Exercise 17

Assume that $\lim_{t\to 0} \frac{g(t)}{t} = 0$. Then

$$\frac{1}{t}\big(I(u+th) - I(u)\big) \geq \frac{1}{t}\Big[L_u(th) + g(t\|h\|)\Big] = L_u(h) + \frac{g(t\|h\|)}{t} \xrightarrow{t\to 0} L_u(h)$$

$\Rightarrow$ I is Gateaux differentiable and $DI(u) = L_u$.

Now let I be Gateaux differentiable with $DI(u) = L_u$. Let $h \in X$ with $\|h\| = 1$.
Then

$$\frac{g(t)}{t} = \frac{1}{t} g(t\|h\|) \leq \frac{1}{t}\big[I(u+th) - I(u) - L_u(th)\big]$$
$$= \frac{1}{t}\big[I(u+th) - I(u)\big] - L_u(h)$$

$\Rightarrow$ $0 \leq \limsup_{t\to 0} \frac{g(t)}{t} \leq DI(u)[h] - L_u(h) = 0.$ $\Rightarrow$ $\lim_{t\to 0} \frac{g(t)}{t} = 0$.

Exercise 18

a) Let $u, v \in X$. Then

$$\mathcal{I}(v) - \mathcal{I}(u) = \|v\|_X^2 - \|u\|_X^2 = \langle v+u, v-u \rangle$$
$$= \langle 2u, v-u \rangle + \langle v-u, v-u \rangle$$
$$= L_u(v-u) + g(\|v-u\|)$$

for $L_u(w) := \langle 2u, w \rangle$, $g(t) = t^2$.

Note: This is in accordance with Exercise 17 since "$D\mathcal{I}(u) = 2u$".

b) $\mathcal{I}$ is convex since for $\lambda \in (0,1)$, $u, v \in X$ the triangle inequality yields

$$\mathcal{I}((1-\lambda)u + \lambda v) = \|(1-\lambda)u + \lambda v\|_X \leq (1-\lambda)\|u\|_X + \lambda \|v\|_X = (1-\lambda)\mathcal{I}(u) + \lambda \mathcal{I}(v).$$

However, $\mathcal{I}$ is not uniformly convex: Assume the opposite, i.e., there is $g: [0,\infty) \to \mathbb{R}$ with $g(0) = 0$ and g strictly increasing such that for all $u \in X$ there is $L_u \in X'$ such that

$$\forall v \in X: \mathcal{I}(v) \geq \mathcal{I}(u) + L_u(u-v) + g(\|u-v\|).$$

Using $v = \lambda u$, we have

$$\mathcal{I}(\lambda u) \geq \mathcal{I}(u) + L_u((1-\lambda)u) + g(\|(1-\lambda)u\|)$$
$$\Leftrightarrow \quad (|\lambda| - 1)\|u\| + (\lambda - 1) L_u(u) \geq g(\|(\lambda-1)u\|)$$

Choosing $\lambda = 1 + \tau$ with $\tau \in (-1,1)$, we obtain

$$\tau \|u\| + \tau L_u(u) \geq g(\tau \|u\|) \geq 0 \qquad (*)$$

For $\tau > 0$, this yields $\|u\| + L_u(u) \geq 0$.

For $\tau < 0$, this yields $\|u\| + L_u(u) \leq 0$.

This implies $\|u\| + L_u(u) = 0$, and $(*)$ yields $g(\tau\|u\|) = 0$.

Now choose $\|u\| \neq 0$. Then this contradicts the strict monotonicity of g.

$\Rightarrow$ $\mathcal{I}$ is not uniformly convex.

Note: This argument applies in any Banach space $X \neq \{0\}$.