COMPLETELY INTEGRABLE CLASSICAL SYSTEMS CONNECTED WITH SEMISIMPLE LIE ALGEBRAS*

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ABSTRACT. The complete integrability of a class of dynamical systems, more general than those considered recently by Moser and Calogero, is proved. It is shown that these systems are connected with semisimple Lie algebras.

1. Let us consider the classical dynamical system with n degrees of freedom that is described by the Hamiltonian

$$H(p_k, q_l) = \frac{1}{2} \sum_{k=1}^{n} p_k^2 + U(q_1, \dots q_n). \tag{1}$$

In his recent paper Moser [2] has found n integrals of motion for such systems with the potential

$$U = g^2 \sum_{i<j} V(q_i - q_j) \tag{2}$$

where

$$V(q) = q^{-2} \tag{3}$$

(Calogero model [3]) and

$$V(q) = a^2 \sin^{-2} aq \tag{4}$$

(Sutherland model [4]). Using similar methods, Calogero in Papers [5] and [6] has found n integrals of motion for the cases

$$V(q) = a^2 \sinh^{-2} aq \tag{5}$$

and

$$V(q) = a^2 \wp(aq) \tag{6}$$

where $\wp(aq)$ is the Weierstrass function. In papers [2], [5] it was proved that these integrals are in involution if $V(q)$ is given by (3)-(5) and, hence, these systems are completely integrable.

* Some of the results of this paper were announced in [1].

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In the present paper we consider the potentials which are given by the expression

$$U = g^2 \sum_{k<l} [V(q_k - q_l) + \epsilon V(q_k + q_l)] + g_1^2 \sum_{k=1}^{n} V(q_k) + g_2^2 \sum_{k=1}^{n} V(2q_k). \tag{7}$$

$$\epsilon = \begin{cases} 0 \text{ or } 1 & \text{for } g_1 = g_2 = 0 \\ 1 & \text{for } g_1^2 + g_2^2 > 0. \end{cases}$$

In addition to the case of functions $V(q)$ given by (3)-(6) we investigate also the case when

$$V(q) = q^{-2} + c^2 q^2 \; * \tag{8}$$

All the above-mentioned results are extended to systems with potential (7) and functions $V(q)$ given by (3)-(6) and (8). The principal result is

THEOREM. The Hamiltonian systems with potentials (7), with restrictions on the constants given by

$$\begin{aligned} & g_1^2 - 2g^2 + \sqrt{2}\, g g_2 = 0 && (g_1 \neq 0) \\ & g, g_2 \quad \text{are arbitrary} && (g_1 = 0) \end{aligned} \tag{7'}$$

and with functions $V(q)$ given by (3)-(6) and (8), possesses n independent integrals of motion. With the exception of the case of $V(q)$ given by (6) such systems are completely integrable.

For the systems of type (8), besides the integrals of motion, we obtain also explicit expressions for the quantities B_k^* and B_k which are a classical analogue of the quantum operators of creation and annihilation B_k^+ and B_k, introduced in [7]. In this paper an algorithm was given for finding the operators B_k^+ and B_k, and an explicit form of the first three of them was found**.

2. It appears that the potentials (3-5) are in close connection with the root systems of the simple Lie algebras***. The root system R is a finite set of vectors $\{\alpha\}$ in Euclidean space E which generates E and satisfy the following conditions

(a) $\dfrac{2(\alpha, \beta)}{(\alpha, \alpha)} =$ integer for all $\alpha, \beta \in R$ $\quad$ ($(\alpha, \beta) =$ scalar product in E).

(b) The map

$$S_\alpha : q \to q - \frac{2(q, \alpha)}{(\alpha, \alpha)} \alpha \qquad (q \in E)$$

* It follows from a remark of Moser in [2] that such a potential was considered also by M. Adler.

** Note that as in [5], in the quantum case $(p_k \to i \frac{\partial}{\partial q_k})$, the operators B_k^+ and B_k are well-defined. Thus we obtain explicit expressions for them.

*** The basic facts about Lie algebras may be found in [8] and [9].

leaves R invariant.

In particular, the following sets of vectors form the root systems ($n = \dim E$, $\{e_i\}$ = Basis in E):

$$
\begin{aligned}
A_{n-1} &= \{\pm(e_i - e_j)\}\ i \neq j \\
B_n &= \{\pm e_i, \pm e_i \pm e_j\}\ i \neq j \\
C_n &= \{\pm 2e_i, \pm e_i \pm e_j\}\ i \neq j \\
D_n &= \{\pm e_i \pm e_j\}\ i \neq j \\
BC_n &= \{\pm 2e_i, \pm e_i, \pm e_i \pm e_j\}\ i \neq j.
\end{aligned}
\tag{9}
$$

These systems are called the root systems of classical type. In addition to these infinite series there are five special root systems. They are not considered here. We can select such a hyperplane in E which divides R into two subsets $R = R_+ \cup R_-$. The subset R_+ is called the subset of positive roots. If we define the constants g_α^2 depending only on the length of the root α, and variables $q_\alpha = (q, \alpha)$ where (q, α) is the scalar product in E, then the potential (7) may be given in a more abstract form by:

$$
U = \sum_{\alpha \in R+} g_\alpha^2\, V(q_\alpha). \tag{10}
$$

The correspondence between the potential (7) and the classical root systems (9) is the following:

$$
\begin{aligned}
A_{n-1} &\Leftrightarrow (\epsilon = g_1 = g_2 = 0) \\
B_n &\Leftrightarrow (\epsilon = 1, g_2 = 0) \\
C_n &\Leftrightarrow (\epsilon = 1, g_1 = 0) \\
D_n &\Leftrightarrow (\epsilon = 1, g_1 = g_2 = 0) \\
BC_n &\Leftrightarrow (\epsilon = 1).
\end{aligned}
$$

3. Let us point out the basic steps of the proof and by the way give an explicit expression for the integrals of motion.

Similarly to [2] we shall use the Lax trick [10]. Namely, let us construct a pair of matrices L and M depending on p and q so that Hamilton's equations:

$$
\dot{q}_k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k} \tag{11}
$$

are equivalent to the matrix equation:

$$
\dot{L} = [L, M]. \tag{12}
$$

Let us require that L is an Hermitian matrix and M is a skew-Hermitian one. It follows from (12) that the eigenvalues of L do not depend on time. Hence we obtain n global integrals of motion. In particular the quantities

$$I_k = \mathrm{Sp}\,(L^k) \tag{13}$$

are integrals of motion.

Let us take these matrices as the operators of a finite-dimensional irreducible representation φ of the simple Lie algebra $\mathfrak{G}$ with the corresponding system of roots.

It is clear that the most general form of the potentials is determined by the system BC_n. So we shall construct operators L and M for this system. Let us take the Lie algebra $\mathfrak{G}$ of the matrix group SU $(n+1, n)$, which leaves invariant the form $\sum_{k=1}^{n} (|z_k|^2 - |z_{-k}|^2) + |z_0|^2$, and consider the representation φ of $\mathfrak{G}$ into the algebra of $(2n+1)\times(2n+1)$ matrices with zero trace. By choosing a corresponding basis in the representation space we construct the operators

$$L = \begin{pmatrix} A_1 & -C_1' & B_1 \\ \bar{C}_1 & O & -\bar{C}_1 \\ -B_1 & -C_1' & -A_1 \end{pmatrix} \tag{14}$$

$$M = \begin{pmatrix} A_2 & C_2' & B_2 \\ -\bar{C}_2 & d_0 & -\bar{C}_2 \\ B_2 & C_2' & A_2 \end{pmatrix} \tag{15}$$

The matrix elements of these operators are expressed through the functions x, y and z depending on q and through the momentum p in the following form

$$\begin{aligned}
&\{A_1\}_{kl} = \delta_{kl} p_k + (1-\delta_{kl})\, igx\,(q_k - q_l) \\
&\{C_1\}_k = ig_1 x(q_k) \\
&\{B_1\}_{kl} = i\sqrt{2}\,\delta_{kl} g_2 x(2q_k) + i(1-\delta_{kl}) gx(q_k + q_l) \\
&\{A_2\}_{kl} = i\,\delta_{kl} d_k + i(1-\delta_{kl}) gy\,(q_k - q_l) \\
&\{B_2\}_{kl} = i\sqrt{2}\,\delta_{kl} g_2 y(2q_k) + i(1-\delta_{kl}) gy\,(q_k + q_l) \\
&\{C_2\}_k = ig_1 y(q_k) \\
&d_k = e_k - c, \qquad d_0 = i(e_0 - c), \qquad i = \sqrt{-1} \\
&e_k = g \sum_{r\neq k} [z(q_k - q_r) + z(q_k + q_r)] + g_1^2 g^{-1} z(q_k) + \sqrt{2} g_2 z(2q_k) \quad (k = 1, \dots n) \\
&e_0 = 2g \sum_{k=1}^{n} z(q_k), \qquad c = \frac{2}{2n+1}\left(\sum_{k=1}^{n} e_k + \frac{1}{2} e_0\right).
\end{aligned} \tag{16}$$

If $x(\xi)$, $y(\xi)$ and $z(\xi)$ are connected by the relations*

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* The Equations (17) and (18) were obtained by Calogero [6] when he studied the systems with the potential (2).

$$y(\eta)x(\xi) - y(\xi)x(\eta) = x(\xi + \eta)\,[z(\eta) - z(\xi)] \tag{17}$$

$$y(\xi) = x'(\xi)$$

and the function $V(\xi)$, defining the potential, is given by

$$V'(\xi) = 2x(\xi)y(\xi) \tag{18}$$

then under the fulfilment of conditions (7′) the systems (11) and (12) are equivalent. This can be proved by direct computation.

Under the condition $V(\xi) \sim \xi^{-2}$ $(\xi \to 0)$ we obtain the following special solutions of the Equation (17)

$$x(\xi) = \xi^{-1}, \qquad a\,\mathrm{cotg}\,a\xi, \qquad a \coth a\xi, \qquad a\frac{\mathrm{cn}(a\xi)}{\mathrm{sn}(a\xi)} \tag{19}$$

connected with $V(\xi)$ given in (3)-(6) according to (18). Any other solution $x(\xi)$ which is also known does not give new functions $V(\xi)$ besides (3)-(6). Therefore we do not get new potentials. If we know $x(\xi)$ we can construct the matrix L (14) and compute the integrals of motion I_k (13).

Excluding the case $V(\xi) = a^2\wp(a\xi)$ * one can show as in [2] and [5] that the integrals I_k are in involution. Thus the systems with the potentials (7) and with the functions $V(\xi)$ (3)-(5) are completely integrable.

In the papers [5], [6] the quantum systems with the potential (2) and $V(q($ given by (3)-(6) were considered. The results of these papers can be extended to the systems considered here.

4. Let us now consider the system with the function $V(q)$ given by (8).

Note first of all that under the fulfilment of the condition $\sum_{j=1}^{n} q_j = 0$ the system of interest described by the Hamiltonian (1) with the potential (10) and $V(q)$ given by (8) is equivalent to the system with the potential

$$U = U_0 + \frac{\omega^2}{2}\sum_{k=1}^{n} q_k^2, \tag{20}$$

where U_0 is given by the formula (10) with $V(q)$ given by (3).

Similarly to Section 3, in order to prove the complete integrability of the system under consideration, we show that Hamilton's Equations (11) are equivalent to the matrix equation

$$\dot{\tilde{L}} = [\tilde{L}, M] - i\omega\tilde{L} \tag{21}$$

where matrices M and $\tilde{L}$ depend on p_k and q_l and the matrix M is skew-Hermitian.

* Note that as in [6] in this case one can introduce four different types of particles by replacement of the variables.

In the case of interest we take

$$\tilde{L} = L - i\omega Q \tag{22}$$

where the matrix Q is obtained from the matrix P * replacing p_k by q_k, and correspondingly the matrix $\tilde{L}$ is obtained from the matrix L replacing p_k by $b_k = p_k - i\omega q_k$.

Then, as it can be easily checked, Equations (21) are the consequence of Equation (22) and of the identity

$$L = P + [M, Q] \tag{23}$$

To check this identity it is necessary to make use of the explicit expressions (14), (15) for the matrices L and M.

Let us now turn to obtaining consequences from Equation (21). Form the quantities

$$B_k = \mathrm{Sp}\,(\tilde{L})^k \quad \text{and} \quad B_k^* = \mathrm{Sp}\,(\tilde{L}^+)^k \tag{24}$$

Then as it follows from Equation (21):

$$\dot{B}_k = -ik\omega B_k, \qquad B_k(t) = \exp(-ik\omega t)\,B_k(0) \tag{25}$$

Thus the quantities B_k not being integrals of motion possess, however, very simple dependence on time. **

Let us now find the integrals of motion. It is easy to see that these are both $B_k^* B_k$ and $\{B_k, B_k^*\}$. Thus, the knowledge of the quantities B_k is sufficient to find the invariants. It will be more convenient for us to use another set of invariants. Let us consider matrices $N_1 = \tilde{L}^+ L$ and $N_2 = \tilde{L}\tilde{L}^+$. It follows from (21) that they satisfy the equation

$$\frac{dN_r}{dt} = [N_r, M], \qquad (r = 1,2). \tag{26}$$

Consequently, the matrix

$$N = \frac{1}{2}(\tilde{L}^+\tilde{L} + \tilde{L}\tilde{L}^+) \tag{27}$$

also satisfies the equation

$$\dot{N} = [N, M]. \tag{28}$$

* The matrix P is the diagonal part of the matrix L.

** The quantities B_k completely determine the evolution of the considered system. They are rational functions of variables p_k and q_l. Expressing the coordinates and the momenta through B_k and B_l^* we get an explicit expression for $p_k(t)$ and $q_l(t)$ and thereby completely integrate the systems of Hamilton's Equation (11).

Therefore, the eigenvalues of the matrix N are not changed with time and consequently the quantities

$$I_{2k} = \mathrm{Sp}\,(N)^k \tag{29}$$

are also constants.

Let us note in conclusion that, as it is easy to check, the quantities B_k are in involution, i.e. $\{B_k, B_l\} = 0$. Here $\{A, B\} = \sum_{k=1}^{n} \left(\frac{\partial A}{\partial p_k} \frac{\partial B}{\partial q_k} - \frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} \right)$ is the Poisson bracket for A and B.

Details of this work will be published in *Inventiones Mathematicae*.

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