

Multiphase solutions and their reductions for a nonlocal nonlinear Schrödinger equation with focusing nonlinearity

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Abstract

A nonlocal nonlinear Schrödinger equation with focusing nonlinearity is considered which has been derived as a continuum limit of the Calogero-Sutherland model in an integrable classical dynamical system. The equation is shown to stem from the compatibility conditions of a system of linear PDEs, assuring its complete integrability. We construct a nonsingular N -phase solution (N : positive integer) of the equation by means of a direct method. The features of the one- and two-phase solutions are investigated in comparison with the corresponding solutions of the defocusing version of the equation. We also provide an alternative representation of the N -phase solution in terms of solutions of a system of nonlinear algebraic equations. Furthermore, the eigenvalue problem associated with the N -phase solution is discussed briefly with some exact results. Subsequently, we demonstrate that the N -soliton solution can be obtained simply by taking the long-wave limit of the N -phase solution. The similar limiting procedure gives an alternative representation of the N -soliton solution as well as the exact results related to the corresponding eigenvalue problem.

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1. Introduction

1.1. Nonlocal nonlinear Schrödinger equation

The nonlocal nonlinear Schrödinger (NLS) equation that we consider here can be written in the form

$$iu_t = u_{xx} - iu(1 - iH)(|u|^2)_x, \quad (1.1a)$$

with

$$Hu(x, t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(y, t)}{y - x} dy, \quad (1.1b)$$

where $u = u(x, t)$ is a complex function, H is the Hilbert transform and the subscripts t and x appended to u denote partial differentiation. Equation (1.1) is characterized by the nonlocal nonlinearity of focusing type. It has been derived as a continuum limit of the Calogero-Sutherland model in an integrable classical dynamical system [1]. We recall that the defocusing version of equation (1.1)

$$iu_t = u_{xx} + iu(1 - iH)(|u|^2)_x, \quad (1.2)$$

has been introduced as a deep-water limit of the intermediate NLS equation of defocusing type which describes the long-term evolution of the envelope of quasi-harmonic internal waves in a stratified fluid of finite depth [2]. A number of results have been obtained for equation (1.2). Among them are multiperiodic and multisoliton solutions [3], linear stability of the multisoliton solutions [4], integrable dynamical systems associated with it [5], asymptotic analysis of periodic solutions in the limit of small dispersion [6], initial value problems [7, 8] and new representation of multiperiodic solutions [9].

On the other hand, only a few outcomes are available for equation (1.1). To be more specific, the one-soliton solution of equation (1.1) has been obtained by performing a limiting procedure to the one-soliton solution of the intermediate NLS equation of focusing type [10]. The construction of the general N -soliton solution (N : positive integer), however, remained open until very recently. In fact, a recent work deals with the Lax integrability and multisoliton solutions as well as their dynamics [11]. As in the case of the NLS equation [12-14], the feature of solutions depends crucially on the types of nonlinearity. Quite recently, the intermediate version of the focusing-defocusing Manakov system was proposed with some numerical solutions [15]. The focusing nonlocal NLS equation is now an interesting issue in search of integrable nonlocal nonlinear partial differential equations (PDEs).

1.2. Integrability

Integrability of given nonlinear PDE may be characterized by the existence of the Lax pair and infinite number of conservation laws. Actually, equation (1.1) was shown to

exhibit a Lax pair structure and associated conservation laws [11]. One can also derive equation (1.1) as the compatibility conditions of the following system of linear PDEs for the eigenfunctions ϕ and ψ^\pm

$$i\phi_x + \lambda\phi + u\psi^+ = 0, \quad (1.3)$$

$$\psi^+ - \sigma\psi^- - u^*\phi = 0, \quad (1.4)$$

$$i\phi_t - 2i\lambda\phi_x + \phi_{xx} - 2iu_x\psi^+ - \kappa\phi = 0, \quad (1.5)$$

$$i\psi_t^\pm - 2i\lambda\psi_x^\pm + \psi_{xx}^\pm - i[(\pm 1 - iH)(|u|^2)_x - i\kappa]\psi^\pm = 0, \quad (1.6)$$

where $\psi^+(\psi^-)$ is the boundary value of function analytic in the upper (lower)-half complex x plane $\text{Im } x > 0$ ($\text{Im } x < 0$) and they have the unique representations $\psi^\pm = \pm \frac{1}{2}(1 \mp iH)\psi$ with ψ being a complex function. Here, the asterisk appended to u denotes the complex conjugation, λ is the spectral parameter and σ and κ are constants related to λ which are fixed by the boundary conditions for ϕ and ψ^\pm . Notice that the corresponding linear system for equation (1.2) can be obtained from (1.3)-(1.6) if one replaces the minus sign of the third term on the left hand-side of (1.4) with the plus one. See, for example [7]. One can see that the linear system (1.3)-(1.6) yields a Lax pair equivalent to that obtained in [11].

1.3. Outline of the paper

The main purpose of the present paper is to construct the multiphase solutions of equation (1.1) and investigate their properties in comparison with those of equation (1.2). To this end, we employ the direct method (or sometimes called the bilinear transformation method) [16, 17] which has been used frequently in obtaining special solutions such as soliton and periodic solutions. The method of solution is elementary in the sense that it does not require the knowledge of the inverse scattering transform (IST) and is based only on an elementary theory of determinants.

The remaining part of this section addresses a summary of notations. In section 2, we first transform equation (1.1) to a set of bilinear equations through appropriate dependent variable transformations. The N -phase solution of the bilinear equations is obtained following the standard procedure of the direct method. It is represented by a quotient of two fundamental determinants (or tau-functions) which play a fundamental role in carrying out the analysis. To assure the nonsingular nature of the solution, we give a nondegeneracy condition for the matrix associated with a tau-function. This justifies the assumption in deriving the bilinear equations. Subsequently, an alternative representation of the N -phase solution is presented which is analogous to the corresponding one [9] for the N -phase solution of equation (1.2). Last, the eigenvalue problems (1.3)-(1.6) associated with the N -phase solution are discussed briefly which would turn out to be a pivot to

solving the general initial value problem of equation (1.1). As a byproduct, we provide another proof of the N -phase solution. In section 3, after parameterizing the N -phase solution in terms of wavenumbers and velocities, the explicit examples of solutions are illustrated for $N = 1, 2$ in comparison with the corresponding solutions of equation (1.2). In section 4, the reduction procedures are performed for the N -phase solution. First, we consider the case in which all the wavenumbers are set to a positive real constant. The resulting solution is shown to exhibit a pole representation whose dynamics obey the completely integrable Calogero-Moser-Sutherland dynamical system. The subsequent analysis reveals that the N -soliton solution can be obtained simply from the N -phase solution by taking the long-wave limit, recovering the N -soliton solution constructed by means of an inverse spectral formula [11]. Then, the one- and two-soliton solutions are displayed, as well as the large-time asymptotic of the general N -soliton solution. Section 5 is devoted to concluding remarks. Most of the technical details are given in appendices. In appendix A, we show that the compatibility conditions for the linear system (1.3)-(1.6) indeed yields equation (1.1). As a consequence, we derive an infinite number of conservation laws. Although they have been obtained by computing the trace of the powers of the Lax operator [11], the present alternative construction uses a linear recursion relation among the eigenfunctions. In appendix B, we prove various differential rules for the tau-functions. In appendix C, the two striking identities are verified for the tau-functions. In appendix D, we provide the proof of the results for the eigenvalue problems (1.3)-(1.6) presented in section 2.

1.4. Notations

We summarize the notations concerning vectors, matrices and bilinear operators which will be used frequently throughout the paper.

1) Row vectors

$$\mathbf{a} = (a_1, a_2, \dots, a_N), \quad \mathbf{b} = (b_1, b_2, \dots, b_N), \quad \mathbf{c} = (c_1, c_2, \dots, c_N), \quad \mathbf{d} = (d_1, d_2, \dots, d_N), \quad (1.7a)$$

$$\mathbf{1} = (\underbrace{1, 1, \dots, 1}_N), \quad \hat{\mathbf{1}} = (\underbrace{1, 1, \dots, 1}_{N-1}), \quad \mathbf{e}_1 = (\underbrace{1, 0, \dots, 0}_N), \quad (1.7b)$$

$$\mathbf{p} = (p_1, p_2, \dots, p_N), \quad \hat{\mathbf{p}} = (p_2, p_3, \dots, p_N), \quad \mathbf{q} = (q_1, q_2, \dots, q_N), \quad \hat{\mathbf{q}} = (q_2, q_3, \dots, q_N), \quad (1.7c)$$

$$\mathbf{P} = (p_1^2, p_2^2, \dots, p_N^2), \quad \mathbf{Q} = (q_1^2, q_2^2, \dots, q_N^2), \quad (1.7d)$$

$$\mathbf{f}_1 = (f_{12}, f_{13}, \dots, f_{1N}), \quad \mathbf{f}_2 = (f_{21}, f_{31}, \dots, f_{N1}), \quad (1.7e)$$

where a_j, b_j, c_j, d_j ($j = 1, 2, \dots, N$) $\in \mathbb{C}$, f_{1j}, f_{j1} ($j = 2, 3, \dots, N$) $\in \mathbb{C}$ and p_j, q_j ($j = 1, 2, \dots, N$) $\in \mathbb{R}$.

2) Matrices and cofactors

$$D = (d_{jk})_{1 \leq j, k \leq N}, \quad D(\mathbf{a}; \mathbf{b}) = \begin{pmatrix} D & \mathbf{b}^T \\ \mathbf{a} & 0 \end{pmatrix},$$

$$D(\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}) = \begin{pmatrix} D & \mathbf{c}^T & \mathbf{d}^T \\ \mathbf{a} & 0 & 0 \\ \mathbf{b} & 0 & 0 \end{pmatrix} \text{ (bordered matrices),} \quad (1.8)$$

$$|D| = \det D, \quad D_{jk} = \partial|D|/\partial d_{jk} \text{ (first cofactor of } d_{jk}), \quad (1.9a)$$

$$D_{jk,lm} = \partial^2|D|/\partial d_{jl}\partial d_{km} \text{ (} j < k, l < m \text{) (second cofactor),} \quad (1.9b)$$

where d_{jk} ($j, k = 1, 2, \dots, N$) $\in \mathbb{C}$ and the symbol T denotes transpose.

3) Bilinear operators

$$D_x^m D_t^n g \cdot f = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n g(x, t) f(x', t') \Big|_{x'=x, t'=t}, \quad (m, n = 0, 1, 2, \dots), \quad (1.10a)$$

$$D_t g \cdot f = g_t f - g f_t, \quad D_x g \cdot f = g_x f - g f_x, \quad D_x^2 g \cdot f = g_{xx} f - 2g_x f_x + g f_{xx}. \quad (1.10b)$$

2. Multiphase solutions

The goal of this section is to establish the following theorem.

Theorem 2.1. *The N -phase solution of equation (1.1) admits a determinantal expression in terms of the tau-functions $f = f(x, t)$ and $g = g(x, t)$*

$$u = \frac{g}{f}, \quad f = |F|, \quad g = g_0 |G|, \quad (2.1)$$

where F and G are $N \times N$ matrices whose elements are given by

$$F = (f_{jk})_{1 \leq j, k \leq N}, \quad f_{jk} = \zeta_j \delta_{jk} + \frac{1}{p_j - q_k}, \quad (2.2a)$$

$$G = (g_{jk})_{1 \leq j, k \leq N}, \quad g_{1k} = 1 \text{ (} k = 1, 2, \dots, N \text{)}, \quad g_{jk} = f_{jk}, \quad (j = 2, 3, \dots, N, k = 1, 2, \dots, N), \quad (2.2b)$$

with

$$g_0 = |g_0| e^{i\chi}, \quad |g_0|^2 = \frac{1}{q_1} (1 - e^{-2q_1 \rho}) \prod_{j=2}^N \frac{p_j}{q_j}, \quad (2.3a)$$

$$\zeta_j = \frac{e^{-i\theta_j + \delta_j}}{p_j - q_j}, \quad \theta_j = (p_j - q_j) \{x - (p_j + q_j)t - x_{j0}\}, \quad (j = 1, 2, \dots, N), \quad (2.3b)$$

$$\delta_j = \phi_j + \frac{1}{2} \sum_{\substack{k=1 \\ (k \neq j)}}^N A_{jk}, \quad \phi_j = -q_1 \rho \delta_{j1}, \quad (j = 1, 2, \dots, N), \quad (2.3c)$$

$$e^{A_{jk}} = \frac{(p_j - p_k)(q_j - q_k)}{(p_j - q_k)(q_j - p_k)}, \quad (j, k = 1, 2, \dots, N; j \neq k). \quad (2.3d)$$

Here, δ_{jk} denotes Kronecker's delta, ρ is a positive constant and x_{j0} ($j = 1, 2, \dots, N$) $\in \mathbb{R}$ and $\chi \in \mathbb{R}$ are phase parameters. The real parameters p_j and q_j are imposed on the conditions

$$p_1 = 0, \quad q_1 < 0 < q_2 < p_2 < \dots < q_N < p_N. \quad (2.4)$$

Remark 2.1. In view of the conditions (2.4), the tau-function f has zeros only in the lower-half complex plane so that (2.1) gives a nonsingular solution of equation (1.1) in the upper-half complex plane. The squared modules of u is expressed simply in terms of f and its complex conjugate f^* as

$$|u|^2 = -\mu - i \frac{\partial}{\partial x} \ln \frac{f^*}{f}, \quad \mu = \sum_{j=1}^N (p_j - q_j). \quad (2.5)$$

Notice that the matrix G is constructed from the matrix F by replacing its first row by the row vector $\mathbf{1}$ from (1.7b). These statements will be proved later in this section. The solutions (2.1) and (2.5) are functions of the N phase variables θ_j ($j = 1, 2, \dots, N$) and hence they are called the N -phase solutions. If the parameters $p_j - q_j$ satisfy the conditions $p_j - q_j = 2\pi m_j / L$ ($j = 1, 2, \dots, N$), where m_j are positive integers such that $0 < m_1 \leq m_2 \leq \dots \leq m_N$ and L is an arbitrary positive constant, then the solutions become periodic functions in the variable x . This specific setting will be considered in section 4.

2.1. Bilinearization

The proof of theorem 2.1 consists of a sequence of steps. The first step is to bilinearize equation (1.1) in accordance with the standard procedure of the direct method [16, 17].

Proposition 2.1 *Through the dependent variable transformations (2.1) and (2.5), equation (1.1) can be transformed to the set of bilinear equations for f and g*

$$iD_t g \cdot f = D_x^2 g \cdot f, \quad (2.6)$$

$$iD_x f^* \cdot f = -\mu f^* f - g^* g. \quad (2.7)$$

Proof. Since $f_x / f (f_x^* / f^*)$ is analytic in $\text{Im } x \geq 0$ ($\text{Im } x \leq 0$), we deduce $H(f_x / f) =$

$\mathrm{i} f_x/f$ and $H(f_x^*/f^*) = -\mathrm{i} f_x^*/f^*$. This gives

$$\frac{1}{2}(1 - \mathrm{i}H)(|u|^2)_x = \mathrm{i} \left(\frac{f_x}{f} \right)_x.$$

If we substitute the above relation and (2.1) into equation (1.1), we have

$$\mathrm{i} \left(\frac{g}{f} \right)_t = \left(\frac{g}{f} \right)_{xx} + 2 \frac{g}{f} \left(\frac{f_x}{f} \right)_x.$$

We can rewrite this equation in terms of the bilinear operators defined by (1.10) to obtain (2.6). The bilinear equation (2.7) follows simply from (2.1) and (2.5). \square

To proceed, we provide the following differential rules for the determinants $|F|$ and $|G|$.

Lemma 2.1.

$$|F|_t = \mathrm{i} \sum_{j=1}^N (p_j^2 - q_j^2) |F| + \mathrm{i} (|F(\mathbf{1}; \mathbf{p})| + |F(\mathbf{q}; \mathbf{1})|), \quad (2.8a)$$

$$|F|_x = -\mathrm{i}\mu |F| - \mathrm{i} |F(\mathbf{1}; \mathbf{1})|, \quad (2.8b)$$

$$|F|_{xx} = -\mu^2 |F| - 2\mu |F(\mathbf{1}; \mathbf{1})| + |F(\mathbf{1}; \mathbf{p})| - |F(\mathbf{q}; \mathbf{1})|, \quad (2.8c)$$

$$|G|_t = \mathrm{i} \sum_{j=1}^N (p_j^2 - q_j^2) |G| - \mathrm{i} |F(\mathbf{Q}; \mathbf{e}_1)| - \mathrm{i} |F(\mathbf{q}, \mathbf{1}; \mathbf{1}, \mathbf{e}_1)|, \quad (2.9a)$$

$$|G|_x = -\mathrm{i} (\mu |G| - |F(\mathbf{q}; \mathbf{e}_1)|), \quad (2.9b)$$

$$|G|_{xx} = -\mu^2 |G| + 2\mu |F(\mathbf{q}; \mathbf{e}_1)| + |F(\mathbf{Q}; \mathbf{e}_1)| - |F(\mathbf{q}, \mathbf{1}; \mathbf{1}, \mathbf{e}_1)|. \quad (2.9c)$$

The proof of lemma 2.1 is given in appendix B.

2.2. Proof of theorem 2.1.

We now show that the N -phase solution given by theorem 1 satisfies the bilinear equations (2.6) and (2.7). The proof is carried out by using the basic formulas of determinants, some of which are summarized in appendix B. Among them, Jacobi's formula (B.3) will play a key role.

2.2.1. Proof of (2.6)

Define the polynomial P in ζ_j ($j = 1, 2, \dots, N$) by

$$g_0 P = D_t g \cdot f - D_x^2 g \cdot f = (\mathrm{i} g_t - g_{xx}) f - (\mathrm{i} f_t + f_{xx}) g + 2 g_x f_x.$$

We then substitute (2.8) and (2.9) into P and see that P reduces to the form

$$P = 2|F(\mathbf{q}, \mathbf{1}; \mathbf{1}, \mathbf{e}_1)||F| + 2|F(\mathbf{q}, \mathbf{1})||G| + 2|F(\mathbf{q}; \mathbf{e}_1)||F(\mathbf{1}; \mathbf{1})|.$$

Using Jacobi's formula (B.3), the first term of P becomes

$$2|F(\mathbf{q}, \mathbf{1}; \mathbf{1}, \mathbf{e}_1)||F| = 2\{|F(\mathbf{q}; \mathbf{1})||F(\mathbf{1}; \mathbf{e}_1)| - |F(\mathbf{q}; \mathbf{e}_1)||F(\mathbf{1}; \mathbf{1})|\},$$

Invoking the definition of the matrix G , one has $|G| = -|F(\mathbf{1}; \mathbf{e}_1)|$. Hence, the second term can be written as $-2|F(\mathbf{q}; \mathbf{1})||F(\mathbf{1}; \mathbf{e}_1)|$. If we substitute the above two expressions into P , P turns out to be zero identically. This proves (2.6). \square

Before proceeding to the proof of (2.7), we prepare the following formulas.

Lemma 2.2. *Let us introduce the determinants $|\bar{F}|$ and $|\bar{G}|$, where $\bar{F} = (\bar{f}_{jk})_{1 \leq j, k \leq N}$ is an $N \times N$ matrix and $\bar{G} = (\bar{g}_{jk})_{2 \leq j, k \leq N}$ is an $N-1 \times N-1$ matrix whose elements are given respectively by*

$$\bar{f}_{jk} = e^{-2\phi_j} \zeta_j \delta_{jk} + \frac{1}{p_j - q_k}, \quad \bar{g}_{jk} = \frac{q_j}{p_j} \zeta_j \delta_{jk} + \frac{1}{p_j - q_k}. \quad (2.10)$$

Then, the determinantal identities hold

$$|\bar{F}| = \exp \left[- \sum_{j=1}^N (\mathrm{i}\theta_j + \phi_j) \right] |F|^*, \quad (2.11)$$

$$|\bar{G}| = \exp \left[-\mathrm{i} \sum_{j=2}^N \theta_j - \frac{1}{2} \sum_{k=2}^N A_{1k} \right] |G|^*. \quad (2.12)$$

The proof of lemma 2.2 is given in appendix C.

2.2.2. Proof of (2.7)

Let $Q = \mathrm{i}D_x f^* \cdot f + \mu f^* f + g^* g$ and $\bar{f} = |\bar{F}|$. Thanks to (2.11), one can rewrite f^* in terms of \bar{f} to obtain

$$\mathrm{i}D_x f^* \cdot f + \mu f^* f = \mathrm{i} \exp \left[\sum_{j=1}^N (\mathrm{i}\theta_j + \phi_j) \right] D_x \bar{f} \cdot f. \quad (2.13)$$

Now, using the differential rule (B.1) and an analogous one

$$\bar{f}_x = -\mathrm{i}\mu|\bar{F}| - \mathrm{i}|\bar{F}(\mathbf{1}; \mathbf{1})|,$$

we deduce

$$\mathrm{i}D_x \bar{f} \cdot f = |\bar{F}(\mathbf{1}; \mathbf{1})||F| - |F(\mathbf{1}; \mathbf{1})||\bar{F}|. \quad (2.14)$$

Since the $(1, 1)$ element of the matrix $\bar{F}(\mathbf{1}; \mathbf{1})$ can be written as $e^{-2\phi_1}\zeta_1 + 1 = \zeta_1 + 1 + (e^{-2\phi_1} - 1)\zeta_1$, an elementary manipulation yields the relation

$$|\bar{F}(\mathbf{1}; \mathbf{1})| = |F(\mathbf{1}; \mathbf{1})| + (e^{-2\phi_1} - 1)\zeta_1|\hat{F}(\hat{\mathbf{1}}; \hat{\mathbf{1}})|.$$

Similarly, one has

$$|\bar{F}| = |F| + (e^{-2\phi_1} - 1)\zeta_1|\hat{F}|,$$

where $\hat{F} = (f_{jk})_{2 \leq j, k \leq N}$ is an $N - 1 \times N - 1$ matrix. Substituting these two expressions into (2.14), we recast it to the form

$$\mathrm{i}D_x \bar{f} \cdot f = (e^{-2\phi_1} - 1)\zeta_1(Q_1 - Q_2), \quad (2.15a)$$

with

$$Q_1 = |\hat{F}(\hat{\mathbf{1}}; \hat{\mathbf{1}})||F|, \quad Q_2 = |F(\mathbf{1}; \mathbf{1})||\hat{F}|. \quad (2.15b)$$

We modify $|F(\mathbf{1}; \mathbf{1})|$ and $|F|$ with the aid of the basic rules for determinants as

$$|F(\mathbf{1}; \mathbf{1})| = |\hat{F}(\hat{\mathbf{1}}; \hat{\mathbf{1}})|f_{11} - (|\hat{F}(\hat{\mathbf{1}}; \mathbf{f}_2| + |\hat{F}|) - |\hat{F}(\mathbf{f}_1; \hat{\mathbf{1}})| + |\hat{F}(\mathbf{f}_1, \hat{\mathbf{1}}; \mathbf{f}_2, \hat{\mathbf{1}})|,$$

$$|F| = |\hat{F}|f_{11} + |\hat{F}(\mathbf{f}_1; \mathbf{f}_2)|,$$

where \mathbf{f}_1 and \mathbf{f}_2 are row vectors defined by (1.7e). By using these expressions as well as Jacobi's formula

$$|\hat{F}(\mathbf{f}_1, \hat{\mathbf{1}}; \mathbf{f}_2, \hat{\mathbf{1}})||\hat{F}| = |\hat{F}(\mathbf{f}_1; \mathbf{f}_2)||\hat{F}(\hat{\mathbf{1}}; \hat{\mathbf{1}})| - |\hat{F}(\mathbf{f}_1; \hat{\mathbf{1}})||\hat{F}(\hat{\mathbf{1}}; \mathbf{f}_2)|,$$

and an identity $|\hat{F}(\hat{\mathbf{1}}; \mathbf{f}_2)| = |G| - |\hat{F}|$, we obtain

$$\begin{aligned} Q_1 - Q_2 &= (|\hat{F}| + |\hat{F}(\mathbf{f}_1; \hat{\mathbf{1}})|)|G| \\ &= \prod_{j=2}^N \frac{p_j}{q_j} |\bar{G}||G|. \end{aligned} \quad (2.16)$$

Note that the second line of (2.16) comes from (C.6). It now follows from (2.13), (2.15) and (2.16) that

$$Q = \exp \left[\sum_{j=1}^N (\mathrm{i}\theta_j + \phi_j) \right] (e^{-2\phi_1} - 1) \zeta_1 \prod_{j=2}^N \frac{p_j}{q_j} |\bar{G}||G| + |g_0|^2 |G|^* |G|.$$

We introduce $|\bar{G}|$ from (2.12) and use the relations

$$\zeta_1 = -\frac{1}{q_1} \exp \left[-\mathrm{i}\theta_1 - \frac{1}{2} \sum_{k=2}^N A_{1k} + \phi_1 \right], \quad \phi_1 = -q_1 \rho, \quad \phi_j = 0 \quad j \geq 2,$$

to reduce the quantity Q to the form

$$Q = \left\{ -\frac{1}{q_1} (1 - e^{-2q_1\rho}) \prod_{j=2}^N \frac{p_j}{q_j} + |g_0|^2 \right\} |G|^* |G|.$$

In view of (2.3a), we finally arrive at the identity $Q = 0$, completing the proof of (2.7). \square

2.3. Analyticity of the tau-function f

The basic assumption in deriving the bilinear equations (2.6) and (2.7) is that the tau-function f has no zeros in the upper-half complex plane. The proposition below describes this statement clearly.

Proposition 2.2. *The matrix F is nondegenerate in $\text{Im } x \geq 0$ provided that the conditions (2.4) are satisfied.*

Proof. We follow the argument developed in [1, 18] for the multiphase solutions of the Benjamin-Ono (BO) equation. First, we put $f_{jk} = \zeta_j \left(\delta_{jk} + \frac{\alpha_j}{p_j - q_k} \right) \equiv \zeta_j a_{jk}$ with $\alpha_j = \zeta_j^{-1}$ and consider the matrix $A = (a_{jk})_{1 \leq j, k \leq N}$. Assume that A is generate for some $x = x_0 \in \mathbb{C}$ and $t \in \mathbb{R}$. It implies that the system of linear algebraic equations for $r_j \in \mathbb{C}$

$$r_j + \sum_{k=1}^N \frac{\alpha_j}{p_j - q_k} r_k = 0, \quad (j = 1, 2, \dots, N), \quad (2.17)$$

has a nonzero solution. Define a meromorphic function $\psi(z) = \sum_{k=1}^N \frac{r_k}{z - q_k}$. Then, the system of equations (2.17) can be written as N conditions $r_j = -\alpha_j \psi(p_j)$ ($j = 1, 2, \dots, N$).

We introduce the function R by

$$R(z) = \psi(z) \psi^*(z^*) \prod_{k=1}^N \frac{z - q_k}{z - p_k}, \quad (2.18)$$

where $\psi^*(z^*) = \sum_{k=1}^N \frac{r_k^*}{z - q_k}$. Let \mathcal{R} be the sum of residues of $R(z)$ at $z = p_j, z = q_j$ ($j = 1, 2, \dots, N$). By a simple computation, we find that

$$\mathcal{R} = \sum_{j=1}^N (p_j - q_j) \frac{|r_j|^2}{|\alpha_j|^2} \prod_{\substack{k=1 \\ (k \neq j)}}^N \frac{p_j - q_k}{p_j - p_k} \left\{ 1 - \frac{|\alpha_j|^2}{(p_j - q_j)^2} \prod_{\substack{k=1 \\ (k \neq j)}}^N \frac{(p_j - p_k)(q_j - q_k)}{(p_j - q_k)(q_j - p_k)} \right\}. \quad (2.19a)$$

Using the relation

$$|\alpha_j|^2 = (p_j - q_j)^2 \exp \left[-2(p_j - q_j) \text{Im } x_0 - \sum_{\substack{k=1 \\ (k \neq j)}}^N A_{jk} + 2\rho q_1 \delta_{j1} \right],$$

which follows from (2.2a) as well as the condition $p_1 = 0$, \mathcal{R} recasts to

$$\mathcal{R} = -q_1 \frac{|r_1|^2}{|\alpha_1|^2} (1 - e^{2q_1(\text{Im } x_0 + \rho)}) \prod_{k=2}^N \frac{q_k}{p_k} + \sum_{j=2}^N (p_j - q_j) \frac{|r_j|^2}{|\alpha_j|^2} (1 - e^{-2(p_j - q_j) \text{Im } x_0}) \prod_{\substack{k=1 \\ (k \neq j)}}^N \frac{p_j - q_k}{p_j - p_k}. \quad (2.19b)$$

On the other hand, since $R(z) \sim z^{-2}$ as $z \rightarrow \infty$, evaluating the residue at $z = \infty$ gives $\mathcal{R} = 0$. To compute \mathcal{R} from (2.19), one has two scenarios according to values of $\text{Im } x_0$. If $\text{Im } x_0 > 0$, then $\mathcal{R} > 0$ by (2.4) and (2.19b) with $\rho > 0, q_1 < 0$, which is a contradiction. Hence, $r_j = 0$ ($j = 1, 2, \dots, N$), implying that the matrix A is nondegenerate, or equivalently $|F| = \prod_{j=1}^N \zeta_j |A| \neq 0$ for $\text{Im } x_0 > 0$.

We consider the second case $\text{Im } x_0 = 0$. It then turns out from (2.19b) that $\mathcal{R} = -q_1 |r_1|^2 / |\alpha_1|^2 (1 - e^{2q_1 \rho}) \prod_{k=2}^N (q_k / p_k) \geq 0$. Hence, \mathcal{R} becomes zero only if $r_1 = 0$, which we address in detail. When $r_1 = 0$, the system of equations (2.17) becomes

$$\sum_{k=2}^N \frac{r_k}{q_k} = 0, \quad r_j + \sum_{k=2}^N \frac{\alpha_j}{p_j - q_k} r_k = 0, \quad (j = 2, 3, \dots, N). \quad (2.20)$$

Eliminating r_N by means of the first equation of (2.20), one can rewrite the second equation in the form

$$\tilde{r}_j + \sum_{k=2}^{N-1} \frac{\tilde{\alpha}_j}{p_j - q_k} \tilde{r}_k = 0, \quad (j = 2, 3, \dots, N-1), \quad (2.21a)$$

with

$$\tilde{r}_j = \frac{q_j - q_N}{q_j} r_j, \quad \tilde{\alpha}_j = \zeta_j^{-1} \frac{p_j(q_j - q_N)}{q_j(p_j - q_N)}, \quad (j = 2, 3, \dots, N-1). \quad (2.21b)$$

We then find the expression $\tilde{\mathcal{R}}$ of the residue corresponding to (2.19a)

$$\tilde{\mathcal{R}} = \sum_{j=2}^{N-1} (p_j - q_j) \frac{|\tilde{r}_j|^2}{|\tilde{\alpha}_j|^2} \prod_{\substack{k=2 \\ (k \neq j)}}^{N-1} \frac{p_j - q_k}{p_j - p_k} \left\{ 1 - \frac{|\tilde{\alpha}_j|^2}{(p_j - q_j)^2} \prod_{\substack{k=2 \\ (k \neq j)}}^{N-1} \frac{(p_j - p_k)(q_j - q_k)}{(p_j - q_k)(q_j - p_k)} \right\}. \quad (2.22)$$

Since $\text{Im } x_0 = 0$, it follows from (2.3b) that

$$|\zeta_j|^{-2} = (p_j - q_j)^2 \exp \left[- \sum_{\substack{k=1 \\ (k \neq j)}}^N A_{jk} \right], \quad (j = 2, 3, \dots, N-1).$$

By using this expression to evaluate $|\tilde{\alpha}_j|^2$, $\tilde{\mathcal{R}}$ becomes

$$\tilde{\mathcal{R}} = - \sum_{j=2}^{N-1} (p_j - q_j)^2 \frac{|\tilde{r}_j|^2}{|\tilde{\alpha}_j|^2} \frac{(p_j + q_j - q_1)p_N q_N - (p_N + q_N - q_1)p_j q_j}{q_j(p_N - p_j)(q_N - p_j)(q_j - q_1)} \prod_{\substack{k=2 \\ (k \neq j)}}^{N-1} \frac{(p_j - q_k)}{(p_j - p_k)}.$$

We add the inequalities $1/p_j > 1/p_N, 1/q_j > 1/q_N, -q_1/(p_j q_j) > -q_1/(p_N q_N)$ ($q_1 < 0, p_j, q_j > 0$ ($j = 2, 3, \dots, N-1$)) together and take into account (2.4) to obtain the inequality

$$\begin{aligned} & (p_j + q_j - q_1)p_N q_N - (p_N + q_N - q_1)p_j q_j \\ &= p_j q_j p_N q_N \left\{ \frac{1}{p_j} + \frac{1}{q_j} - \frac{q_1}{p_j q_j} - \left(\frac{1}{p_N} + \frac{1}{q_N} - \frac{q_1}{p_N q_N} \right) \right\} > 0. \end{aligned}$$

It turns out from the above observation and (2.4) that $\tilde{\mathcal{R}} < 0$. Since $\tilde{\mathcal{R}} = 0$, this leads to a contradiction. Consequently, $\tilde{r}_j = 0$ ($j = 2, 3, \dots, N-1$), which yields $\tilde{r}_N = 0$ by virtue of the first equation of (2.20). When we plug this result into the condition $\tilde{r}_1 = 0$, we find $\tilde{r}_j = 0$ ($j = 1, 2, \dots, N$) and hence $|F| \neq 0$ for $\text{Im } x_0 = 0$. Consequently, $f = |F| \neq 0$ for $\text{Im } x_0 \geq 0$. Thus, we finish the proof of proposition 2.2. \square

Remark 2.2. The tau-function f has zeros in the lower-half complex plane. Actually, it can be inferred from (2.19b) that \mathcal{R} may become zero in the range of the parameter $-\rho < \text{Im } x_0 < 0$. Although a number of numerical computations confirm this conjecture, its verification needs the rigorous analysis.

2.4. Alternative representation of the N -phase solution

As in the case of the N -phase solution of equation (1.2) [3, 9], the N -phase solution constructed in theorem 2.1 has an alternative representation which provides an analog of Dubrobin's formulation for the multiphase solutions of the Korteweg-de Vries equation [19]. We describe it by the following proposition.

Proposition 2.3. *The squared modules of the N -phase solution of equation (1.1) admits a representation*

$$|u|^2 = \sum_{j=1}^N (p_j + q_j) - i \sum_{j=1}^N (\mu_j - \mu_j^*), \quad (2.23)$$

where the functions $\mu_j = \mu_j(x, t)$ solve the system of nonlinear algebraic equations

$$\frac{\prod_{\substack{k=1 \\ (k \neq j)}}^N (q_j - q_k)}{\prod_{k=1}^N (p_j - q_k)} \frac{\prod_{k=1}^N (p_j - i\mu_k)}{\prod_{k=1}^N (q_j - i\mu_k)} = -\zeta_j, \quad (j = 1, 2, \dots, N). \quad (2.24)$$

They also satisfy the system of nonlinear PDEs

$$\sum_{k=1}^N \frac{\mu_{k,x}}{(p_j - i\mu_k)(q_j - i\mu_k)} = -1, \quad \sum_{k=1}^N \frac{\mu_{k,t}}{(p_j - i\mu_k)(q_j - i\mu_k)} = p_j + q_j, \quad (j = 1, 2, \dots, N). \quad (2.25)$$

Proof. First, consider the system of linear algebraic equations $i \sum_{j=1}^N f_{jk} \Phi_k = 1$ for Φ_j ($j = 1, 2, \dots, N$). By Cramer's rule, the solution is found to be

$$\Phi_j = -i \frac{\sum_{k=1}^N F_{kj}}{|F|}, \quad (j = 1, 2, \dots, N). \quad (2.26)$$

Referring to formula (2.8b), we obtain

$$i \sum_{j=1}^N \Phi_j = \sum_{j=1}^N (p_j - q_j) - i \frac{|F|_x}{|F|},$$

which, plugged into its complex conjugate expression, gives

$$i \sum_{j=1}^N (\Phi_j - \Phi_j^*) = 2 \sum_{j=1}^N (p_j - q_j) - i \frac{\partial}{\partial x} \ln \frac{|F|}{|F|^*}.$$

Using this relation, one can rewrite $|u|^2$ from (2.5) in the form

$$|u|^2 = \sum_{j=1}^N (p_j - q_j) - i \sum_{j=1}^N (\Phi_j - \Phi_j^*). \quad (2.27)$$

Introduce the functions $G = G(x, t, \lambda)$ and $\mu_j = \mu_j(x, t)$ ($j = 1, 2, \dots, N$) by

$$G(x, t, \lambda) = 1 - i \sum_{j=1}^N \frac{\Phi_j}{\lambda - q_j} \quad (2.28a)$$

$$= \frac{\prod_{j=1}^N (\lambda - i\mu_j)}{\prod_{j=1}^N (\lambda - q_j)}. \quad (2.28b)$$

It follows from (2.28a) by using the equation for Φ_j that

$$G(x, t, p_j) = i\zeta_j \Phi_j, \quad (2.29)$$

and from (2.28b) with $\lambda = p_j$ that

$$G(x, t, p_j) = \frac{\prod_{k=1}^N (p_j - i\mu_k)}{\prod_{k=1}^N (p_j - q_k)}. \quad (2.30)$$

On the other hand, by computing the residue at $\lambda = q_j$ in (2.28), we find

$$\Phi_j = i \frac{\prod_{k=1}^N (q_j - i\mu_k)}{\prod_{\substack{k=1 \\ (k \neq j)}}^N (q_j - q_k)}. \quad (2.31)$$

If we equate (2.29) with (2.30) and then insert Φ_j from (2.31) into the resultant expression, we establish (2.24). The system of PDEs (2.25) for μ_j follows by taking the logarithmic derivative of (2.24) with respect to x and t , respectively. Expanding (2.28a) and (2.28b) in inverse powers of λ and comparing the coefficient of λ^{-1} , we obtain the relation $\sum_{j=1}^N \Phi_j = \sum_{j=1}^N (\mu_j + iq_j)$. Introducing this expression and its complex conjugate into (2.27) yields (2.23). \square

2.5. Eigenvalue problem for the N -phase solution

In developing the formulation of IST for equation (1.1), the spectral analysis of the spatial part of the associated eigenvalue equations plays a central role. Here, we demonstrate shortly that the eigenvalue equations (1.3), (1.5) and (1.6) can be solved exactly for the N -phase solution (or time dependent N -phase potential). A full analysis for the generic potentials will be devoted to a future work. See [7] for an analogous work on the eigenvalue problem associated with the N -soliton solution of equation (1.2). The following proposition describes the main result.

Proposition 2.4. *Let ϕ_j, ψ_j^+ and λ_j be the eigenfunctions and corresponding eigenvalue for equation (1.3) with u being the N -phase solution (2.1). If ϕ_j and ψ_j^+ satisfy the system of linear algebraic equations*

$$\sum_{k=1}^N f_{jk} \phi_k = g_0 \delta_{j1}, \quad (j = 1, 2, \dots, N), \quad (2.32a)$$

$$\sum_{k=1}^N f_{jk} \psi_k^+ = -1, \quad (j = 1, 2, \dots, N), \quad (2.32b)$$

then they solve the eigenvalue equations

$$i\phi_{j,x} + \lambda_j \phi_j + u\psi_j^+ = 0, \quad (j = 1, 2, \dots, N), \quad (2.33)$$

$$i\phi_{j,t} - 2i\lambda_j \phi_{j,x} + \phi_{j,xx} - 2iu_x \psi_j^+ = 0, \quad (j = 1, 2, \dots, N), \quad (2.34)$$

$$i\psi_{j,t}^+ - 2i\lambda_j \psi_{j,x}^+ + \psi_{j,xx}^+ - i[(1 - iH)(|u|^2)_x] \psi_j^+ = 0, \quad (j = 1, 2, \dots, N), \quad (2.35)$$

with $\lambda_j = -q_j$ ($j = 1, 2, \dots, N$).

The proof of proposition 2.4 is given in appendix D. As will be shown in Appendix A, the compatibility conditions of equations (2.33)-(2.35) yield equation (1.1). Hence, the proposition above provides an alternative proof of the N -phase solution.

3. Explicit examples

3.1. New parameterization of the N -phase solution

The N -phase solution (2.1) is characterized by the parameters ρ, χ, p_j, q_j and x_{j0} ($j = 1, 2, \dots, N$). The parameters p_j and q_j are constrained by the conditions (2.4) to assure the analyticity of the solution. In order to clarify the physical implication of the solution, it is useful to introduce the wavenumber k_j and the velocity v_j according to the relations

$$k_j = p_j - q_j, \quad v_j = p_j + q_j, \quad (j = 1, 2, \dots, N). \quad (3.1)$$

Then, the expressions (2.1)-(2.3) and (2.5) can be rewritten in the form

$$u = \frac{g}{f}, \quad |u|^2 = - \sum_{j=1}^N k_j - i \frac{\partial}{\partial x} \ln \frac{f^*}{f}, \quad f = |F|, \quad g = g_0 |G|, \quad (3.2)$$

where

$$F = (f_{jl})_{1 \leq j, l \leq N}, \quad f_{jl} = \zeta_j \delta_{jl} + \frac{2}{v_j - v_l + k_j + k_l}, \quad (3.3a)$$

$$G = (g_{jk})_{1 \leq j, k \leq N}, \quad g_{1k} = 1 \quad (k = 1, 2, \dots, N), \quad g_{jk} = f_{jk}, \quad (j = 2, 3, \dots, N, k = 1, 2, \dots, N), \quad (3.3b)$$

with

$$g_0 = |g_0| e^{i\chi}, \quad |g_0|^2 = \frac{1 - e^{-2k_1\rho}}{k_1} \prod_{j=2}^N \frac{v_j + k_j}{v_j - k_j}, \quad (3.4a)$$

$$\zeta_j = \frac{e^{-i\theta_j + \delta_j}}{k_j} = \frac{e^{-ik_j \xi_j + \delta_j}}{k_j}, \quad \xi_j = x - v_j t - x_{j0}, \quad (j = 1, 2, \dots, N), \quad (3.4b)$$

$$\delta_j = \phi_j + \frac{1}{2} \sum_{\substack{k=1 \\ (k \neq j)}}^N A_{jk}, \quad \phi_j = k_1 \rho \delta_{j1}, \quad (j = 1, 2, \dots, N), \quad (3.4c)$$

$$e^{A_{jl}} = \frac{(v_j - v_l)^2 - (k_j - k_l)^2}{(v_j - v_l)^2 - (k_j + k_l)^2}, \quad (j, l = 1, 2, \dots, N; j \neq l). \quad (3.4d)$$

The conditions (2.4) become

$$k_j > 0, \quad (j = 1, 2, \dots, N), \quad v_1 < 0, \quad v_j > 0, \quad v_j > k_j, \quad (2 \leq j \leq N). \quad (3.5)$$

Remark 3.1. With the parameterization (3.1), the N -phase solution of equation (1.2) can be written in the form [3, 9]

$$u = \frac{g}{f}, \quad |u|^2 = 1 + \sum_{j=1}^N k_j + i \frac{\partial}{\partial x} \ln \frac{f^*}{f}, \quad f = |F|, \quad g = g_0 |G|, \quad (3.6)$$

where

$$F = (f_{jl})_{1 \leq j, l \leq N}, \quad f_{jl} = \tilde{\zeta}_j \delta_{jl} + \frac{2}{v_j - v_l + k_j + k_l}, \quad (3.7a)$$

$$G = (g_{jl})_{1 \leq j, l \leq N}, \quad g_{jl} = e^{\tilde{\psi}_j - \tilde{\phi}_j} \tilde{\zeta}_j \delta_{jl} + \frac{2}{v_j - v_l + k_j + k_l}, \quad (3.7b)$$

with

$$g_0 = |g_0| e^{i\chi}, \quad |g_0|^2 = \prod_{j=1}^N \frac{v_j - k_j}{v_j + k_j}, \quad (3.8a)$$

$$\tilde{\zeta}_j = \frac{e^{-i\theta_j + \tilde{\delta}_j}}{k_j} = \frac{e^{-ik_j \xi_j + \tilde{\delta}_j}}{k_j}, \quad \xi_j = x - v_j t - x_{j0}, \quad (j = 1, 2, \dots, N), \quad (3.8b)$$

$$\tilde{\delta}_j = \tilde{\phi}_j + \frac{1}{2} \sum_{\substack{k=1 \\ (k \neq j)}}^N \tilde{A}_{jk}, \quad e^{2\tilde{\phi}_j} = \frac{(v_j + k_j + 2)(v_j - k_j)}{(v_j - k_j + 2)(v_j + k_j)}, \quad e^{\tilde{\psi}_j} = \frac{v_j + k_j}{v_j - k_j} e^{\tilde{\phi}_j}, \quad (j = 1, 2, \dots, N), \quad (3.8c)$$

$$e^{\tilde{A}_{jl}} = \frac{(v_j - v_l)^2 - (k_j - k_l)^2}{(v_j - v_l)^2 - (k_j + k_l)^2}, \quad (j, l = 1, 2, \dots, N; j \neq l), \quad (3.8d)$$

and the parameters k_j and v_j are imposed on the conditions

$$k_j > 0, \quad v_j + k_j < 0, \quad v_j - k_j + 2 > 0, \quad (j = 1, 2, \dots, N). \quad (3.9)$$

3.2. Examples of solutions

3.2.1. One-phase solution

The one-phase solution is the fundamental constituent among the class of periodic solutions. It reads in the form

$$u = \frac{g_0}{e^{-ik_1 \xi_1 + k_1 \rho} + 1}, \quad (3.10a)$$

$$|u|^2 = \frac{k_1 \sinh k_1 \rho}{\cos k_1 \xi_1 + \cosh k_1 \rho}, \quad (3.10b)$$

where

$$g_0 = |g_0| e^{i\chi}, \quad |g_0|^2 = (e^{2k_1 \rho} - 1)/k_1, \quad (k_1 > 0, \rho > 0), \quad \xi_1 = x - v_1 t - x_{10}. \quad (3.10c)$$

The one-phase solution represents a nonlinear periodic traveling wave with the period $2\pi/k_1$. The tau-function f associated with it has zeros only in the lower-half complex plane whose imaginary part is given by $-\rho$. See remark 2.2.

It is instructive to compare the one-phase solution with the corresponding one for the defocusing nonlocal NLS equation (1.2). Explicitly, it follows from (3.6) and (3.7) with $N = 1$ that

$$u = g_0 \frac{e^{-ik_1\xi_1 + \tilde{\psi}_1} + 1}{e^{-ik_1\xi_1 + \tilde{\phi}_1} + 1}, \quad (3.11a)$$

$$|u|^2 = 1 - \frac{k_1 \sinh \tilde{\phi}_1}{\cos k_1\xi_1 + \cosh \tilde{\phi}_1}, \quad (3.11b)$$

where

$$g_0 = |g_0| e^{i\chi}, \quad |g_0|^2 = \frac{v_1 - k_1}{v_2 - k_2}, \quad \xi_1 = x - v_1 t - x_{10}, \quad (3.11c)$$

$$e^{2\tilde{\phi}_1} = \frac{(v_1 + k_1 + 2)(v_1 - k_1)}{(v_1 - k_1 + 2)(v_1 + k_1)}, \quad e^{\tilde{\psi}_1} = \frac{v_1 + k_1}{v_1 - k_1} \tilde{\phi}_1. \quad (3.11d)$$

Note that in accordance with (3.9), the conditions $k_1 > 0$, $v_1 + k_1 < 0$ and $v_1 - k_1 + 2 > 0$ must be imposed on the parameters k_1 and v_1 . We can see that while the allowable values of the velocity in the expression of the solution (3.10) lie in the interval $-\infty < v_1 < 0$ as indicated by (3.5), the corresponding velocity in the solution (3.11) has an upper limit $|v_1| = 2$. Actually, its value enters into the region surrounded by the three straight lines $k_1 = 0$, $v_1 = -k_1$, $v_1 = k_1 - 2$ in the (k_1, v_1) plane. It is interesting to observe that when $v_1 = -1$, the minimum value of $|u|^2$ evaluated by (3.11b) becomes zero regardless of the value of k_1 . Figure 1 plots the profiles of the modulus $|u|$ for (3.10b) and (3.11b).

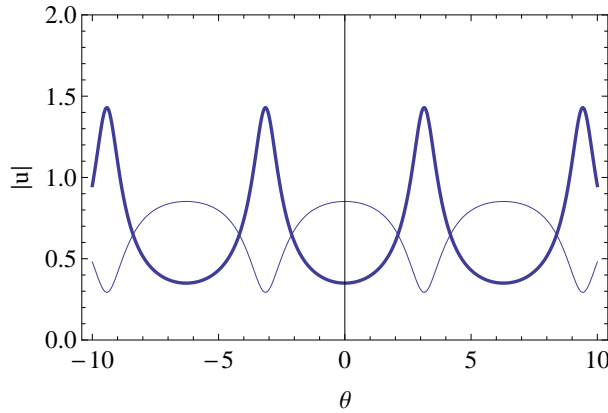


Figure 1. The profiles of the modulus $|u|$ are plotted for the one-phase solutions (3.10b) (thick curve) and (3.11b) (thin curve) as function of the phase variable $\theta (= k_1\xi_1)$. The parameters are set to $\rho = 1$, $k_1 = 1/2$ for (3.10b) and $k_1 = 1$, $v_1 = -3/4$ for (3.11b).

3.2.2. Two-phase solution

The two-phase solution from (3.2) with $N = 2$ can be written in the form

$$u = g_0 \frac{|G|}{|F|}, \quad (3.12a)$$

$$|u|^2 = -(k_1 + k_2) - i \frac{\partial}{\partial x} \ln \left(\frac{|F|^*}{|F|} \right), \quad (3.12b)$$

with

$$|F| = \zeta_1 \zeta_2 + \frac{\zeta_1}{k_2} + \frac{\zeta_2}{k_1} + \frac{1}{k_1 k_2} \frac{(v_1 - v_2)^2 - (k_1 - k_2)^2}{(v_1 - v_2)^2 - (k_1 + k_2)^2}, \quad (3.13a)$$

$$|G| = \zeta_2 + \frac{1}{k_2} \frac{v_1 - v_2 + k_1 - k_2}{v_2 - v_1 + k_1 + k_2}, \quad (3.13b)$$

$$g_0 = |g_0| e^{ix}, \quad |g_0|^2 = \frac{v_2 + k_2}{k_1(v_2 - k_2)} (e^{2k_1 \rho} - 1). \quad (3.13c)$$

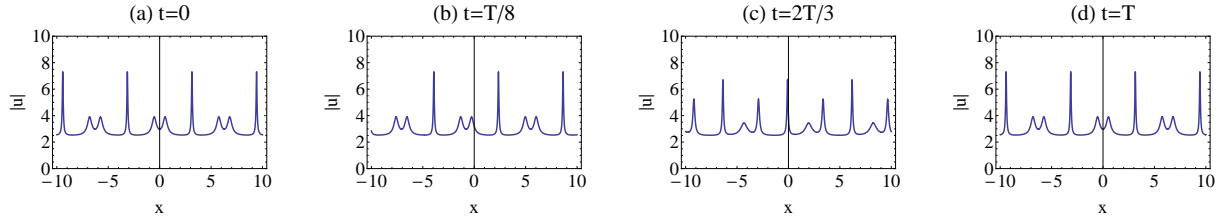


Figure 2 The profiles of the modulus $|u|$ are plotted for the two-phase solution (3.12b) as function of x . The parameters are set to $\rho = 1, k_1 = 1, k_2 = 2, v_1 = -1, v_2 = 3, x_{10} = x_{20} = 0$. The period is $T = 2\pi$. The recurrence time is $T_R = \pi/4 = T/8$.

Figure 2 depicts the profiles of $|u|$ from (3.12b) for four different times. The two-phase solution consists of two waves interacting with each other repeatedly. The period of the slow wave is given by $T_1 = 2\pi/(k_1|v_1|) = 2\pi$ whereas that of the fast one is $T_2 = 2\pi/(k_2|v_2|) = \pi/3$. As a result, the initial profile is recovered at $t = T_1 = 2\pi$. One can see that at a time $t = T_R = 2\pi(k_2 - k_1)/\{k_1 k_2 |v_2 - v_1|\} = \pi/4$, the initial profile is recovered, but with a phase shift. This formula for the recurrence time T_R can be derived by eliminating the spatial coordinate x from the relations $\theta_1 = 2\pi, \theta_2 = 2\pi$ with $x_{10} = x_{20} = 0$.

The two-phase solution of equation (1.2) is given by (3.6) with $N = 2$. It reads in the form

$$u = g_0 \frac{|G|}{|F|}, \quad (3.14a)$$

$$|u|^2 = 1 + k_1 + k_2 + i \frac{\partial}{\partial x} \ln \left(\frac{|F|^*}{|F|} \right), \quad (3.14b)$$

with

$$|F| = \tilde{\zeta}_1 \tilde{\zeta}_2 + \frac{\tilde{\zeta}_1}{k_2} + \frac{\tilde{\zeta}_2}{k_1} + \frac{1}{k_1 k_2} \frac{(v_1 - v_2)^2 - (k_1 - k_2)^2}{(v_1 - v_2)^2 - (k_1 + k_2)^2}, \quad (3.15a)$$

$$|G| = \frac{(v_1 + k_1)(v_2 + k_2)}{(v_1 - k_1)(v_2 - k_2)} \tilde{\zeta}_1 \tilde{\zeta}_2 + \frac{v_1 + k_1}{v_1 - k_1} \frac{\tilde{\zeta}_1}{k_2} + \frac{v_2 + k_2}{v_2 - k_2} \frac{\tilde{\zeta}_2}{k_1} + \frac{1}{k_1 k_2} \frac{(v_1 - v_2)^2 - (k_1 - k_2)^2}{(v_1 - v_2)^2 - (k_1 + k_2)^2}, \quad (3.15b)$$

$$g_0 = |g_0| e^{i\chi}, \quad |g_0|^2 = \frac{(v_1 - k_1)(v_2 - k_2)}{(v_1 + k_1)(v_2 + k_2)}. \quad (3.15c)$$

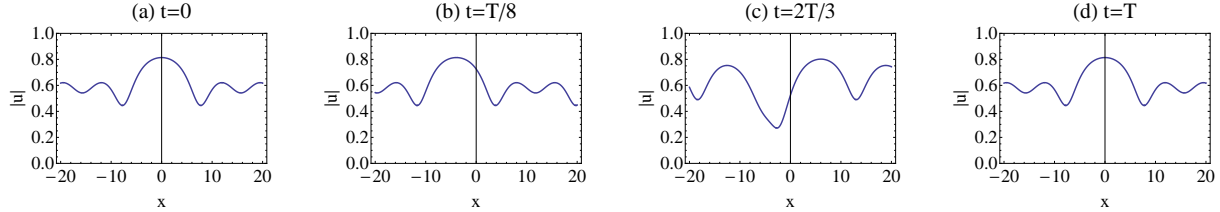


Figure 3. The profiles of the modulus $|u|$ are plotted for the two-phase solution (3.14b) as function of x . The parameters are set to $k_1 = 0.2, k_2 = 0.4, v_1 = -0.3, v_2 = -1.5, x_{10} = x_{20} = 0$. The period is $T = 100\pi/3$. The recurrence time is $T_R = T/8$.

Figure 3 illustrates the profiles of $|u|$ from (3.14b). In this example, $T_1 = 100\pi/3, T_2 = 10\pi/3$ and $T_R = 25\pi/6$. The period T turns out to be T_1 .

4. Reductions of the N -phase solution

Here, we present some results arising from the reductions of the N -phase solution constructed in section 2. First, we consider the special case of the solution in which all the wavenumbers coincide. The resulting solution is shown to exhibit a pole representation whose pole dynamics obey an integrable Calogero-Moser-Sutherland dynamical system. The second reduction is concerned with the rational (or algebraic) N -soliton solution. It is demonstrated that the N -soliton solution can be reduced simply from the N -phase solution via an appropriate limiting procedure.

4.1. Special case of the N -phase solution

The N -phase solution takes a simple structure when all the wavenumbers k_j ($j = 1, 2, \dots, N$) are equal to a positive constant k , for example. In this setting, the tau-function f takes the form

$$f = |F|, \quad F = (f_{jl})_{1 \leq j, l \leq N}, \quad f_{jl} = \zeta_j \delta_{jl} + \frac{2}{v_j - v_l + 2k}, \quad (4.1a)$$

$$\zeta_j = \frac{e^{-i\theta_j + \delta_j}}{k} = \frac{e^{-ik\xi_j + \delta_j}}{k}, \quad \xi_j = x - v_j t - x_{j0}, \quad (j = 1, 2, \dots, N), \quad (4.1b)$$

$$e^{A_{jl}} = \frac{(v_j - v_l)^2}{(v_j - v_l)^2 - 4k^2}, \quad (j, l = 1, 2, \dots, N; j \neq l), \quad (4.1c)$$

whereas the tau-function g is constructed simply from f in accordance with a prescription given in (2.2). It turns out that the solution u can be represented by a rational function of e^{ikx} . In particular, the tau-function f becomes an N th-order polynomial of e^{ikx} . This fact makes it possible to introduce an ansatz for the solution. The proposition below characterizes the structure of a special class of the N -phase solution.

Proposition 4.1. *The periodic solution (2.1) with (4.1) admits a pole representation*

$$u = \beta + i \sum_{j=1}^N c_j V(x - x_j), \quad V(x) = \frac{k}{2} \cot \frac{kx}{2}, \quad \beta \in \mathbb{R}, \quad \text{Im } x_j < 0, \quad (j = 1, 2, \dots, N). \quad (4.2)$$

The pole $x_j = x_j(t)$ and the residue $c_j = c_j(t)$ evolve according to the system of nonlinear ordinary differential equations (ODEs)

$$\dot{x}_j = \frac{2\beta}{c_j} + \frac{2i}{c_j} \sum_{\substack{k=1 \\ (k \neq j)}}^N c_k V'(x_j - x_k), \quad (j = 1, 2, \dots, N), \quad (4.3)$$

$$\dot{c}_j = -2i \sum_{\substack{k=1 \\ (k \neq j)}}^N c_k^* V'(x_j - x_k), \quad (j = 1, 2, \dots, N), \quad (4.4)$$

where the dot denotes the differentiation with respect to t and $V'(x) = dV(x)/dx$. In addition, the N constraints

$$1 - \beta c_j + i c_j \sum_{k=1}^N V(x_j - x_k^*) c_k^* = 0, \quad (j = 1, 2, \dots, N), \quad (4.5)$$

are imposed on c_j and x_j ($j = 1, 2, \dots, N$).

Proof. Substituting (4.2) into equation (1.1) and comparing the coefficients of $V'(x - x_j)$, $V(x - x_j)$ and $V^3(x - x_j)$ at the pole $x = x_j$, we obtain (4.3), (4.4) and (4.5), respectively. \square

Remark 4.1. The similar system has been derived for the periodic solutions of equation (1.2) [5]. It is important that the constraints (4.5) are preserved if they hold at an initial time. With the aid of (4.5), one can eliminate the variables c_j from (4.3) and (4.4). The

resulting system of second order ODEs for x_j turns out to be the completely integrable Calogero-Moser-Sutherland dynamical system [20-23]

$$\frac{d^2 x_j}{dt^2} = -\frac{\partial}{\partial x_j} \sum_{\substack{k=1 \\ (k \neq j)}}^N \frac{k^2}{\sin^2 \left[\frac{k}{2}(x_j - x_k) \right]}, \quad (j = 1, 2, \dots, N). \quad (4.6)$$

An exact method of solution developed in [5] for solving an analogous system of equations associated with equation (1.2) can be applicable as well to the above system of equations. In this approach, the major nontrivial issue to be addressed is to establish that the poles $x_j(t)$ stay in the lower-half complex plane as time evolves. Fortunately, this problem has been resolved in section 2 by constructing explicitly an N -phase solution together with its analyticity in the upper-half complex plane.

4.2. N -soliton solution

The N -soliton solution is produced in the long-wave limit of the N -phase solution, as we now demonstrate.

Proposition 4.2. *The N -soliton solution of equation (1.1) can be represented by the determinantal formulas*

$$u = \frac{\tilde{g}}{\tilde{f}}, \quad |u|^2 = -i \frac{\partial}{\partial x} \ln \frac{\tilde{f}^*}{\tilde{f}}, \quad \tilde{f} = |\tilde{F}|, \quad \tilde{g} = \tilde{g}_0 |\tilde{G}|, \quad (4.7a)$$

where

$$\tilde{F} = (\tilde{f}_{jk})_{1 \leq j, k \leq N}, \quad \tilde{f}_{jk} = (\xi_j + i\rho\delta_{j1})\delta_{jk} - \frac{2i}{v_j - v_k}(1 - \delta_{jk}), \quad (4.7b)$$

$$\tilde{G} = (\tilde{g}_{jk})_{1 \leq j, k \leq N}, \quad \tilde{g}_{1k} = 1 \quad (k = 1, 2, \dots, N), \quad \tilde{g}_{jk} = \tilde{f}_{jk} \quad (j = 2, 3, \dots, N, k = 1, 2, \dots, N). \quad (4.7c)$$

$$\tilde{g}_0 = -i\sqrt{2\rho}e^{i\chi}, \quad \xi_j = x - v_j t - x_{j0}, \quad (v_1 = 0, \quad v_j > 0, \quad j = 2, 3, \dots, N), \quad (4.7d)$$

The tau-functions \tilde{f} and \tilde{g} satisfy the bilinear equations

$$iD_t \tilde{g} \cdot \tilde{f} = D_x^2 \tilde{g} \cdot \tilde{f}, \quad (4.8a)$$

$$iD_x \tilde{f}^* \cdot \tilde{f} = -\tilde{g}^* \tilde{g}. \quad (4.8b)$$

Proof. The N -soliton solution is derived simply from the long-wave limit of the N -phase solution. To be more specific, one first shifts the phase variables x_{j0} as $x_{j0} \rightarrow x_{j0} + \pi/k_j$ ($j = 1, 2, \dots, N$) and then takes the limits $k_j \rightarrow 0$ while fixing the velocities

v_j ($j = 1, 2, \dots, N$). The leading-order asymptotics of the parameters given in (3.4) are found to be as

$$\zeta_j \sim -\frac{1}{k_j} + i(\xi_j + i\rho\delta_{j1}), \quad \delta_j \sim \sum_{\substack{l=1 \\ (l \neq j)}}^N \frac{2k_j k_l}{(v_j - v_l)^2} + k_1 \rho \delta_{j1}, \quad e^{A_{jl}} \sim 1 + \frac{4k_j k_l}{(v_j - v_l)^2}.$$

Consequently, the asymptotics of f_{jl} , $|F|$ and $|G|$ from (3.3) become

$$f_{jj} \sim i(\xi_j + i\rho\delta_{j1}), \quad f_{jl} \sim \frac{2}{v_j - v_l} \quad (j \neq l), \quad |F| \sim i^N |\tilde{F}|,$$

$$|g_0| \sim \sqrt{2\rho}, \quad |G| \sim i^{N-1} |\tilde{G}|.$$

Substituting these expressions into (3.2) and taking the limits $k_j \rightarrow 0$, we obtain (4.7a). The bilinear equations (4.8) stem simply from (2.6) and (2.7) by introducing the above asymptotic expressions of $|F|$ and $|G|$. \square

Remark 4.2. The N -soliton solution given above recovers the corresponding one obtained in [11] by using an inverse spectral formula for the Lax operator. Note that since $p_1 = 0$, $v_1 = -k_1 \rightarrow 0$ as $k_1 \rightarrow 0$. The pole representation of the N -soliton solution subjected to the boundary condition $u \rightarrow 0$, $|x| \rightarrow \infty$ follows from (4.2)-(4.4) by taking the limit $k \rightarrow 0$. It reads in the form

$$u = \sum_{j=1}^N \frac{c_j}{x - x_j}. \quad (4.9)$$

The time evolution of x_j and c_j is governed by the system of ODEs

$$\dot{x}_j = \frac{2i}{c_j} \sum_{\substack{k=1 \\ (k \neq j)}}^N \frac{c_k}{x_j - x_k}, \quad \dot{c}_j = 2i \sum_{\substack{k=1 \\ (k \neq j)}}^N \frac{c_j - c_k}{(x_j - x_k)^2}, \quad (j = 1, 2, \dots, N), \quad (4.10)$$

under the constraints

$$1 + ic_j \sum_{j=1}^N \frac{c_k^*}{x_j - x_k^*} = 0, \quad (j = 1, 2, \dots, N). \quad (4.11)$$

When plugged into (4.11), the system of ODEs (4.10) for x_j can be recast to the Calogero-Moser system [20, 22, 23]

$$\frac{d^2 x_j}{dt^2} = -4 \frac{\partial}{\partial x_j} \sum_{\substack{k=1 \\ (k \neq j)}}^N \frac{1}{(x_j - x_k)^2}, \quad (j = 1, 2, \dots, N). \quad (4.12)$$

Remark 4.3. The long-wave limit of the N -phase solution (3.4) with (3.5) for equation (1.2) takes the form [3, 9]

$$u = \frac{\tilde{g}}{\tilde{f}}, \quad |u|^2 = 1 + i \frac{\partial}{\partial x} \ln \frac{\tilde{f}^*}{\tilde{f}}, \quad \tilde{f} = |\tilde{F}|, \quad g = \tilde{g}_0 |\tilde{G}|, \quad (4.13a)$$

where

$$\tilde{F} = (\tilde{f}_{jk})_{1 \leq j, k \leq N}, \quad \tilde{f}_{jk} = \left(\xi_j + \frac{i}{a_j} \right) \delta_{jk} - \frac{2i}{v_j - v_k} (1 - \delta_{jk}), \quad (4.13b)$$

$$\tilde{G} = (\tilde{g}_{jk})_{1 \leq j, k \leq N}, \quad \tilde{g}_{jk} = \left\{ \xi_j + i \left(\frac{1}{a_j} + \frac{2}{v_j} \right) \right\} \delta_{jk} - \frac{2i}{v_j - v_k} (1 - \delta_{jk}), \quad (4.13c)$$

$$\tilde{g}_0 = e^{ix}, \quad \xi_j = x - v_j t - x_{j0}, \quad (j = 1, 2, \dots, N). \quad (4.13d)$$

The amplitude parameters a_j are related to the velocity parameters v_j by the relations

$$v_j(v_j + 2) + 2a_j = 0, \quad 0 < a_j < \frac{1}{2}, \quad (j = 1, 2, \dots, N). \quad (4.14)$$

4.3. Examples of solutions

We explore the properties of soliton solutions in comparison with those of equation (1.2). Specifically, we consider the one- and two-soliton solutions.

4.3.1. One-soliton solution

The one-soliton solution of equation (1.1) takes the form

$$u = \frac{-i\sqrt{2\rho} e^{ix}}{x - x_{10} + i\rho}, \quad (4.15a)$$

$$|u|^2 = \frac{2\rho}{(x - x_{10})^2 + \rho^2}, \quad (4.15b)$$

and the corresponding solution of equation (1.2) is written in the form

$$u = \frac{e^{ix}}{x - v_1 t - x_{10} + \frac{i}{a_1}}, \quad (4.16a)$$

$$|u|^2 = 1 - \frac{2a_1}{a_1^2(x - v_1 t - x_{10})^2 + 1}, \quad (4.16b)$$

where $v_1(v_1 + 2) + 2a_1 = 0$ and $0 < a_1 < 1/2$.

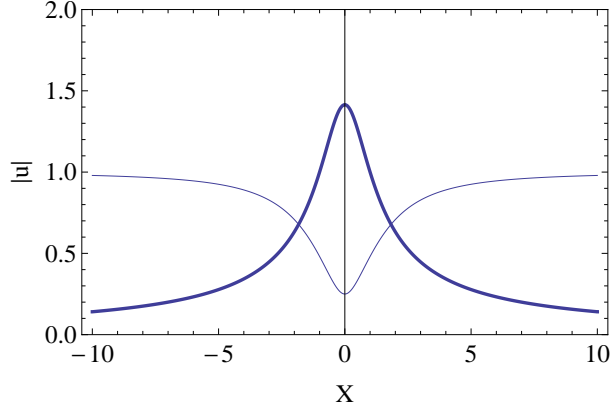


Figure 4. The long-wave limit of the one-phase solutions (3.9) (thick curve) and (3.10) (thin curve) as function of the traveling wave coordinate $X = x - v_1 t - x_{10}$. The parameters are same as those used in figure 1. Note from (4.14) with $v_1 = -3/4$ that $a_1 = 15/32$.

Figure 4 depicts the long-wave limit of one-phase solutions which are given respectively by (4.15a) and (4.16a). The former solution takes the form of a bright soliton whereas the latter one shows a dark soliton on a constant background. It is interesting to observe that the velocity of the bright soliton becomes zero since $v_1 \rightarrow 0$ in the long-wave limit. This reflects the fact that the eigenvalue problem (1.3) for the N -soliton potential has zero eigenvalue [11]. Note that in view of the invariance of equation (1.1) under the Galilean transformation, $u(x, t) \rightarrow e^{ivx - iv^2 t} u(x - 2vt, t)$, one can rewrite (4.15a) in a time-dependent form presented in [10].

4.3.2. Two-soliton solution

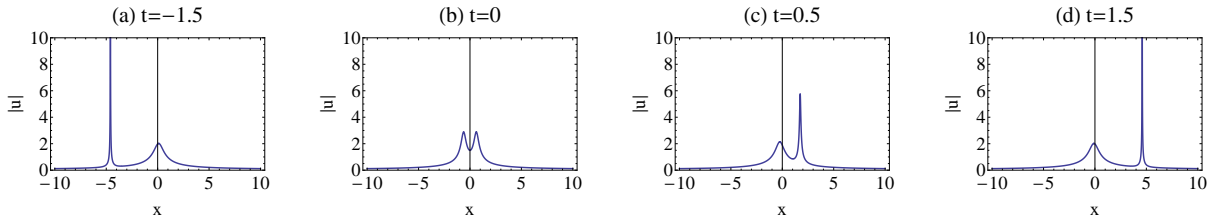


Figure 5. The two-soliton solution of equation (1.1) as function of x . The parameters are the same as those used in figure 2 except $v_1 = 0$.

Figure 5 shows the two-soliton solution of equation (1.1) which has been obtained by the long-wave limit of the two-phase solution depicted in figure 2. See (4.7). It represents the interaction of two solitons. The asymptotic profile of the solution consists of a superposition of the two solitons, one is a bright soliton with zero velocity and the other one takes

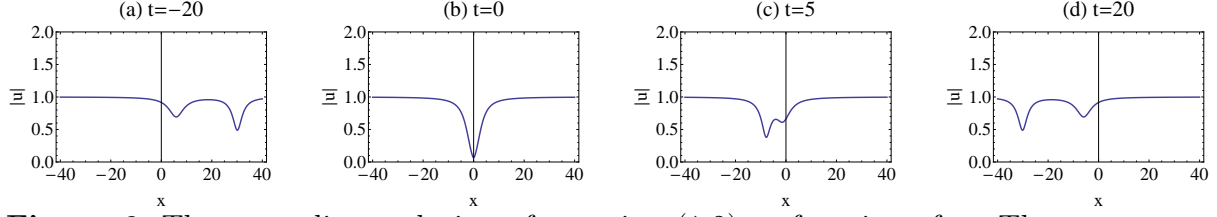


Figure 6. The two-soliton solution of equation (1.2) as function of x . The parameters are the same as those used in figure 3.

the form of the delta function. Explicitly,

$$|u|^2 \sim \frac{2\rho}{(x - x_{10})^2 + \rho^2} + 2\pi\delta(x - v_2t - x_{20}), \quad t \rightarrow \pm\infty. \quad (4.17)$$

We recall that this intriguing feature of the solution has been found in analyzing the structure of the two-soliton solution of equation (1.1) [11].

Figure 6 shows the two-soliton solution of equation (1.2) reduced from the long-wave limit of the two-phase solution depicted in figure 3. See (4.13). The asymptotic profile is simply a superposition of two dark solitons. It reads

$$|u|^2 \sim 1 - \sum_{j=1}^2 \frac{2a_j}{a_j^2(x - v_jt - x_{j0})^2 + 1}, \quad t \rightarrow \pm\infty. \quad (4.18)$$

In the above two examples, solitons exhibit no phase shifts after their interaction. The similar feature has been found for the first time in the interaction process of rational solitons of the BO equation [24, 25].

Remark 4.4. An explicit expression of the N -soliton solution of equation (1.1) has been derived in the analysis of the spectral problems of the Lax operator [11]. The detailed investigation of the dynamics of solitons was performed as well. In particular, the asymptotic form of the N -soliton solution at large time was shown to be represented by a superposition of a single bright soliton with zero velocity and a train of $N - 1$ pulses with delta function profiles. To be more specific,

$$|u|^2 \sim \frac{2\rho}{(x - x_{10})^2 + \rho^2} + 2\pi \sum_{j=2}^N \delta(x - v_jt - x_{j0}), \quad t \rightarrow \pm\infty. \quad (4.19)$$

Obviously, this expression is a generalization of the asymptotic of the two-soliton solution (4.17).

We recall that the similar behavior of solution has been observed in the interaction process of solitons of the following sine-Hilbert equation [26-31]

$$H\theta_t = -\sin \theta, \quad \theta = \theta(x, t). \quad (4.20)$$

The asymptotic form of the N -soliton solution has been found to be as

$$\theta_x \sim \frac{2a_1}{(x - a_1 t - b_1 + a_2/a_1)^2 + a_1^2} + 2\pi \sum_{j=1}^{N-1} \delta(x - \alpha_j), \quad t \rightarrow \pm\infty, \quad (4.21)$$

where $a_1 > 0, a_2, b_1, \alpha_j (j = 1, 2, \dots, N-1) \in \mathbb{R}$. Unlike the asymptotic form (4.19), a single soliton propagates with a constant velocity and remaining $N-1$ pulses take the form of the delta functions with zero velocity. It is interesting to see that the velocity of the soliton is inversely proportional to its amplitude so that the small soliton propagates more rapidly than the large soliton.

4.4. Alternative representation of the N -soliton solution

The N -soliton solution presented in proposition 4.2 has an alternative representation which is given in the proposition.

Proposition 4.3. *The squared modules of the N -soliton solution of equation (1.1) admits a representation*

$$|u|^2 = \sum_{j=1}^N v_j - i \sum_{j=1}^N (\tilde{\mu}_j - \tilde{\mu}_j^*), \quad (4.22)$$

where the functions $\tilde{\mu}_j = \tilde{\mu}_j(x, t)$ solve the system of nonlinear algebraic equations

$$\sum_{k=1}^N \frac{2i}{v_j - 2i\tilde{\mu}_k} = \xi_j + i\rho\delta_{j1} + \sum_{\substack{k=1 \\ (k \neq j)}}^N \frac{2i}{v_j - v_k}, \quad (j = 1, 2, \dots, N). \quad (4.23)$$

They satisfy the system of nonlinear PDEs

$$\sum_{k=1}^N \frac{\tilde{\mu}_{k,x}}{(v_j - 2i\tilde{\mu}_k)^2} = -\frac{1}{4}, \quad \sum_{k=1}^N \frac{\tilde{\mu}_{k,t}}{(v_j - 2i\tilde{\mu}_k)^2} = \frac{v_j}{4}, \quad (j = 1, 2, \dots, N). \quad (4.24)$$

Proof. If we apply the long-wave limit prescribed in proposition 4.2 to proposition 2.3, the above results follow immediately. \square

4.5. Eigenvalue problem for the N -soliton solution

The eigenvalue problem for the N -soliton solution has been resolved by the spectral analysis of the Lax operator associated with equation (1.1). For completeness, we reproduce the result by means of the reduction procedure.

Proposition 4.4. *Let ϕ_j, ψ_j^+ and λ_j be the eigenfunctions and corresponding eigenvalue for equation (1.3) with the N -soliton solution (4.7). If ϕ_j and ψ_j^+ satisfy the system of linear algebraic equations*

$$\sum_{k=1}^N \tilde{f}_{jk} \phi_k = g_0 \delta_{j1}, \quad (j = 1, 2, \dots, N), \quad (4.25a)$$

$$\sum_{k=1}^N \tilde{f}_{jk} \psi_k^+ = i, \quad (j = 1, 2, \dots, N), \quad (4.25b)$$

then they solve the eigenvalue equations

$$i\phi_{j,x} + \lambda_j \phi_j + u\psi_j^+ = 0, \quad (j = 1, 2, \dots, N), \quad (4.26)$$

$$i\phi_{j,t} - 2i\lambda_j \phi_{j,x} + \phi_{j,xx} - 2iu_x \psi_j^+ = 0, \quad (j = 1, 2, \dots, N), \quad (4.27)$$

$$i\psi_{j,t}^+ - 2i\lambda_j \psi_{j,x}^+ + \psi_{j,xx}^+ - i[(1 - iH)(|u|^2)_x] \psi_j^+ = 0, \quad (j = 1, 2, \dots, N), \quad (4.28)$$

with $\lambda_j = -\frac{v_j}{2}$ ($j = 1, 2, \dots, N$).

Proof. If we take the long-wave limit of (2.32a, b) by employing the leading-order asymptotics of the elements f_{jk} given in the proof of proposition 4.2, then (4.25a, b) follow immediately. Note in this limit that $\lambda_j = -q_j = -(v_j - k_j)/2 \rightarrow -v_j/2$, as indicated. The proof of (4.26)-(4.28) can be done in parallel with the proof of equations (2.33)-(2.35). Actually, after performing the limiting procedure, it is found that they take the same forms as (2.33)-(2.35). \square

Remark 4.5. An intriguing result extracted from proposition 4.4 is that $v_1 = 0$, as a consequence of the relations $v_1 = p_1 + k_1, p_1 = k_1 = 0$. Thus, among N solitons, one soliton does not propagate. One can observe this phenomenon in the interaction process of two-solitons. See figure 5. The eigenfunction ϕ_1 for the eigenvalue $\lambda_1 = 0$ takes the form $\phi_1 = c(\tilde{f} - \tilde{f}^*)/\tilde{f}$, where $c \in \mathbb{C}$ is a normalization constant. Indeed, it follows from (1.3) and (1.4) that ϕ_1 satisfies the equation $i\phi_{1,x} + uP_+(u^*\phi_1) = 0$ with $P_+ = \frac{1}{2}(1 - iH)$ being a projection operator. One can show that if ϕ_1 and u^* from (4.7a) are substituted into the undermentioned equation for ϕ_1 , then it recasts to the bilinear equation (4.8b).

5. Concluding remarks

In this paper, we were concerned with the focusing nonlocal NLS equation. Specifically, we constructed the N -phase solution by means of the direct method. The long-wave limit was then taken for the N -phase solution to deduce the N -soliton solution. We illustrated a few examples of both the periodic and soliton solutions in comparison with

those corresponding to the defocusing nonlocal NLS equation and clarified their novel features.

There are still many problems we have to settle. Among them, the initial value problem will be the most important issue which may be tackled with the aid of IST. As a first step toward this goal, the eigenfunctions of the linear system (1.3)-(1.6) were obtained for both the N -phase and N -soliton potentials. Recall that the method has been applied to the defocusing nonlocal NLS equation to solve the direct and inverse scattering problems for the N -soliton potential [7, 8].

Equation (1.2) has been derived formally by using an asymptotic multiscale expansion method to the BO equation [2]. It is interesting to explore whether a similar procedure is applicable for deriving equation (1.1) as well starting from an integrable nonlocal evolution equation.

Appendix A. Integrability and conservation laws

Here, we first show that equation (1.1) can be obtained as the compatibility conditions of the system of linear PDEs (1.3)-(1.6). Then, we derive an infinite number of conservation laws by solving successively the recursion relation for the eigenfunctions of the linear system.

A.1. Integrability

We differentiate (1.3) by t and use (1.5) and (1.6) to eliminate ϕ_t , ϕ_{xt} and ψ_t^+ , respectively. Rearranging the resultant expression gives

$$\begin{aligned} -\phi_{xxx} + 3i\lambda\phi_{xx} + (2\lambda^2 + \kappa)\phi_x - i\kappa\lambda\phi + iu\psi_{xx}^+ + 2i(u_x\psi^+)_x + 2\lambda(u\psi^+)_x \\ + u\{(1 - iH)(|u|^2)_x - i\kappa\}\psi^+ + u_t\psi^+ = 0. \end{aligned} \quad (A.1)$$

It follows by using (1.3) repeatedly that

$$-\phi_{xxx} + 3i\lambda\phi_{xx} + (2\lambda^2 + \kappa)\phi_x - i\kappa\lambda\phi = -i(u\psi^+)_{xx} - 2\lambda(u\psi^+)_x + i\kappa u\psi^+. \quad (A.2)$$

If we substitute (A.2) into (A.1), we have

$$\{u_t + iu_{xx} + u(1 - iH)(|u|^2)_x\}\psi^+ = 0,$$

which, divided by ψ^+ , yields equation (1.1). We note that equation (1.4) and the time evolution of ψ^- from (1.6) have not been employed in the derivation process.

In a similar manner, we can derive the evolution equation for u^* . Actually, differentiating (1.4) by t and using (1.5) and (1.6) to eliminate ϕ_t and ψ_t^\pm , we deduce

$$\begin{aligned} i(\psi_{xx}^+ - \sigma\psi_{xx}^-) + 2\lambda(\psi_x^+ - \sigma\psi_x^-) - i\kappa(\psi^+ - \sigma\psi^- - u^*\phi) + (1 - iH)(|u|^2)_x\psi^+ \\ + 2\sigma(1 + iH)(|u|^2)_x\psi^- - u^*(i\phi_{xx} + 2\lambda\phi_x + 2u_x\psi^+) - u_t^*\phi = 0. \end{aligned} \quad (A.3)$$

We use (1.4) for the first three terms in (A.3) to simplify it to

$$\begin{aligned} i(u^*\phi)_{xx} + 2\lambda(u^*\phi)_x + (1 - iH)(|u|^2)_x\psi^+ + \sigma(1 - iH)(|u|^2)_x\psi^- \\ - u^*(i\phi_{xx} + 2\lambda\phi_x + 2u_x\psi^+) - u_t^*\phi = 0. \end{aligned} \quad (A.4)$$

The derivatives ϕ_x and ϕ_{xx} in (A.4) can be eliminated with the aid of (1.3). After a few manipulations, (A.4) recasts to

$$\{-u_t^* + iu_{xx}^* - u^*(1 + iH)(|u|^2)_x\}\phi = 0.$$

Dividing this expression by ϕ , we obtain the complex conjugate expression of equation (1.1).

A.2. Conservation laws

We consider the function u analytic in the upper-half complex plane and vanishes rapidly at infinity. To begin with, introduce the eigenfunctions of the linear system (1.3)-(1.6) subjected to the boundary conditions $\phi \rightarrow 0$, $\psi^\pm \rightarrow 1$ as $|x| \rightarrow \infty$. Let them be $\bar{\phi}$ and $\bar{\psi}^\pm$, respectively. Then, equations (1.3)-(1.5) can be written in the form

$$i\bar{\phi}_x + \lambda\bar{\phi} + u\bar{\psi}^+ = 0, \quad (\text{A.5})$$

$$\bar{\psi}^+ - \bar{\psi}^- - u^*\bar{\phi} = 0, \quad (\text{A.6})$$

$$i\bar{\phi}_t - 2i\lambda\bar{\phi}_x + \bar{\phi}_{xx} - 2iu_x\bar{\psi}^+ = 0, \quad (\text{A.7})$$

where we have put $\sigma = 1$ and $\kappa = 0$ as is consistent with the boundary conditions. Applying the projection operator $P_+ = \frac{1}{2}(1 - iH)$ to (A.6) and using the boundary condition for $\bar{\psi}^+$ as well as the relations $P_+(\bar{\psi}^+ - 1) = \bar{\psi}^+ - 1$, $P_+(\bar{\psi}^- - 1) = 0$, we deduce $\bar{\psi}^+ = 1 + P_+(u^*\bar{\phi})$. Substituting this expression into (A.5) and (A.7), respectively, we rewrite them as

$$i\bar{\phi}_x + \lambda\bar{\phi} + uP_+(u^*\bar{\phi}) + u = 0, \quad (\text{A.8})$$

$$i\bar{\phi}_t - 2i\lambda\bar{\phi}_x + \bar{\phi}_{xx} - 2iu_xP_+(u^*\bar{\phi}) - 2iu_x = 0. \quad (\text{A.9})$$

We differentiate (A.8) by x and use it to replace the second term of (A.9). Consequently, equation (A.9) becomes

$$i\bar{\phi}_t - \bar{\phi}_{xx} + 2iuP_+(u^*\bar{\phi})_x = 0. \quad (\text{A.10})$$

It follows from (A.10) and equation (1.1) that

$$i(u^*\bar{\phi})_t = (u^*\bar{\phi}_x - u_x^*\bar{\phi})_x - i(u^*|u|^2\bar{\phi})_x + H(|u|^2)_xu^*\bar{\phi} - |u|^2H(u^*\bar{\phi})_x. \quad (\text{A.11})$$

Integrating (A.11) with respect to x and taking into account the formula $\int_{-\infty}^{\infty} fHg_x dx = \int_{-\infty}^{\infty} gHf_x dx$, we find that the quantity $I \equiv \int_{-\infty}^{\infty} u^*\bar{\phi} dx$ is conserved in time. This can be interpreted as a generating function for the conserved quantities. Specifically, if we expand $\bar{\phi}$ in inverse powers of λ as $\bar{\phi} = \sum_{j=1}^{\infty} (-1)^j \bar{\phi}_j \lambda^{-j}$, insert this series into (A.8) and then set the coefficients of λ^{-j} to zero, we obtain the linear recursion relation for $\bar{\phi}_j$

$$\bar{\phi}_1 = u, \quad \bar{\phi}_{j+1} = i\bar{\phi}_{j,x} + uP_+(u^*\bar{\phi}_j), \quad (j = 1, 2, \dots). \quad (\text{A.12})$$

The j th conserved quantity I_j is then generated via the series expansion

$$I = \sum_{j=1}^{\infty} (-1)^j I_j \lambda^{-j}, \quad I_j = \int_{-\infty}^{\infty} u^*\bar{\phi}_j dx. \quad (\text{A.13})$$

The explicit forms of $\bar{\phi}_2$ and $\bar{\phi}_3$ are found to be as

$$\bar{\phi}_2 = \mathrm{i}u_x + uP_+(|u|^2), \quad (\text{A.14})$$

$$\bar{\phi}_3 = -u_{xx} + \mathrm{i}(uP_+|u|^2)_x + u \left[\mathrm{i}P_+(u^*u_x) + P_+\{|u|^2P_+(|u|^2)\} \right]. \quad (\text{A.15})$$

The conserved quantities I_j for $j = 1, 2, 3$ are computed by using (A.13). They read

$$I_1 = \int_{-\infty}^{\infty} |u|^2 dx, \quad (\text{A.16})$$

$$I_2 = \int_{-\infty}^{\infty} \left[\frac{1}{2}|u|^4 + \frac{\mathrm{i}}{2}(u^*u_x - u_x^*u) \right] dx, \quad (\text{A.17})$$

$$I_3 = \int_{-\infty}^{\infty} \left[\frac{1}{3}|u|^6 + \frac{\mathrm{i}}{2}(u^*u_x - u_x^*u)|u|^2 + \frac{1}{2}|u|^2 H(|u|^2)_x + u_x^*u_x \right] dx. \quad (\text{A.18})$$

Appendix B. Proof of lemma 2.1.

First, we enumerate the basic formulas of determinants which are used frequently in the proof. Among them, Jacobi's identity will play an important role. See, for example [32].

$$\frac{\partial |D|}{\partial x} = \sum_{j,k=1}^N \frac{\partial d_{jk}}{\partial x} D_{jk}, \quad (\text{B.1})$$

$$\begin{vmatrix} D & \mathbf{a}^T \\ \mathbf{b} & z \end{vmatrix} = |D|z - \sum_{j,k=1}^N D_{jk}a_jb_k, \quad (\text{B.2})$$

$$|D(\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d})||D| = |D(\mathbf{a}; \mathbf{c})||D(\mathbf{b}; \mathbf{d})| - |D(\mathbf{a}; \mathbf{d})||D(\mathbf{b}; \mathbf{c})|, \quad (\text{Jacobi's identity}), \quad (\text{B.3})$$

$$\delta_{jk}|D| = \sum_{l=1}^N d_{jl}D_{kl} = \sum_{l=1}^N d_{lj}D_{lk}, \quad (\text{B.4})$$

$$D_{jk} = \sum_{l=1}^N d_{lm} D_{jl,km} \quad (k \neq m) \quad (\text{B.5a})$$

$$= \sum_{m=1}^N d_{lm} D_{jl,km} \quad (j \neq l), \quad (\text{B.5b})$$

B.1. Proof of (2.8a)

We differentiate the element f_{jk} of the matrix F from (2.2a) to obtain

$$\begin{aligned}(f_{jk})_t &= \mathbf{i}(p_j^2 - q_j^2)\zeta_j\delta_{jk} \\ &= \mathbf{i}(p_j^2 - q_k^2)f_{jk} - \mathbf{i}(p_j + q_k).\end{aligned}$$

Referring to (B.1) and (B.4), we deduce

$$\begin{aligned}|F|_t &= \mathbf{i} \sum_{j,k=1}^N (p_j^2 - q_k^2)f_{jk}F_{jk} - \mathbf{i} \sum_{j,k=1}^N (p_j + q_k)F_{jk} \\ &= \mathbf{i} \sum_{j=1}^N (p_j^2 - q_j^2)|F| + \mathbf{i}(|F(\mathbf{1}; \mathbf{p})| + |F(\mathbf{q}; \mathbf{1})|).\end{aligned}$$

B.2. Proof of (2.8b)

The proof is performed in parallel with the proof of (2.8a). Specifically,

$$\begin{aligned}(f_{jk})_x &= -\mathbf{i}(p_j - q_j)\zeta_j\delta_{jk} \\ &= -\mathbf{i}(p_j - q_k)f_{jk} + \mathbf{i}.\end{aligned}$$

$$\begin{aligned}|F|_x &= -\mathbf{i} \sum_{j,k=1}^N (p_j - q_k)f_{jk}F_{jk} + \mathbf{i} \sum_{j,k=1}^N F_{jk} \\ &= -\mathbf{i}\mu|F| - \mathbf{i}|F(\mathbf{1}; \mathbf{1})|.\end{aligned}$$

B.3. Proof of (2.8c)

It follows by differentiating (2.8b) with respect to x that

$$|F|_{xx} = -\mathbf{i}\mu|F|_x + \mathbf{i} \sum_{j,k=1}^N (F_{jk})_x.$$

Applying (B.1) and (B.5) to the cofactor F_{jk} , we obtain

$$\begin{aligned}(F_{jk})_x &= \sum_{l,m=1}^N (f_{lm})_x F_{lj,mk} \\ &= -\mathbf{i} \sum_{l,m=1}^N (p_l - q_m)f_{lm}F_{lj,mk} + \mathbf{i} \sum_{l,m=1}^N F_{lj,mk}.\end{aligned}$$

Since $|F| \neq 0$, Jacobi's formula (B.3) can be used to give

$$F_{lj,mk} = \frac{1}{|F|}(F_{lm}F_{jk} - F_{lk}F_{jm}).$$

If we substitute this into the expression of $(F_{jk})_x$ and use (B.4), we obtain

$$\begin{aligned} (F_{jk})_x &= -\frac{i}{|F|} \sum_{l,m=1}^N (p_l - q_m) f_{lm} (F_{lm}F_{jk} - F_{lk}F_{jm}) + \frac{i}{|F|} \sum_{l,m=1}^N (F_{lm}F_{jk} - F_{lk}F_{jm}) \\ &= -i\mu F_{jk} + i(p_j - q_k)F_{jk} + \frac{i}{|F|} \sum_{l,m=1}^N (F_{lm}F_{jk} - F_{lk}F_{jm}). \end{aligned}$$

Thus, we find, after introducing the above expression into the right-hand side of $|F|_{xx}$ and noting the identity, $\sum_{j,k,l,m=1}^N (F_{lm}F_{jk} - F_{lk}F_{jm}) = 0$ that

$$|F|_{xx} = -i\mu|F|_x + \mu \sum_{j,k=1}^N F_{jk} - \sum_{j,k=1}^N (p_j - q_k)F_{jk}.$$

Last, formula (2.8c) follows by substituting $|F|_x$ from (2.8b) and referring to (B.2).

B.4. Proof of (2.9a)

It follows by applying (B.1) and (B.4) that

$$\begin{aligned} |G|_t &= \sum_{j,k=1}^N (g_{jk})_t G_{jk} \\ &= i \sum_{j=2,k=1}^N (p_j^2 - q_k^2) g_{jk} G_{jk} - i \sum_{j=2,k=1}^N (p_j + q_k) G_{jk} \\ &= i \sum_{j=1}^N (p_j^2 - q_j^2) |G| + i \sum_{k=1}^N q_k^2 g_{1k} G_{1k} - i \sum_{j=2,k=1}^N (p_j + q_k) G_{jk}. \end{aligned}$$

Applying the cofactor expansion to the second term with $g_{1k} = 1$ ($k = 1, 2, \dots, N$) and taking into account the definitions (1.7) and (1.8), we obtain

$$\sum_{k=1}^N q_k^2 g_{1k} G_{1k} = -|F(\mathbf{Q}; \mathbf{e}_1)|.$$

We compute the third term while taking into account the relation $p_1 = 0$

$$\begin{aligned} \sum_{j=2,k=1}^N (p_j + q_k) G_{jk} &= \sum_{j=1,k=1}^N (p_j + q_k) G_{jk} + \sum_{k=1}^N q_k G_{1k} \\ &= -|G(\mathbf{1}; \mathbf{p})| - |G(\mathbf{q}; \mathbf{1})| + |G(\mathbf{q}; \mathbf{e}_1)|. \end{aligned}$$

Since $p_1 = 0$, the first term in the above expression becomes zero. The sum of the second and third terms is shown to be $|F(\mathbf{q}, \mathbf{1}; \mathbf{1}, \mathbf{e}_1)|$. Summing up these results, we confirm (2.9a).

B.5. Proof of (2.9b)

In view of the relations $p_1 = 0$ and $g_{1k} = 1$ ($k = 1, 2, \dots, N$), we deduce

$$\begin{aligned} |G|_x &= -i \sum_{j=2, k=1}^N (p_j - q_k) g_{jk} G_{jk} + i \sum_{j=2, k=1}^N G_{jk} \\ &= -i\mu |G| - i \sum_{k=1}^N q_k G_{1k} + i \sum_{j, k=1}^N G_{jk} - i \sum_{k=1}^N G_{1k}. \end{aligned}$$

The sum of the third and fourth terms is found to be zero whereas the second term can be written as $i|F(\mathbf{q}; \mathbf{e}_1)|$, which, plugged into the first term, leads to (2.9b).

B.6. Proof of (2.9c)

Define an $N \times N$ matrix \hat{G} by

$$\hat{G} = (\hat{g}_{jk})_{1 \leq j, k \leq N}, \quad \hat{g}_{1k} = q_k, \quad \hat{g}_{jk} = f_{jk}, \quad (j = 2, 3, \dots, N, k = 1, 2, \dots, N).$$

Then, $|\hat{G}| = -|F(\mathbf{q}; \mathbf{e}_1)|$. We compute $|\hat{G}|_x$ by using the rule (B.1) and obtain

$$\begin{aligned} |\hat{G}|_x &= \sum_{j, k=1}^N (\hat{g}_{jk})_x \hat{G}_{jk} \\ &= -i \sum_{j=2, k=1}^N (p_j - q_k) \hat{g}_{jk} \hat{G}_{jk} + i \sum_{j=2, k=1}^N \hat{G}_{jk}. \end{aligned}$$

Referring to (B.4), we can see that

$$\sum_{j=2, k=1}^N q_k \hat{g}_{jk} \hat{G}_{jk} = \sum_{k=1}^N q_k |\hat{G}| - \sum_{k=1}^N q_k^2 \hat{G}_{1k},$$

which, inserted into the first term of $|\hat{G}|_x$, gives

$$\begin{aligned} |\hat{G}|_x &= -i \left\{ \sum_{j=1}^N (p_j - q_j) |\hat{G}| + \sum_{k=1}^N q_k^2 \hat{G}_{1k} \right\} + i \sum_{j=2, k=1}^N \hat{G}_{jk} \\ &= -i\mu |\hat{G}| + i|F(\mathbf{Q}; \mathbf{e}_1)| + i(-|\hat{G}(\mathbf{1}; \mathbf{1})| + |G|), \end{aligned}$$

where we have applied (B.2) in passing to the second line. It is now straightforward to show that

$$-|\hat{G}(\mathbf{1}; \mathbf{1})| + |G| = -|F(\mathbf{q}, \mathbf{1}; \mathbf{q}, \mathbf{e}_1)|.$$

If we substitute the above formulas and $|G|_x$ from (2.9b) into the expression $|G|_{xx} = -i(\mu|G|_x + |\hat{G}|_x)$, we finally arrive at (2.9c).

Appendix C. Proof of lemma 2.2.

C.1. Proof of (2.11)

Introduce the Cauchy matrix C and its inverse $\tilde{C}(=C^{-1})$

$$C = (c_{jk})_{1 \leq j, k \leq N}, \quad \tilde{C} = (\tilde{c}_{jk})_{1 \leq j, k \leq N}, \quad (C.1)$$

where

$$c_{jk} = \frac{1}{p_j - q_k}, \quad \tilde{c}_{jk} = (p_k - q_j)A_k(q_j)B_j(p_k),$$

with

$$A_k(q_j) = \prod_{\substack{l=1 \\ (l \neq k)}}^N \frac{q_j - p_l}{p_k - p_l}, \quad B_j(p_k) = \prod_{\substack{l=1 \\ (l \neq j)}}^N \frac{p_k - q_l}{q_j - q_l}, \quad (j, k = 1, 2, \dots, N).$$

The following formulas are well-known [33]

$$\sum_{j=1}^N \tilde{c}_{jk} = \frac{\prod_{l=1}^N (p_k - q_l)}{\prod_{\substack{l=1 \\ (l \neq k)}}^N (p_k - p_l)} = (p_k - q_k)\tilde{c}_k, \quad \tilde{c}_k = \prod_{\substack{l=1 \\ (l \neq k)}}^N \frac{p_k - q_l}{p_k - p_l}, \quad (k = 1, 2, \dots, N), \quad (C.2)$$

$$|\tilde{C}| = \prod_{j=1}^N (p_j - q_j) \prod_{\substack{j, k=1 \\ (j > k)}}^N \frac{(p_j - q_k)(q_j - p_k)}{(p_j - p_k)(q_j - q_k)} = \prod_{j=1}^N (p_j - q_j) \prod_{\substack{j, k=1 \\ (j > k)}}^N e^{-A_{jk}}. \quad (C.3)$$

Now, we are going to start the proof. First, we compute

$$\begin{aligned} (F^* \tilde{C})_{jk} &= \sum_{l=1}^N \left(\zeta_j^* \delta_{jl} + \frac{1}{p_j - q_l} \right) \tilde{c}_{lk} \\ &= \zeta_j^* \tilde{c}_{jk} + \delta_{jk}. \end{aligned}$$

and evaluate the determinant of the matrix $F^* \tilde{C}$

$$|F^* \tilde{C}| = \left| (\zeta_j^* \tilde{c}_{jk} + \delta_{jk})_{1 \leq j, k \leq N} \right|.$$

We substitute the relation

$$\zeta_j^* = \frac{e^{i\theta_j + \delta_j}}{p_j - q_j} = \frac{e^{2(\delta_j - \phi_j)}}{(p_j - q_j)^2 e^{-2\phi_j} \zeta_j},$$

into the above expression and then extract the factor $e^{2\phi_j}/\zeta_j$ from the j th row for $j = 1, 2, \dots, N$ and obtain

$$|F^* \tilde{C}| = \prod_{j=1}^N \frac{e^{2\phi_j}}{\zeta_j} \left| \left(e^{2(\delta_j - \phi_j)} \frac{\tilde{c}_{jk}}{(p_j - q_j)^2} + e^{-2\phi_j} \zeta_j \delta_{jk} \right)_{1 \leq j, k \leq N} \right|.$$

To modify this further, we insert the relations

$$e^{2(\delta_j - \phi_j)} = \prod_{\substack{l=1 \\ (l \neq j)}}^N \frac{(p_j - p_l)(q_j - q_l)}{(p_j - q_l)(q_j - p_l)}, \quad \exp \left[- \sum_{j=1}^N (i\theta_j + \phi_j) \right] = \prod_{j=1}^N \frac{(p_j - q_j) \zeta_j}{e^{\delta_j + \phi_j}},$$

as well as the expression of \tilde{c}_{jk} from (C.1). This gives

$$\begin{aligned} & |F^* \tilde{C}| \exp \left[- \sum_{j=1}^N (i\theta_j + \phi_j) \right] \\ &= \prod_{j=1}^N \{ (p_j - q_j) e^{\phi_j - \delta_j} \} \left| \left(\frac{p_k - q_k}{(p_j - q_j)(p_k - q_j)} \prod_{\substack{l=1 \\ (l \neq j)}}^N \frac{p_j - p_l}{p_j - q_l} \prod_{\substack{l=1 \\ (l \neq k)}}^N \frac{p_k - q_l}{p_k - p_l} + e^{-2\phi_j} \zeta_j \delta_{jk} \right)_{1 \leq j, k \leq N} \right| \\ &= \prod_{j=1}^N \{ (p_j - q_j) e^{\phi_j - \delta_j} \} \left| \left(e^{-2\phi_j} \zeta_j \delta_{jk} + \frac{1}{p_j - q_k} \right)_{1 \leq j, k \leq N} \right|, \end{aligned} \tag{C.4}$$

where in passing to the second line, we have extracted the factor $(p_j - q_j)^{-1} \prod_{\substack{l=1 \\ (l \neq j)}}^N (p_j - p_l)/(p_j - q_l)$ from the j th row and the factor $(p_k - q_k) \prod_{\substack{l=1 \\ (l \neq k)}}^N (p_k - q_l)/(p_k - p_l)$ from the k th column, respectively for $j, k = 1, 2, \dots, N$ and used the formula $|\bar{F}^T| = |\bar{F}|$. Last, dividing both sides of (C.4) by $|\tilde{C}|$ and taking into account the relations (C.3) and $\prod_{j=1}^N e^{\phi_j - \delta_j} = \prod_{\substack{j,k=1 \\ (j > k)}}^N e^{-A_{jk}}$, we establish (2.11).

C.2. Proof of (2.12)

We compute the product of the two determinants $|G|^*$ and $|\tilde{C}|$ and express it in the form of a bordered determinant

$$|G|^*|\tilde{C}| = - \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1N} & 1 \\ b_{21} & b_{22} & \cdots & b_{2N} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{N1} & b_{N2} & \cdots & b_{NN} & 0 \\ \sum_{j=1}^N \tilde{c}_{j1} & \sum_{j=1}^N \tilde{c}_{j2} & \cdots & \sum_{j=1}^N \tilde{c}_{jN} & 0 \end{vmatrix},$$

where $B = F^* \tilde{C} = (b_{jk})_{1 \leq j, k \leq N}$ is an $N \times N$ matrix with elements

$$b_{jk} = \frac{1}{p_j - q_j} \frac{e^{2\phi_j}}{\zeta_j} \frac{\tilde{c}_k}{\tilde{c}_j} \frac{p_k - q_k}{p_k - q_j} + \delta_{jk}.$$

After rewriting the $(N+1)$ th row by the formula (C.2), we extract the factor $e^{2\phi_j}/\{(p_j - q_j)\zeta_j \tilde{c}_j\}$ from the j th row and the factor $(p_k - q_k)\tilde{c}_k$ from the k th column, respectively for $j = 2, 3, \dots, N$ and $k = 1, 2, \dots, N$. The resulting expression can be written as

$$|G|^*|\tilde{C}| = - \prod_{j=2}^N \frac{e^{2\phi_j}}{\zeta_j} \frac{1}{\tilde{c}_j} \frac{1}{p_j - q_j} \prod_{k=1}^N \tilde{c}_k (p_k - q_k) \times \begin{vmatrix} (p_1 - q_1)^{-1} \tilde{c}_1^{-1} b_{11} & (p_2 - q_2)^{-1} \tilde{c}_2^{-1} b_{12} & \cdots & (p_N - q_N)^{-1} \tilde{c}_N^{-1} b_{1N} & 1 \\ \bar{f}_{12} & \bar{f}_{22} & \cdots & \bar{f}_{N2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{f}_{1N} & \bar{f}_{2N} & \cdots & \bar{f}_{NN} & 0 \\ 1 & 1 & \cdots & 1 & 0 \end{vmatrix},$$

where the elements \bar{f}_{jk} have been defined by (2.10). Interchanging the first row and $(N+1)$ th row and then expanding the determinant with respect to $(N+1)$ th column, we deduce

$$|G|^*|\tilde{C}| = (p_1 - q_1) \tilde{c}_1 \prod_{j=2}^N \frac{e^{2\phi_j}}{\zeta_j} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \bar{f}_{12} & \bar{f}_{22} & \cdots & \bar{f}_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{f}_{1N} & \bar{f}_{2N} & \cdots & \bar{f}_{NN} \end{vmatrix},$$

Since $|\tilde{C}| \neq 0$ by (C.3) and (2.4), we can divide the above expression by $|\tilde{C}|$. If we insert the relations

$$\phi_j = 0, \quad \zeta_j = \frac{1}{p_j - q_j} \exp \left[-i\theta_j + \frac{1}{2} \sum_{\substack{k=1 \\ (k \neq j)}}^N A_{jk} \right], \quad (j = 2, 3, \dots, N), \quad p_1 = 0, \quad \tilde{c}_1 = \prod_{j=2}^N \frac{q_j}{p_j},$$

into the result and take into account the relations $\bar{f}_{jk} = f_{jk}$ ($j = 1, 2, \dots, N, k = 2, 3, \dots, N$), we find

$$|G|^* = \prod_{j=2}^N \frac{q_j}{p_j} \exp \left[i \sum_{j=2}^N \theta_j + \frac{1}{2} \sum_{k=2}^N A_{1k} \right] d, \quad (C.5)$$

where a factor d is the determinant of a matrix constructed from F^T by replacing its first row by the row vector $\mathbf{1}$. Explicitly,

$$d = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_{12} & f_{22} & \cdots & f_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ f_{1N} & f_{2N} & \cdots & f_{NN} \end{vmatrix}.$$

After a few manipulations, it can be recast to

$$d = \begin{vmatrix} f_{22} & \cdots & f_{2N} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ f_{N2} & \cdots & f_{NN} & 1 \\ -\frac{1}{q_2} & \cdots & -\frac{1}{q_N} & 1 \end{vmatrix},$$

where we have used $f_{1j} = -1/q_j$ ($j = 2, 3, \dots, N$). Then, after extracting the factor $1/q_{k+1}$ from the k th column for $k = 1, 2, \dots, N-1$, we add the N th column to the j th column for $j = 1, 2, N-1$, giving

$$\begin{aligned} d &= \prod_{j=2}^N \frac{p_j}{q_j} \left| \left(\frac{q_j}{p_j} \zeta_j \delta_{jk} + \frac{1}{p_j - q_k} \right)_{2 \leq j, k \leq N} \right| \\ &= \prod_{j=2}^N \frac{p_j}{q_j} |\bar{G}|. \end{aligned} \quad (C.6)$$

Consequently, substitution of (C.6) into (C.5) yields (2.12).

D. Proof of proposition 2.4

Let L_j and l_j be

$$L_j = \sum_{k=1}^N f_{jk} l_k, \quad l_j = i\phi_{j,x} + \lambda_j \phi_j + u\psi_j^+, \quad (j = 1, 2, \dots, N). \quad (D.1)$$

By employing Cramer's rule, the solution of the linear system (2.32a) can be written as $\phi_j = g_0 F_{1j}/|F|$. If we take into account the definition of the matrix G from (2.2b), we have

$$\sum_{j=1}^N \phi_j = g_0 \frac{|G|}{|F|} = u. \quad (D.2)$$

Differentiating (2.32a) with respect to x and using the relation

$$f_{jk,x} = -i(p_j - q_k)f_{jk} + i, \quad (D.3)$$

which follows from (2.2a), we deduce

$$\begin{aligned} \sum_{k=1}^N f_{jk}\phi_x &= -\sum_{k=1}^N f_{jk,x}\phi_k \\ &= i\sum_{k=1}^N (p_j - q_k)f_{jk}\phi_k - i\sum_{k=1}^N \phi_k \\ &= ip_j g_0 \delta_{j1} - i\sum_{k=1}^N q_k f_{jk}\phi_k - iu \\ &= -i\sum_{k=1}^N q_k f_{jk}\phi_k - iu, \end{aligned} \quad (D.4)$$

where we have used (D.2) and the relation $p_j \delta_{j1} = p_1 \delta_{j1} = 0$ with $p_1 = 0$. Thus, substitution of (D.4) into (D.1) gives

$$L_j = \sum_{k=1}^N q_k f_{jk}\phi_k + u + \sum_{k=1}^N \lambda_k f_{jk}\phi_k + u \sum_{k=1}^N f_{jk}\psi_k^+.$$

Referring to (2.32b) and the relation $\lambda_j = -q_j$, we see that $L_j = 0$ for $j = 1, 2, \dots, N$. Since the coefficient matrix F is nonsingular for $x \in \mathbb{R}$ (See proposition 2.2), this homogeneous system for l_j has only the trivial solution $l_j = 0$ ($j = 1, 2, \dots, N$), which proves (2.33).

Next, define M_j and m_j by

$$M_j = \sum_{k=1}^N f_{jk}m_k, \quad m_j = i\phi_{j,t} - 2i\lambda_j\phi_{j,x} + \phi_{j,xx} - 2iu_x\psi_j^+, \quad (j = 1, 2, \dots, N). \quad (D.5)$$

Use (D.2) and the relation

$$f_{jk,t} = i(p_j^2 - q_k^2)f_{jk} - i(p_j + q_k), \quad (D.6)$$

to verify

$$\begin{aligned} \sum_{k=1}^N f_{jk}\phi_{k,t} &= -\sum_{k=1}^N f_{jk,t}\phi_k \\ &= -ip_j^2 g_0 \delta_{j1} + i\sum_{k=1}^N q_k^2 f_{jk}\phi_k + ip_j u + i\sum_{k=1}^N q_k \phi_k. \end{aligned} \quad (D.7)$$

Note that the first term on the right-hand side of (D.7) vanishes. It now follows from (2.33) with $\lambda_j = -q_j$ that

$$\phi_{k,x} = i(-q_k\phi_k + u\psi_k^+), \quad (D.8)$$

$$\phi_{k,xx} = -q_k^2\phi_k + iu\psi_{k,x}^+ + (iu_x + q_ku)\psi_k^+. \quad (D.9)$$

By employing (D.8) and (D.9), we compute

$$-2i\lambda_k\phi_{k,x} + \phi_{k,xx} - 2iu_x\psi_k^+ = q_k^2\phi_k + iu\psi_{k,x}^+ - (iu_x + q_ku)\psi_k^+. \quad (D.10)$$

Introducing (D.7) and (D.10) into (D.5), M_j becomes

$$M_j = -p_ju - \sum_{k=1}^N q_k\phi_k + iu \sum_{k=1}^N f_{jk}\psi_{k,x}^+ + iu_x - u \sum_{k=1}^N f_{jk}q_k\psi_k^+. \quad (D.11)$$

We differentiate (2.32b) by x and use (D.3) and (2.32b) to obtain

$$\begin{aligned} \sum_{k=1}^N f_{jk}\psi_{k,x}^+ &= - \sum_{k=1}^N f_{jk,x}\psi_k^+ \\ &= -ip_j - i \sum_{k=1}^N f_{jk}q_k\psi_k^+ - i \sum_{k=1}^N \psi_k^+. \end{aligned} \quad (D.12)$$

Substituting (D.12) into (D.11), M_j reduces to

$$\begin{aligned} M_j &= - \sum_{k=1}^N q_k\phi_k + u \sum_{k=1}^N \psi_k^+ + iu_x \\ &= -i \sum_{k=1}^N \phi_{k,x} + iu_x, \end{aligned}$$

where in passing to the second line, we have used (D.8). This expression becomes zero by virtue of (D.2), implying that $l_j = 0$ ($j = 1, 2, \dots, N$). This proves (2.34).

Last, define N_j and n_j by

$$N_j = \sum_{k=1}^N f_{jk}n_k, \quad n_j = i\psi_{j,t}^+ - 2i\lambda_j\psi_{j,x}^+ + \psi_{j,xx}^+ - i(1 - iH)(|u|^2)_x\psi_j^+, \quad (j = 1, 2, \dots, N). \quad (D.13)$$

It follows from (2.32b) that

$$\begin{aligned}
\sum_{k=1}^N f_{jk} \psi_{k,t}^+ &= - \sum_{k=1}^N f_{jk,t} \psi_k^+ \\
&= - \sum_{k=1}^N \{i(p_j^2 - q_k^2) f_{jk} - i(p_j + q_k)\} \psi_k^+ \\
&= ip_j^2 + i \sum_{k=1}^N q_k^2 f_{jk} \psi_k^+ + ip_j \sum_{k=1}^N \psi_k^+ + i \sum_{k=1}^N q_k \psi_k^+. \tag{D.14}
\end{aligned}$$

If we differentiate (2.32b) twice by x and use (D.3), we can derive the relation

$$\begin{aligned}
\sum_{k=1}^N f_{jk} \psi_{k,xx}^+ &= - \sum_{k=1}^N f_{jk,xx} \psi_k^+ - 2 \sum_{k=1}^N f_{jk,x} \psi_{k,x}^+ \\
&= \sum_{k=1}^N \{(p_j - q_k)^2 f_{jk} - (p_j - q_k)\} \psi_k^+ \\
&\quad - 2i \sum_{k=1}^N \{-(p_j - q_k) f_{jk} + 1\} \psi_{k,x}^+. \tag{D.15}
\end{aligned}$$

Substituting (D.14) and (D.15) into (D.13) and using (2.32b), we recast it to

$$\begin{aligned}
N_j &= -p_j^2 - 2p_j \sum_{k=1}^N \psi_k^+ + \sum_{k=1}^N p_j(p_j - 2q_k) f_{jk} \psi_k^+ \\
&\quad - 2i \sum_{k=1}^N (-p_j f_{jk} + 1) \psi_{k,x}^+ + i(1 - iH)(|u|^2)_x. \tag{D.16}
\end{aligned}$$

With (D.3), a term in (B.16) is modified as

$$\sum_{k=1}^N f_{jk} \psi_{k,x}^+ = - \sum_{k=1}^N \{-i(p_j - q_k) f_{jk} + i\} \psi_k^+.$$

After a few computations, N_j simplifies considerably as shown below:

$$N_j = -2i \sum_{k=1}^N \psi_{k,x}^+ + i(1 - iH)(|u|^2)_x. \tag{D.17}$$

To proceed, we note the relation

$$\sum_{k=1}^N \psi_k^+ = \frac{|F(\mathbf{1}; \mathbf{1})|}{|F|}, \tag{D.18}$$

which can be verified by solving the linear system (2.32b) by means of Cramer's rule to obtain $\psi_k^+ = -\sum_{l=1}^N F_{lk}/|F|$ and then invoking the formula (B.2). Furthermore, the formula

$$\mathrm{i}(1 - \mathrm{i}H)(|u|^2)_x = -2 \left(\frac{f_x}{f} \right)_x, \quad (D.19)$$

follows from (2.5) and an analyticity property of $f(= |F|)$. If we substitute (D.18) and (D.19) into (D.17) and use (2.8b), we can see that $N_j = 0$ and hence $n_j = 0$ for $j = 1, 2, \dots, N$, which implies (2.35). Three results established above now complete the proof of proposition (2.4).

References

- [1] Abanov A G, Bettelheim E and Wiegmann P 2009 Integrable hydrodynamics of Calogero-Sutherland model: bidirectional Benjamin-Ono equation *J. Phys. A: Math. Gen.* **42** 135201
- [2] Pelinovsky D 1995 Intermediate nonlinear Schrödinger equation for internal waves in a fluid of finite depth *Phys. Lett. A* **197** 401-6
- [3] Matsuno Y 2000 Multiperiodic and multisoliton solutions of a nonlocal nonlinear Schrödinger equation for envelope waves *Phys. Lett. A* **278** 53-8
- [4] Matsuno Y 2001 Linear stability of dark solitary wave solutions of a nonlocal nonlinear Schrödinger equation for envelope waves *Phys. Lett. A* **285** 286-92
- [5] Matsuno Y 2002 Calogero-Moser-Sutherland dynamical systems associated with nonlocal nonlinear Schrödinger equation for envelope waves *J. Phys. Soc. Japan* **71** 1415-18
- [6] Matsuno Y 2003 Asymptotic solutions of the nonlocal nonlinear Schrödinger equation in the limit of small dispersion *Phys. Lett. A* **309** 83-9
- [7] Matsuno Y 2002 Exactly solvable eigenvalue problems for a nonlocal nonlinear Schrödinger equation *Inverse Problems* **18** 1101-25
- [8] Matsuno Y 2004 A Cauchy problem for the nonlocal nonlinear Schrödinger equation *Inverse Problems* **20** 437-45
- [9] Matsuno Y 2004 New representations of multiperiodic and multisoliton solutions for a class of nonlocal soliton equations *J. Phys. Soc. Japan* **73** 3285-93
- [10] Tutiya Y 2006 Bright N -solitons for the intermediate nonlinear Schrödinger equation *J. Nonl. Math. Phys.* **16** 7-23
- [11] Gérard P and Lenzmann E 2022 The Calogero-Moser derivative nonlinear Schrödinger equation (arXiv: 2208.04105v1[math.AP])
- [12] Zakharov V E and Shabat A B 1972 Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media *Sov. Phys.-JETP* **34** 62-9
- [13] Zakharov V E and Shabat A B 1973 Interaction between solitons in a stable medium *Sov. Phys.-JETP* **37** 823-8

- [14] Fadeev L D and Takhtajan L A 2007 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)
- [15] Berntson B K and Fagerlund A 2022 A focusing -defocusing intermediate nonlinear Schrödinger system (arXiv: 2212.03751v1[nlin.SI])
- [16] Matsuno Y 1984 *Bilinear Transformation Method* (New York: Academic)
- [17] Hirota R 2004 *The Direct Method in Soliton Theory* (Cambridge: Cambridge University Press)
- [18] Dobrokhotov S Yu and Krichever I M 1991 Multiphase solutions of the Benjamin-Ono equation and their averaging *Math. Notes* **49** 583-94
- [19] Dubrovin B A 1975 Periodic problems of the Korteweg-de Vries equation in the class of finite-zone potentials *Func. Anal. Appl.* **9** 215-23
- [20] Calogero F 1971 Solution of the one-dimensional N -body problems with quadratic and/or inversely quadratic pair potentials *J. Math. Phys.* **12** 419-36
- [21] Sutherland B 1972 Exact results for a quantum many-body problems in one dimension. II *Phys. Rev. A* **5** 1372-76
- [22] Moser J 1975 Three integrable Hamiltonian systems connected to isospectral deformations *Adv. Math.* **16** 197-220
- [23] van Diejen J F and Vinet L (ed) 2000 *Calogero-Moser-Sutherland models (CRM Series in Mathematical Physics)* (New York: Springer)
- [24] Matsuno Y 1979 Exact multi-soliton solution of the Benjamin-Ono equation *J. Phys. A: Math. Gen.* **12** 619-21
- [25] Matsuno Y 1980 Interaction of the Benjamin-Ono solitons *J. Phys. A: Math. Gen.* **13** 1519-36
- [26] Matsuno Y 1986 N -soliton solution for the sine-Hilbert equation *Phys. Lett. A* **119** 229-33
- [27] Matsuno Y 1987 Periodic problem for the sine-Hilbert equation *Phys. Lett. A* **120** 187-90
- [28] Matsuno Y 1987 Kinks of the sine-Hilbert equation and their dynamical motions *J. Phys. A: Math. Gen.* **20** 3587-606

- [29] Santini P M, Ablowitz M J and Fokas A S 1987 On the initial value problem for a class of nonlinear integral evolution equations including the sine-Hilbert equation *J. Math. Phys.* **28** 2310-16
- [30] Santini P M 1993 Integrable singular integral evolution equations *Important Developments in Soliton Theory* ed A S Fokas and V E Zakharov (New York: Springer) 147-77
- [31] Matsuno Y 1995 Dynamics of interacting algebraic solitons *Int. National. J. Mod. Phys. B* **9** 1985-2081
- [32] Vein R and Dale P 1999 *Determinants and Their Applications in Mathematical Physics* (New York: Springer)
- [33] Schechter S 1959 On the inversion of certain matrices *Mathematical Tables and Other Aids to Computation* **13** 73-7.