Energy cascades for multisoliton solutions of the Calogero–Moser Derivative Nonlinear Schrödinger equation

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The Calogero–Moser derivative NLS equation

Sobolev–Hardy spaces on the line

$H^s_+(\mathbb{R}) := \{f \in H^s(\mathbb{R}) : \mathrm{supp}(\hat{f}) \subset [0, +\infty[\}$, $s \in \mathbb{R}$, $\Pi_+ : H^s(\mathbb{R}) \to H^s_+(\mathbb{R})$.

Setting $D := -i\partial_x$, the equation

$$(CMDNLS) \quad i\partial_t u + \partial_x^2 u + (D + |D|)(|u|^2)u = 0 \ , \ u(t,.) \in H^s_+(\mathbb{R}) \ ,$$

was introduced by Abanov, Bettelheim and Wiegmann in 2009 as a continuous limit in Calogero–Moser systems of classical particles.

Defocusing version previously derived by Pelinovski and Grimshaw (1995)

$$(CMDNLS) \quad i\partial_t u + \partial_x^2 u - (D + |D|)(|u|^2)u = 0 \ , \ u(t,.) \in H^s_+(\mathbb{R}) \ ,$$

from the intermediate long wave equation.

Invariances

- Gauge, translations and Dilatations $\mathrm{e}^{i\theta}\lambda^{1/2}u(\lambda^2(t-t_0),\lambda(x-x_0))$ (mass critical)
- Galilean transformations $\mathrm{e}^{i\eta x - it\eta^2}u(t, x-2\eta t)$
  Caution : $\eta \geq 0$ in order to preserve H^s_+
- Pseudoconformal transformations

$$t^{-1/2}\mathrm{e}^{ix^2/(4t)}u(-t^{-1}, x/t)$$

Caution : acts on L^2 but not on H^0_+

Outline

For (CMDNLS), one can prove

- Lax pair structure.
- Rational solitary waves.
- For every N, class of global in time rational N–soliton solutions.
- For $N \geq 2$, N-solitons display energy cascades : for every $s > 0$, $\|u(t)\|_{H^s} \simeq |t|^{2s}$ as $t \to \infty$.

  Compare with recent results on the usual DNLS equation :

$$i\partial_t u + \partial_x^2 u + i\partial_x(|u|^2 u) = 0$$

Bahouri–Leslie–Perelman, Harrop–Griffiths–Killip–Ntekoume–Vişan.

Local wellposedness

Proposition

For every $R > 0$, there exists $T(R) > 0$ such that, for every $u_0 \in H^2_+(\mathbb{R})$ such that $\|u_0\|_{H^2} \leq R$, the problem

$$i\partial_t u + \partial_x^2 u + (D + |D|)(|u|^2)u = 0 \ , \ u(0) = u_0 \ .$$

has a unique solution $u \in C([-T(R), T(R)], H^2_+(\mathbb{R}))$.

Proof. Kato's iteration scheme.

Can be extended to $s > 1/2$ (Moura–Pilod, 2010) by means of a normal form *à la Tao*.

The Lax pair

Recall the notation $T_b(f) := \Pi_+(bf)\ ,\ f \in H^0_+(\mathbb{R}), b \in L^\infty(\mathbb{R})\ .$
(Toeplitz operators)

Theorem (Lenzmann–PG)

If $u \in C(I, H^2_+(\mathbb{R}))$ is a solution of (CMDNLS) with s large enough, then

$$\frac{d}{dt} L_u = [B_u, L_u]\ ,$$

where L_u, B_u are the following operators on $H^0_+(\mathbb{R})$,

$$L_u := D - T_u T_{\overline{u}}\ ,\ B_u = T_u T_{\partial_x \overline{u}} - T_{\partial_x u} T_{\overline{u}} + i(T_u T_{\overline{u}})^2\ .$$

Conservation laws

Notice that $L_u = D - T_u T_{\overline{u}}$ is a selfadjoint operator on H^0_+.

Corollary (Lenzmann–PG)

- *The spectrum of $L_u = D - T_u T_{\overline{u}}$ is conserved by the (CMDNLS) dynamics.*
- *For every measurable function f on the spectrum of L_u, the quantity $\langle f(L_u)u, u\rangle$ is a conservation law of (CMDNLS).*

If $u_0 \in H^2_+(\mathbb{R})$ with $\|u_0\|^2_{L^2} < 2\pi$, then u_0 generates a global solution of (CMDNLS), globally bounded in H^2.

The last statement follows from the sharp inequality

$$\|T_{\overline{u}} f\|^2_{L^2} \leq (2\pi)^{-1}\|u\|^2_{L^2}\langle Df, f\rangle \ , \ u \in H^0_+(\mathbb{R}) \ , \ f \in H^{1/2}_+(\mathbb{R}) \ .$$

Explicit formula

Theorem (Lenzmann–PG)

If $u \in C(I_{\max}, H^2_+)$ is the solution with $u(0) = u_0$, then

$$\forall z \in \mathbb{C}_+ \ , \ u(t,z) = \frac{1}{2i\pi} I_+[(G + 2tL_{u_0} - z\mathrm{Id})^{-1} u_0] \ ,$$

where, on $H^0_+(\mathbb{R})$, $G := x^*$, $I_+(f) = \hat{f}(0^+)$ if $\hat{f}_{|]0,\delta[} \in H^1(]0,\delta[)$ for some $\delta > 0$.

Similar approach for the Benjamin–Ono equation (PG, 2022).

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Similar approach for the Benjamin–Ono equation (PG, 2022).

Despite this explicit formula, global existence is still an open problem for general $u_0 \in H^2_+$. Issue : prove that the L^2 norm of the right hand side is the L^2 norm of u_0 (work in progress).

Solitary waves

Theorem (Lenzmann–PG)

The equality $\langle L_u(u), u\rangle = 0$ for $\|u\|_{L^2}^2 = 2\pi$ is exactly satisfied by

$$R(x) = \mathrm{e}^{i\theta}\frac{\sqrt{2\mathrm{Im}p}}{x+p}\ ,\ p \in \mathbb{C}_+\ ,\ \theta \in \mathbb{R}.$$

The functions R are stationary solutions of (CMDNLS).

In addition, one can prove that

- Up to the symmetries, these solutions are the only solitary traveling waves of (CMDNLS).
- No finite time blow up solution with critical mass 2π.

N soliton potentials

Proposition

Given $u \in H^2_+(\mathbb{R})$ and $N \in \mathbb{N}_{\geq 1}$, the following properties are equivalent and are preserved by the (CMDNLS) dynamics.

1. *The sum $\mathcal{E}_{pp}(u)$ of the L_u eigenspaces is invariant by the semigroup $(S(\eta)^*)_{\eta \geq 0}$, is N–dimensional and contains u.*
2. *There exists a polynomial Q of degree N, with all its zeroes in $\mathbb{C}_-$, and a polynomial P of degree at most $N-1$, such that*

$$u(x) = \frac{P(x)}{Q(x)} \ , \ P(x)\overline{P}(x) = i(Q'(x)\overline{Q}(x) - \overline{Q}'(x)Q(x))$$

Key for the equivalence :

$$\theta(x) := \frac{\overline{Q}(x)}{Q(x)} \ , \ \mathcal{E}_{pp}(u) = K_\theta := (\theta H^0_+)^\perp = \frac{\mathbb{C}_{\leq N-1}[x]}{Q(x)} \ .$$

The case $N = 2$

Generic case (simple poles)

$$u(x) = \frac{a_1}{x - z_1} + \frac{a_2}{x - z_2} \ , \ y_j := -\mathrm{Im} z_j > 0 \ ,$$

$$a_1 = \pm i \left(\frac{2y_1}{1 \pm 2\frac{\sqrt{y_1 y_2}}{|z_1 - \bar{z}_2|}} \right)^{\frac{1}{2}} \frac{z_1 - \bar{z}_2}{|z_1 - \bar{z}_2|} \mathrm{e}^{i\alpha} \ , \ a_2 = \left(\frac{2y_2}{1 \pm 2\frac{\sqrt{y_1 y_2}}{|z_1 - \bar{z}_2|}} \right)^{\frac{1}{2}} \mathrm{e}^{i\alpha} \ ,$$

$$\lambda_0 = 0 \ , \ \lambda_1 = \frac{\mp(y_1 + y_2)}{\sqrt{y_1 y_2}|z_1 - \bar{z}_2|} \ .$$

The explicit formula and N–soliton dynamics

Theorem (Lenzmann–PG)

Every N-soliton potential u such that L_u has eigenvalues $\lambda_0 = 0, \lambda_1, \dots, \lambda_{N-1}$ can be recovered as

$$u(x) = \frac{I_+(u)}{2i\pi}\langle (M - x\mathrm{Id})^{-1}X, Y\rangle_{\mathbb{C}^N} \ , \ \mathrm{Im} x > 0 \ ,$$

where $X := (1, \dots, 1)^T$, $Y := (1, 0, \dots, 0)^T$ *and, for* $0 \le j, k \le N-1$,

$$M_{jk} = \frac{i}{\lambda_j - \lambda_k} \ , \ j \neq k \ , \ M_{jj} = \gamma_j - i\rho\delta_{j0} \ ,$$

$$\rho := \frac{|I_+(u)|^2}{8\pi^2} \ , \ \gamma_j := \mathrm{Re}\langle G\psi_j, \psi_j\rangle \ , \ L_u\psi_j = \lambda_j\psi_j \ , \ \langle u, \psi_j\rangle = \sqrt{2\pi} \ .$$

Furthermore, the CMDNLS evolution reads as

$$\frac{d}{dt}I_+(u) = 0 \ , \ \frac{d}{dt}\gamma_j = 2\lambda_j \ , \ j = 0, \dots, N-1 \ .$$

Global existence of the N soliton dynamics

Proposition

Let $\lambda_0 = 0, \lambda_1, \dots, \lambda_{N-1}$, be pairwise distinct real numbers, ρ be a strictly positive number, and $\gamma_0, \gamma_1, \dots, \gamma_{N-1}$ be real numbers. We consider the matrix M defined by

$$M_{jk} = \frac{i}{\lambda_j - \lambda_k} \ , \ 0 \le j \ne k \le N-1 \ , \ M_{jj} = \gamma_j - i\rho\delta_{j0} \ , \ j = 0, \dots, N-1 \ .$$

Then the eigenvalues of M have strictly negative imaginary part, and they stay away from the real line as $(\gamma_0, \gamma_1, \dots, \gamma_{N-1})$ vary in a bounded subset of $\mathbb{R}^N$.

Corollary (PG–Lenzmann)

N-soliton solutions of (CMDNLS) are defined for all time.

Sketch of the proof

$$\frac{(M-M^*)_{jk}}{2i} = -\rho\delta_{0j}\delta_{0k} \ , \ [M, \mathrm{diag}(\lambda_0, \ldots, \lambda_{N-1})] = i\mathrm{Id} - i\langle ., \mathbf{1}\rangle_{\mathbb{C}^N}\mathbf{1} \ .$$

Assume $Mv = \mu v \ , \ \mu \in \mathbb{R}$. From the first identity,

$$0 = \frac{1}{2i}\langle (M-M^*)v, v\rangle_{\mathbb{C}^N} = -\rho|v_0|^2$$

Project the equation $Mv = \mu v$ on the mode 0 $\Rightarrow \displaystyle\sum_{k=1}^{N-1} \frac{v_k}{\lambda_k} = 0$ or $\langle w, \mathbf{1}\rangle = 0 \ , \ w := \left(0, \frac{v_1}{\lambda_1}, \ldots, \frac{v_{N-1}}{\lambda_{N-1}}\right)$. Apply the second identity,

$$\mathrm{diag}(\lambda_0, \ldots, \lambda_{N-1})(\mu I - M)w = i(w - \langle w, \mathbf{1}\rangle\mathbf{1}) = iw \ .$$

Inner product of both sides with w leads to $0 = i|w|^2$ hence $v = 0$.

Energy cascade for $N = 2$

From the explicit formula in the special case $N = 2$, we get

$$u(t,x) = C\,\frac{\gamma_1 + 2\lambda_1 t + i\lambda_1^{-1} - x}{x^2 - (\gamma_0 - i\rho + \gamma_1 + 2\lambda_1 t)x + (\gamma_0 - i\rho)(\gamma_1 + 2\lambda_1 t) - \lambda_1^{-2}}\ .$$

As $t \to \infty$, one of the two poles has a finite limit in $\mathbb{C}_-$, while the other one goes to infinity with

$$\mathrm{Im}(z(t)) \sim -\frac{\rho}{\lambda_1^4 t^2}\ .$$

This yields, for every $s > 0$,

$$\|u(t)\|_{H^s} \simeq |t|^{2s}\ ,\ t \to \infty.$$

Can be extended to every $N \geq 2$.

Perspectives and connected works

- Long time behaviour for general solutions ?
  Finite time blow up ? Weak soliton resolution? Link to the inverse scattering theory for the pair $(A = L_u, B = D)$? Work in progress.
- Similar problem on the torus (Calogero–Sutherland DNLS).
  Rana Badreddine : explicit formula. The CSDNLS flow map continuously extends to the open ball of $H^0_+(\mathbb{T})$ with subcritical mass, with relatively compact trajectories. The multi–soliton theory is more complicated.
- Defocusing case : global wellposedness both on the line and on the torus. On the line, scattering theory is expected.