FINITELY MANY MASS POINTS ON THE LINE UNDER THE INFLUENCE OF AN EXPONENTIAL POTENTIAL -- AN INTEGRABLE SYSTEM

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1. Analogue of the Toda Lattice for Finitely Many Mass Points

We consider the analogue of the Toda lattice [8] where only a finite number of mass points are admitted which move freely on the real axis. Denoting the position of the mass points by x_k, $k = 1,\ldots,n$, we form the Hamiltonian

(1.1)
$$H = \frac{1}{2}\sum_{k=1}^{n} y_k^2 + \sum_{k=1}^{n-1} e^{(x_k - x_{k+1})}$$

with the differential equations

(1.2)
$$\begin{aligned}
\dot{x}_k &= H_{y_k} = y_k , & & , \quad k = 1,2,\ldots,n \\
\dot{y}_k &= -H_{x_k} = e^{x_{k-1}-x_k} - e^{x_k - x_{k+1}} & & , \quad k = 2,\ldots,n-1 \\
\dot{y}_1 &= -H_{x_1} = \phantom{e^{x_{k-1}-x_k}} - e^{x_1 - x_2} & & \\
\dot{y}_n &= -H_{x_n} = e^{x_{n-1}-x_n} & &
\end{aligned}$$

Thus we can write our system (1.2) as

(1.2')
$$x_k = e^{x_{k-1}-x_k} - e^{x_k - x_{k+1}} , \qquad k = 1,\ldots,n$$

if we set $e^{x_0 - x_1} = 0$ and $e^{x_n - x_{n+1}} = 0$, that is we have the formal boundary condition

(1.3)
$$x_0 = -\infty , \qquad x_{n+1} = +\infty .$$

It is the aim to study completely the flow determined by this

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system of differential equations and relate the solution to the existence of n integrals of the motion. These integrals are essentially the same as those found by Henon [4] and Flaschka [1] for the same system of differential equations (1.2') under periodic boundary conditions, say

$$x_{k+n} = x_k + 1 \qquad\qquad k = 0, \pm 1, \ldots . \tag{1.3'}$$
$$y_{k+n} = y_k$$

The crucial difference between the two problems is that the boundary condition (1.3') gives rise to a compact energy surface and the solutions are expected to be quasiperiodic, lying on tori, as one is familiar from integrable Hamiltonian systems. If we impose the boundary condition (1.3) instead of (1.3') the energy surface is noncompact, as the particles can run to infinity. In fact, we will show, as is intuitively clear, that for any initial configuration mutual distances between all particles grow indefinitely, i.e. $x_{k-1} - x_k \to \infty$ for $k = 2,\ldots,n$; and they behave asymptotically like free particles depending linearly on time. This suggests the scattering problem: To determine the relation between this asymptotic motion for the past and the future. This can be done explicitly here and one finds that $y_{n-k+1}(+\infty) = y_k(-\infty)$, so that at $t = +\infty$ the first particle has the velocity of the last at $t = -\infty$ etc. as in a familiar experiment of collision of steel balls. Moreover, the phase relation can also be determined explicitly and we will show that

$$x_{n-k+1}(t) - x_k(-t) - 2y_k^- t \to \sum_{j<k} \log (y_j^- - y_k^-)^2 - \sum_{j>k} \log (y_j^- - y_k^-)^2 ,$$

where $y_j^- = y_j(-\infty)$ are assumed ordered according to size. Thus the particles behave asymptotically as if they interacted just pairwise! This will be derived in Section 4.

In the limit $t \to +\infty$, the y_k, $k = 1,2,\ldots,n$, or their symmetric functions, are t-independent integrals of the motion, and one may ask for integrals of the given system which asymptotically agree with these integrals. This is indeed possible, and Henon's construction of integrals was based on this idea, even though in the periodic case this idea is not really justified and was only a guiding principle for the construction of integrals. For the noncompact case, i.e. boundary condition (1.3), the free system is indeed the limit state and this approach quite natural. On the other hand, the noncompact case is, of course, much less complicated, as the solutions have no recurrence property and the flow has the nature of parallel flow. In fact, we will show that (1.2) can be mapped into the following system of differential equations,

$$\frac{d\lambda_k}{dt} = 0 \quad , \quad k = 1,\ldots,n, \qquad \frac{dr_k}{dt} = -\frac{\partial V}{\partial r_k} \tag{1.4}$$

where

$$V = \frac{\sum_{k=1}^{n} \lambda_k r_k^2}{2 \sum_{k=1}^{n} r_k^2} \tag{1.4'}$$

and the variables are restricted to the (2n-1) dimensional domain

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n \; ; \quad \sum_{k=1}^{n} r_k^2 = 1 \; , \qquad r_k > 0 \; . \tag{1.5}$$

Clearly, the solutions run from the maximum of V at $r_k = \delta_{kn}$ to the minimum of V of $r_k = \delta_{k1}$ as t runs from $-\infty$ to $+\infty$, and $\lambda_1,\lambda_2,\ldots,\lambda_n$ are integrals of the motion, while $r_1,\ldots,r_{n-1}$ can be viewed as parameters on the surfaces $\lambda_k = \text{const}$. The mapping taking x,y into the variables λ_k, r_k on (1.5) is up to translation of the x_k one to one and will be given explicitly. The inverse mapping illustrates the inverse method of spectral theory.

Thus this note does not claim any new idea and should be considered as providing a simple model illustrating the construction of integrals and its connection with the inverse method of spectral theory in extreme simplicity, yet with all rigor. On the other hand, it leads immediately to an unsolved problem if one wants to carry out this approach for the periodic boundary condition (1.3'). Although the integrals I_k for this problem are well known, no parameters are known on the level surfaces $I_k = c_k$ which determine the x_k (mod 1), y_k uniquely. This is related to the lack of an inverse theory for the Hill's equation $-u'' + q(x)u = \lambda u$, $q(x+1) = q(x)$ under periodic boundary conditions $u(x+1) = u(x)$ where the problem consists in finding a set of quantities which together with the eigenvalues allow one to determine $q(x)$. One can hope to shed some light on this question if one could solve the above finite dimensional problem.

2. Flaschka's Form of the Differential Equation and Asymptotic Behavior

We set, with Flaschka,

$$a_k = \frac{1}{2} e^{(x_k - x_{k+1})/2} \quad , \quad b_k = -\frac{1}{2} y_k \tag{2.1}$$

so that the differential equations (1.2) go into

$$\begin{aligned} \dot{a}_k &= a_k(b_{k+1} - b_k) \quad , \quad k = 1,2,\ldots,n-1 \\ \dot{b}_k &= 2(a_k^2 - a_{k-1}^2) \quad , \quad k = 1,2,\ldots,n \end{aligned} \tag{2.2}$$

with the boundary conditions (1.3) being

$$a_0 = 0 \ , \quad a_n = 0 \ . \tag{2.3}$$

Observe that (2.1) provides a transformation of the (x,y) variables into the (a,b)-variables. We identify points (x,y), $(\tilde{x},\tilde{y})$ if $x_k - \tilde{x}_k$ is independent of k, and call the equivalence class a "configuration". It is characterized by the $2n-1$ numbers $x_k - x_n$, $(k = 1,\ldots,n-1$, and y_k, $k = 1,\ldots,n$. Thus (2.1) defines an invertible transformation of the $(2n-1)$-dimensional space of configurations into the domain

$$D = \{a,b \mid a_k > 0 \ , \ k = 1,\ldots,n-1)\} \ ,$$

and it remains to study the flow given by the quadratic differential equation (2.2) in D. The energy is given by

$$H = 4\left\{ \sum_{k=1}^{n-1} a_k^2 + \frac{1}{2} \sum_{k=1}^{n} b_k^2 \right\} . \tag{2.4}$$

We show first that for any solution in D

$$a_k(t) \to 0 \quad \text{for} \quad t \to \pm\infty \quad \text{and} \quad k = 1,\ldots,n-1 \ , \tag{2.5}$$

which amounts to the assertion that $x_{k+1}-x_k \to \infty$ as $t \to \pm\infty$. To prove this we consider the system (2.2) with prescribed $a_0(t), a_n(t) \in L^2(-\infty,+\infty)$ so that $\int_{-\infty}^{+\infty} (a_0^2 + a_n^2)\, dt < \infty$ and prove the

Lemma. For any solution of (2.2) with this modified boundary condition we have

$$\int_{-\infty}^{\infty} (a_1^2 + a_{k-1}^2)\, dt < \infty \ .$$

Proof: Consider the function

$$\phi(t) = b_1 - b_n$$

for which

$$\frac{d\phi}{dt} = \dot{b}_1 - \dot{b}_n = 2(a_0^2 + a_n^2) - 2(a_1^2 + a_{n-1}^2) \ .$$

Thus

$$\psi = \frac{1}{2}\phi - \int_{-\infty}^{t} (a_0^2 + a_n^2)\, dt$$

satisfies

$$\frac{d\psi}{dt} = -(a_1^2 + a_{n-1}^2) .$$

Since by the energy relation ϕ and hence ψ is bounded, also

$$\int_{-T}^{T} (a_1^2 + a_{n-1}^2)\, dt = \psi(-T) - \psi(T)$$

is bounded for $T \to \pm\infty$, proving the lemma.

We can apply this argument, in particular, to $a_0 = a_n = 0$. Applying this lemma to the reduced system where the first and last equations in the first and second line of (2.2) are cancelled we conclude that $\int_{-\infty}^{+\infty} (a_n^2 + a_{n-2}^2)\, dt < \infty$ and inductively that

$$\sum_{k=1}^{n-1} \int_{-\infty}^{+\infty} a_k^2\, dt < \infty . \tag{2.6}$$

Since on the other hand $|\dot{p}| \leq 2 \sum a_k^2 |b_k - b_{k+1}| \leq M$ is bounded it follows that $p = \sum_1^{n-1} a_k^2 \to 0$ for $t \to \pm\infty$. Indeed, otherwise there would exist a sequence $|t_k| \to \infty$ with $p(t_k) \geq \delta > 0$. We may assume that the sequence is so selected that $|t_{k+1} - t_k| \geq \delta/M$. Since $p(t) \geq \delta/2$ in the disjoint intervals $|t - t_k| < \frac{1}{2}\frac{\delta}{M}$ it cannot be integrable, contradicting (2.6). This proves (2.5). Moreover, we conclude from (2.2) that b_k tends to a limit $b_k(\infty)$ as $t \to +\infty$.

Flaschka [1,2] noted that the above system (2.2) can be expressed in matrix form

$$\frac{d}{dt} L = BL - LB \tag{2.7}$$

where

$$L = \begin{pmatrix} b_1 & a_1 & & & 0 \\ a_1 & b_2 & & & \\ & & \ddots & & \\ & & & b_{n-1} & a_{n-1} \\ 0 & & & a_{n-1} & b_n \end{pmatrix} ; \quad B = \begin{pmatrix} 0 & a_1 & & & 0 \\ -a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_{n-1} \\ 0 & & & -a_{n-1} & 0 \end{pmatrix}$$

Thus if $U = U(t)$ is the orthogonal matrix satisfying

$$\frac{dU}{dt} = BU \ ; \qquad U(0) = I$$

then by (2.7)

$$\frac{d}{dt}(U^{-1}L\,U) = 0$$

hence

$$U^{-1}L\,U = L(0) \ .$$

Thus, $L(t)$ is similar to $L(0)$ and the eigenvalues λ_k of the Jacobi matrix L, which are real and distinct, are independent of t. This description of the integrals as eigenvalues of a linear operator is due to Lax [5] and Flaschka's derivation was based on his approach.

Thus the characteristic polynomial

$$\Delta_n(\lambda) = \det(\lambda I - L) = \prod_{k=1}^{n} (\lambda - \lambda_k) = \sum_{k=0}^{n} I_k \lambda^{n-k} \tag{2.8}$$

as well as the coefficients $I_1, \dots, I_n$ are constants of the motion (2.2). For definiteness we order the eigenvalues according to their size,

$$\lambda_1 < \lambda_2 < \dots < \lambda_n \quad .$$

Notice that $L(t) \to L(\infty)$ as $t \to +\infty$ where $L(\infty)$ is a diagonal matrix whose diagonal elements must be the eigenvalues λ_k in appropriate order. From

$$\frac{\dot{a}_k}{a_k} \sim b_{k+1}(\infty) - b_k(\infty)$$

and (2.5) we conclude that $b_{k+1}(\infty) < b_k(\infty)$ or

$$b_k(\infty) = \lambda_{n-k+1}$$

i.e.

$$L(\infty) = \mathrm{diag}(\lambda_n, \lambda_{n-1}, \dots, \lambda_1) \ .$$

Using that the t-reversing substitution

$$t \to -t \ ; \quad a_k \to a_{n-k} \ ; \quad b_k \to b_{n+1-k}$$

leaves the system invariant, we conclude that

$$L(-\infty) = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

i.e. $L(\infty)$, $L(-\infty)$ differ just in the order of the diagonal elements. The physical interpretation of this result is: If for $t \to -\infty$ the particles x_k approach the velocities $y_k = -2\lambda_k$ where $y_1 < y_2 < \dots < y_n$ then for $t \to +\infty$ the particles x_k have the velocities y_{n-k+1} so that the particles exchange their velocities.

This describes the flow for our problem (2.2), (2.3). Still we will find another set of variables, $r_k > 0$, $k = 1,\dots,n-1$, which together with the λ_k form a set of coordinates, and represent the differential equations in these new variables.

3. Partial Fractions and Continued Fractions

Let

$$R(\lambda) = (\lambda I - L)^{-1}$$

where we suppress the dependence in t. This is an n by n matrix and we single out the element in the last row and last column

(3.1) $R_{nn}(\lambda) = (R(\lambda)e_n, e_n) = f(\lambda)$ where $e_n = (0,0,\dots,0,1)$,

and $f(\lambda)$ is hereby defined. Since L is symmetric it follows that $f(\lambda)$ is an analytic function for $\mathrm{Im}\,\lambda \neq 0$ and

$$\operatorname{Im} f(\lambda) > 0 \quad \text{for} \quad \operatorname{Im} \lambda > 0 .$$

Moreover, it is rational with simple poles at the eigenvalues λ_k and so admits the partial fraction expansion

$$f(\lambda) = \sum_{k=1}^{n} \frac{r_k^2}{\lambda-\lambda_k}, \qquad r_k > 0 , \tag{3.2}$$

with positive residua r_k^2. Moreover, for $|\lambda| \to \infty$ one has $\lambda f(\lambda) \to 1$ and

$$\sum_{k=1}^{n} r_k^2 = 1 .$$

Thus we have a mapping ϕ associating with every point in

$$D = \{a_1,\ldots,a_{n-1},b_1,\ldots,b_n \text{ with } a_k > 0\} \tag{3.3}$$

a point in

$$\Lambda = \{\lambda_1,\ldots,\lambda_n,r_1,\ldots,r_n \text{ with } \lambda_1 < \lambda_2 < \ldots < \lambda_n, \quad \sum_{k=1}^{n} r_k^2 = 1,\ r_k > 0\} . \tag{3.4}$$

We claim that this mapping $\phi\colon D \to \Lambda$ is one to one and onto. We will view it as a coordinate transformation and then describe the differential equations in the new variables. The fact that the mapping ϕ has an inverse $\phi^{-1}\colon \Lambda \to D$ corresponds to the inverse method of spectral theory, which in the elementary form described here goes back to Stieltjes [3]. It is based on the fact that $f(\lambda)$ admits a continued fraction expansion

$$f(\lambda) = \cfrac{1}{\lambda - b_n - \cfrac{a_{n-1}^2}{\lambda - b_{n-1} \ddots - \cfrac{a_1^2}{\lambda - b_1}}} \tag{3.5}$$

where the entries a_k, b_k agree precisely with those of L.

To prove this we establish the identity

$$f(\lambda) = \frac{\Delta_{n-1}}{\Delta_n} \tag{3.6}$$

where Δ_n is the characteristic polynomial of $(\lambda I-L)$, see (2.8), and Δ_k the k by k subdeterminant obtained by canceling the last n-k rows and columns of $(\lambda I-L)$. Expanding Δ_k by the last row one finds

$$\Delta_k = (\lambda-b_k)\,\Delta_{k-1} - a_{k-1}^2\,\Delta_{k-2} \tag{3.7}$$

for $k = 3,4,\ldots,n$; it holds also for $k=1,2$ if we set

$$\Delta_{-1} = 0\ , \quad \Delta_0 = 1\ .$$

Thus the ratios $s_k = \Delta_k/\Delta_{k-1}$ satisfy the recursion formula

$$s_k = \lambda - b_k - \frac{a_{k-1}^2}{s_{k-1}} \quad \text{for} \quad k = 2,3,\ldots,n$$

which leads to a finite continued fraction for $s_n = \Delta_n/\Delta_{n-1} = f^{-1}(\lambda)$.

Thus the representation (3.5) follows from (3.6) which we prove now. For this purpose we compute the last column

$$R\,e_n = z \quad \text{of} \quad R = R(\lambda)\ .$$

We find

$$\left\{\begin{aligned} z_{k+1} &= \frac{\Delta_k}{\Delta_n}\,a_{k+1}\cdots a_{n-1} \quad \text{for} \quad k = 0,1,\ldots,n-2 \\ z_n &= \frac{\Delta_{n-1}}{\Delta_n}\ . \end{aligned}\right. \tag{3.8}$$

Indeed z is the solution of

$$(\lambda I-L)z = e_n$$

and using the recursion formula one readily verifies (3.8). Thus

$$f(\lambda) = R_{nn} = z_n = \frac{\Delta_{n-1}}{\Delta_n}$$

as we wanted to show.

Thus for a given matrix L we can compute the rational function $f(\lambda)$ which has n simple real poles with positive residua, since $\mathrm{Im}\, f(\lambda) > 0$ for $\mathrm{Im}\,\lambda > 0$. Ordering these poles according to size we have defined the mapping ϕ taking D into Λ (see (3.3), (3.4)).

We come to the "inverse problem" which requires that we determine ϕ^{-1}. With any point in Λ we associate $f(\lambda)$ by (3.2). Then $\mathrm{Im}\, f > 0$ for $\mathrm{Im}\,\lambda > 0$ and $\lambda f(\Lambda) \to 1$ for $|\lambda| \to \infty$. Thus

$$\frac{1}{f(\lambda)} = \lambda + A - g(\lambda)$$

where A is a real constant and $g(\lambda)$ is a rational function which satisfies

$$\mathrm{Im}\, g(\lambda) = \mathrm{Im}\,\lambda + \frac{\mathrm{Im}\, f}{|f|^2} > 0 \quad \text{for} \quad \mathrm{Im}\,\lambda > 0 \ .$$

Thus $g(\lambda)$ has only simple poles on the real axis and their number is n-1. One computes easily $-A = \sum\limits_1 \lambda_k r_k^2$, $\lambda g(\lambda) \to \sum \lambda_k^2 r_k^2 - (\sum \lambda_k r_k^2)^2 > 0$. Thus $g = B\, f_{n-1}$ with $B > 0$, and $\lambda f_{n-1} \to 1$ for $|\lambda| \to \infty$. Thus

$$f(\lambda) = \frac{1}{\lambda + A - Bf_{n-1}}$$

and by induction we get a unique continued fraction of the form (3.5) with $A = -b_n$, $B = a_{n-1}^2 > 0$, etc. This shows that ϕ maps D one to one onto Λ.

Finally we express the differential equation (2.2) in these new variables. For this purpose we deduce from (2.7)

$$\frac{d}{dt} R = R \frac{dL}{dt} R = BR - RB$$

and taking the last element $R_{nn} = f$ in R we find

$$\frac{df}{dt} = (e_n, (BR-RB)e_n) = -2(e_n, Ra_{n-1}e_{n-1}) = -2a_{n-1} R_{n,n-1} \ .$$

Since $R_{n,n-1}$ agrees with z_{n-1} in (3.8) we obtain

$$\frac{df}{dt} = -2\, a_{n-1}^2 \frac{\Delta_{n-2}}{\Delta_n} .$$

This formula allows us to determine the desired differential equations. Since we established already that $d\lambda_k/dt = 0$ we have

$$\frac{df}{dt} = \sum_{k=1}^{n} \frac{2r_k \dot{r}_k}{\lambda-\lambda_k}$$

Comparing the residue of the last two expressions we get

$$2r_k\dot{r}_k = -2\, a_{n-1}^2 \left.\frac{\Delta_{n-2}}{\Delta_n'}\right|_{\lambda=\lambda_k} .$$

By the recursion formula (3.7) we have

$$\Delta_n = (\lambda-b_n)\Delta_{n-1} - a_{n-1}^2\Delta_{n-2}$$

or, since $\Delta_n(\lambda_k) = 0$,

$$\Delta_{n-2}(\lambda_k) = \frac{\lambda_k-b_n}{a_{n-1}^2}\, \Delta_{n-1}(\lambda_k) ,$$

hence

$$2r_k\dot{r}_k = -2(\lambda_k-b_n) \left.\frac{\Delta_{n-1}}{\Delta_n'}\right|_{\lambda=\lambda_k} .$$

A similar comparison of the residua of

$$f(\lambda) = \frac{\Delta_{n-1}}{\Delta_n} = \sum_{k=1}^{n} \frac{r_k^2}{\lambda-\lambda_k}$$

gives

$$\left.\frac{\Delta_{n-1}}{\Delta_n'}\right|_{\lambda=\lambda_k} = r_k^2$$

hence

$$2r_k\dot{r}_k = -2(\lambda_k-b_n)r_k^2 .$$

Since $\sum_{k=1}^{n} r_k^2 = 1$ we find

$$0 = \sum r_k\dot{r}_k = -\sum \lambda_k r_k^2 + b_n$$

and so, $b_n = \sum \lambda_k r_k^2$, as we had seen before, and

$$\dot{r}_k = -(\lambda_k - \sum \lambda_j r_j^2) r_k$$

which gives the differential equation (1.4) of Section 1. These differential equations represent the vector field along the gradient of the function V(r) (see (1.4')) restricted to the part of the unit sphere lying in the positive quadrant. Thus every solution approaches for $t \to +\infty$ the minimum: $r(t) \to e_1$ and for $t \to -\infty$ the maximum: $r(t) \to e_n$. Of course, it is also possible to give an analytical representation for the solutions, since they are obtained by projecting the linear differential equations $\dot{r}_k = -\lambda_k r_k$ on the unit sphere. Thus we find

$$\begin{cases} \lambda_k(t) = \lambda_k(0) \\ r_k^2(t) = \dfrac{r_k^2(0)\, e^{-2\lambda_k t}}{\sum\limits_{j=1}^{n} r_j^2(0)\, e^{-2\lambda_j t}} \end{cases} .$$

To summarize our result we consider the r_k as homogeneous variables, still positive, and set, accordingly,

$$f(\lambda) = \left(\sum_{k=1}^{n} \frac{r_k^2}{\lambda - \lambda_k} \right) \left(\sum_{k=1}^{n} r_k^{\,2} \right)^{-1} . \tag{3.9}$$

From (3.5) and the calculation of continued fractions it is clear that the a_k^2, b_k are rational functions of r_j, λ_j, of degree 0 in the r_j. One verifies that a_k, b_k are of degree 1 in λ_j. Thus we have the following rational transformation

$$\begin{aligned} a_k^2 &= A_k(r,\lambda) , \qquad && k = 1,2,\dots,n-1 \\ b_k^2 &= B_k(r,\lambda) , \qquad && k = 1,2,\dots,n \end{aligned} \tag{3.10}$$

of $\lambda_1 < \dots < \lambda_n$; $r_k > 0$ into the domain D. If we identify two proportional vectors r the mapping is one to one.

In these homogeneous coordinates r_k the differential equations become linear

$$\frac{d\lambda_k}{dt} = 0 \ ; \qquad \frac{dr_k}{dt} = -\lambda_k r_k \ . \tag{3.11}$$

Thus the solutions of (2.2) can be represented as rational functions of the n constants λ_j and n exponential functions $e^{-\lambda_j t}$.

As we mentioned in Section 1 the flow is particularly simple in this case since no periodic or recurrent solutions are present. In the more interesting case of the periodic boundary condition one has quasiperiodic solutions and the task of finding coordinates for the integral surfaces which are tori, is more difficult. The main problem is the "inverse problem" which consists in recovering L -- which is then a cyclic matrix -- from its eigenvalues and appropriately chosen quantities. This problem seems unsolved as yet.

4. Solution of the Scattering Problem

From the results of Section 2 it follows that the asymptotic behavior of the solutions of our problem (1.2') is given by

$$\begin{aligned} x_k(t) &= \alpha_k^+ t + \beta_k^+ + O(e^{-\delta t}) \\ x_k(-t) &= -\alpha_k^- t + \beta_k^- + O(e^{-\delta t}) \end{aligned} \tag{4.1}$$

for $t \to +\infty$ with some $\delta > 0$. Moreover, we found

$$\alpha_k^+ = \lim_{k \to k+\infty} y_k = -2\lambda_{n-k+1} \quad \text{and} \quad \alpha_k^- = -2\lambda_k$$

i.e.

$$\alpha_{n-k+1}^+ = \alpha_k^- \tag{4.2}$$

which expresses that the $(n-k+1)^{st}$ particle has then for $t \to +\infty$ the velocity which the k^{th} particle had in the past.

Our goal is to determine the relation between the phases β_k^+, β_k^- which can be given explicitly too. This remarkable fact is also a consequence of the integrable character of the system and the representation of $e^{x_k - x_{k+1}}$, y_k as rational functions of λ_j, $e^{-\lambda_j t}$ given by (3.10),(3.11). An explicit calculation seems prohibitive; nevertheless the following argument, which uses just rudimentary properties of rational functions will lead to the goal. The result is

$$\beta_{n-k+1}^+ = \beta_k^- + \sum_{j \neq k} \phi_{jk}(\alpha^-) \tag{4.3}$$

where

$$\phi_{jk}(\alpha) = \begin{cases} \log\, (\alpha_j^- - \alpha_k^-)^2 & \text{for} \quad j < k \\ -\log\, (\alpha_j^- - \alpha_k^-)^2 & \text{for} \quad j > k \end{cases} \tag{4.4}$$

For $n=2$ this amounts to

$$\begin{aligned} \beta_2^+ &= \beta_1^- - \log\, (\alpha_1^- - \alpha_2^-)^2 \\ \beta_1^+ &= \beta_2^- + \log\, (\alpha_1^- - \alpha_2^-)^2 \end{aligned} \tag{4.5}$$

Thus ϕ_{jk} represents the phase shift between two particles with velocities α_j^-, α_k^- at $t = -\infty$. The result (4.3) can therefore be interpreted as follows: The particles are scattered just as if their interaction takes place two at a time! This was suggested to me by M. Kruskal who described an analogous phenomenon for solutions of the Korteweg-de Vries equation (see [6], Theorem 3.7) and by P. D. Lax. This phenomenon which had been discovered by Zakharov et al. (see [6] for references) is obviously intimately related to our result and it is conceivable that one can be derived from the other -- but we have not pursued this point.

We illustrate the statement in Figures 1, 2. Figure 1 illustrates the case $n = 2$, which is given explicitly in terms of $\cosh\, (\lambda_2 - \lambda_1)t$. The asymptotic behavior can be interpreted as the elastic reflection of two rods of length $\phi_{21} = \log\, (\alpha_2^- - \alpha_1^-)^2$,

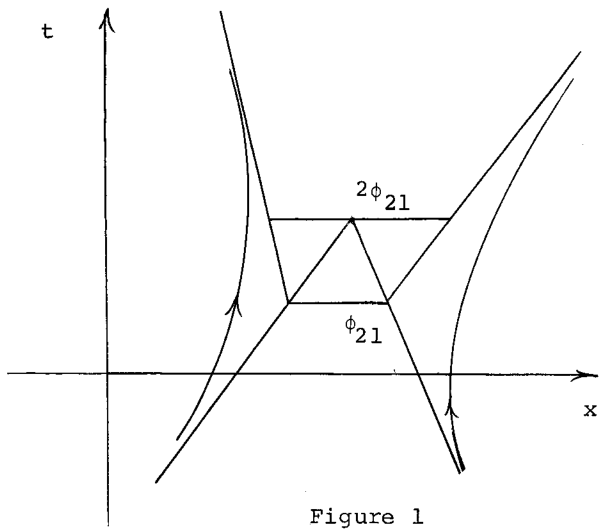

Figure 1

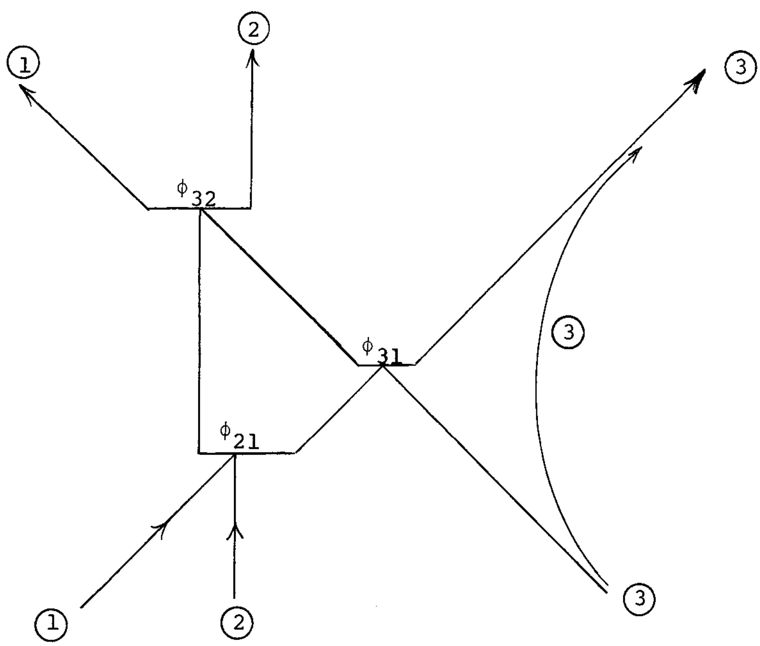

Figure 2

provided this number is positive. For negative values of ϕ_{21} the particles reflect only after passing each other. However, this interpretation is somewhat misleading, especially if $n > 2$, since the length of the rods depends on their velocity, not on the label. We indicate schematically the construction of the scattering for $n = 3$ in Figure 2.

To prove (4.3) first translate it into an asymptotic statement for (2.2). For this purpose we note that on account of the linear t-dependence of the center of mass we have

$$\sum_{k=1}^{n} \beta_k^+ = \sum_{k=1}^{n} \beta_k^-$$

and therefore it suffices to prove (4.3) for the differences $\beta_{k+1}^- - \beta_k^-$, i.e. it suffices to establish

$$\beta_{n-k}^+ - \beta_{n-k+1}^+ = \beta_{k+1}^- - \beta_k^- - \sum_{j\neq k} \phi_{jk} + \sum_{j\neq k+1} \phi_{j,k+1} \, .$$

Using (2.1), (4.1) this amounts to

$$\lim_{t\to+\infty} a_{n-k}(t)\, a_k(-t)\, e^{2(\lambda_{k+1}-\lambda_k)t} = C_k(\lambda) \, , \qquad k = 1,2,\ldots,n-1 \tag{4.6}$$

with

$$\log (4C_k)^2 = - \sum_{j\neq k} \phi_{j,k} + \sum_{j\neq k+1} \phi_{j,k+1} \, .$$

Finally, with $\alpha_k^- = -2\lambda_k$ and (4.4) this gives for C_k the expression

$$C_k = \frac{\prod_{j>k} (\lambda_j - \lambda_k)}{\prod_{j<k} (\lambda_k - \lambda_j)} \cdot \frac{\prod_{j<k+1} (\lambda_{k+1} - \lambda_j)}{\prod_{j>k+1} (\lambda_j - \lambda_{k+1})} \tag{4.7}$$

where empty products are to be set equal to 1.

Thus it suffices to prove (4.6) with (4.7). For $n = 2$ this is easily verified. In that case our transformation (3.10) takes the explicit form

$$b_1 = \frac{\lambda_2 r_1^2 + \lambda_1 r_2^2}{r_1^2 + r_2^2}\ , \qquad b_2 = \frac{\lambda_1 r_1^2 + \lambda_2 r_2^2}{r_1^2 + r_2^2}$$

(4.8)

$$a_1 = \frac{(\lambda_2-\lambda_1) r_1 r_2}{r_1^2+r_2^2} = (\lambda_2 - \lambda_1)\left(\frac{r_1}{r_2} + \frac{r_2}{r_1}\right)^{-1}$$

Since $r_k(t) = r_k(0)e^{-\lambda_k t}$ we find

$$a_1(t)\ a_1(-t)\ e^{2(\lambda_2-\lambda_1)t} \to (\lambda_2-\lambda_1)^2 \tag{4.9}$$

which corresponds to (4.6) for $k = 1$, $n = 2$.

For $n = 3$ one can still, with some effort, verify the above statement by an explicit calculation but for general n this seems a hopeless approach. Therefore we proceed as follows. We know that for every solution

$$\left.\begin{aligned} a_{n-k}(t) &\sim c^+_{n-k}\ e^{-(\lambda_{k+1}-\lambda_k)t} \\ a_k(-t) &\sim c^-_k\ e^{-(\lambda_{k+1}-\lambda_k)t} \end{aligned}\right\} \quad \text{as} \quad t \to +\infty \tag{4.10}$$

with positive constants c^+_{n-k}, c^-_k.

(i) First we establish that c^+_{n-k}, c^-_k depend real analytically on the initial data $a_j(0)$, $b_j(0)$ of the solution. (ii) Second, we show that

$$\begin{aligned} c^+_{n-k} &= \frac{r_{k+1}(0)}{r_k(0)}\ C^+_{n-k}(\lambda) \\ c^-_k &= \frac{r_k(0)}{r_{k+1}(0)}\ C^-_k(\lambda) \end{aligned} \tag{4.11}$$

with C^+_{n-k}, C^-_k depending on λ only. This shows that the limit (4.6) is equal to

$$C_k(\lambda) = C^+_{n-k}(\lambda)\ C^-_k(\lambda)\ , \tag{4.11'}$$

and therefore independent of the initial condition of r_j. This makes the actual determination of $C_k(\lambda)$ easy if we consider various limit situations for the initial conditions, which will be the third step (iii).

To begin with the analytic dependence of the constants c^+_{n-k}, c^-_k on the initial conditions we fix a solution $a_j(t)$, $b_j(t)$ of (2.2) and describe a nearby one by

$$\tilde{a}_j = a_j e^{u_j} , \qquad \tilde{b}_j = b_j + v_j$$

where $u_j(0)$, $v_j(0)$ are small. The differential equations for u,v are then

$$\begin{aligned} \dot{u}_k &= v_{k+1} - v_k \\ \dot{v}_k &= 2\left(a_k^2(e^{2u_k} - 1) - a_{k-1}^2(e^{2u_{k-1}} - 1)\right) . \end{aligned}$$

The asymptotic behavior of the solutions is given by

$$(4.12) \qquad \begin{aligned} u_k &= (v_{k+1}(\infty) - v_k(\infty))t + \gamma_k + O(e^{-\delta t}) \\ v_k &= v_k(+\infty) + O(e^{-\delta t}) \end{aligned} \qquad \text{for} \quad t \to +\infty$$

if the initial data are small enough. It is sufficient to show that $v_k(\infty)$, γ_k depend real analytically on $u_j(0)$, $v_j(0)$ if these are close to zero. For this purpose we permit complex initial values and show that the above asymptotic description holds for a complex neighborhood of the origin. This requires some simple a priori estimates:

Obviously it suffices to establish the analytic dependence on the initial values $u_k(\tau)$, $v_k(\tau)$ for some fixed positive τ. The fixed real solution satisfies an estimate

$$0 < a_k(t) < c_1 e^{-\delta t} \quad \text{for} \quad 0 \le t < \infty$$

with some positive constants δ, c_1; we may assume $\delta < 1 < c_1$.

With

$$0 < \eta < \frac{\delta}{8}$$

and some τ, to be determined later, we consider complex initial values in

(4.13) $$|u_k(\tau)| < \eta \,, \qquad |v_k(\tau)| < \eta$$

Let $M(t) = \max_k |v_k(t)|$ and consider this function in an interval $\tau \le t < \tau'$ in which $M(t) \le 2\eta$. Then we get from the differential equations for $\tau \le t < \tau'$

$$|u_k(t)| \le \eta + 2 \int_\tau^t M(t)\, dt \le \eta(1 + 4(t-\tau)) \,.$$

Using the inequality

$$|e^{2u} - 1| \le 2|u|\, e^{2|u|}$$

and setting $s = t - \tau$ we get for $\tau \le t < \tau'$

$$|v_k(t)| \le \eta + 8c_1^2\eta \int_\tau^t e^{-2\delta t}(1 + 4(t-\tau))\, e^{2\eta(1+4(t-\tau))}\, dt$$

hence

$$M(t) \le \eta\left(1 + c_2\, e^{-2\delta\tau} \int_0^\infty e^{-(2\delta-8\eta)s}\, (1 + 4s)\, ds\right)$$

with $c_2 = 8c_1^2 e^2$. Since $2\delta - 8\eta > \delta$ we get

$$M(t) \le \eta\left(1 + c_2\, e^{-2\delta\tau}\, 5\delta^{-2}\right) \,.$$

Now we fix τ so that

$$5c_2\, e^{-2\delta\tau}\, \delta^{-2} < 1$$

so that

$$M(t) < 2\eta \quad \text{for} \quad \tau \le t < \tau' \,.$$

Thus we can take $\tau' = \infty$ and have the estimate

$$|v_k(t)| \leq 2\eta \quad \text{for all} \quad t \geq \tau ,$$

and all complex initial data in the polydisk (4.13).

Since $v_k(t)$ depends analytically on those initial data and converges on the real axis for $t \to +\infty$ the limit function $v_k(\infty)$ is analytic in (4.13). From the differential equation we obtain

$$|v_k(t) - v_k(\infty)| \leq c_4 \, e^{-\delta t} \quad \text{for} \quad t \geq \tau$$

and

$$u_k(t)-(v_{k+1}(\infty)-v_k(\infty))t = u_k(0) + \int_0^t \{v_{k+1}(t)-v_{k+1}(\infty)-v_k(t)+v_k(\infty)\} \, dt$$

converges for $t \to +\infty$ with a uniform bound. Hence its limit γ_k is analytic in (4.13), completing the proof of (i).

To prove (ii) we use the representation (3.10) of the solutions by

$$a^2_{n-k}(t) = A_{n-k}(r,\lambda) \quad \text{with} \quad r_j = r_j(0) \, e^{-\lambda_j t}, \quad \lambda_j = \lambda_j(0) \ .$$

Here A_{n-k} is a rational function in r, λ, say,

$$A_{n-k} = \frac{P}{Q}$$

with P, Q being polynomials in r, λ. They are homogeneous in the r_j, both of the same degree, since A_{n-k} is of degree 0. To study its asymptotic behavior for $t \to +\infty$ we assume first that λ_{j+1}/λ_j is sufficiently large for all $j = 1,2,\ldots,n-1$. Then the dominant term in P is the one which comes first in lexicographical ordering of the exponents of r_j. Let $P_0 = \prod_{j=1}^{n} r_j^{p_j}$ be this term in P and $Q_0 \prod_{j=1}^{n} r_j^{q_j}$ the dominant term in Q. Then P_0, Q_0 are polynomials in λ and

$$A_{n-k} \sim \frac{P_0}{Q_0} \prod_{j=1}^{n} r_j^{(p_j-q_j)} \ .$$

Since on the other hand

$$a_{n-k}^2(t) \sim \text{const. } e^{-2(\lambda_{k+1}-\lambda_k)t}, \quad k=1,\dots,n-1,$$

we conclude that $p_k - q_k = -2$, $p_{k+1} - q_{k+1} = +2$, $p_j = q_j$ otherwise, and

$$A_{n-k} \sim \frac{P_0}{Q_0}\left(\frac{r_{k+1}}{r_k}\right)^2 \quad \text{for} \quad t \to +\infty .$$

Here the coefficient P_0/Q_0 is positive for $\lambda_1 < \lambda_2 < \dots < \lambda_n$. This proves the first line of (4.11) with

$$(c_{n-k}^+)^2 = \frac{P_0}{Q_0}$$

at least for large values of λ_{j+1}/λ_j. Since the coefficient is real analytic for all real λ in $\lambda_1 < \lambda_2 < \dots < \lambda_n$ this equation holds for all those λ.* The second equation of (4.11) follows in just the same manner.

This shows that the coefficient $C_k(\lambda)$ in (4.11') is independent of $r_j(0)$, and that $C_k^2(\lambda)$ is a rational function of λ. We will determine $C_k(\lambda)$ by induction on n. Incidentally, this will show that even $C_k(\lambda)$ is rational. For n = 2 the formula (4.7) or equivalently (4.3), (4.4) was verified. We prefer to prove the statement in the form (4.3). From our argument we know that

$$\beta_{n-k+1}^+ = \beta_k^- + \Phi_k(\alpha), \qquad k = 1,2,\dots,$$

where $\Phi_k(\alpha)$ is a real analytic function of $\alpha = (\alpha_1^-,\dots,\alpha_n^-)$, independent of β_j^-. This follows from the fact that $C_k(\lambda)$ is independent of $r_j(0)$, and depends on $\lambda_j = -\frac{1}{2}\alpha_j^-$ only. For n+1

* In general, the limit of such a rational function of exponentials may even be discontinuous, e.g. for

$$\frac{Ar_1r_3 + Br_2^2}{Cr_1r_3 + Dr_2^2} \to \frac{A}{C} \quad \text{for} \quad \frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_2} > 1,$$

but

$$\to \frac{B}{D} \quad \text{for} \quad \frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_2} < 1 .$$

particles we denote the corresponding function by $\Psi_k(\tilde{\alpha})$, $\tilde{\alpha} = (\alpha_1^-,\ldots,\alpha_{n+1}^-)$. The induction proof requires the verification of

$$(4.14) \qquad \Psi_k(\tilde{\alpha}) = \Phi_k(\alpha) + \phi_{n+1,k} \quad \text{for} \quad k = 1,2,\ldots,n \; .$$

The determination of $\Psi_{n+1}(\tilde{\alpha})$ follows then from

$$\sum_{k=1}^{n+1} \Psi_k(\tilde{\alpha}) = 0$$

which is a consequence of the linear t-dependence of the center of mass. Thus it suffices to prove (4.14). Since both sides are independent of the β_j^- we may, and will, choose β_{n+1}^- very large positive, so that the n particles $x_1,x_2,\ldots,x_n$ have already undergone their mutual interaction and are very far apart by the time x_{n+1} interacts with any of them. In other words, when $x_{n+1}(t)$ exerts some force on $x_1,\ldots,x_n$ there are already close to

$$x_k \sim \alpha_k^+ t + \beta_k^+ \; , \qquad \text{where} \quad \beta_k^+ = \beta_{n-k+1}^- + \Phi_{n-k+1}$$

and t is large positive. Thus the interaction of x_{n+1} with $x_k(t)$, $k \leq n$ takes essentially place pairwise. Since before the interaction with x_{n+1} we have

$$x_{n-k+1} \sim \alpha_k^- t + \beta_k^- + \Phi_k \; , \qquad k = 1,2,\ldots,n,$$

we obtain after interaction

$$x_{n-k+1} \sim \alpha_k^- t + \beta_k^- + \Phi_k + \phi_{n+1,k}$$

which shows that $\Psi_k \sim \Phi_k + \phi_{n+1,k}$ if $\beta_{n+1}^- \to \infty$. But since Ψ_k, Φ_k are independent of β^- the assertions (4.14) follow. The situation is depicted in Figure 3.

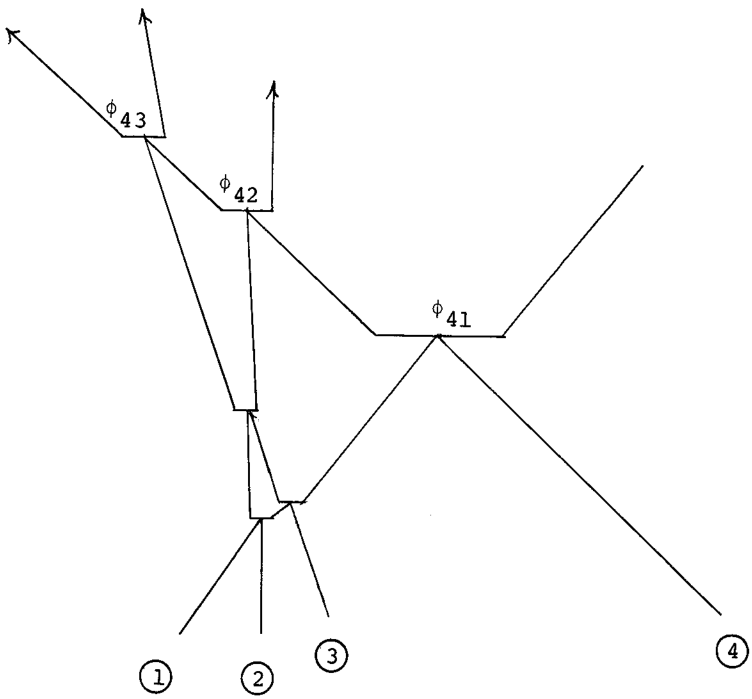

Figure 3

5. Associated Differential Equations

The above Hamiltonian system (1.2) possesses, according to the above, n integrals $I_1, I_2, \ldots, I_n$, which incidentally are polynomials in y_k and $e^{x_k - x_{k+1}}$. One may show, which we will not do here, that these integrals are in involution, i.e. the Poisson bracket for any two of these vanishes. Using the integrals as new Hamiltonians one can introduce n new vector fields which possess the same integrals and commute with each other. This makes the manifolds I_k = const. into commutative groups. These are well known facts for integrable Hamiltonian systems (see, for example, Appendix 26 of [7]) which we will verify here directly.

As a starting point we take the differential equation (2.7) which represents a deformation of the Jacobi matrix L leaving the spectrum fixed. But there are many such isospectral deformations corresponding to different choices of B. We restrict ourselves to skew symmetric matrices B giving rise to orthogonal similarity transformations. But instead of permitting only one pair of off diagonals we allow several. Let B_p stand for a skew symmetric matrix with p off diagonals above and adjacent to the diagonal. Thus the matrix B defined below (2.7) would be denoted by B_1. We claim that for every p in $1 \leq p < n$ we can find nontrivial matrices B_p such that $\dot{L} = B_p L - L B_p$ defines a meaningful differential equation, that is that the commutator $B_p L - L B_p$ has only one off diagonal above the diagonal, while the others all vanish. We will establish this assertion below but point out first that all these differential equations have the eigenvalues of L as integrals and can therefore be transformed into the variables r_k, λ_k as $\dot{\lambda}_k = 0$, $\dot{r}_k = f_k(\lambda, r)$. For p = 2 one finds the matrix

$$B_2 = \begin{pmatrix} 0 & \beta_1 & \gamma_1 & & & \\ -\beta_1 & 0 & \beta_2 & \gamma_2 & & \\ -\gamma_1 & & & \ddots & & \\ & & \ddots & & & \gamma_{n-2} \\ & \ddots & & & & \beta_{n-1} \\ & & -\gamma_{n-2} & -\beta_{n-1} & 0 & 0 \end{pmatrix}$$

where

$$(5.1)\qquad \begin{aligned} \beta_k &= (b_k+b_{k+1})a_k \;, & k &= 1,2,\ldots,n-1 \\ \gamma_k &= a_k\, a_{k+1} \;, & k &= 1,2,\ldots,n-2 \end{aligned}$$

and the differential equation $\dot{L} = B_2L - L\,B_2$ takes the explicit form

$$(5.2)\qquad \begin{aligned} \dot{a}_k &= a_k(a_{k+1}^2 - a_{k-1}^2 + b_{k+1}^2 - b_k^2) \;, & k &\le n-1, \\ \dot{b}_k &= 2\,b_k(a_k^2 - a_{k-1}^2) + 2b_{k+1}a_k^2 - b_{k-1}a_{k-1}^2 \;, & k &\le n \;, \end{aligned}$$

where we set $a_0 = 0$, $a_n = 0$. Introducing r, λ again by the transformation (3.10) we find the differential equation

$$(5.3)\qquad \frac{d\lambda_k}{dt} = 0 \;, \qquad \frac{dr_k}{dt} = -\,\lambda_k^2 r_k \;.$$

We just indicate the calculation. First restricting r_k to $\sum r_k^2 = 1$ we have

$$(5.4)\qquad \frac{df}{dt} = \sum \frac{2r_k\dot{r}_k}{\lambda - \lambda_k}$$

On the other hand

$$\begin{aligned} \frac{df}{dt} &= (\dot{R}(\lambda)e_n, e_n) = ((B_2R - R\,B_2)e_n,\, e_n) \\ &= -2(R\,B_2e_n,\, e_n) = -2(R(\gamma_{n-2}e_{n-2} + \beta_{n-1}e_{n-1}), e_n) \\ &= -2(\gamma_{n-2}R_{n,n-2} + \beta_{n-1}R_{n,n-1}) \;. \end{aligned}$$

With (5.1) and

$$R_{n,n-1} = \frac{\Delta_{n-2}}{\Delta_n} a_{n-1}\ , \qquad R_{n,n-2} = \frac{\Delta_{n-3}}{\Delta_n} a_{n-2}\, a_{n-1}$$

we get

$$\frac{df}{dt} = -\ 2\ \frac{a_{n-1}^2}{\Delta_n} (a_{n-2}^2\ \Delta_{n-3} + (b_{n-1} + b_n)\ \Delta_{n-2})\ .$$

Using the recursion formulae (3.7) for $k = n-1,n$ we find

$$\frac{df}{dt} = -\ 2(\lambda^2 - b_n^2 - a_{n-1}^2)\ \frac{\Delta_{n-1}}{\Delta_n}\ .$$

Comparing the residue of these expressions at λ_k with those of (5.3) we find

$$\dot{r}_k = -\ (\lambda_k^2 - b_n^2 - a_{n-1}^2) r_k\ ,$$

or using r_k as homogeneous coordinates

$$\dot{r}_k = -\ \lambda_k^2 r_k\ .$$

Thus the solutions of (5.2) can be expressed as rational functions of λ_k and $e^{-\lambda_k^2 t}$ and the asymptotic behavior of its solutions is also completely understood from the results of the previous sect ons.

It is interesting to observe that the differential equations (5.2) possess $b_k = 0$, $k = 1,\dots,n$, as an invariant manifold on which they reduce to

$$\dot{a}_k = a_k(a_{k+1}^2 - a_{k-1}^2)\ , \qquad k = 1,\dots,n-1, \tag{5.4}$$

which are the deformation equations for a Jacobi matrix L with a zero diagonal.*

To understand which of the solutions of (5.3) corresponds to (5.4) we consider again the continued fraction expansion (3.5), denoting the left-hand side by $f(\lambda,a,b)$. One easily verifies that

* These equations for $n = \infty$ were recently studied by M. Kac and van Moerbeke, according to a letter from M. Kac.

the involution $b_k \to -b_k$, $a_k \to a_k$ gives rise to

$$- f(-\lambda,a,-b) = f(\lambda,a,b) = \sum_{k=1}^{n} \frac{r_k^2}{\lambda-\lambda_k} .$$

Hence, since the eigenvalues λ_k are ordered according to size, the above involution corresponds to

$$\lambda_k \to -\lambda_{n-k+1} , \qquad r_k \to r_{n-k+1} .$$

The fixed points of this involution are the points (a,b) with $b_k = 0$ in the first representation and the points (λ,r) with

$$\lambda_k + \lambda_{n-k+1} = 0 , \quad r_k = r_{n-k+1} . \tag{5.5}$$

This is evident also from the fact that the symmetric Jacobi matrices with zero diagonal have a spectrum symmetric with respect to the origin. Thus the solutions of (5.4) are given by precisely those rational functions in λ_j , $e^{-\lambda_j^2 t}$ for which the λ_j satisfy (5.5)

Using (2.1) it is easy to rewrite the system (5.2) in the variables x_k, y_k and one finds a Hamiltonian system

$$\dot{x}_k = \frac{\partial H_2}{\partial y_k} , \qquad \dot{y}_k = - \frac{\partial H_2}{\partial x_k} , \quad k = 1,2,\ldots,n,$$

with

$$H_2 = - \frac{1}{6} \sum_{k=1}^{n} y_k^3 - \frac{1}{2} \sum_{k=1}^{n} y_k (e^{x_{k-1}-x_k} + e^{x_k-x_{k+1}}) .$$

Again, in the above system one has to set $x_0 = -\infty$, $x_{n+1} = +\infty$. Although this system has no physical interpretation one has a full description of the scattering problem.

If one expresses the above Hamiltonian H_2 in terms of a,b one finds readily

$$H_2 = \frac{4}{3} \operatorname{tr} L^3 = \frac{4}{3} \sum_{k=1}^{n} \lambda_k^3 .$$

Since our original Hamiltonian (1.1) is given by

$$H = 2 \operatorname{tr} L^2 = 2 \sum_{k=1}^{n} \lambda_k^{\,2},$$

one can expect that the further differential equations are associated with Hamiltonian proportional to $\operatorname{tr}(L^{p+1})$.

We will not follow this up but conclude with establishing the existence of the matrices B_p for $p = 1,2,\ldots,n-1$. It is convenient to write the matrices as difference operators. Let ξ stand for a double infinite sequence with components ξ_k (k integers) and let σ denote the shift operator

$$(\sigma\ \xi)_k = \xi_{k+1} .$$

We will assume that $\xi_k = 0$ if $k \leq 0$ or $k > n$ and write the matrix L in the form

$$L\ \xi = a(\sigma\ \xi) + b\ \xi + \sigma^{-1}(a\ \xi) .$$

Here a, b stand for sequences with components a_k and $(a\xi)_k = a_k\xi_k$. Thus $\sigma(a\xi) = \sigma(a)\cdot\sigma(\xi)$.

In this notation B_p will be presented by

$$(5.6)\quad B_p\xi = \gamma\sigma^p\xi + \ldots + \beta(\sigma^q\xi) + \ldots - \sigma^{-q}(\beta\ \xi) + \ldots - \sigma^{-p}(\gamma\ \xi) ,$$

where the qth order term indicates a typical term, $1 \leq q < p$.

The commutator $[B_p, L] = B_pL - L\,B_p$ contains σ, σ^{-1} to powers up to $p+1$. In fact, the highest order terms of this commutator are given by

$$[B_p, L] = \{\gamma\sigma^p(a) - a\sigma(\gamma)\}\ \sigma^{p+1} + \ldots$$

and we determine the $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ so that

$$(5.7)\qquad (\gamma\sigma^p(a) - a\sigma(\gamma))\xi = 0$$

i.e.

$$\gamma_k a_{k+p} - a_k \gamma_{k+1} = 0 , \qquad k = 1,2,\ldots n-p-1.$$

This can be satisfied by

$$\gamma_k = a_k a_{k+1} \cdots a_{k+p-1} , \qquad k = 1,2,\ldots,n-p.$$

Now we proceed inductively and determine the coefficients β of σ^q in (5.6) to remove the terms of order q+1 in $[B_p,L]$, decreasing q from q = p to q = 1. Analogously to (5.7) this gives an equation of the form

$$(\beta\sigma^q(a) - a\sigma(\beta))\xi = g\cdot\xi$$

where g is a given sequence. In components

$$\beta_k a_{k+q} - \beta_{k+1} a_k = g_k , \qquad k = 1,2,\ldots n-q-1.$$

These are n-q-1 equations for n-q unknowns. The solution is therefore not unique, but if β_1 is fixed arbitrarily, these equations can be solved recursively and uniquely, since $a_k > 0$.

Thus B_p can be so determined that in $[B_p,L]$ all coefficients of σ^{q+1} for $q = 1,2,\ldots,p$ vanish. Since $[B_p,L]$ is symmetric it is a Jacobi matrix, giving rise to the desired differential equation

$$\dot{L} = [B_p,L] . \tag{5.8}$$

These are clearly the analogues of the higher order Korteweg-de Vries equations.

Finally, it is obvious that the multiplication

$$(r \otimes r)_k = r_k \tilde{r}_k$$

introduces a group structure into the manifolds λ_k = const. making the n-1 dimensional manifold of Jacobi matrices L with fixed spectrum into an Abelian group. This group action commutes with the vector field (2.2), and more generally with the vector fields (5.8) for p = 1,2,...n-1.

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