Exercise 7

a) We have

$$DI(u)[v-u] = \lim_{t\to 0} \frac{1}{t}\Big[I(\underbrace{u+t(v-u)}_{=(1-t)u+tv}) - I(u)\Big]$$

$$\leq \lim_{t\to 0} \frac{1}{t}\Big[(1-t)I(u) + tI(v)) - I(u)\Big]$$

$$= \lim_{t\to 0} \frac{1}{t}\Big[-tI(u) + tI(v)\Big]$$

$$= I(v) - I(u),$$

so that for a critical point u_* we obtain

$$I(v) \geq I(u_*) + \underbrace{DI(u_*)[v-u_*]}_{=0} = I(u^*)$$

for all $v\in X$. Hence u_* is a global minimizer.

b) By assumption, u_* satisfies the Euler-Lagrange equations for the functional I. Hence u_* is a critical point of I.
Moreover, I is convex since f is convex, and (a) implies that u_* is a minimizer of I since we minimize I over the affine linear space

$$\{v\in C^2(\bar\Omega) \mid v = u_* \text{ on } \partial\Omega\} = u_* + C_0^2(\bar\Omega).$$

Now assume that $\tilde u \in C^1(\bar\Omega)$ is another minimizer of I. Then $I(\tilde u) = I(u_*)$, and $u_*\neq\tilde u$ implies that there is an open set $\tilde\Omega\subset\Omega$ such that

$$u_*(x) \neq \tilde u(x) \qquad \text{for all } x\in\tilde\Omega.$$

Now the strict convexity of f (with $\lambda = \frac{1}{2}$) implies

$$I\left(\tfrac{1}{2}u_* + \tfrac{1}{2}\tilde u\right) = \int_{\tilde\Omega} f\left(\tfrac{1}{2}\nabla u_* + \tfrac{1}{2}\nabla\tilde u\right)dx + \int_{\Omega\setminus\tilde\Omega} f\left(\tfrac{1}{2}\nabla u_* + \tfrac{1}{2}\nabla\tilde u\right)dx$$

$$< \int_{\tilde\Omega} \tfrac{1}{2}f(\nabla u_*) + \tfrac{1}{2}f(\nabla\tilde u)\,dx + \int_{\Omega\setminus\tilde\Omega} \tfrac{1}{2}f(\nabla u_*) + \tfrac{1}{2}f(\nabla\tilde u)\,dx$$

$$= \tfrac{1}{2}I(u_*) + \tfrac{1}{2}I(\tilde u) = I(u_*).$$

This contradicts the fact that u_* is a minimizer.

$\Rightarrow$ u_* is the unique minimizer of I.

c) From the Lecture, we know that the problem corresponds to minimizing the functional

$$I(u) = \int_a^b \sqrt{1+u'(x)^2}\, dx \quad \text{on} \quad M = \{ u \in C^1([a,b]) \mid u(a) = A,\ u(b) = B \}.$$

In Example 2.9 we have seen that

$$u_*(x) = B\frac{x-a}{b-a} + A\frac{b-x}{b-a} \qquad (\hat{=} \text{ straight line from } (a,A) \text{ to } (b,B))$$

is the solution to the Euler-Lagrange equations and thus a critical point.

Let $f(A) = \sqrt{1+A^2}$. Then f is strictly convex if $f''(A) > 0$ for all $A \in \mathbb{R}$.

[This follows from Taylor's theorem:

$$f((1-\lambda)A_0 + \lambda A_1) = f(A_0 + \lambda(A_1 - A_0))$$
$$= f(A_0) + \lambda f'(A_0)(A_1 - A_0) + \int_0^\lambda \tau^2 \underbrace{f''(A_0 + \tau(A_1 - A_0))(A_1 - A_0)^2}_{>0 \text{ if } A_1 \neq A_0} d\tau \Bigg]$$

We compute

$$f'(A) = \frac{A}{\sqrt{1+A^2}}, \qquad f''(A) = \frac{1}{\sqrt{1+A^2}} - \frac{2A^2}{2(\sqrt{1+A^2})^3} = \frac{1}{(\sqrt{1+A^2})^3} > 0$$

$\Rightarrow$ f strictly convex $\overset{b)}{\Rightarrow}$ u_* unique minimizer.

Exercise 8

a) We have

$$\frac{d}{d\varphi}[R_\varphi y]\Big|_{\varphi=0} = \begin{pmatrix} -\sin\varphi\, y_1 - \cos\varphi\, y_2 \\ \cos\varphi\, y_1 - \sin\varphi\, y_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\Bigg|_{\varphi=0} = \begin{pmatrix} -y_2 \\ y_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This implies (with $R_0 y = y$)

$$\frac{d}{d\varphi} f(t, R_\varphi u, R_\varphi A)\Big|_{\varphi=0} = \left[\sum_{i=1}^d \frac{\partial f}{\partial u_i}(t, R_\varphi u, R_\varphi A) \cdot \frac{\partial}{\partial\varphi}(R_\varphi u)_i + \sum_{i=1}^d \frac{\partial f}{\partial A_i}(t, R_\varphi u, R_\varphi A) \cdot \frac{\partial}{\partial\varphi}(R_\varphi A)_i\right]\Bigg|_{\varphi=0}$$
$$= \frac{\partial f}{\partial u_1}(t,u,A)(-u_2) + \frac{\partial f}{\partial u_2}(t,u,A)u_1 + \frac{\partial f}{\partial A_1}(t,u,A)(-A_2) + \frac{\partial f}{\partial A_2}(t,u,A)A_1$$

Since $f(t, R_\varphi u, R_\varphi A) = f(t,u,A)$, we also have $\frac{d}{d\varphi} f(t, R_\varphi u, R_\varphi A) = 0$,

so that

$$0 = \frac{\partial f}{\partial u_2}(t,u,A)u_1 - \frac{\partial f}{\partial u_1}(t,u,A)u_2 + \frac{\partial f}{\partial A_2}(t,u,A)A_1 - \frac{\partial f}{\partial A_1}(t,u,A)A_2. \qquad (*)$$

Now let u satisfy the Euler-Lagrange equation for I:

$$-\frac{d}{dt}\left[\partial_A f(t,u(t),\dot u(t))\right] + \partial_u f(t,u(t),\dot u(t)) = 0, \qquad \text{(EL)}$$

and define the quantity

$$E(t,u,A) := u_1 \partial_{A_2} f(t,u,A) - u_2 \partial_{A_1} f(t,u,A).$$

Then we have

$$\begin{aligned}
\frac{d}{dt} E(t,u(t),\dot u(t)) &= \dot u_1(t)\, \partial_{A_2} f(t,u(t),\dot u(t)) + u_1(t) \frac{d}{dt}\left[\partial_{A_2} f(t,u(t),\dot u(t))\right] \\
&\qquad - \dot u_2(t)\, \partial_{A_1} f(t,u(t),\dot u(t)) - u_2(t) \frac{d}{dt}\left[\partial_{A_2} f(t,u(t),\dot u(t))\right] \\
&\overset{\text{(EL)}}{=} \dot u_1(t)\, \partial_{A_2} f(t,u(t),\dot u(t)) + u_1(t)\, \partial_{u_2} f(t,u(t),\dot u(t)) \\
&\qquad - \dot u_2 \partial_{A_1} f(t,u(t),\dot u(t)) - u_2(t)\, \partial_{u_1} f(t,u(t),\dot u(t)) \\
&\overset{(*)}{=} 0.
\end{aligned}$$

b) For $R_\varphi = e^{\varphi B}$ we have

$$\frac{d}{d\varphi}\left[R_\varphi y\right]\Big|_{\varphi=0} = \left[B e^{\varphi B} y\right]\Big|_{\varphi=0} = By.$$

Similarly to above, we thus obtain

$$\begin{aligned}
0 = \frac{d}{d\varphi}\left[f(t, R_\varphi u, R_\varphi A)\right]\Big|_{\varphi=0} &= \left[\sum_{i=1}^d \frac{\partial f}{\partial u_i}(t, R_\varphi u, R_\varphi A)\frac{d}{d\varphi}\left[R_\varphi u\right] + \sum_{i=1}^d \frac{\partial f}{\partial A_i}(t, R_\varphi u, R_\varphi A)\frac{d}{d\varphi}\left[R_\varphi A\right]\right]\Big|_{\varphi=0} \\
&= \sum_{i=1}^d \frac{\partial f}{\partial u_i}(t,u,A)(Bu)_i + \sum_{i=1}^d \frac{\partial f}{\partial A_i}(t,u,A)(BA)_i \\
&= (Bu)\cdot \partial_u f(t,u,A) + (BA)\cdot \partial_A f(t,u,A)
\end{aligned}$$

With (EL), this implies

$$\begin{aligned}
0 &= (Bu(t))\cdot \partial_u f(t,u(t),\dot u(t)) + (B\dot u(t))\cdot \partial_A f(t,u(t),\dot u(t)) \\
&= (Bu(t))\cdot \frac{d}{dt}\left[\partial_A f(t,u(t),\dot u(t))\right] + (B\dot u(t))\cdot \partial_A f(t,u(t),\dot u(t)) \\
&= \frac{d}{dt}\Big[\underbrace{(Bu(t))\cdot \partial_A f(t,u(t),\dot u(t))}_{=:\, J(t,u(t),\dot u(t))}\Big]
\end{aligned}$$

c) We have

$$0 = -m\ddot{u}(t) + F(u(t)) = -\frac{d}{dt}[m\dot{u}(t)] - \nabla V(u(t))$$

$$= -\frac{d}{dt}[\partial_A f(u(t), \dot{u}(t))] + \partial_u f(u(t), \dot{u}(t))$$

for $f(u, A) = \frac{1}{2}m|A|^2 - V(u) = \frac{1}{2}m|A|^2 - \tilde{V}(|u|)$

Hence, Newton's law corresponds to the Euler-Lagrange equations for

$$I(u) = \int_0^T f(u(t), \dot{u}(t))\,dt.$$

For any rotational matrix $R_\varphi = e^{\varphi B}$, $B = -B^T \in \mathbb{R}^{3\times 3}$, we have $|e^{\varphi B} y| = |y|$ for all $y \in \mathbb{R}^3$, so that

$$f(R_\varphi u, R_\varphi A) = \frac{1}{2}m|R_\varphi A|^2 - \tilde{V}(|R_\varphi u|) = \frac{1}{2}m|A|^2 - \tilde{V}(|u|) = f(u, A).$$

Hence, b) implies that

$$J_B(u(t), \dot{u}(t)) = (Bu(t)) \cdot \partial_A f(u(t), \dot{u}(t)) = Bu(t) \cdot m\dot{u}(t)$$

is a conserved quantity for any $B \in \mathbb{R}^{d\times d}$.

The choices

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

yield $L(u)_j = J_{B_j}(u, \dot{u})$, so that $L(u) = mu \times \dot{u}$ is a conserved quantity.