Exercise 9

$$I : \begin{cases} C^1(\bar{\Omega}; \mathbb{R}^m) \to \mathbb{R} \\ u \mapsto \int_\Omega f(x, u(x), \nabla u(x))\, dx \end{cases}$$

where $f \in C^2(\bar{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m \times d})$

We have estimates for some $u_0 \in C^1(\bar{\Omega}; \mathbb{R}^n)$

(1) $$\int_\Omega D_A^2 f(x, u_0(x), \nabla u_0(x))[\nabla w(x)]\, dx \geq \gamma_1 \int_\Omega |\nabla w(x)|^2 dx$$

(2) $$D^2 I(u)[w] \geq \gamma_2 \int_\Omega |w(x)|^2 dx$$

for all $w \in C^1(\bar{\Omega}; \mathbb{R}^m)$ (was missing in exercise statement)

(a) We have

$$D^2 I(u_0)[w] = \int D_A^2 f(x, u_0, \nabla u_0)[\nabla w] + 2 D_A D_u f(x, u_0, \nabla u_0)[w, \nabla w] + D_u^2 f(x, u_0, \nabla u_0)[w]\, dx$$

(see lecture, Lemma 2.3)

The estimate in (1) yields

$$D^2 I(u_0)[w] \geq \int_\Omega \gamma_1 |\nabla w|^2 + 2 D_A D_u f(\dots)[w, \nabla w] + D_u^2 f(\dots)[w]\, dx$$

We have

$$\sup_{x \in \overline{\Omega}} |D_A D_u f(x, u_0(x), \nabla u_0(x))| \leq M_1 < \infty$$

$$\sup_{x \in \overline{\Omega}} |D_u^2 f(x, u_0(x), \nabla u_0(x))| \leq M_2 < \infty$$

why? ($f \in C^2(\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m \times d})$
$u_0 \in C^1(\overline{\Omega}; \mathbb{R}^n)$)

We estimate

$$D^2 I(u_0)[w] \geq \int_\Omega \gamma_1 |\nabla w|^2 \underline{- 2 M_1 |\nabla w| |w|} - M_2 |w|^2 dx$$

We need to take care of the <u>mixed term</u>

Trick use <u>Young's inequality</u>

$$a \cdot b = \left(\sqrt{2\varepsilon}\, a\right)\left(\frac{1}{\sqrt{2\varepsilon}} b\right) \leq \varepsilon\, a^2 + \frac{1}{4\varepsilon} b^2$$

$\forall\, a, b \geq 0, \quad \varepsilon > 0$

For $a = |\nabla w|$, $b = 2M_1 |w|$

we get

$$\int_\Omega 2M_1 |\nabla w||w|\, dx \leq \int_\Omega \varepsilon |\nabla w|^2 + \frac{4M_1^2 |w|^2}{4\varepsilon}\, dx$$

Thus, we proceed in the above estimate

$$D^2 I(u_0)[w] \geq \int_\Omega (\gamma_1 - \varepsilon)|\nabla w|^2 - \left(M_2 + \frac{M_1}{\varepsilon}\right)|w|^2 dx$$

With $\varepsilon = \frac{\gamma_1}{2}$, we get

$$D^2 I(u_0)[w] \geq \int \frac{\gamma_1}{2} |\nabla w|^2 - \left(M_2 + \frac{2M_1}{\gamma_1}\right)|w|^2 dx$$

i.e., we get the assertion of the exercise with $C^* = \left(M_2 + \frac{2M_1}{\gamma_1}\right) \geq 0$

(b) We can decompose the second variation

$$D^2 I(u_\omega)[w] = (1-\alpha) D^2 I(u_0)[w] + \alpha D^2 I(u_0)[w]$$

$$\alpha \in [0, \infty)$$

Using (a) for the first term and (2) for the second yields

$$D^2 I(u_0)[w] \geq (1-\alpha)\left(\frac{\gamma_1}{2} \|\nabla w\|_{L^2}^2 - C^* \|w\|_{L^2}^2 \right) + \alpha \gamma_2 \|w\|_{L^2}^2$$

$$= \frac{(1-\alpha)\gamma_1}{2} \|\nabla w\|_{L^2}^2 + \left[\alpha \gamma_2 - (1-\alpha) C^*\right) \|w\|_{L^2}^2$$

α interpolates between $\gamma_2 > 0$ and $-C^* \leq 0$

$1 > \alpha > \dfrac{C^*}{\gamma_2 + C^*} \geq 0$ yields the result.

Remark: Consider quadratic form

$$J(w) = \int_\Omega M(x)\nabla w : \nabla w + 2B(x)w : \nabla w + C(x)w \cdot w \, dx$$

where $M(x) := \partial_A^2 f(x, u_0(x), \nabla u_0(x)) \in \mathrm{Lin}(\mathbb{R}^{m \times d}; \mathbb{R}^{m \times d})$

$$B(x) := \partial_A \partial_u f(x, u_0(x), Du_0(x)) \in \mathrm{Lin}(\mathbb{R}^m; \mathbb{R}^{m \times d})$$

and $$C(x) := \partial_u^2 f(x, u_0(x), Du_0(x)) \in \mathbb{R}^{m \times m}.$$

Euler-Lagrange-Equation for J is called Jacobi equation associated with $D^2 I$.

Compare with course on linear PDE (Riesz, Lax-Milgram, etc.)

<u>Exercise 10</u> $M = C^1([a,b])$

$$I : \begin{cases} M \longrightarrow \mathbb{R} \\ u \longmapsto \int_a^b g(u'(x)) + h(u(x))\, dx \end{cases}$$

with $g, h \in C^2(\mathbb{R})$

$$f(x, u, A) = g(A) + h(u)$$

a) E-L equation

$$-(\partial_A g(u'(x)))' + \partial_u h(u(x)) = 0 \quad \text{in } [a,b]$$

$$\partial_A g(u'(x)) = 0 \quad \text{in } \{a,b\}$$

Solutions of the form $u(x) = u_* = \text{const}$ $(u' \equiv 0)$ exist if

$\boxed{0 + \partial_u h(u_*) = 0}$ AND $\boxed{\partial_A g(0) = 0}$

i.e. u_* is critical point of $\boxed{h}$

$\quad\;\; 0$ " " " " $\boxed{g}$

(b) We assume $u \equiv u_* = \text{const}$ is critical point of I, i.e. u_* is critical for h and 0 for g.

With $g''(0) = \partial_A^2 g(0) \geq \gamma_1 > 0$ and

$h''(u_*) = \partial_u h(u_*) \geq \gamma_2 > 0$

we get for the second variation

$$D^2 I(u)[w] = \int_a^b g''(0)(w'(x))^2 + h''(u_*)\, w(x)^2 \, dx$$

$$\geq \gamma_1 \|w'\|^2_{L^2} + \gamma_2 \|w\|^2_{L^2}$$

Theorem 2.15 (b) $\Rightarrow$ $u \equiv u_*$ is strict weak local minimizer

(recall what that means)

(c) We now assume that

(i) $g(A) \geq 0 = g(0) \Rightarrow 0$ global minimizer of g

(ii) u^0 local minimizer of h, i.e.

$$\exists \varepsilon > 0 \ \forall \tilde{u} \in B_\varepsilon(u^0): \ h(\tilde{u}) \geq h(u^0)$$

consider $\tilde{u} \in C^1([a,b])$ with $\|\tilde{u} - u^0\|_{C^0} \leq \varepsilon$

$$I(\tilde{u}) = \int_a^b g(\tilde{u}') + h(\tilde{u})\,dx$$

$$\overset{(i)}{\geq} \int_a^b g(0) + h(\tilde{u})\,dx$$

$$\overset{(ii)}{\geq} \int_a^b 0 + h(u^0)\,dx = I(u^0)$$

$\Rightarrow u^0$ is strong local minimizer

We see from the last step that $\|\tilde{u} - u^0\|_{C^0} \leq \varepsilon$ can be dropped if u^0 is global minimizer of h.