Exercise 40. Rank-one affinity

$f : \mathbb{R}^{m\times 2} \longrightarrow \mathbb{R}$ rank-one affine, i.e.

$\forall A, B \in \mathbb{R}^{m\times 2}$ such that $B - A = \xi \otimes \eta$ with $\xi \in \mathbb{R}^m$, $\eta \in \mathbb{R}^2$ and all $t \in [0,1]$

a) $f(A + t\,\xi\otimes\eta) = (1-t) f(A) + t f(\underbrace{A + \xi\otimes\eta}_{=B})$

$$\Rightarrow \quad \frac{f(A + t\,\xi\otimes\eta) - f(A)}{t} = f(A+\xi\otimes\eta) - f(A)$$

$$\overset{t\downarrow 0}{\Rightarrow} \quad \nabla f(A) : \xi\otimes\eta = f(A+\xi\otimes\eta) - f(A)$$

Write $A = (a_1 \mid a_2)$ with $a_i \in \mathbb{R}^m$

choose $\eta = (1,0) \in \mathbb{R}^2$ and define

$$g_{a_2}(a_1) = f(a_1, a_2)$$

$$\Rightarrow g_{a_2}(a_1 + t\xi) = (1-t) g_{a_2}(a_1) + t g_{a_2}(a_1 + \xi)$$

$$\Rightarrow \nabla g_{a_2}(a_1)\cdot\xi = g_{a_2}(a_1+\xi) - g_{a_2}(a_1)$$

pick $a_1 = 0 \in \mathbb{R}^m$ and set

$$\tilde{b}_1(a_2) := \nabla g_{a_2}(0) \in \mathbb{R}^m$$

we conclude that g_{a_2} is affine for each $a_2 \in \mathbb{R}^m$

$$g_{a_2}(\xi) = \tilde{b}_1(a_2) \cdot \xi + g_{a_2}(0)$$

define $h_0(a_2) = g_{a_2}(0) = f(0, a_2)$

with the same arguments as above, we find that h_0 is affine:

$$h_0(a_2) = b_2 \cdot a_2 + f(0,0)$$

Summarizing, we get

$$f(a_1, a_2) = g_{a_2}(a_1) = \tilde{b}_1(a_2) \cdot a_1 + b_2 \cdot a_2 + f(0,0)$$

we fix now $a_1 \in \mathbb{R}^m$:

$$h_{a_1}(a_2) = f(a_1, a_2) = a_1 \cdot \tilde{b}_1(a_2) + b_2 \cdot a_2 + f(0,0)$$

is affine, thus $\tilde{b}_1 : \mathbb{R}^m \to \mathbb{R}^m$ is affine

$$\tilde{b}_1(a_2) = b_1 + B a_2, \quad B \in \mathbb{R}^{m \times m}$$

$$\Rightarrow f(a_1, a_2) = a_1 \cdot b_1 + a_1 \cdot B a_2 + b_2 \cdot a_2 + \underbrace{f(0,0)}_{=: \beta}$$

b) $B \in \mathbb{R}^{m \times m}$ is skew-symmetric

we have $t \mapsto f(A + t\, \xi \otimes \eta)$ is linear

$$\Rightarrow \frac{d^2}{dt^2} f(A + t\, \xi \otimes \eta) = 0$$

$$\Rightarrow \frac{d^2}{dt^2}\Big((a_1 + t\, \xi \eta_1) \cdot b_1 + (a_2 + t\, \xi \eta_2) \cdot b_2$$

$$+ (a_1 + t\, \xi \eta_1) \cdot B (a_2 + t\, \xi \eta_2) \Big) = 0$$

$$\Rightarrow \xi \eta_1 \cdot B \xi \eta_2 = 0 \quad \forall \xi \in \mathbb{R}^n, \eta_i \in \mathbb{R}$$

$$\Rightarrow \xi \cdot B \xi = 0$$

$$\Rightarrow B = -B^T$$

we can always decompose a matrix B into symmetric and skew-symmetric part $B = \frac{1}{2}(B + B^T) + \frac{1}{2}(B - B^T)$

$$\underbrace{}_{B_{sym}} \qquad \underbrace{}_{B_{skew}}$$

$$\xi \cdot B \xi = \xi B_{sym} \xi + \underbrace{\xi \cdot B_{skew} \xi}_{\frac{1}{2}(\xi \cdot B \xi - \xi \cdot B^T \xi) = 0} = \xi \cdot B_{sym} \cdot \xi$$

c) Since B is skew-symmetric, we have

$$B_{ij} = -B_{ji} \quad \forall\, i,j = 1,\dots,m$$

in particular $B_{ii} = 0 \quad \forall\, i = 1,\dots,m$

write $$a_1 \cdot B a_2 = \sum_{i \neq j} (a_1)_i B_{ij} (a_2)_j$$

$$= \sum_{1 \le i < j \le m} \Big\{ \underbrace{(a_1)_i (a_2)_j - (a_2)_i (a_1)_j}_{\det \begin{pmatrix} (a_1)_i & (a_2)_i \\ (a_2)_j & (a_2)_j \end{pmatrix}} \Big\} B_{ij}$$

$$= \boldsymbol{\beta} \cdot T_2(A)$$

$\square$