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Ex 28

Sufficient conditions for weak lower semicontinuity

In Thm. 3.44, we require that

$f: \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \longrightarrow \mathbb{R}$ satisfies

(i) Carathéodory property

(ii) $A \mapsto f(x, u, A)$ is convex f.a.e. $x \in \Omega$ and for all $u \in \mathbb{R}^m$

(iii) $f(x, u, A) \geq \gamma(x)$ for some $\gamma \in L^1(\Omega)$.

to conclude that $I(u) = \int_\Omega f(x, u, Du)\,dx$ is weak lower semicontinuous on $W^{1,p}(\Omega; \mathbb{R}^m)$

We replace (iii) with the weaker condition

$\exists \gamma \in L^1(\Omega), c \in \mathbb{R}$ such that

$$f(x, u, A) \geq \gamma(x) + c|u|^q$$

where $\frac{1}{q} \geq \frac{d-p}{dp}$

② In particular, if $p > d$ then $q \in [1, \infty[$

To show that I is still w.l.s.c. let us consider the function

$$\tilde{f}(x,u,A) = f(x,u,A) - \gamma(x) - c|u|^q,$$

which satifies (i), (ii) and $\tilde{f} \geq 0$

Thus, the Theorem 3.44 can be applied to obtain that $\tilde{I}(u) = \int_\Omega \tilde{f}(x,u,Du)\,dx$ is w.l.s.c., i.e.

$$\tilde{I}(u) \leq \liminf_{n\to\infty} \tilde{I}(u_n) \text{ when } u_n \rightharpoonup u \text{ in } W^{1,p}$$

Moreover,

$$\tilde{I}(u) = I(u) - J(u)$$

with $J(u) = -\int_\Omega \gamma + c|u|^q\,dx$

J is weakly continuous on $W^{1,p}$ since $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ implies via Rellichs theorem that $u_n \longrightarrow u$ in $L^q(\Omega)$

Thus $\liminf_{n\to\infty} I(u) = \liminf_{n\to\infty}(\tilde{I}(u_n) + J(u_n))$

$$= \liminf_{n\to\infty} \tilde{I}(u_n) + \lim_{n\to\infty} J(u_n) = \tilde{I}(u) + J(u)$$
$$= I(u)$$

where we have used that

$$\liminf_{n\to\infty} (a_n + b_n) = \liminf_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

if $b_n \to b$ in $\mathbb{R}$.

<u>Remark:</u> we could also weaken the assumption to

$$f(x,u,A) \geq \gamma(x) + c|u|^q + \beta(x) : A$$
for $\beta \in L^{p'}(\Omega; \mathbb{R}^{m\times d})$.

<u>Ex 29</u> Lavrentiev gap phenomenon

(a) Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$ Carathéodory function with $|f(x,u,A)| \leq C(1 + |u|^p + |A|^p)$.

We show that

$$\inf_{u\in W^{1,p}(\Omega;\mathbb{R}^m)} I(u) = \inf_{\tilde{u}\in C^\infty(\bar{\Omega};\mathbb{R}^m)} I(\tilde{u})$$

Note that we do not ask whether infimum is actually attained. We neglected boundary conditions. However, they can also be considered using the trace operator.

First, note that the growth condition gives the continuity of I on $W^{1,p}(\Omega;\mathbb{R}^m)$ w.r.t. the strong topology, see Exercise 16, Sheet 5.

Second, since $C^\infty(\Omega;\mathbb{R}^m)$ is dense in $W^{1,p}(\Omega;\mathbb{R}^m)$ we can argue as follows:

If u_n is such that

$$\left|\inf_{u\in W^{1,p}} I - I(u_n)\right| \le 1/n$$

then there for each $\varepsilon>0$ there exists $v_n^\varepsilon \in C^\infty$ such that $|I(u_n) - I(v_n^\varepsilon)| \le 1/n$. Thus,

$\left|\inf_{u\in W^{1,p}} I - I(v_n^\varepsilon)\right| \le 2/n$, which proves the claim

b) Example by Manià (Basilio, 1909-1939)

$$I(u) = \int_0^1 (u^3 - x)^2 |u'|^6 \, dx$$

for $u(0) = 0$, $u(1) = 1$

$$f(x,u,A) = (u^3 - x)^2 |A|^6$$

Caratheodory, convex, nonnegative,

<u>no</u> coercivity (can be cured by adding $\frac{\varepsilon}{2}|A|^2$)

$\boxed{\text{Step 1}}$

In particular, $I(u) \geq 0$ and for $u_*(x) = x^{1/3}$ we have $I(u_*) = 0$.

Since $u_*'(x) = \frac{1}{3} x^{-2/3}$ it holds that $u_* \in W^{1,1}(0,1)$ but $u_* \notin W^{1,\infty}(0,1)$.

We conclude $\inf\limits_{u \in W^{1,1}(0,1)} I(u) = 0$.

We show that $\inf\limits_{u \in W^{1,\infty}(0,1)} I(u) > 0$

$\boxed{\text{Step 2.}}$ If $u \in W^{1,\infty}(0,1)$, we know from Exercise 26 that u can be assumed to be Lipschitz continuous.

With $u(0)=0$ and $u(1)$ there exists $x_0 > 0$ such that

$$u(x) \le \frac{x^{1/3}}{2} =: w(x) \quad \text{for all } x \in [0, x_0]$$

$$\text{and} \quad u(x_0) = w(x_0)$$

$$(w \text{ is } \underline{\text{not}} \text{ Lipschitz})$$

$\boxed{\text{Step 3}}$ For $x \in [0, x_0]$ we deduce that

$$u(x)^3 - x \le \underbrace{w(x)^3}_{= \frac{x}{8}} - x = -\frac{7}{8} x \le 0$$

$$\text{thus} \quad (u^3 - x)^2 \ge (w^3 - x)^2 = \frac{7^2}{8^2} x^2$$

and

$$I(u) \ge \int_0^{x_0} (u^3 - x)^2 |u'|^6 dx$$

$$\ge \frac{7^2}{8^2} \int_0^{x_0} x^2 |u'|^6 dx .$$

$\boxed{\text{Step 4.}}$ We use Hölder's inequality for $p=6 \Rightarrow p' = \frac{p}{p-1} = \frac{6}{5}$

$$\int_0^{x_0} u'(x)\,dx = \int_0^{x_0} x^{-\frac{1}{3}} x^{\frac{1}{3}} u'(x)\,dx$$

$$\le \underbrace{\left(\int_0^{x_0} x^{-2/5}\,dx\right)^{5/6}}_{\left(\frac{5}{3} x_0^{3/5}\right)^{5/6}} \cdot \left(\int_0^{x_0} x^2 |u'|^6 dx\right)^{1/6}$$

$$= \left(\frac{5}{3}\right)^{5/6} x_0^{1/2} \left(\int_0^{x_0} x^2 |u'|^6 dx\right)^{1/6}$$

Thus, with Step 3 we conclude

$$I(u) \ge \frac{7^2}{8^2} \frac{3^5}{5^5} \frac{1}{x_0^3} \left(\int_0^{x_0} u'\,dx\right)^6$$

$$= \frac{7^2}{8^2} \frac{3^5}{5^5} \frac{1}{x_0^3} \Big(\underbrace{u(x_0)}_{=\frac{1}{2} x_0^{1/3}} - \underbrace{u(0)}_{=0}\Big)^6$$

$$= \frac{7^2}{8^2} \frac{3^5}{5^5} \frac{1}{2^6} \frac{x_0^2}{x_0^3} > 0 = I(u_*)$$

This holds for all $u \in W^{1,\infty}(0,1)$ with .

$u(0)=0$ and $u(1)>1$.

$\square$

Message: The set $W^{1,\infty}$ is much smaller than $W^{1,1}$. In particular, functions in $W^{1,\infty}$ are more regular (Lipschitz) than those in $W^{1,1}$. There are less competitors available than in $W^{1,1}$. Minimizers in $W^{1,1}$ must be less regular to achieve $I(u_*)=0$.