Exercise 19. We consider $f: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ given for a matrix $B \in \mathbb{R}^{m \times m}$ via

$$f(u,A) = \frac{1}{2}|A|^2 + (Bu)\cdot A + \frac{1}{2}|u|^2$$

B is not symmetric!

f is quadratic, i.e., $f(u,A) = \frac{1}{2}\begin{pmatrix} u \\ A \end{pmatrix} \cdot \mathbb{M}\begin{pmatrix} u \\ A \end{pmatrix}$

where $\mathbb{M} \in \mathbb{R}^{2m \times 2m}$, $\mathbb{M} = \begin{pmatrix} I & B^T \\ B & I \end{pmatrix}$.

a) We show that f convex $\Longleftrightarrow \|B\| \leq 1$

Let $\lambda \in]0,1[$ and fix $(u_0, A_0), (u_1, A_1)$, then for $u_\lambda = (1-\lambda)u_0 + \lambda u_1$, $A_\lambda = (1-\lambda)A_0 + \lambda A_1$ we have (since f is quadratic)

$$f(u_\lambda, A_\lambda) = (1-\lambda)^2 f(u_0, A_0) + \lambda^2 f(u_1, A_1)$$
$$+ \frac{(1-\lambda)\lambda}{2}\left(\begin{pmatrix} u_0 \\ A_0 \end{pmatrix}\mathbb{M}\begin{pmatrix} u_1 \\ A_1 \end{pmatrix} + \begin{pmatrix} u_1 \\ A_1 \end{pmatrix}\mathbb{M}\begin{pmatrix} u_0 \\ A_0 \end{pmatrix}\right)$$

$$= (1-\lambda) f(u_0, A_0) + \lambda f(u_1, A_1)$$

$$+ (\underbrace{1 - 2\lambda + \lambda^2 - 1 + \lambda}_{= \lambda(\lambda - 1)}) f(u_0, A_0)$$

$$+ (\underbrace{\lambda^2 - \lambda}_{= \lambda(\lambda-2)}) f(u_1, A_1)$$

$$+ \frac{(1-\lambda)\lambda}{2} \left(\begin{pmatrix} u_0 \\ A_0 \end{pmatrix} \mathbb{M} \begin{pmatrix} u_1 \\ A_1 \end{pmatrix} + \begin{pmatrix} u_1 \\ A_1 \end{pmatrix} \mathbb{M} \begin{pmatrix} u_0 \\ A_0 \end{pmatrix} \right)$$

$$= (1-\lambda) f(u_0, A_0) + \lambda f(u_1, A_1)$$

$$+ \frac{(1-\lambda)\lambda}{2} \left(\begin{pmatrix} u_0 \\ A_0 \end{pmatrix} \mathbb{M} \begin{pmatrix} u_1 - u_0 \\ A_1 - A_0 \end{pmatrix} + \begin{pmatrix} u_1 \\ A_1 \end{pmatrix} \mathbb{M} \begin{pmatrix} u_0 - u_1 \\ A_0 - A_1 \end{pmatrix} \right)$$

$$= (1-\lambda) f(u_0, A_0) + \lambda f(u_1, A_1)$$

$$\boxed{- \frac{(1-\lambda)\lambda}{2} \left(\begin{pmatrix} u_1 - u_0 \\ A_1 - A_0 \end{pmatrix} \mathbb{M} \begin{pmatrix} u_1 - u_0 \\ A_1 - A_0 \end{pmatrix} \right)}$$

Thus, we have convexity if and only

if $(*)\ \begin{pmatrix} u \\ A \end{pmatrix} \mathbb{M} \begin{pmatrix} u \\ A \end{pmatrix} \geq 0 \quad \forall (u, A) \in \mathbb{R}^{2n}$

(essentially, we derived the Taylor expansion of f and found that the Hessian has to be positive semidefinite)

$(*)$ holds iff $\forall (u, A)$

$$|u|^2 + |A|^2 + 2(Bu)\cdot A \geq 0$$

$$\Longleftrightarrow \quad |Bu| \leq |u| \quad \forall u \in \mathbb{R}.$$

$\boxed{\text{why?}}$

b) Let us assume that I is convex if $\|B - B^T\| \leq 2$

Note $\quad B = \underbrace{\tfrac{1}{2}(B + B^T)}_{\substack{\text{symmetric} \\ \text{part}}} + \underbrace{\tfrac{1}{2}(B - B^T)}_{\substack{\text{skew-symmetric} \\ \text{part}}}$

In particular, if $B = B^T$ then I is

convex, even if $\|B\| > 1$ and hence f is not convex!

The crucial point is that the symmetric part of B doesn't contribute to the functional!

Indeed, let $C = \frac{1}{2}(B + B^T) = \mathrm{sym}(B)$, then

$$\int_0^\ell Cu \cdot u' \, dx = \int_0^\ell \frac{1}{2} \frac{d}{dx}(Cu \cdot u)\, dx$$

used symmetry here!

$$= (Cu \cdot u)\Big|_{x=0}^{\ell} = 0$$

boundary conditions.

This means that, we have

$$I(u) = \int_0^\ell \tilde{f}(u, u')\, dx \quad \text{with}$$

$$\tilde{f}(u,A) = \frac{1}{2}|u|^2 + \frac{1}{2}|A|^2 + (B_{skew}u)\cdot A$$

skew-symmetric part.

if $B_{skew} = \frac{1}{2}(B - B^T)$ satisfies condition from a), i.e., $\|B - B^T\| \leq 2$, then $\tilde{f}$ is convex and therefore also I.