Explicit Solution of the Calogero Model in the Classical Case and Geodesic Flows on Symmetric Spaces of Zero Curvature.

M. A. OLSHANETSKY and A. M. PERELOMOV
Institute for Theoretical and Experimental Physics - Moscow, U.S.S.R.

(ricevuto il 2 Aprile 1976)

The Calogero model describes the one-dimensional motion of n particles, with co-ordinates $\{q_k\}$ and momenta $\{p_k\}$ characterized by the Hamiltonian

$$H = \tfrac{1}{2}\sum_{k=1}^{n} p_k^2 + U(q)\,, \tag{1}$$

where the potential

$$U(q) = g^2 \sum_{k<l} (q_k - q_l)^{-2} + \frac{\omega^2}{2}\sum_{k=1}^{n} q_k^2\,. \tag{2}$$

This model was investigated in the quantum case by CALOGERO (1), and in the classical case ($n = 3$) by MARCHIORO (2). MOSER (3) by using the technique of isospectral deformation proved the complete integrability in the classical case with $\omega = 0$. In the papers (4,5) the Moser method was generalized to the case $\omega \neq 0$. Actually in the last paper more general potentials than (2) were considered. Namely, if $R = \{\alpha\}$ is the root system (6) in the space $\mathscr{H}$, R_+ is the subset of positive roots, $q_\alpha = (q, \alpha)$ is the scalar product of the co-ordinate vector q and the root α, and g_α^2, ω_α^2 are constants depending only on the length of the root α, then the Hamiltonian system (1) with the potential (7)

$$U(q) = \sum_{\alpha \in R_+} (g_\alpha^2 q_\alpha^{-2} + \omega_\alpha^2 q_\alpha^2) \tag{3}$$

(1) F. CALOGERO: *Journ. Math. Phys.*, **12**, 419 (1971).
(2) C. MARCHIORO: *Journ. Math. Phys.*, **11**, 2193 (1970).
(3) J. MOSER: *Adv. in Math.*, **16**, 197 (1975).
(4) M. ADLER: *A new integrable system and a conjecture by Calogero*, preprnt (1975).
(5) A. PERELOMOV: preprint ITEP-27 (1976).
(6) For the main results about the root systems, Lie algebras and the symmetric spaces see, *e.g.*, S. HELGASON: *Differential Geometry and Symmetric Spaces* (New York, N. Y., 1962).
(7) If one assumes the centre-of-mass at the origin: $\sum_{k=1}^{n} q_k = 0$ then $U(q) = \sum_{\alpha \in R_+} g_\alpha^2 q_\alpha^{-2} + (\omega^2/2)\sum_{k=1}^{n} q_k^2$.

is also completely integrable for the classical root systems. This class of potentials, connected with the root systems of semi-simple Lie algebras, was introduced in [8] and [9]. In particular the root system of type $A_{n-1}(q_\alpha = q_k - q_l)$ corresponds to the potential (2).

In the present paper the explicit solution for the system with potential (3) is constructed. If $\omega = 0$ then this solution was obtained from the motion along straight lines (the geodesics) in the symmetric space with zero curvature. If $\omega \neq 0$ then we have the motion along a closed curve in this space. In other words we reduce our non-linear problem to a linear one. In particular we prove the Marchioro hypothesis [2] about the lack of phase shift in the scattering process in the case $\omega = 0$ [10]. Note that in the limit $g_\alpha \to 0$ we thus have the solution of the billiard problem in the Weyl chamber with the potential $\sum\limits_{\alpha \in R_+} \omega_\alpha^2 q_\alpha^2$.

Let us first at all provide some information about symmetric spaces with zero curvature.

Let $\mathscr{G}$ be a real semi-simple Lie algebra, $\mathscr{K}$ a maximal compact subalgebra in it and $\mathscr{L}$ the orthogonal complement to $\mathscr{K}$ in $\mathscr{G}$ in the sense of the Cartan scalar product

$$\mathscr{G} = \mathscr{K} + \mathscr{L} . \tag{4}$$

If K is the Lie group corresponding to $\mathscr{K}$, then $\mathscr{L}$ is the invariant space under the adjoint representation

$$\mathrm{Ad}\,(u)\,\mathscr{L} \subset \mathscr{L} .$$

The affine transformations F of $\mathscr{L}$ have the form

$$x \to \mathrm{Ad}\,(u)x + a , \qquad u \in K , \quad a, x \in \mathscr{L} . \tag{5}$$

The homogeneous space $\mathscr{L} = F/K$ is the symmetric space with zero curvature.

Let $\mathscr{H}$ be the maximal commutative subalgebra in $\mathscr{L}$ (Cartan subalgebra) and A the positive Weyl chamber in $\mathscr{H}$ with regard to some ordering. Then almost every point $x \in \mathscr{L}$ can be decomposed as follows:

$$x = \mathrm{Ad}\,(u)\,h , \tag{6}$$

where $h \in A$ is the «radial part» of x and u is the «angular part» of x.

In particular if the potential $U(q)$ has the form (2) (the root system of type A_{n-1}), then $\mathscr{G}$ is the algebra of $n \times n$ matrices with zero trace, $\mathscr{K}$ are skew-Hermitian matrices, $\mathscr{L}$ are Hermitian matrices (see (4)), K are unitary matrices, $\mathscr{H}$ are real diagonal matrices and A is the subset in $\mathscr{H}$ characterized by the condition $(q_1 > q_2 > \ldots > q_n)$.

Thus the metric tensor is a constant on $\mathscr{L}$ then the equation for the geodesic can be written as follows:

$$\ddot{x}(t) = 0 . \tag{7}$$

([8]) M. OLSHANETSKY and A. PERELOMOV: *Lett. Math. Phys.* in press.
([9]) M. OLSHANETSKY and A. PERELOMOV: *Invent. Math.* in press.
([10]) This fact was proved recently for potential (2). P. KULISH: preprint IHEP OTP 75-123 (1975).

The solution of this equation has the form

$$x(t) = at + b\,, \qquad a, b \in \mathscr{L}\,. \tag{8}$$

Let us now consider the dynamical systems, connected with the geodesic flow (8). Let $q(t) \in A$ be the radial part of $x(t)$

$$x(t) = \operatorname{Ad}(u)q(t) \tag{9}$$

and

$$p(t) = \dot{q}(t)\,. \tag{10}$$

We shall say, that a dynamical system admits the Lax representation if it is equivalent to the operator equation

$$\dot{L} = [L, M]\,, \tag{11}$$

where L is an Hermitian operator and M is a Skew-Hermitian operator.

Let us now prove the following

Proposition 1. The geodesic flow (8) admits the Lax representation (11) with the L, M operators related to each other as follows:

$$L = [M, q] + p\,, \tag{12}$$

where p and q are defined in (9) and (10) and $L \in \mathscr{L}$, $M \in \mathscr{K}$.

Proof. Differentiate (9) with respect to t

$$\dot{x}(t) = \operatorname{Ad}(u)[u^{-1}\dot{u}q - qu^{-1}\dot{u} + p] \tag{13}$$

and set

$$M = u^{-1}\dot{u}\,. \tag{14}$$

Then note that $M \in \mathscr{K}$ and $L = [M, q] + p \in \mathscr{L}$ (see (12)). The basis in $\mathscr{G}$ can be chosen so that $\mathscr{L}$ be the subset of Hermitian matrices, $\mathscr{K}$ be the subset of skew-Hermitian matrices, and p and q the diagonal matrices.

Equation (13) may be rewritten in the form

$$\dot{x}(t) = \operatorname{Ad}(u)\big([M, q] + p\big)\,, \tag{15}$$

or, by means of (8) and (12),

$$a = \operatorname{Ad}\big(u(t)\big)L(t)\,. \tag{16}$$

If we differentiate this equation with respect to t we get

$$0 = \operatorname{Ad}\big(u(t)\big)\big([M, L] + \dot{L}\big)\,.$$

Equation (11) follows immediately from this relation.

Let us consider the Hamiltonian system with the potential (3) with $\omega = 0$ [11].

Proposition 2. The Hamiltonian systems of type I ($\omega = 0$) are equivalent to certain geodesic flow given by formula (8).

Proof. As it was proved in ref. [9] these systems admit a Lax representation with the operators L and M of the form

$$L = i\sum_{\alpha\in R_+} g_\alpha q_\alpha^{-1}(E_\alpha - E_{-\alpha}) + p\,, \tag{17}$$

$$M = i\sum_{\alpha\in R_+} g_\alpha q_\alpha^{-2}(E_\alpha + E_{-\alpha}) + d\,. \tag{18}$$

Here E_α are properly normalized matrices, $d = d(q)$ belongs to $\mathscr{K}$ and

$$[d, h] = 0 \qquad \text{for any } h\in\mathscr{H}\,. \tag{19}$$

From (18) there follows that

$$[M, q] = [d, q] + i \sum_{\alpha\in R_+} g_\alpha q_\alpha^{-2}\{[E_\alpha, q] + [E_{-\alpha}, q]\}\,.$$

Since $[d, q] = 0$ (see eq. (19)) and $[q, E_\alpha] = q_\alpha E_\alpha$, comparison of this formula with eq. (17) yields $-[q, M] + p = L$. Hence proposition 2 follows from proposition 1.

Note that we do not get the most general geodesic flow on $\mathscr{L}$, but only that for which the L-M pair has the special form given by eqs. (17) and (18).

Let us now show how by means of the equivalence proved in proposition 2 one can find the explicit solutions of Hamilton's equations with the potential (3). Consider specifically the Cauchy problem. Assume that $q(0) = q^0$ and $p(0) = p^0$ are given. Then from eqs. (8) (9) and (16) we get

$$x(0) = b = \mathrm{Ad}\,(u(0))q^0\,,$$
$$\dot{x}(0) = a = \mathrm{Ad}\,(u(0))L(0)\,,$$

where $L(0)$ is constructed according to formula (17) and we may take $u(0)$ as unit of the group K. According to eq. (8) at time t we have $x(t) = at + b$. Then we need only find the «radial» $q(t)$ and «angular» $u(t)$ components of the point $x(t)$ (see eq. (6)). This problem is just the standard diagonalization problem for Hermitian matrices. Once the matrix $u(t)$ is known one can also find $L(t) = \mathrm{Ad}\,(u^{-1}(t))a$ (see (16)) and $p(t)$ (17). Thus the solution of the Cauchy problem is given by the formulae [12]

$$q(t) = \mathrm{Ad}\,(u^{-1}(t))[q^0 + L(0)\,t]\,, \tag{20}$$

$$p(t) = -[M(t), q(t)] + L(t)\,, \tag{21}$$

$$L(t) = \mathrm{Ad}\,(u^{-1}(t))L(0)\,. \tag{22}$$

---

[11] In accordance with the notations introduced in ref. [9] the Hamiltonian systems with the potential (3) ($\omega = 0$) are called systems of type I. The systems with potential (3) ($\omega \neq 0$) shall be called systems of type V.

[12] From this it follows that the operator $q(t) - tL(t)$ undergoes an isospectral deformation. This fact was noticed by MOSER (see ref. [4]).

In other words the components $q_k(t)$ of $q(t)$ are eigenvalues of the matrix

$$q^0 + L(0)\,t = q^0 + \Big(p^0 + i \sum_{\alpha\in R_+} g_\alpha (q^0_\alpha)^{-1}(E_\alpha - E_{-\alpha})\Big)\,t\,.$$

Let us discuss now the scattering process in systems of type I ($\omega = 0$). The potential $U(q)$ given by formula (3) vanishes in the limit $q_\alpha \to \infty$, so that

$$q(t) = p^- t + \beta^- + o(1) \qquad \text{as } t \to -\infty\,, \tag{23}$$

$$q(t) = p^+ t + \beta^+ + o(1) \qquad \text{as } t \to +\infty\,. \tag{24}$$

Proposition 3. Let s be an element of the Weyl group [13] such as

$$sA = -A\,, \tag{25}$$

where A is the Weyl chamber [14]. Then [15]

$$p^+ = sp^-\,, \tag{26}$$

$$\beta^+ = s\beta^-\,. \tag{27}$$

Proof. The proof of formula (26) coincides in fact with the proof given in ref. [3], where the systems with the potential (2) were considered. Formula (27) for those systems was obtained in ref. [10] but the proof given here is essentially different.

In ref. [9] it is proved that the integrals of motion $I_k = Sp(L^k)$ in the $t \to \pm\infty$ limit become polinomials $P_k(p^\pm)$ on $\mathscr{H}$ which are invariant under the transformations of the Weyl group. Hence from the relations $P_k(p^-) = P_k(p^+)$ there follows that $p^+ = sp^-$. Let us show that the element s has the form (25). Since the potential diverges to positive infinity on the boundary on the Weyl chamber ($q_\alpha = 0$), the trajectories remain always inside this chamber. Hence if $p^+ \in A$, also $-p^- \in A$. Thus we have

$$p^+ = sp^- \in A\,, \qquad -p^- \in A\,.$$

From this relation formula (25) follows. Let us now prove relation (27). We introduce the notation: $\hat{u}h = \mathrm{Ad}\,(u^{-1})\,h = u^{-1}hu$.

From (16) there follows that

$$a = \hat{u}^{-1}(\infty)\,L(\infty) = \hat{u}^{-1}(-\infty)\,L(-\infty)\,. \tag{28}$$

But $L(\pm\infty) = p^\pm \in \mathscr{H}$ (see eq. (17)). The unique transformation of K, that leaves $\mathscr{H}$ invariant, is the transformation of the group $W \subset K$. Thus from (16) and (28) there follows

$$\hat{u}(\infty)\,\hat{u}^{-1}(-\infty) = s\,. \tag{29}$$

---

[13] The Weyl group is the group acting on $\mathscr{H}$. This group is generated by the reflections in the hyperplanes orthogonal to the roots. For the systems with the potential (2) this group is isomorphic to the permutation group.

[14] Since the group W acts simply transitively on the set of chambers, the element s is unique.

[15] This formula is valid also for the systems of type II ($U = \sum_{\alpha\in R_+} g_\alpha^2 \sinh^{-2} q_\alpha$).

On the other hand from (8), (9), (12) and (16) we get

$$q(t) = \hat{u}(t)x(t) = pt + [M, q]t + \hat{u}(t)b \,. \tag{30}$$

From this there follows that $\beta^{\pm} = \hat{u}(\pm\infty)b$. Hence $\beta^{+} = \hat{u}(\infty)\hat{u}^{-1}(-\infty)\beta^{-} = s\beta$. The proposition is thus proved.

Let now $\omega \neq 0$ (system of type V) [7]. In this case we must consider a harmonic (instead of a uniform) motion in $\mathscr{L}$:

$$\ddot{x} + \omega^2 x = 0 \,. \tag{31}$$

The solution of this equation has the form

$$x = a\cos\omega t + b\sin\omega t \,, \qquad a, b \in \mathscr{L} \,. \tag{32}$$

Proposition 4. The dynamical system (31) admits the modified Lax representation $\dot{\tilde{L}} = [\tilde{L}, M] - i\omega\tilde{L}$. Here $\tilde{L} = L - i\omega q$ and the operators L and M are related to one other by relation (12).

Proposition 5. The Hamiltonian systems of type V are equivalent to a certain flow of type (32).

The proof of these propositions coincides with the proof of propositions 1 and 2 and is based on the results of ref. [4,5].

The proposition 5 gives the possibility to construct the solutions of Hamilton's equations for the systems of type V analogously to the case of the systems of type I. In this case the formulae analogous to (20)-(22) are

$$q(t) = \mathrm{Ad}\,(u^{-1}(t))[q^0\cos\omega t + \omega^{-1}L(0)\sin\omega t] \,, \tag{20'}$$

$$p(t) = -[M(t), q(t)] + L(t) \,, \tag{21'}$$

$$L(t) = \mathrm{Ad}\,(u^{-1}(t))[L(0)\cos\omega t - \omega q^0\sin\omega t] \,, \tag{22'}$$

i.e. the components $q_k(t)$ of $q(t)$ are the eigenvalues of the matrix

$$q^0\cos\omega t + \omega^{-1}p^0\sin\omega t + i\omega^{-1}\sin\omega t\cdot\sum_{\alpha\in R_+} g_\alpha(q^0_\alpha)^{-1}(E_\alpha - E_{-\alpha}) \,.$$

It follows from (9) that

$$Sp\,[q(t)]^k = Sp\,[x(t)]^k \,. \tag{33}$$

But $Sp\,[q(t)]^k$ is the polinomial on $\mathscr{H}$ which is invariant under transformations of the Weyl group. Hence we obtain from (8), (32) and (33):

Corollary 1 [16]. The invariant polinomial in q_k on $\mathscr{H}$ of degree k is a polinomial of degree k in t $(\omega = 0)$ and $\sin\omega t$, $\cos\omega t$ $(\omega \neq 0)$.

---

[16] This result was obtained in ref. [4] for potential (2).

Remark I. It follows from the form of the Hamiltonian and from eq. (32), that the eigenvalues of the matrix

$$\frac{\omega^{-1}\boldsymbol{L}^2(t) + \omega q(t)^2}{2}$$

are the action variables and at nonvanishing ω, logarithms of the eigenvalues of the matrix

$$\frac{1}{\sqrt{2\omega}}\left(i\boldsymbol{L}(t) + \omega q(t)\right),$$

are the angle variables.