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In [7]:
```
# load in basic libraries and autograd wrapped numpy
from autograd import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
import sys
sys.path.append('../')
import matplotlib.pyplot as plt
datapath = '../mlrefined_datasets/superlearn_datasets/'

# imports from custom library
from mlrefined_libraries import basics_library as baslib
from mlrefined_libraries import calculus_library as calib
from mlrefined_libraries import superlearn_library as superlearn
from mlrefined_libraries import math_optimization_library as optlib
import autograd.numpy as np

# demos for this notebook
regress_plotter = superlearn.lin_regression_demos
optimizers = optlib.optimizers
static_plotter = optlib.static_plotter.Visualizer();
plotter = superlearn.multi_outupt_plotters

# this is needed to compensate for matplotlib notebook's tendancy to blow up imag
es when plotted inline
%matplotlib notebook
from matplotlib import rcParams
rcParams['figure.autolayout'] = True
```

Exercise 7.1. One-versus-All classification pseudo-code

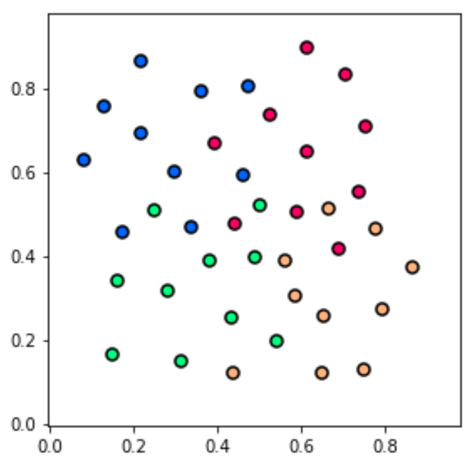
Below we provide a formal pseudo-code summarizing these steps.

Algorithm 1 One-versus-All multi-class classification algorithm

1: Input: multiclass dataset $\left\{\left(\mathbf{x}_p, y_p\right)\right\}_{p=1}^{P}$ where $y_p \in \{0, ..., C-1\}$, two-class classification scheme and optimizer
2: for $j = 0, ..., C-1$
3: form temporary labels $\tilde{y}_p = \begin{cases} +1 & \text{if } y_p = j \\ -1 & \text{if } y_p \neq j \end{cases}$
4: solve two-class subproblem on $\left\{\left(\mathbf{x}_p, \tilde{y}_p\right)\right\}_{p=1}^{P}$ to find weights $\mathbf{w}_j$
5: normalize classifier weights by magnitude of feature-touching portion $\mathbf{w}_j \longleftarrow \frac{\mathbf{w}_j}{\left\|\omega_j\right\|_2}$
6: end for
7: To assign label y to a point $\mathbf{x}$, apply the fusion rule: $y = \underset{c=0,...,C-1}{\text{argmax}} \; \mathring{\mathbf{x}}^T \mathbf{w}_c$
=0

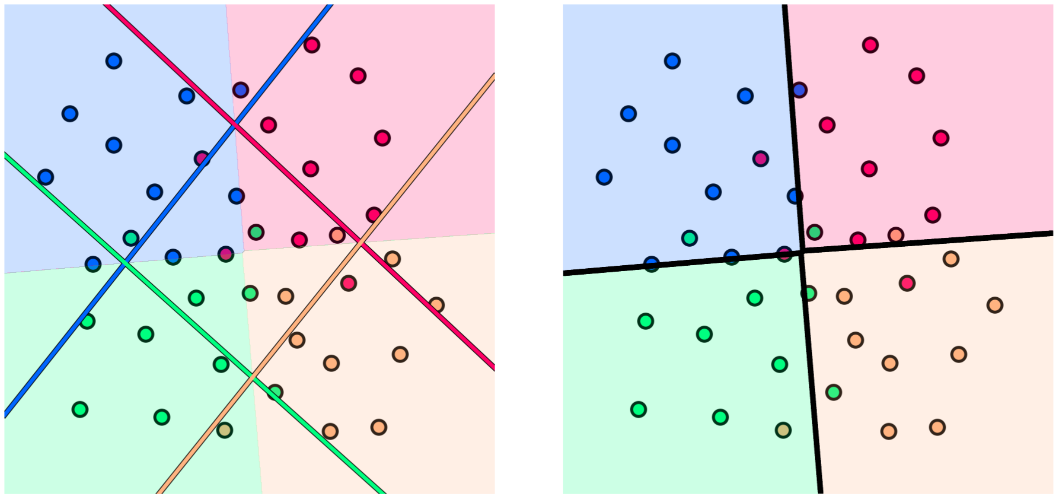
Exercise 7.2. One-versus-All classification

In [2]:
```
# load in dataset
csvname = datapath + '4class_data.csv'
data = np.loadtxt(csvname,delimiter = ',');

# create an instance of the ova demo
demo = superlearn.ova_illustrator.Visualizer(data);

# visualize dataset
demo.show_dataset();
```



In [5]:
```
# solve the 2-class subproblems
demo.solve_2class_subproblems()
```

In [18]:
```
# classify and color the entire space using our individual classifiers and the fu
sion rule
demo.show_complete_coloring()
```

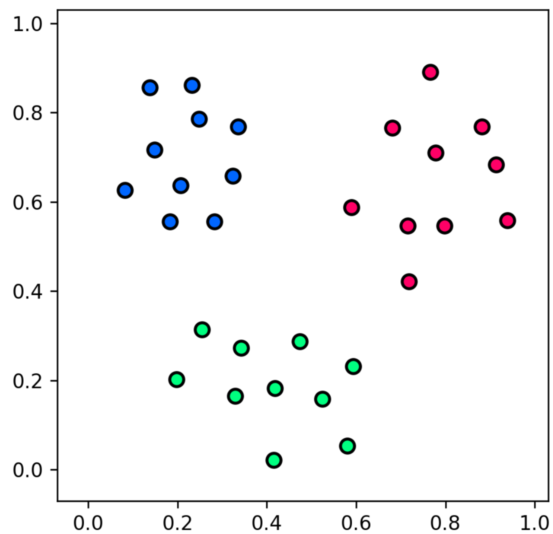


Exercise 7.3. Multi-class Perceptron

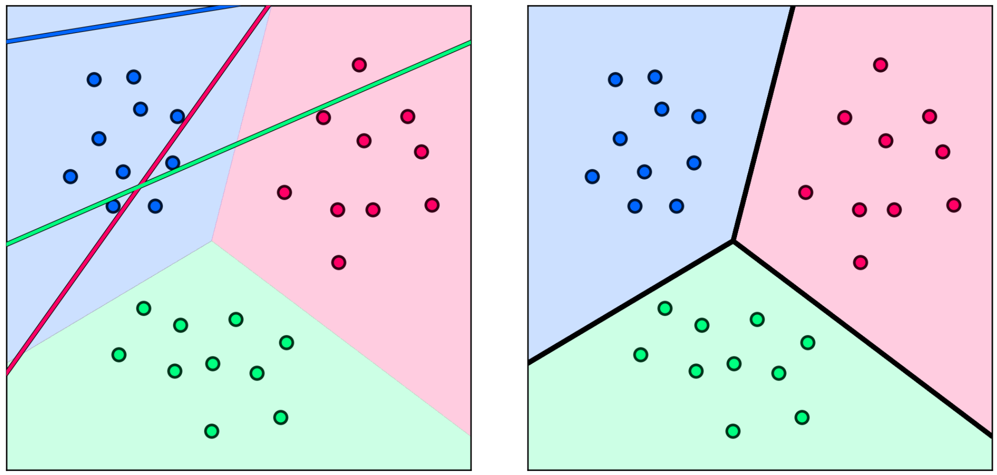
In [19]:
```
# compute C linear combinations of input point, one per classifier
def model(x,w):
    a = w[0] + np.dot(x.T,w[1:])
    return a.T
```

In [20]:
```
lam = 10**-5  # our regularization paramter
def multiclass_perceptron(w):
    # pre-compute predictions on all points
    all_evals = model(x,w)

    # compute maximum across data points
    a = np.max(all_evals,axis = 0)

    # compute cost in compact form using numpy broadcasting
    b = all_evals[y.astype(int).flatten(),np.arange(np.size(y))]
    cost = np.sum(a - b)

    # add regularizer
    cost = cost + lam*np.linalg.norm(w[1:,:],'fro')**2

    # return average
    return cost/float(np.size(y))
```

In [21]:
```
# This code cell will not be shown in the HTML version of this notebook
# load in dataset
data = np.loadtxt(datapath + '3class_data.csv',delimiter = ',')

# create an instance of the ova demo
demo = superlearn.multiclass_illustrator.Visualizer(data)

# get input/output pairs
x = data[:-1,:]
y = data[-1:,:]

# visualize dataset
demo.show_dataset()

# run gradient descent to minimize cost
g = multiclass_perceptron; w = 0.1*np.random.randn(3,3); max_its = 1000; alpha_ch
oice = 10**(-1);
weight_history,cost_history = optimizers.gradient_descent(g,alpha_choice,max_its,
w)
```



In [22]:
```
# This code cell will not be shown in the HTML version of this notebook
# plot classification of space, individual learned classifiers (left panel) and j
oint boundary (middle panel), and cost-function panel in the right panel
demo.show_complete_coloring(weight_history, cost = multiclass_perceptron)
```



Exercise 7.4. The multi-class and two-class Perceptrons

When $C = 2$ the multi-class perceptron cost in equation (4) reduces to

$$g\left(w_0^{(0)},\, \mathbf{w}^{(0)},\, w_0^{(1)},\, \mathbf{w}^{(1)}\right) = \sum_{p=1}^{P}\left[\max\left(w_0^{(0)} + \mathbf{x}_p^T\mathbf{w}^{(0)},\, w_0^{(1)} + \mathbf{x}_p^T\mathbf{w}^{(1)}\right) - \left(w_0^{(y_p)} + \mathbf{x}_p^T\mathbf{w}^{(y_p)}\right)\right]$$

which, using the following equality for any real values a, b, and c

$$\max(a, b) - c = \max(a - c, b - c)$$

can be written equivalently as

$$g\left(w_0^{(0)},\, \mathbf{w}^{(0)},\, w_0^{(1)},\, \mathbf{w}^{(1)}\right) = \sum_{p=1}^{P}\max\left(w_0^{(0)} + \mathbf{x}_p^T\mathbf{w}^{(0)} - \left(w_0^{(y_p)} + \mathbf{x}_p^T\mathbf{w}^{(y_p)}\right),\, w_0^{(1)} + \mathbf{x}_p^T\mathbf{w}^{(1)} - \left(w_0^{(y_p)} + \mathbf{x}_p^T\mathbf{w}^{(y_p)}\right)\right)$$

Grouping the summands according to their labels, we have

$$g\left(w_0^{(0)},\, \mathbf{w}^{(0)},\, w_0^{(1)},\, \mathbf{w}^{(1)}\right) = \sum_{p:\, y_p=0}\max\left(0,\, w_0^{(1)} - w_0^{(0)} + \mathbf{x}_p^T\left(\mathbf{w}^{(1)} - \mathbf{w}^{(0)}\right)\right)$$
$$+ \sum_{p:\, y_p=1}\max\left(0,\, w_0^{(0)} - w_0^{(1)} + \mathbf{x}_p^T\left(\mathbf{w}^{(0)} - \mathbf{w}^{(1)}\right)\right)$$

Re-assigning the labels $y_p = 0 \rightarrow y_p = -1$ and $y_p = 1 \rightarrow y_p = +1$ to match the label values we used in deriving the two-class perceptron, we can write

$$g\left(w_0^{(-1)},\, \mathbf{w}^{(-1)},\, w_0^{(+1)},\, \mathbf{w}^{(+1)}\right) = \sum_{p:\, y_p=-1}\max\left(0,\, w_0^{(+1)} - w_0^{(-1)} + \mathbf{x}_p^T\left(\mathbf{w}^{(+1)} - \mathbf{w}^{(-1)}\right)\right)$$
$$+ \sum_{p:\, y_p=+1}\max\left(0,\, w_0^{(-1)} - w_0^{(+1)} + \mathbf{x}_p^T\left(\mathbf{w}^{(-1)} - \mathbf{w}^{(+1)}\right)\right)$$

Letting $w_0 = w_0^{(+1)} - w_0^{(-1)}$ and $\mathbf{w} = \mathbf{w}^{(+1)} - \mathbf{w}^{(-1)}$, the above can be written as

$$g\left(w_0, \mathbf{w}\right) = \sum_{p:\, y_p=-1}\max\left(0,\, w_0 + \mathbf{x}_p^T\mathbf{w}\right) + \sum_{p:\, y_p=+1}\max\left(0,\, -w_0 - \mathbf{x}_p^T\mathbf{w}\right)$$

This can be written more compactly to arrive at the familiar two-class perceptron cost function

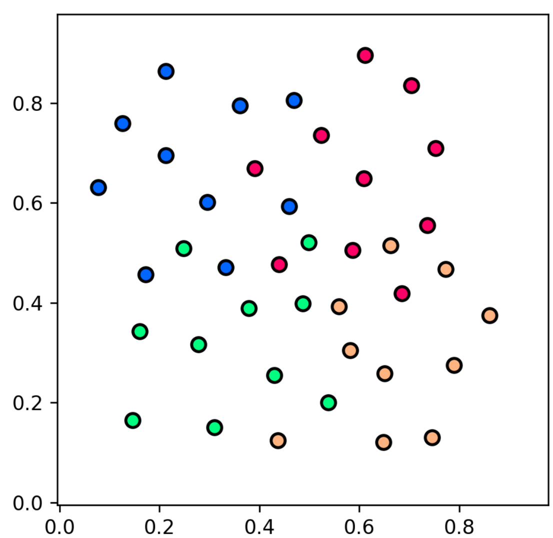
$$g\left(w_0, \mathbf{w}\right) = \sum_{p=1}^{P}\max\left(0, -y_p\left(w_0 + \mathbf{x}_p^T\mathbf{w}\right)\right)$$

Exercise 7.5. Multi-class Softmax

In [23]:
```
# compute C linear combinations of input point, one per classifier
def model(x,w):
    a = w[0] + np.dot(x.T,w[1:])
    return a.T
```

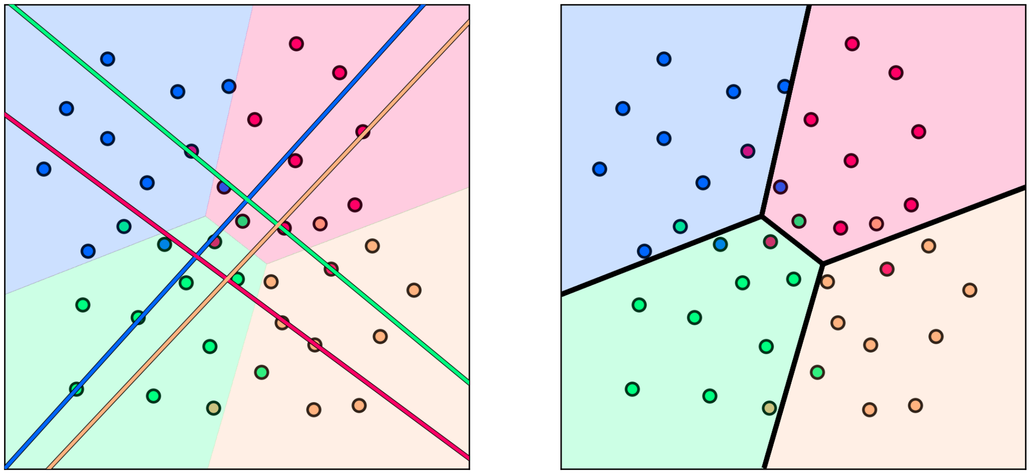
In [24]:
```
# multiclass softmaax regularized by the summed length of all normal vectors
lam = 10**(-5)  # our regularization paramter
def multiclass_softmax(w):
    # pre-compute predictions on all points
    all_evals = model(x,w)

    # compute softmax across data points
    a = np.log(np.sum(np.exp(all_evals),axis = 0))

    # compute cost in compact form using numpy broadcasting
    b = all_evals[y.astype(int).flatten(),np.arange(np.size(y))]
    cost = np.sum(a - b)

    # add regularizer
    cost = cost + lam*np.linalg.norm(w[1:,:],'fro')**2

    # return average
    return cost/float(np.size(y))
```

In [25]:
```
# load in dataset
data = np.loadtxt(datapath + '4class_data.csv',delimiter = ',')

# get input/output pairs
x = data[:-1,:]
y = data[-1:,:]

# create an instance of the ova demo
demo = superlearn.multiclass_illustrator.Visualizer(data)

# visualize dataset
demo.show_dataset()

# run gradient descent to minimize cost
g = multiclass_softmax; w = 0.1*np.random.randn(3,4); max_its = 5;
weight_history,cost_history = optimizers.newtons_method(g,max_its,w)
```



Finally, we plot our results, as in the previous Example.

In [26]:
```
# This code cell will not be shown in the HTML version of this notebook
# plot classification of space, individual learned classifiers (left panel) and j
oint boundary (middle panel), and cost-function panel in the right panel
demo.show_complete_coloring(weight_history, cost = multiclass_softmax)
```



Exercise 7.6. Show the multi-class Softmax reduces to two-class Softmax when $C = 2$

With $C = 2$ the multiclass softmax cost

$$\sum_{c=1}^{C}\sum_{p\in\Omega_c}\log\left(1+\sum_{\substack{j=1\\ j\neq c}}^{C}e^{\left(b_j-b_c\right)+\mathbf{x}_p^T\left(\mathbf{w}_j-\mathbf{w}_c\right)}\right),$$

reduces to \noindent

$$\sum_{p\in\Omega_1}\log\left(1+e^{\left(b_2-b_1\right)+\mathbf{x}_p^T\left(\mathbf{w}_2-\mathbf{w}_1\right)}\right)+\sum_{p\in\Omega_2}\log\left(1+e^{\left(b_1-b_2\right)+\mathbf{x}_p^T\left(\mathbf{w}_1-\mathbf{w}_2\right)}\right).$$

Now note that because we have that $y_p = \begin{cases} -1 & p\in\Omega_1 \\ +1 & p\in\Omega_2\end{cases}$, the cost in (???) can be written equivalently as \noindent

$$\sum_{p\in\Omega_1}\log\left(1+e^{-y_p\left(\left(b_2-b_1\right)+\mathbf{x}_p^T\left(\mathbf{w}_2-\mathbf{w}_1\right)\right)}\right)+\sum_{p\in\Omega_2}\log\left(1+e^{-y_p\left(\left(b_2-b_1\right)+\mathbf{x}_p^T\left(\mathbf{w}_2-\mathbf{w}_1\right)\right)}\right),$$

which can then be written in a more compact form as \noindent

$$\sum_{p=1}^{P}\log\left(1+e^{-y_p\left(\left(b_2-b_1\right)+\mathbf{x}_p^T\left(\mathbf{w}_2-\mathbf{w}_1\right)\right)}\right).$$

Finally letting $b = b_2 - b_1$ and $\mathbf{w} = \mathbf{w}_2 - \mathbf{w}_1$, we arrive at the familiar two-class softmax cost function \noindent

$$\sum_{p=1}^{P} \log\left(1 + e^{-y_p\left(b+\mathbf{x}_p^T\mathbf{w}\right)}\right).$$

Exercise 7.7. Hand-calculations with the multi-class Softmax cost

```
In [ ]:
```

Exercise 7.8. The multi-class Perceptron and Softmax costs are convex

Notice we always have that:

I. Addition of two (or more) convex functions is always convex.

II. Linear and affine functions are convex.

III. The max, exponential, and negative logarithm functions are all convex.

IV. Composition of two convex functions remains convex.

Each of the statements above can be verified easily using the following definition of convexity:

A function g is convex if and only if for all $\mathbf{w}_1$ and $\mathbf{w}_2$ in the domain of g and all $\lambda \in [0, 1]$, we have

$$g\left(\lambda \mathbf{w}_1 + (1-\lambda)\,\mathbf{w}_2\right) \leq \lambda g\left(\mathbf{w}_1\right) + (1-\lambda)\, g\left(\mathbf{w}_2\right)$$

With these four statements at hand, it is straight-forward to prove convexity of multi-class perceptron and softmax cost functions.

Exercise 7.9. Balanced accuracy in the multi-class setting

Suppose we have formed the $(C-1)\times(C-1)$ confusion matrix $\mathbf{Q}$ for a general C-class classification, where the (i, j)th entry $Q_{i,j}$ is the number of datapoints in class i that have been assigned label j by the classifier.

Then the accuracy for class i can be found as

$$\mathcal{A}_i = \frac{Q_{i,i}}{\sum_{j=0}^{C-1} Q_{i,j}}$$

Averaging the accuracies for all C classes we have

$$\text{balanced accuracy} = \frac{\sum_{i=0}^{C-1} \mathcal{A}_i}{C} = \frac{1}{C}\sum_{i=0}^{C-1} \frac{Q_{i,i}}{\sum_{j=0}^{C-1} Q_{i,j}}$$

Exercise 7.10. Weighted multi-class Softmax

In [ ]:
```
## This code cell will not be shown in the HTML version of this notebook
# imports from custom library
import sys
sys.path.append('../../')
from mlrefined_libraries import superlearn_library as superlearn
from mlrefined_libraries import math_optimization_library as optlib

# demos for this notebook
regress_plotter = superlearn.lin_regression_demos
optimizers = optlib.optimizers
static_plotter = optlib.static_plotter.Visualizer()
datapath = '../../mlrefined_datasets/superlearn_datasets/'

# import autograd functionality to bulid function's properly for optimizers
import autograd.numpy as np

# import timer
from datetime import datetime

# this is needed to compensate for %matplotlib notebook's tendancy to blow up ima
ges when plotted inline
%matplotlib notebook
from matplotlib import rcParams
rcParams['figure.autolayout'] = True
```

- Remember: while the base of all regressions is to make the following hold by tuning $\mathbf{w}$

$$\text{model}\left(\mathbf{x}_p,\mathbf{w}\right)\approx y_p$$

and for two-class classification (using ± 1 labels)

$$\text{sign}\left(\text{model}\left(\mathbf{x}_p,\mathbf{w}\right)\right)\approx y_p$$

- The analagous desire for multiclass classification is given by the *fusion rule*

$$y_p = \underset{j\,=\,0,\ldots,C-1}{\text{argmax}}\ \ \text{model}_j\left(\mathbf{x}_p,\mathbf{w}^{(j)}\right)$$

- Here we have a `model` for each of our C classifiers (in the simplest instance these are linear)

- When we use a shared architecture / model $\mathbf{w}^{(j)}$ denotes the weights of the classifier-unique linear combination

- Weighting here works just as with regression / two-class classification, e.g., weighting a multiclass softmax / logistic regression cost looks like

$$g\left(\mathbf{w}\right) = -\frac{1}{P}\sum_{p=1}^{P}\log\left(\frac{e^{\text{model}_{y_p}\left(\mathbf{x}_p,\mathbf{w}^{(y_p)}\right)}}{\sum_{j=0}^{C-1}e^{\text{model}_j\left(\mathbf{x}_p,\mathbf{w}^{(j)}\right)}}\right)$$

- The weighted version looks precisely as previous - an individual weight β_p controls the contribution of the p^{th} point in the summand

$$g\left(\mathbf{w}\right) = -\frac{1}{P}\sum_{p=1}^{P}\beta_p\log\left(\frac{e^{\text{model}_{y_p}\left(\mathbf{x}_p,\mathbf{w}^{(y_p)}\right)}}{\sum_{j=0}^{C-1}e^{\text{model}_j\left(\mathbf{x}_p,\mathbf{w}^{(j)}\right)}}\right)$$

- This is done for the same reasons listed with weighted two-class classification - commonly to deal with large class imbalances

- Our weightings determine *how important each datapoint is* in the training of the model

In [ ]:
```
# weighted multiclass softmax
def multiclass_softmax(self,w,x,y,beta,iter):
    # get subset of points
    x_p = x[:,iter]
    y_p = y[:,iter]
    beta_p = beta[:,iter]

    # pre-compute predictions on all points
    all_evals = model(x_p,w)

    # compute softmax across data points
    a = np.log(np.sum(np.exp(all_evals),axis = 0))

    # compute cost in compact form using numpy broadcasting
    b = all_evals[y_p.astype(int).flatten(),np.arange(np.size(y_p))]
    cost = np.sum(beta_p*(a - b))

    # return average
    return cost/float(np.size(y_p))
```

Exercise 7.11. Recognizing handwritten digits

Load in data.

In [80]:
```
# get MNIST data from online repository
from sklearn.datasets import fetch_openml
x, y = fetch_openml('mnist_784', version=1, return_X_y=True)

# convert string labels to integers
y = np.array([int(v) for v in y])[:,np.newaxis]
```

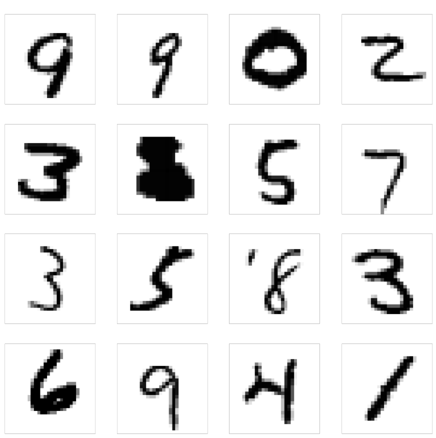
In [81]:
```
print("input shape = " , x.shape)
print("output shape = ", y.shape)
```

```
input shape =  (784, 70000)
output shape =  (1, 70000)
```

Randomly sample input / output pairs.

In [82]:
```
# sample indices
num_sample = 10000
inds = np.random.permutation(y.shape[1])[:num_sample]
x_sample = x[:,inds]
y_sample = y[:,inds]
```

In [105]:
```
from matplotlib import gridspec
from matplotlib.gridspec import GridSpec

fig = plt.figure(figsize=(10,4))
gs=GridSpec(4,10)

for i in range(0, 16):
    fig.add_subplot(gs[i%4 + int(i/4)*10])
    plt.imshow(np.max(x_sample[:,i]) - np.reshape(x_sample[:,i],(28,28)), cmap='g
ray')
    plt.xticks([]), plt.yticks([])
    plt.axis('off')
plt.show()
```



Implementation of multi-class cost and gradient descent optimizer that takes in mini-batches.

In [58]:
```
# compute C linear combinations of input point, one per classifier
def model(x,w):
    a = w[0] + np.dot(x.T,w[1:])
    return a.T

# multiclass perceptron
def multiclass_perceptron(w,x,y,iter):
    # get subset of points
    x_p = x[:,iter]
    y_p = y[:,iter]

    # pre-compute predictions on all points
    all_evals = model(x_p,w)

    # compute maximum across data points
    a =  np.max(all_evals,axis = 0)

    # compute cost in compact form using numpy broadcasting
    b = all_evals[y_p.astype(int).flatten(),np.arange(np.size(y_p))]
    cost = np.sum(a - b)

    # return average
    return cost/float(np.size(y_p))
```

In [59]:
```
from autograd.misc.flatten import flatten_func
from autograd import grad as gradient
from timeit import default_timer as timer

# minibatch gradient descent
def gradient_descent(g,w,x_train,y_train,alpha,max_its,batch_size,**kwargs):
    verbose = True
    if 'verbose' in kwargs:
        verbose = kwargs['verbose']

    # flatten the input function, create gradient based on flat function
    g_flat, unflatten, w = flatten_func(g, w)
    grad = gradient(g_flat)

    # record history
    num_train = y_train.size
    w_hist = [unflatten(w)]
    train_hist = [g_flat(w,x_train,y_train,np.arange(num_train))]

    # how many mini-batches equal the entire dataset?
    num_batches = int(np.ceil(np.divide(num_train, batch_size)))

    # over the line
    for k in range(max_its):
        # loop over each minibatch
        start = timer()
        train_cost = 0
        for b in range(num_batches):
            # collect indices of current mini-batch
            batch_inds = np.arange(b*batch_size, min((b+1)*batch_size, num_trai
n))

            # plug in value into func and derivative
            grad_eval = grad(w,x_train,y_train,batch_inds)
            grad_eval.shape = np.shape(w)

            # take descent step with momentum
            w = w - alpha*grad_eval

        end = timer()

        # update training and validation cost
        train_cost = g_flat(w,x_train,y_train,np.arange(num_train))

        # record weight update, train and val costs
        w_hist.append(unflatten(w))
        train_hist.append(train_cost)

        if verbose == True:
            print ('step ' + str(k+1) + ' done in ' + str(np.round(end - start,
1)) + ' secs, train cost = ' + str(np.round(train_hist[-1][0],4)))

    if verbose == True:
        print ('finished all ' + str(max_its) + ' steps')
        #time.sleep(1.5)
        #clear_output()
    return w_hist,train_hist
```

Run minimization.

In [68]:
```
# parameters for general run
g = multiclass_perceptron
alpha = 10**(-3)
max_its = 5
N = x_sample.shape[0]
C = len(np.unique(y_sample))
w = 0.1*np.random.randn(N+1,C)

# make first run
batch_size = 200
weight_history_1,cost_history_1 = gradient_descent(g,w,x_sample,y_sample,alpha,ma
x_its,batch_size,verbose=False)

# make second run
batch_size = y_sample.shape[1]
weight_history_2,cost_history_2 = gradient_descent(g,w,x_sample,y_sample,alpha,ma
x_its,batch_size,verbose=False)
```

In [77]:
```
# the import statement for matplotlib
import matplotlib.pyplot as plt

# cost function history plotter
def plot_cost_histories(cost_histories,labels):
    # create figure
    plt.figure(figsize=(9,3))

    # loop over cost histories and plot each one
    for j in range(len(cost_histories)):
        history = cost_histories[j]
        label = labels[j]
        plt.plot(history,label = label)
    plt.legend(loc='upper right')
    plt.title('cost function history comparison')
    plt.xlabel('iteration')
    plt.ylabel('cost function value',rotation = 90)
    plt.show()
```

In [78]:
```
cost_histories = [cost_history_1,cost_history_2]
labels = ['minibatch','full batch']
plot_cost_histories(cost_histories,labels)
```