1. (i) Alternatively, potentials $u \in \mathcal{U}_N$ with $N \geq 1$ can be written in the form (cf. formula (7.9) below)

(7.4)
$$u(x) = \sum_{k=1}^{\infty} (\sum_{j=0}^{N-1} q_j^k) e^{ikx} + \sum_{k=1}^{\infty} (\sum_{j=0}^{N-1} \overline{q_j}^k) e^{-ikx}$$

where $1/q_0, \ldots, 1/q_{N-1} \in \mathbb{C}^*$ denote the roots of the polynomial Q(z) of degree N, defined in (7.6) below. Note that the above formula for u can also be expressed in terms of the Poisson kernel $P_r(x) = \frac{1-r^2}{1-2r\cos(x)+r^2}$,

$$u(x) = \sum_{j=0}^{N-1} (P_{r_j}(x + \alpha_j) - 1), \quad q_j = r_j e^{i\alpha_j} \ \forall 0 \le j \le N - 1.$$

(ii) Real coordinates on \mathcal{U}_N are given by the real and imaginary parts $(Q_{j,1})_{1\leq j\leq N}$, $(Q_{j,2})_{1\leq j\leq N}$, of the coefficients $(Q_j)_{1\leq j\leq N}$ of the polynomial $Q\in \mathbb{C}_N^+[z]$ normalized by Q(0)=1, $Q(z)=1+\sum_{j=1}^N(Q_{j,1}+iQ_{j,2})z^j$. The corresponding potential u can then be written as

$$u(x) = -e^{ix} \frac{Q'(e^{ix})}{Q(e^{ix})} - e^{-ix} \frac{P'(e^{-ix})}{P(e^{-ix})}$$

where

$$P(z) = 1 + \sum_{j=1}^{N} P_j z^j$$
, $P_j := Q_{j,1} - iQ_{j,2}$.

Note that this formula for u is real analytic in the coordinates $Q_{j,1}$, $Q_{j,2}$, $1 \leq j \leq N$. Complex valued coefficients $Q_{j,1}$, $Q_{j,2}$, $1 \leq j \leq N$ then will lead to complex valued potentials.

Furthermore note that by Proposition 4.1, for any $n \geq 1$, the real part $\zeta_{n,1}$ and the imaginary part $\zeta_{n,2}$ of ζ_n are real analytic functionals on $L^2_{r,0}$ and so are their restrictions to \mathcal{U}_N . It then follows that Φ_N is real analytic when viewed as map $\mathcal{U}_N \to \mathbb{R}^{2(N-1)} \times \mathbb{R}^2 \setminus \{(0,0)\}$.

Proof. We first prove that any element $u \in \mathcal{U}_N$ has the claimed form. To simplify notation within this proof, we do not indicate the dependence of various quantities on u. Our starting point is the inverse formula (4.5), stated in Lemma 4.1,

$$\Pi u(z) = \langle (Id - zM)^{-1}X|Y\rangle_{\ell^2}, \quad z \in \mathbb{C}, \ |z| < 1,$$

where $X = -(\lambda_p \langle 1|f_p \rangle)_{p \geq 0}$ and $Y = (\langle 1|f_n \rangle)_{n \geq 0}$. According to the computations in the proof of Proposition 4.2, the n-th row of M with n > N in the case at hand satisfies

$$(M_{np})_{p\geq 0}=(\delta_{n+1,p})_{p\geq 0} \quad \forall n\geq N.$$

Since the coefficients of $X = (-\lambda_p \langle 1 | f_p \rangle)_{p \geq 0}$ with $p \geq N + 1$ vanish, $(\xi_n(z))_{n \geq 0} := (Id - zM)^{-1}[X]$ satisfies

$$\xi_N(z) - z\xi_{N+1}(z) = -\lambda_N \langle 1|f_N\rangle, \quad \xi_n(z) - z\xi_{n+1}(z) = 0, \quad \forall n \ge N+1,$$
 or

$$\xi_{N+1}(z) = z^{n-N-1}\xi_n(z) , \quad \forall n \ge N+1.$$

Combined with the fact that $\sum_{n\geq 0} |\xi_n(z)|^2 < \infty$ for any |z| < 1, we infer that

$$\xi_n(z) = 0$$
, $n \ge N + 1$, $\xi_N(z) = -\lambda_N \langle 1 | f_N \rangle$.

This means that

(7.5)
$$\Pi u(z) = \langle (Id - zM_N)^{-1} X_N | Y_N \rangle_{\mathbb{C}^{N+1}},$$

where

$$X_N = (-\lambda_p \langle 1|f_p \rangle)_{0 \le p \le N}, \quad Y_N = (\langle 1|f_n \rangle)_{0 \le n \le N}, \quad M_N = (M_{np})_{0 \le n, p \le N}.$$

Notice that the last row of M_N is 0. With the same arguments used to derive the formula for $\Pi u(z)$ (cf. (4.3)) one obtains a similar formula for the extension $f_n(z)$ of the eigenfunctions f_n of L_u to the unit disc. For $0 \le n \le N$ and |z| < 1 one has

$$f_n(z) = \langle (Id - zM_N)^{-1} \mathbf{1}_n | Y_N \rangle_{\mathbb{C}^{N+1}}$$

where $\mathbf{1}_n := (\delta_{pn})_{0 \leq p \leq N}$. We thus conclude that Πu and f_0, \ldots, f_N belong to the \mathbb{C} -vector space

(7.6)
$$\mathcal{R}_N := \left\{ \frac{P(z)}{Q(z)} : P \in \mathbb{C}_{\leq N}[z] \right\}, \qquad Q(z) := \det(Id - zM_N),$$

and that f_0, \ldots, f_N are N+1 linearly independent elements in \mathcal{R}_N and thus form a basis. As a consequence, 1/Q(z) belongs to the Hardy space of the unit disc, hence Q(z) cannot vanish at any point z with $|z| \leq 1$. Furthermore one has Q(0) = 1. To see that $\deg(Q) = N$ first note that the last row of M_N is identically zero and thus

(7.7)
$$\det(Id - zM_N) = \det(Id - zM_{N-1}),$$

where $M_{N-1}=(M_{np})_{0\leq n,p\leq N-1}$. It is to prove that $\det(M_{N-1})\neq 0$. The formula (4.7) for the coefficients M_{np} for $\zeta_{n+1}\neq 0$ and the definition (4.9) of μ_{n+1} imply that in the case at hand the coefficients M_{np} with $0\leq n,p\leq N-1$ read

(7.8)
$$M_{np} = \begin{cases} \delta_{p,n+1} & \text{if } \zeta_{n+1} = 0, \\ \sqrt{\mu_{n+1}} \gamma_{n+1} \frac{\langle f_p | 1 \rangle}{\langle \lambda_p - \lambda_n - 1 \rangle \langle f_{n+1} | 1 \rangle} & \text{if } \zeta_{n+1} \neq 0. \end{cases}$$

Expanding $\det(M_{N-1})$ by the formula for Cauchy determinants, one sees that

$$|\det(M_{N-1})| = \left| \det\left(\sqrt{\mu_n} \gamma_n \frac{\langle f_p | 1 \rangle}{(\lambda_p - \lambda_{n-1} - 1) \langle f_n | 1 \rangle}\right)_{n \in J, p \in \tilde{J}} \right|$$

$$= \left(\prod_{n \in J} \sqrt{\mu_n} \gamma_n\right) \frac{\langle f_0 | 1 \rangle}{|\langle f_N | 1 \rangle|} \left| \det\left(\frac{1}{\lambda_{n-1} + 1 - \lambda_p}\right)_{n \in J, p \in \tilde{J}} \right| \neq 0$$

where $J := \{1 \leq j \leq N \mid \gamma_j > 0\}, \tilde{J} := \{0\} \cup J \setminus \{N\}$. Thus we showed that $\deg(Q) = N$.

Next we prove that $\Pi u(z) = -zQ'(z)/Q(z)$. Indeed since $\langle u|1\rangle = 0$, there exists a polynomial $R \in \mathbb{C}_{\leq N-1}[z]$ so that $\Pi u(z) = zR(z)/Q(z)$. Furthermore, since $L_u f_n = \lambda_n f_n$ and since f_0, \ldots, f_N is a basis of \mathcal{R}_N , L_u leaves \mathcal{R}_N invariant. Let us look at the action of L_u on \mathcal{R}_N in more detail. For any $h \in L^2_+$, one has

$$T_u h = (\Pi u)h + H_h(\Pi u)$$
, $H_h(f) := \Pi(h\overline{f})$

By elementary properties of Hankel operators — see e.g. [8], Prop. 11—, for every $h \in \mathcal{R}_N$, the range of the Hankel operator H_h is included in \mathcal{R}_N . Hence the linear map $h \mapsto L_u h - H_h(\Pi(u)) = -i\partial_x h - (\Pi u)h$ leaves \mathcal{R}_N invariant. Expressed in an alternative way, it means that

$$z\partial_z - z\frac{R(z)}{Q(z)}$$

leaves $\frac{1}{Q(z)}\mathbb{C}_{\leq N}[z]$ invariant. Using the latter fact, one verifies in a straightforward way that R(z) = -Q'(z), showing that u has indeed

the claimed form

$$\Pi(u)(z) = -z \frac{Q'(z)}{Q(z)}.$$

Now let us prove the converse. Assume that $Q \in \mathbb{C}_N^+[z]$ with Q(0) = 1. Thus Q is of the form

$$Q(z) = \prod_{j=0}^{N-1} (1 - q_j z), \quad q_j \in \mathbb{C}, 0 < |q_j| < 1, \quad \forall \, 0 \le j \le N - 1,$$

and in turn

(7.9)
$$\Pi u(z) = \sum_{j=0}^{N-1} \frac{q_j z}{1 - q_j z}.$$

Let

$$f(z) := \prod_{j=0}^{N-1} \frac{z - \overline{q}_j}{1 - q_j z}$$
.

Clearly, f is in \mathcal{R}_N . Writing $L_u(f)$ as $z\partial_z f - (\Pi u)f - H_f(\Pi u)$ one computes

$$L_u(f) = f \sum_{j=0}^{N-1} \left(\frac{z}{z - \overline{q}_j} + \frac{q_j z}{1 - q_j z} - \frac{q_j z}{1 - q_j z} - \frac{\overline{q}_j}{z - \overline{q}_j} \right) = Nf.$$

It means that f is an eigenfunction of L_u with eigenvalue N. Furthermore, since for any |z| = 1,

$$f(z) = z^N \overline{Q(z)}/Q(z) ,$$

one verifies that for any $k \geq 1$, $S^k f$ is orthogonal to \mathcal{R}_N and thus in particular to Πu . By Lemma 2.1 this implies that for any $k \geq 1$, $S^k f$ is an eigenfunction of L_u with eigenvalue N+k. Taking into account that \mathcal{R}_N is of dimension N+1 and that the eigenvalues λ_n satisfy (3.13) with $\langle \Pi u | 1 \rangle = 0$ one concludes that $\lambda_{N+k} = N+k$ for any $k \geq 0$ and $\gamma_{N+k} = 0$ for any $k \geq 1$. On the other hand, $\langle 1 | f \rangle = (-1)^N \prod_{j=0}^{N-1} q_j \neq 0$ implying that $\gamma_N \neq 0$. Altogether this shows that $u \in \mathcal{U}_N$.

Having established identity (7.3) we know that $\Pi(\mathcal{U}_N)$ is a connected, complex manifold of dimension N, parametrized by the open subset of \mathbb{C}^N described by the coefficients of the polynomials $Q \in \mathbb{C}_N^+[z]$ satisfying Q(0) = 1. Furthermore, since for any $u, v \in L_{r,0}^2$,

$$\omega(u,v) = \langle u | \partial_x^{-1} v \rangle = -2 \text{Im} \langle D^{-1} \Pi u | \Pi v \rangle$$

 $\Pi(\mathcal{U}_N)$ is a Kähler manifold with Hermitian form $\langle D^{-1}\Pi u|\Pi v\rangle$ and hence \mathcal{U}_N is a real analytic symplectic submanifold of $L^2_{r,0}$ of real dimension 2N. We claim that, for every $1 \leq n \leq N$, γ_n does not identically vanish on \mathcal{U}_N . Indeed, denote by J the set of such indices n. By the definition of \mathcal{U}_N , $\gamma_N(u) \neq 0$ for any $u \in \mathcal{U}_N$ and hence $N \in J$. Since each γ_n is a real analytic functional and since \mathcal{U}_N is connected, $G_J \cap \mathcal{U}_N$ is an open dense subset of \mathcal{U}_N . From the proof of Proposition 4.2 we

know that the coefficients M_{np} of the matrix M_N can be expressed in terms of the Birkhoff coordinates $(\zeta_j)_{1 \leq j \leq N}$. Similarly, this is the case for the vectors X_N and Y_N , introduced above. Hence the right hand side of the identity (7.5) locally extends analytically, yielding a real analytic map Ψ_N on $(\mathbb{C}^*)^J$ so that for any u in the open subset $G_J \cap \mathcal{U}_N$ of \mathcal{U}_N ,

$$\Psi_N \circ \Phi_N(u) = \Pi(u)$$
.

This implies that the rank of the map Φ_N at any point in $G_J \cap \mathcal{U}_N$ has to be 2N. Since in view of the definition of J it cannot be bigger than $2\sharp J$, we infer that $J = \{1, 2, \ldots, N\}$. As a consequence, $\mathcal{O}_N = G_{\{1,\ldots,N\}} \cap \mathcal{U}_N$ is an open dense subset of \mathcal{U}_N and a symplectic submanifold of \mathcal{U}_N of dimension 2N. On \mathcal{O}_N , the smooth functions $\gamma_1, \ldots, \gamma_N, \varphi_1, \ldots, \varphi_N$, referred to as actions and angles, are well defined. From Corollary 6.1 and identity (6.9), we know that

(7.10)
$$\{\gamma_n, \gamma_p\} = 0, \{\gamma_n, \varphi_p\} = \delta_{pn}, 1 \le n, p \le N.$$

This implies that Φ_N is a local diffeomorphism on \mathcal{O}_N and therefore, since Φ_N is one to one, a diffeomorphism onto its range which is contained in $(\mathbb{C}^*)^N$. In order to prove that the range is $(\mathbb{C}^*)^N$, we observe that $(\mathbb{C}^*)^N$ is connected and Φ_N proper (cf. Proposition 4.1). Being a local diffeomorphism, Φ_N is open and hence $\Phi_N(\mathcal{O}_N) = (\mathbb{C}^*)^N$. To prove that Φ_N is symplectic on \mathcal{O}_N introduce the two-form

$$\tilde{\omega} = \omega - \sum_{n=1}^{N} d\gamma_n \wedge d\varphi_n \ .$$

In view of the commutation relations (7.10),

$$\partial_{\varphi_p} \, \, \, \, \, \, \tilde{\omega} = 0 \, \, , \quad \forall \, 1 \leq p \leq N \, \, .$$

Since moreover $\tilde{\omega}$ is a closed two form there exist smooth functions a_{np} so that

(7.11)
$$\tilde{\omega} = \sum_{1 \le n$$

On the other hand, the pullback of $\sum_{n=1}^{N} d\gamma_n \wedge d\varphi_n$ to the submanifold $\varphi_1 = \cdots = \varphi_N = 0$ vanishes and by the inverse formula (7.5), on this submanifold, $\Pi u(\overline{z}) = \overline{\Pi u(z)}$, implying that u is even, u(-x) = u(x). Therefore, on this submanifold, formula (7.1) leads to $\omega = 0$. Altogether we thus have shown that $\tilde{\omega} = 0$. By (7.11) we then infer that $\tilde{\omega}$ vanishes identically on \mathcal{O}_N , showing that Φ_N is symplectic on \mathcal{O}_N .

Note that

$$d\zeta_n = \frac{1}{2\sqrt{\gamma_n}} e^{i\varphi_n} d\gamma_n + \sqrt{\gamma_n} e^{i\varphi_n} i d\varphi_n$$

and thus $d\zeta_n \wedge d\overline{\zeta}_n = -id\gamma_n \wedge d\varphi_n$. When expressed in the coordinates ζ_n , the pull back of the symplectic form ω to \mathcal{O}_N is therefore

$$\omega = i \sum_{n=1}^{N} d\zeta_n \wedge d\overline{\zeta}_n .$$

Since \mathcal{O}_N is dense in \mathcal{U}_N , this identity holds on all of \mathcal{U}_N . Using again that Φ_N is one to one and proper, we conclude that Φ_N is a symplectic diffeormorphism from \mathcal{U}_N onto $\mathbb{C}^{N-1} \times \mathbb{C}^*$. By Remark 7.1 (ii), Φ_N is a real analytic map and hence the proof of Theorem 3 is complete. \square

As an important application of Theorem 3, we prove that the Birkhoff map Φ is onto. Recall that $u \in L^2_{r,0}$ is a finite gap potential if $J(u) = \{n \geq 1 \mid \gamma_n(u) > 0\}$ is finite. The set of finite gap potentials in $L^2_{r,0}$ is thus given by $\{0\} \cup \bigcup_{N>1} \mathcal{U}_N$.

Corollary 7.1. The map $\Phi: L^2_{r,0} \longrightarrow h^{1/2}_+, u \mapsto (\zeta_n)_{n\geq 1}$ is a homeomorphism and the set of finite gap potentials is dense in $L^2_{r,0}$. Furthermore, the identities

$$\{\zeta_n, \zeta_m\} = 0$$
, $\{\zeta_n, \overline{\zeta}_m\} = -i\delta_{nm}$, $n, m \ge 1$,

are valid on all of $L_{r,0}^2$.

Proof. We already know that Φ is continuous and proper (cf. Proposition 4.1) and one to one (cf. Proposition 4.2). To prove that Φ is onto, let $\zeta := (\zeta_n)_{n\geq 1}$ be any nonzero sequence in $h_+^{1/2} \setminus \bigcup_{N\geq 1} \Phi(\mathcal{U}_N)$ where \mathcal{U}_N is defined by (7.2). Then there exists an increasing sequence $(N_k)_{k\geq 1}$ with $N_k \to \infty$ such that $\zeta_{N_k} \neq 0$ for any $k \geq 1$. We approximate ζ by the truncated sequence

$$\zeta_n^{(k)} = \begin{cases} \zeta_n & \text{if } n \le N_k \\ 0 & \text{if } n > N_k \end{cases}.$$

By Theorem 3, there exists a unique element $u^{(k)} \in \mathcal{U}_{N_k}$ such that $\Phi(u^{(k)}) = \zeta^{(k)}$. Since Φ is proper and continuous, $u^{(k)}$ has a limit point u in $L_{r,0}^2$ which satisfies

$$\Phi(u) = \zeta .$$

Thus we have proved that Φ is onto. Since Φ is proper and continuous it then follows that Φ^{-1} is continuous as well and in turn $\bigcup_{N\geq 1}\mathcal{U}_N$ is dense in $L^2_{r,0}$. It remains to verify the claimed Poisson bracket relations. By Proposition 5.1, for any $n, m \geq 1$, the Poisson brackets $\{\zeta_n, \zeta_m\}$ and $\{\zeta_n, \overline{\zeta_m}\}$ are real analytic functionals on $L^2_{r,0}$. Since by Theorem 3 the claimed Poisson bracket relations hold on $\bigcup_{N\geq 1}\mathcal{U}_N$ and $\bigcup_{N\geq 1}\mathcal{U}_N$ is dense in $L^2_{r,0}$ it then follows that they hold on all of $L^2_{r,0}$.

8. Proof of Theorem 1 and Theorem 2

In this section, we make a detailed synopsis of the proof of Theorem 1. As an application, we derive formulas for the BO frequencies at the end of this section (cf. Proposition 8.1) and prove Theorem 2.

Proof of Theorem 1 By Corollary 7.1, Φ is a homeomorphism and the Poisson bracket identities (B2) hold. It remains to express the BO Hamiltonian $\mathcal{H}(u)$, defined for $u \in H_{r,0}^{1/2}$, in terms of the Birkhoff coordinates ζ_n , $n \geq 1$. For $u \in H_{r,0}^{1/2}$ one computes, using that $\Pi u = -\sum_{n=0}^{\infty} \lambda_n \langle 1 | f_n \rangle f_n$,

$$\mathcal{H}(u) = \frac{1}{2} \langle |D|u|u\rangle - \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{3} u^3 dx = \langle L_u(\Pi u)|\Pi u\rangle = \sum_{n=0}^{\infty} \lambda_n^3 |\langle 1|f_n\rangle|^2.$$

With the help of $\tilde{\mathcal{H}}_{\varepsilon}$, defined by (3.8) as

$$\tilde{\mathcal{H}}_{\varepsilon} = \sum_{n=0}^{\infty} \frac{|\langle 1|f_n \rangle|^2}{1 + \varepsilon \lambda_n} ,$$

one can express $\mathcal{H}(u)$ as follows,

(8.1)
$$\mathcal{H}(u) = -\frac{1}{6} \frac{d^3}{d\varepsilon^3} \Big|_{\varepsilon=0} \tilde{\mathcal{H}}_{\varepsilon} = -\frac{1}{6} \frac{d^3}{d\varepsilon^3} \Big|_{\varepsilon=0} \log \tilde{\mathcal{H}}_{\varepsilon}.$$

Using the formula for $\frac{d}{d\varepsilon} \log \tilde{\mathcal{H}}_{\varepsilon}$ derived in Section 3 (cf. (3.11)), one then obtains

$$\mathcal{H}(u) = \frac{1}{3}\lambda_0^3 + \frac{1}{3}\sum_{n=1}^{\infty} \gamma_n \left(\lambda_n^2 + (\lambda_{n-1} + 1)^2 + \lambda_n(\lambda_{n-1} + 1)\right)$$

or using that by (3.13), $\lambda_n = n - s_{n+1}$ with $s_{n+1} := \sum_{k=n+1}^{\infty} \gamma_k, n \ge 0$,

(8.2)
$$\mathcal{H}(u) = -\frac{1}{3}s_1^3 + \sum_{n=1}^{\infty} \gamma_n ((n-s_n)^2 + \gamma_n (n-s_n) + \frac{1}{3}\gamma_n^2).$$

This proves Theorem 1.

Actually, formula (8.2) of the BO Hamiltonian $\mathcal{H}(u)$ can be further simplified as follows.

Proposition 8.1. For any $u \in H_{r,0}^{1/2}$,

(8.3)
$$\mathcal{H}(u) = \sum_{n=1}^{\infty} n^2 \gamma_n - \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \gamma_k\right)^2.$$

As a consequence, the BO frequencies $\omega_n = \frac{\partial \mathcal{H}}{\partial \gamma_n}$, $n \geq 1$, read

(8.4)
$$\omega_n = n^2 - 2\sum_{k=1}^{\infty} \min(k, n)\gamma_k .$$

Thus for any $u_0 \in L^2_{r,0}$, the solution u of the BO equation (1.1) with $u(0) = u_0$, when expressed in Birkhoff coordinates, is given by

(8.5)
$$\zeta_n(u(t)) = \zeta_n(u_0) e^{i\omega_n(u_0)t}, \quad t \in \mathbb{R}.$$

Remark 8.1. (i) Note that the right hand side of (8.4) is well defined for any $u \in L^2_{r,0}$. Hence the BO frequencies continuously extend to $L^2_{r,0}$. (ii) Note that by the trace formula of Proposition 3.1 (ii), the formula (8.4) for ω_n can be written as

$$\omega_n = n^2 - ||u||^2 + 2 \sum_{k=n+1}^{\infty} (k-n)\gamma_k$$
.

The sum $2\sum_{k=n+1}^{\infty}(k-n)\gamma_k$ satisfies the asymptotics

$$2\sum_{k=n+1}^{\infty} (k-n)\gamma_k = o(1) \quad as \quad n \to \infty$$

and hence in particular, $\lim_{n\to\infty}(\omega_n-n^2)=-\|u\|^2$.

Proof. We view $\mathcal{H}(u)$ as a function of the actions γ_n , $n \geq 1$, given by formula (8.2) and write the claimed identity (8.3) as

(8.6)
$$\mathcal{H}(u) = \sum_{n=1}^{\infty} n^2 \gamma_n - \sum_{n=1}^{\infty} s_n^2.$$

Note that both sides are real analytic functionals on the space $\ell^{1,2}(\mathbb{N},\mathbb{R})$, defined as the Banach space of real sequences $(\xi_n)_{n\geq 1}$ satisfying

$$\sum_{n=1}^{\infty} n^2 |\xi_n| < +\infty .$$

By (8.2), the partial derivative $\frac{\partial \mathcal{H}}{\partial \gamma_r}$, $r \geq 1$, is given by

$$\frac{\partial \mathcal{H}}{\partial \gamma_r} = -s_1^2 + (r - s_r)^2 - \sum_{n < r} \gamma_n (2(n - s_n) + \gamma_n) ,$$

and hence for any $m \geq r$,

$$\frac{\partial^2 \mathcal{H}}{\partial \gamma_r \partial \gamma_m} = -2s_1 - 2(r - s_r) + 2\sum_{n \le r} \gamma_n = -2r ,$$

while the corresponding second partial derivatives of the right hand side of (8.6) all equal -2r as well. Therefore the difference of the left hand side and the right hand side of (8.6) is a linear function of the γ_r , which moreover vanishes at $\gamma = 0$. Since the first derivative of this difference with respect to γ_r at $\gamma = 0$ is clearly 0, this proves (8.6). Finally, by a straightforward computation one has $\omega_n = n^2 - 2\sum_{k=1}^n s_k$. Since

$$\sum_{k=1}^{n} s_k = n \sum_{j=n+1}^{\infty} \gamma_j + \sum_{k=1}^{n} \sum_{j=k}^{n} \gamma_j = \sum_{j=1}^{\infty} \min(j, n) \gamma_j$$

the claimed formula (8.4) follows. Finally, formula (8.5), expressing the BO solution u(t) with initial data u_0 in terms of Birkhoff coordinates, is an immediate consequence of the properties of the Birkhoff map, applied to the BO Hamiltonian \mathcal{H} , and the wellposedness result of [21].

As a further application of the properties of Birkhoff map we show that the isospectral set of any potential $u \in L^2_{r,0}$,

$$Iso(u) = \{ v \in L_{r,0}^2 : \mathcal{H}_{\lambda}(v) = \mathcal{H}_{\lambda}(u) \ \forall \lambda \in \mathbb{C} \setminus \{ \lambda_n(u) : n \ge 0 \} \} .$$

is an invariant torus in $L_{r,0}^2$ of the BO equation. To state this result in more detail, introduce for any sequence $(\zeta_n)_{n\geq 1}$ in $h_+^{1/2}$

$$\operatorname{Tor}((\zeta_n)_{n>1}) := \{(z_n)_{n>1} \in h_+^{1/2} : |z_n| = |\zeta_n| \ \forall n \ge 1\}.$$

It then follows from the construction of the Birkhoff map in a straightforward way that the following result holds:

Corollary 8.1. For any $u \in L^2_{r,0}$,

$$\Phi(\operatorname{Iso}(u)) = \operatorname{Tor}(\Phi(u)).$$

In particular, Iso(u), being homeomorphic to a countable product of circles, is compact and connected. Since Iso(u) is invariant by the flow of the BO equation, we say – by a slight abuse of terminology – that Iso(u) is an invariant torus. Note that $L_{r,0}^2$ is the union of such tori.

Finally we prove Theorem 2.

Proof of Theorem 2 The solutions with initial data in $L_{r,0}^2$ can be constructed using Theorem 1 and Proposition 8.1. Note that for any $u_0 \in L_{r,0}^2$, the solution $t \mapsto u(t)$ with $u(0) = u_0$ stays on $\operatorname{Iso}(u_0)$ and hence by Proposition 3.3 (cf. also Proposition 8.1), the orbit of the solution is relatively compact in $L_{r,0}^2$. In order to prove that $t \in \mathbb{R} \mapsto u(t) \in L_{r,0}^2$ is almost periodic, we appeal to Bochner's characterization of such functions (cf. e.g. [17]): a bounded continuous function $f: \mathbb{R} \to X$ with values in a Banach space X is almost periodic if and only if the set $\{f_{\tau}, \tau \in \mathbb{R}\}$ of functions defined by $f_{\tau}(t) := f(t+\tau)$ is relatively compact in the space $C_b(\mathbb{R}, X)$ of bounded continuous functions on \mathbb{R} with values in X. Since Φ is a homeomorphism, in the case at hand, it suffices to prove that for every sequence $(\tau_k)_{k\geq 1}$ of real numbers, the sequence $f_{\tau_k}(t) := \Phi(u_{\tau_k}(t)), k \geq 1$, in $C_b(\mathbb{R}, h_+^{1/2})$ admits a subsequence which converges uniformly in $C_b(\mathbb{R}, h_+^{1/2})$. Notice that

$$f_{\tau_k}(t) = \left(\zeta_n(u(0))e^{i\omega_n(t+\tau_k)}\right)_{n\geq 1}.$$

By Cantor's diagonal process and since the circle is compact, there exists a subsequence of $(\tau_k)_{k\geq 1}$, again denoted by $(\tau_k)_{k\geq 1}$, so that for any $n\geq 1$, $\lim_{k\to\infty} \mathrm{e}^{i\omega_n\tau_k}$ exists, implying that the sequence of functions f_{τ_k} converges uniformly in $\mathcal{C}_b(\mathbb{R}, h_+^{1/2})$.

APPENDIX A. ON THE LAX PAIR FOR THE BO EQUATION

The purpose of this appendix is to derive the Lax pair formulation of the Benjamin-Ono equation, described in Section 1. The corresponding computations for the Benjamin-Ono equation on the line can be found in [32, Appendix A]. In order to be comprehensive, we include its derivation.

Recall that for any $u \in L_r^2$, we introduced the (unbounded) operators L_u and B_u on L_+^2 ,

$$L_u = -i\partial_x - T_u$$
, $B_u := -i\partial_x^2 + 2T_{\partial_x(\Pi u)} - 2\partial_x T_u$

where $T_u: L^2_+ \to L^2_+$ denotes the Toeplitz operator given by $T_u f = \Pi(uf)$ and $\Pi: L^2 \to L^2_+$ the Szegő projector. We claim that for any smooth function u(t,x) on $\mathbb{R} \times \mathbb{T}$, and any $h \in L^2_+$

(A.1)
$$\left(\frac{d}{dt} L_u + [L_u, B_u]\right) h = -\Pi((\partial_t u + 2u\partial_x u - H\partial_x^2 u)h)$$

To verify this identity, we write

$$[L_u, B_u] = -i[\partial_x, B_u] - [T_u, B_u] = Q_0 + Q_1 + Q_2$$

where for any $0 \le k \le 2$, Q_k is an operator, homogenous of degree k in u. One computes $Q_0 = -[\partial_x, \partial_x^2] = 0$ and

$$Q_1 = -2i[\partial_x, T_{\partial_x(\Pi u)} - \partial_x T_u] + i[T_u, \partial_x^2],$$

$$Q_2 = -2[T_u, T_{\partial_x(\Pi u)} - \partial_x T_u].$$

Let us first discuss Q_2 . For any $h \in L^2_+$, one has

$$T_u(T_{\partial_x(\Pi u)}h) = \Pi(\partial_x(\Pi u)uh), \quad T_{\partial_x(\Pi u)}(T_uh) = \Pi(\partial_x(\Pi u)\Pi(uh)).$$

Since $\Pi(I - \Pi) = 0$ one then obtains

$$[T_u, T_{\partial_x(\Pi u)}]h = \Pi(\Pi(\partial_x u)(I - \Pi)(uh)) = \Pi(\partial_x u)(I - \Pi)(uh)).$$

On the other hand, $[T_u, \partial_x T_u]h = -\Pi(\partial_x u\Pi(uh))$. Combining the two results then leads to

$$Q_2 h = -2\Pi ((\partial_x u)uh) = -\Pi ((2u\partial_x u)h).$$

Concerning the operator Q_1 , note that

$$-2i[\partial_x,T_{\partial_x(\Pi u)}] = -2iT_{\partial_x^2(\Pi u)}\,,\quad 2i[\partial_x,\partial_x T_u] = 2i\partial_x T_{\partial_x u}\,,$$

and

$$i[T_u, \partial_x^2] = -iT_{\partial_x^2 u} - 2iT_{\partial_x u}\partial_x.$$

Combining the three results then leads to

$$Q_1 h = -2iT_{\partial_x^2(\Pi u)} h + iT_{\partial_x^2 u} h = -2i\Pi(\Pi(\partial_x^2 u)h) + i\Pi((\partial_x^2 u)h)$$

or $Q_1h = -\Pi(i(2\Pi - I)(\partial_x^2 u)h)$. Since $i(2\Pi - I)(\partial_x^2 u) = -H\partial_x^2 u$ we conclude that

$$Q_1 h = \Pi ((H\partial_x^2 u)h).$$

Finally, one has $\partial_t L_u h = -\Pi(\partial_t u h)$. Altogether, we thus have proved that

$$\left(\frac{d}{dt}L_u + [L_u, B_u]\right)(h) = -\Pi((\partial_t u + 2u\partial_x u - H\partial_x^2 u)h).$$

APPENDIX B. TRAVELING WAVES AND ONE GAP POTENTIALS

In this appendix we identify traveling wave solutions of the Benjamin Ono equation as one gap potentials, which we describe in more detail, obtaining a new proof of the result of Amick-Toland in [2]. As we already observed, we may restrict our analysis to solutions with average 0.

Proposition B.1. The nonzero traveling wave solutions of the Benjamin-Ono equation coincide with the solutions with initial data given by one gap potentials.

Proof. A traveling wave solution is of the form $u(t, x) = u_0(x + ct)$ for some $c \in \mathbb{R}$, or, with our notation, $u(t) = Q_{ct}u_0$. In view of identity (2.17), we have

$$\zeta_n(Q_\tau v) = e^{i\tau n} \zeta_n(v) , \ v \in L^2_{r,0} .$$

Comparing with the general expression (8.5) of the BO solution, we infer

$$e^{icnt}\zeta_n(u_0) = e^{i\omega_n(u_0)t}\zeta_n(u_0) .$$

Hence for any $n \geq 1$ with $\zeta_n(u_0) \neq 0$, one has $cn = \omega_n(u_0)$. In view of formula (8.4), the mapping $n \mapsto \omega_n/n$ is strictly increasing. Consequently, there cannot be more than one n with $\zeta_n(u_0) \neq 0$. If u_0 is not identically 0, this precisely means that u_0 is a one gap potential. \square

Let $N \geq 1$ be given and introduce $\mathcal{O}_{\{N\}} := G_{\{N\}} \cap L^2_{r,0}$. Without further reference, we use the notation introduced in the main body of the paper. Denote by $\Phi_{\{N\}}$ the restriction of the Birkhoff map Φ to $\mathcal{O}_{\{N\}}$. It is given by

$$\Phi_{\{N\}}: \mathcal{O}_{\{N\}} \to \mathbb{C}^*, \ u \mapsto \zeta_N(u) = \frac{1}{\sqrt{\kappa_N}} \langle 1 \mid f_N \rangle$$

where $\kappa_N = 1/(N + \gamma_N)$ and hence

(B.1)
$$\langle 1 | f_N \rangle = \frac{1}{\sqrt{N + \gamma_N}} \zeta_N .$$

By Theorem 3 in Section 7,

$$\Pi u(z) = -zQ'(z)/Q(z)$$

where by (7.7), $Q(z) = \det(Id - zM_{N-1})$ and M_{N-1} is the $N \times N$ matrix $(M_{np})_{0 \le n, p \le N-1}$ with coefficients M_{np} given by

$$M_{np} = \begin{cases} \delta_{p,n+1} & \text{if } \zeta_{n+1} = 0, \\ \sqrt{\mu_{n+1}} \gamma_{n+1} \frac{\langle f_p, 1 \rangle}{(\lambda_p - \lambda_n - 1)\langle f_{n+1} | 1 \rangle} & \text{if } \zeta_{n+1} \neq 0. \end{cases}$$

Since $u \in G_{\{N\}}$ and hence $\langle 1|f_p\rangle = 0$ for any $p \neq 0, N$ one computes

$$Q(z) = 1 + z^{N} \frac{\sqrt{\mu_{N}} \gamma_{N}}{N} \frac{\langle f_{0} | 1 \rangle}{\langle f_{N} | 1 \rangle}$$

where by (4.9)

$$\mu_N = 1 - \frac{\gamma_N}{\lambda_N - \lambda_0} = \frac{N}{N + \gamma_N}.$$

To compute $\langle 1|f_0\rangle$, use again that $\langle 1|f_k\rangle = 0$ for $k \neq 0, N$ to infer $1 = \langle 1|f_0\rangle f_0 + \langle 1|f_N\rangle f_N$. By the definition of f_0 one has $\langle 1|f_0\rangle > 0$ and hence

(B.2)
$$\langle 1|f_0\rangle = (1 - |\langle 1|f_N\rangle|^2)^{1/2} = \frac{\sqrt{N}}{\sqrt{N + \gamma_N}}.$$

Since $\zeta_N = \sqrt{\gamma_N} e^{i\varphi_N}$ and $\langle f_N | 1 \rangle = \overline{\langle 1 | f_N \rangle}$, one obtains, using (B.1) and (B.2),

$$Q(z) = 1 + z^N \frac{\sqrt{\mu_N} \gamma_N}{N} \frac{\sqrt{N}}{\zeta_N} = 1 + z^N \frac{1}{\sqrt{N + \gamma_N}} \zeta_N.$$

Combining the results obtained leads to the following formula

$$\Pi u(z) = -zQ'(z)/Q(z) = N \frac{wz^N}{1 - wz^N}, \qquad w := -\langle 1|f_N \rangle = -\frac{1}{\sqrt{N + \gamma_N}} \zeta_N$$

or by substituting e^{ix} for z,

$$u(x) = N \frac{we^{iNx}}{1 - we^{iNx}} + N \frac{\overline{w}e^{-iNx}}{1 - \overline{w}e^{-iNx}}.$$

Note that $0 < |w| = \frac{\sqrt{\gamma_N}}{\sqrt{N + \gamma_N}} < 1$, showing that the potential u(x + iy) is analytic in a strip $|y| < y_0$.

For any $u \in \mathcal{O}_{\{N\}}$, the eigenvalues of L_u are given by

$$\lambda_n = n - \gamma_N$$
, $\forall 0 \le n < N$, $\lambda_n = n$, $\forall n \ge N$,

and

$$\gamma_N = N \frac{|w|^2}{1 - |w|^2}, \qquad \gamma_n = 0, \quad \forall n \neq N.$$

In a straightforward way one verifies that the eigenfunctions f_n , $n \ge 0$, corresponding to the eigenvalues of λ_n , $n \ge 0$, are given by

$$f_n(x) = e^{inx} \frac{\sqrt{1 - |w|^2}}{1 - we^{iNx}}, \quad \forall 0 \le n < N ,$$

$$f_n(x) = e^{i(n-N)x} \left(\frac{(1 - |w|^2)e^{iNx}}{1 - we^{iNx}} - \overline{w} \right), \quad \forall n \ge N .$$

APPENDIX C. ON THE SPECTRUM OF L_u

By Proposition 2.1, for any $u \in L_r^2$, the spectrum of L_u is discrete and the eigenvalues $\lambda_n(u)$, $n \geq 0$, of L_u satisfy $\lambda_0(u) < \lambda_1(u) < \cdots$ with $\gamma_n(u) = \lambda_n(u) - \lambda_{n-1}(u) - 1 \geq 0$ for any $n \geq 1$. The purpose of this appendix is to provide a spectral interpretation of the real numbers $\gamma_n(u)$, $n \geq 1$, by considering the operator L_u on the real line \mathbb{R} . Denote by $L^2(\mathbb{R}) \equiv L^2(\mathbb{R}, \mathbb{C})$ the standard L^2 – space with corresponding inner product $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx$ and for any $f \in L^2(\mathbb{R})$ by $\mathcal{F}(f)$ its Fourier transform,

$$\mathcal{F}(f)(\eta) = \int_{-\infty}^{\infty} f(x)e^{-ix\eta}dx .$$

Furthermore, let $L^2_+(\mathbb{R})$ be the closed subspace of $L^2(\mathbb{R})$ given by

$$L_+^2(\mathbb{R}) = \{ f \in L^2(\mathbb{R}) : \mathcal{F}(f)(\eta) = 0 \quad \forall \eta < 0 \} ,$$

 $\Pi^{\mathbb{R}}: L^2(\mathbb{R}) \to L^2_+(\mathbb{R})$ the corresponding orthogonal projector, $H^1(\mathbb{R}) \equiv H^1(\mathbb{R}, \mathbb{C})$ the standard H^1 -Sobolev space and $H^1_+(\mathbb{R}) := H^1(\mathbb{R}) \cap L^2_+(\mathbb{R})$. For any $u \in L^2_r$, denote by $L^\mathbb{R}_u$ the operator on $L^2_+(\mathbb{R})$ with domain $H^1_+(\mathbb{R})$ given by

$$L_u^{\mathbb{R}} f = -i\partial_x f - \Pi^{\mathbb{R}}(uf) , \quad \forall f \in H^1_+(\mathbb{R}) .$$

To see that $L_u^{\mathbb{R}}$ is selfadjoint, we first need to establish the following

Lemma C.1. For any $\epsilon > 0$, $u \in L_r^2$, and $f \in H^1(\mathbb{R})$,

$$\int_{-\infty}^{\infty} |uf|^2 dx \le ||u||^2 \left(\epsilon ||f'||_{L^2(\mathbb{R})}^2 + \left(1 + \frac{1}{\epsilon} \right) ||f||_{L^2(\mathbb{R})}^2 \right) .$$

Proof. For any $u \in L^2_r$ and $f \in H^1(\mathbb{R})$ one has

$$\int_{-\infty}^{\infty} |u(x)f(x)|^2 dx = \int_{0}^{2\pi} |u(x)|^2 \sum_{n=-\infty}^{+\infty} |f(x+2n\pi)|^2 dx.$$

Let $g(t;x) := f(x+2\pi t)$. It then suffices to show that for any $0 \le x \le 2\pi$,

$$\sum_{n=-\infty}^{+\infty} |g(n;x)|^2 \le \epsilon ||f'||_{L^2(\mathbb{R})}^2 + \left(1 + \frac{1}{\epsilon}\right) ||f||_{L^2(\mathbb{R})}^2.$$

For any given $0 \le x \le 2\pi$, $n \in \mathbb{Z}$, and $t \in [n, n+1]$,

$$|g(t;x)|^2 - |g(n;x)|^2 = \int_{t}^{t} 2\operatorname{Re}(\overline{g(s;x)}\partial_s g(s;x)) ds$$

and hence

$$\left| |g(t;x)|^2 - |g(n;x)|^2 \right| \le \int_n^{n+1} \left(\varepsilon |\partial_s g(s;x)|^2 + \frac{1}{\varepsilon} |g(s;x)|^2 \right) ds.$$

Integrating in $t \in [n, n+1]$, one gets

$$|g(n;x)|^2 \le \int_{n}^{n+1} \left(\varepsilon |\partial_t g(t;x)|^2 + \left(1 + \frac{1}{\varepsilon}\right) |g(t;x)|^2 \right) dt$$

and then summing over $n \in \mathbb{Z}$ yields the claimed inequality.

Clearly, $-i\partial_x$ is a selfadjoint operator on $L^2_+(\mathbb{R})$ with domain $H^1_+(\mathbb{R})$. By the Kato-Rellich theorem (cf. [27]) one then infers from Lemma C.1 that $L^{\mathbb{R}}_u$ is selfadjoint as well.

Proposition C.1. For any $u \in L_r^2$, the spectrum $spec(L_u^{\mathbb{R}})$ of $L_u^{\mathbb{R}}$ is absolutely continuous and consists of a union of bands,

$$spec(L_u^{\mathbb{R}}) = \bigcup_{n=0}^{\infty} [\lambda_n(u), \lambda_n(u) + 1].$$

Hence, for any $n \geq 1$, $\gamma_n(u)$ is the length of the nth gap in the spectrum of $L_u^{\mathbb{R}}$.

To prove Proposition C.1, we first need to make some preliminary considerations. Introduce the following L^2 -space

$$L^{2}(\mathbb{T}\times[0,1]) \equiv L^{2}(\mathbb{T}\times[0,1],\mathbb{C};\frac{1}{4\pi^{2}}dx\,d\xi)$$

which is clearly isometric to $L^2([0,1],L^2(\mathbb{T});\frac{1}{2\pi}d\xi)$. Furthermore for any $f\in L^2(\mathbb{R})$, denote by $\psi[f]\in L^2(\mathbb{T}\times[0,1])$ the function

$$\psi[f](x,\xi) = \sum_{n \in \mathbb{Z}} e^{inx} \mathcal{F}(f)(\xi + n) .$$

Since

(C.1)
$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{4\pi^2} \int_{0}^{2\pi} \left(\int_{0}^{1} |\psi[f](x,\xi)|^2 d\xi \right) dx ,$$

(C.2)
$$f(x) = \frac{1}{2\pi} \int_{0}^{1} e^{ix\xi} \psi[f](x,\xi) d\xi,$$

it follows that the linear map $L^2(\mathbb{R}) \to L^2(\mathbb{T} \times [0,1])$, $f \mapsto \psi[f]$ is unitary. Restricting it to the Sobolev space $H^1(\mathbb{R}) \equiv H^1(\mathbb{R}, \mathbb{C})$ one sees that $H^1(\mathbb{R})$ is isometric to $L^2([0,1],H^1(\mathbb{T});\frac{1}{2\pi}d\xi)$. Finally, we claim that for any $0 \le \xi < 1$,

(C.3)
$$\psi[\Pi^{\mathbb{R}} f](\cdot, \xi) = \Pi(\psi[f](\cdot, \xi)).$$

Indeed, for any $0 \le \xi < 1$, $\psi[\Pi^{\mathbb{R}} f](x,\xi) = \sum_{n \in \mathbb{Z}} e^{inx} \mathcal{F}(\Pi^{\mathbb{R}} f)(\xi + n)$ is given by

$$\psi[\Pi^{\mathbb{R}} f](x,\xi) = \sum_{n=0}^{\infty} e^{inx} \mathcal{F}(f)(\xi+n) = \Pi(\psi[f](\cdot,\xi))(x) .$$

(Note that for $\xi = 1$, one has $\psi[\Pi^{\mathbb{R}} f](x,1) = \sum_{n=-1}^{\infty} e^{inx} \mathcal{F}(f)(1+n)$ which equals $\Pi(\psi[f](\cdot,1))(x) + e^{-ix} \mathcal{F}(f)(0)$.)

Lemma C.2. For any $f \in H^1_+(\mathbb{R})$ and $0 \le \xi < 1$,

$$\psi[L_u^{\mathbb{R}}f](\cdot,\xi) = (\xi + L_u)(\psi[f](\cdot,\xi)) .$$

Proof. Let $f \in H^1_+(\mathbb{R})$ and $0 \le \xi < 1$. In view of the definition of $\psi[f]$,

$$\psi[Df](x,\xi) = (\xi + D_x)\psi[f](x,\xi) .$$

Since u is 2π periodic, one has $\psi[uf](x,\xi) = u(x)\psi[f](x,\xi)$ and hence one infers from (C.3) that

$$\psi[\Pi^{\mathbb{R}}(uf)](\cdot,\xi) = \Pi(\psi[uf](\cdot,\xi)) = \Pi(u\psi[f](\cdot,\xi)) = T_u\psi[f](\cdot,\xi) .$$

Combining these computations one obtains the claimed formula. \Box

Proof of Proposition C.1. Let $u \in L_r^2$. For any $f \in L_+^2(\mathbb{R})$ introduce the sequence $(\psi_n^u[f](\xi))_{n\geq 0}$ where for any $0\leq \xi \leq 1$,

$$\psi_n^u[f](\xi) := \langle \psi[f](\cdot, \xi) | f_n(\cdot, u) \rangle$$
.

Then

$$L^{2}_{+}(\mathbb{R}) \to \ell^{2}\left(\mathbb{Z}_{n\geq 0}, L^{2}\left([0,1], \frac{1}{2\pi}d\xi\right)\right), f \mapsto (\psi_{n}^{u}[f])_{n\geq 0}$$

is a unitary operator, and, for every $f \in H^1_+(\mathbb{R})$, Lemma C.2 implies

(C.4)
$$\psi_n^u[L_u^{\mathbb{R}}f](\xi) = (\xi + \lambda_n(u))\psi_n^u[f](\xi) , \ 0 \le \xi < 1 .$$

Consequently, the spectral measure μ_f of a function $f \in H^1_+(\mathbb{R})$, acting on any given continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support, $\int_{\mathbb{R}} \varphi(\lambda) d\mu_f(\lambda) = \langle \varphi(L_u^{\mathbb{R}}) f | f \rangle$, can be computed as

$$\int_{-\infty}^{\infty} \varphi(\lambda) d\mu_f(\lambda) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_{0}^{1} \varphi(\xi + \lambda_n(u)) |\psi_n^u[f](\xi)|^2 d\xi$$
$$= \frac{1}{2\pi} \sum_{n=0}^{\infty} \int_{\lambda_n(u)}^{\lambda_n(u)+1} \varphi(\lambda) |g_n^f(\lambda)|^2 d\lambda ,$$

where for any $\lambda_n(u) \leq \lambda < \lambda_n(u) + 1$, $g_n^f(\lambda)$ is given by $\psi_n^u[f](\lambda - \lambda_n(u))$. This proves that the spectrum of $L_u^{\mathbb{R}}$ is absolutely continuous and is contained in the union

$$\bigcup_{n=0}^{\infty} [\lambda_n(u), \lambda_n(u) + 1] .$$

In order to prove that $spec(L_u^{\mathbb{R}})$ coincides with the latter union it is enough to show that any measurable subset E of [0,1] with the property that, for some n,

(C.5)
$$\mathbf{1}_{E}(\xi) \psi_{n}^{u}[f](\xi) = 0 , \quad \forall 0 \le \xi \le 1, f \in L^{2}(\mathbb{R}) ,$$

has Lebesgue measure 0. Note that (C.5) says that

$$\int_{0}^{2\pi} \int_{0}^{1} \mathbf{1}_{E}(\xi) \psi[f](x,\xi) \overline{f_{n}}(x,u) dx d\xi = 0 , \qquad \forall f \in L^{2}(\mathbb{R}) ,$$

or, since ψ is unitary,

$$\psi^{-1}[f_n \otimes \mathbf{1}_E] = 0 .$$

In view of (C.2), one then concludes that

$$f_n(x) \int_0^1 \mathbf{1}_E(\xi) e^{ix\xi} d\xi = 0 , \quad \forall x \in \mathbb{R} .$$

Since f_n is continuous and is not identically 0, the Fourier transform of $\mathbf{1}_E$ then vanishes on some nonempty open set of \mathbb{R} . Since the Fourier transform of $\mathbf{1}_E$ is an entire function, it then must vanish identically, implying that E has measure 0. This completes the proof of Proposition C.1.

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