ENERGY CALCULATIONS

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Theorem 0.1 (Energy of a Soliton)**.** *Consider an N-soliton of the form*

$$u(t,x) = \frac{P(t,x)}{Q(t,x)}$$

where $Q(t)$ is polynomial in x of degree N and $P(t)$ is polynomial in x of degree at most $N-1$. Then

$$E(u) = \frac{1}{2}\left\|\frac{P_x}{Q}\right\|_{L_x^2}^2$$

Proof. We recall that by the definition of a N-soliton,

$$|P|^2 = i(Q_x\overline{Q} - \overline{Q_x}Q).$$

Then by definition of the energy,

$$\begin{aligned} E(u) &= \frac{1}{2}\int \left|u_x - i\Pi_+(|u|^2)u\right|^2 \\ &= \frac{1}{2}\int \left|\frac{QP_x - PQ_x}{Q^2} + \Pi_+\left(\frac{Q_x}{Q} - \frac{\overline{Q}_x}{\overline{Q}}\right)\frac{P}{Q}\right|^2 \end{aligned}$$

Since Q (resp. $\overline{Q}$) has N zeros in the lower (resp. upper) half plane, the definition of Π_+ implies that

$$\begin{aligned} E(u) &= \frac{1}{2}\int \left|\frac{QP_x - PQ_x}{Q^2} + \frac{Q_xP}{Q^2}\right|^2 \\ &= \frac{1}{2}\left\|\frac{P_x}{Q}\right\|_{L_x^2}^2 \end{aligned}$$

as desired. $\square$

Theorem 0.2 (Energy of Traveling Wave)**.** *Consider a traveling solitary wave*

$$u(t,x) = e^{i\omega t}\mathcal{R}_{v,w}(x-vt) = e^{i\omega t}e^{\frac{i}{2}v(x-vt)}e^{i\theta}\frac{\sqrt{2\lambda}}{\lambda x - \lambda vt + y + i}$$

for $\omega, \theta, v, y \in \mathbb{R}$ and $\lambda > 0$. Then

$$E(u) = \frac{\pi|v|^2}{4}.$$

Proof. Define

$$\tilde{u}(t,x) = e^{i\omega t}e^{\frac{-i}{2}v^2t}e^{i\theta}\frac{\sqrt{2\lambda}}{\lambda x - \lambda vt + y + i}$$

so that $u(t,x) = e^{\frac{i}{2}vx}\tilde{u}(t,x)$. Note that for fixed t, $\tilde{u}(t), u(t)$ have the profiles of 1-solitons. In particular, $\tilde{u}_x - i\Pi_+(|\tilde{u}|^2)\tilde{u} = 0$ and $\|u\|_2^2 = 2\pi$. This implies

$$\begin{aligned} E(u) &= \frac{1}{2}\int \left|\frac{i}{2}vu + e^{\frac{i}{2}vx}\tilde{u}_x - \Pi_+\left(|\tilde{u}|^2\right)e^{\frac{i}{2}vx}\tilde{u}\right|^2 \\ &= \frac{|v|^2}{8}\int |u|^2 \\ &= \frac{\pi|v|^2}{4} \end{aligned}$$

as desired. □

Theorem 0.3 (2-Soliton Energy). *Consider a 2-soliton $u(t,x)$ of the general form*

$$u(t,x) = \frac{e^{i\varphi}\sqrt{2\rho}(\gamma_1 + 2\lambda t + i\lambda^{-1} - x)}{x^2 - (\gamma_0 - i\rho + \gamma_1 + 2\lambda t)x + (\gamma_0 - i\rho)(\gamma_1 + 2\lambda t) - \lambda^{-2}}$$

for $\varphi, \gamma_0, \gamma_1 \in \mathbb{R}$, $\rho > 0$, and $\lambda \in \mathbb{R}\setminus\{0\}$. Then

$$E(u) = \pi\lambda^2.$$

Proof. Let $z_\pm(t)$ denote the zeros of the denominator of $u(t,x)$. Recall that asymptotically,

$$z_+(t) \to \gamma_0 - i\rho, \text{ and } z_-(t) \to 2\lambda t + i\frac{-\rho}{4\lambda^4 t^2} + O(1).$$

Then by direct calculation and theorem **??**,

$$E(u) = \frac{1}{2}\left\|\frac{-e^{i\varphi}\sqrt{2\rho}}{(x-z_+)(x-z_-)}\right\|_{L_x^2}^2 = \rho\left\|\frac{1}{(x-z_+)(x-z_-)}\right\|_{L_x^2}^2$$

A standard contour shift then yields

$$\begin{aligned} E(u) &= \rho\int \frac{1}{(x-z_+)(x-z_-)(x-\overline{z_+})(x-\overline{z_-})} \\ &= 2\pi i\rho\left(\frac{1}{(\overline{z_+}-z_+)(\overline{z_+}-z_-)(\overline{z_+}-\overline{z_-})} + \frac{1}{(\overline{z_-}-z_+)(\overline{z_-}-z_-)(\overline{z_-}-\overline{z_+})}\right) \end{aligned}$$

Since $E(u)$ is conserved, we may work asymptotically to find

$$\begin{aligned} E(u) &= \lim_{t\to\infty} 2\pi i\rho\left(\frac{1}{(2i\rho)(2\lambda t + O(1))(2\lambda t + O(1))} + \frac{1}{(2\lambda t + O(1))(\frac{2i\rho}{4\lambda^4 t^2} + O(1))(2\lambda t + O(1))}\right) \\ &= \pi\lambda^2 \end{aligned}$$

as desired. □

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