## Exercise 26

First, let $u \in C^{0,1}(\overline{\Omega})$ with Lipschitz constant $L > 0$. Set $\Omega_\varepsilon := \{ x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon \}$ and define the difference quotients

$$D_j^h u(x) := \frac{1}{h}\big( u(x + h e_j) - u(x) \big) \chi_{\Omega_{|h|}}(x)$$

for $j \in \{1, \dots, d\}$ and $h \in \mathbb{R} \setminus \{0\}$. Then $\|D_j^h u\|_{L^\infty(\Omega)} \leq L$, so that $(D_j^h u)_{h>0}$ is bounded. We identify $L^\infty(\Omega) = (L^1(\Omega))'$. Then the Banach-Alaoglu Theorem implies that there is a weakly* convergent subsequence with $D_j^{h_k} u \xrightarrow{*} w_j$ as $k \to \infty$ for some $w_j \in L^\infty(\Omega)$.

For $\varphi \in C_c^\infty(\Omega)$ this implies

$$\int_\Omega u \, \partial_j \varphi \, dx \overset{\text{Lebesgue}}{=} \lim_{k\to\infty} \int_\Omega u(x) \frac{\varphi(x - h_k e_j) - \varphi(x)}{-h} dx$$

$$= \lim_{k\to\infty} \int_\Omega - \frac{u(x + h_k e_j) - u(x)}{h} \varphi(x) \, dx = - \int_\Omega w_j \varphi \, dx,$$

so that $u$ is weakly differentiable with $\partial_j u = w_j \in L^\infty(\Omega)$. Hence $u \in W^{1,\infty}(\Omega)$.

Now let $u \in W^{1,\infty}(\Omega)$. Since $\Omega$ is bounded, we then have $u \in W^{1,p}(\Omega)$ for $p \in (d, \infty)$, and the Sobolev embedding theorem implies $u = v$ a.e. for some $v \in C^0(\overline{\Omega})$.

Identify $u$ with $v$ in what follows.

Since $\Omega$ has Lipschitz boundary, the connected components of $\Omega$ cannot have intersecting boundary, and it suffices to consider a connected domain $\Omega$.

Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a mollifier and set $u_\varepsilon := \varphi_\varepsilon * u$, where $u$ is extended to $\mathbb{R}^d$ by $0$. Let $x, y \in \Omega$. Then there exists a path $\gamma \in C^1([0,1];\Omega)$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Therefore,

$$u_\varepsilon(x) - u_\varepsilon(y) = \int_0^1 \frac{d}{dt}\left[ u_\varepsilon(\gamma(t))\right] dt = \int_0^1 \nabla u_\varepsilon(\gamma(t)) \cdot \gamma'(t)\, dt.$$

We show below that we can estimate the length of $\gamma$ by $|x-y|$, i.e.,

$$\boxed{L(\gamma) = \int_0^1 |\gamma'(t)|\, dt \leq C_\Omega |x-y|}$$

for a constant $C_\Omega > 0$ independent of $x$ and $y$.

Then

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C_\Omega \|\nabla u_\varepsilon\|_{L^\infty(\Omega)} |x-y|. \qquad (*)$$

Since $u \in C(\bar\Omega)$, we have $u_\varepsilon \to u$ uniformly in $\bar\Omega$. Moreover, for $x \in \Omega$ we obtain

$$\partial_j u_\varepsilon(x) = \int_{\mathbb{R}^d} \partial_j \varphi_\varepsilon(x-y)\, u(y)\, dy = \int_{\mathbb{R}^d} \varphi_\varepsilon(x-y)\, \partial_j u(y)\, dy$$

by the definition of the weak derivative. Therefore, if $\varepsilon > 0$ is so small that $B_\varepsilon(x) \subset \Omega$, we obtain

$$|\partial_j u_\varepsilon(x)| \leq \int_{\mathbb{R}^d} |\varphi_\varepsilon(x-y)|\, |\partial_j u(y)|\, dy \leq \|\nabla u\|_{L^\infty(\Omega)} \underbrace{\int_{\mathbb{R}^d} \varphi_\varepsilon(x-y)\, dy}_{=1}.$$

Using this estimate and passing to the limit $\varepsilon \to 0$ in $(*)$ leads to

$$|u(x) - u(y)| \leq C_\Omega \|\nabla u\|_{L^\infty(\Omega)} |x-y|,$$

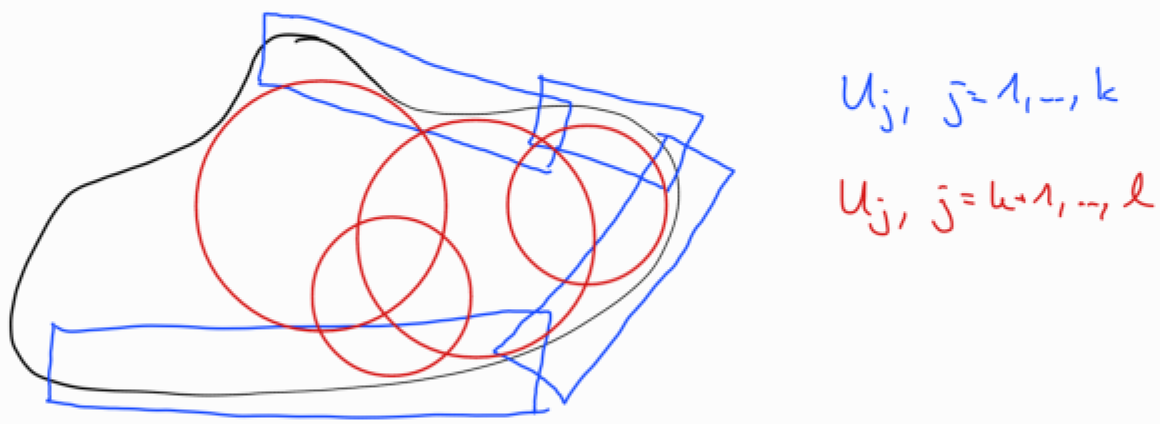
so that $u \in C^{0,1}(\bar\Omega)$.

## • Uniform estimate of the length of the curve $\gamma$.

For the construction of suitable curves, we use the geometry of $\Omega$, i.e., that $\Omega$ has a Lipschitz boundary.

Since $\partial\Omega$ is compact, there exists a finite number of open (convex) sets $U_1, \dots, U_k$ such that $U_j \cap \partial\Omega$ is the graph of a Lipschitz function.

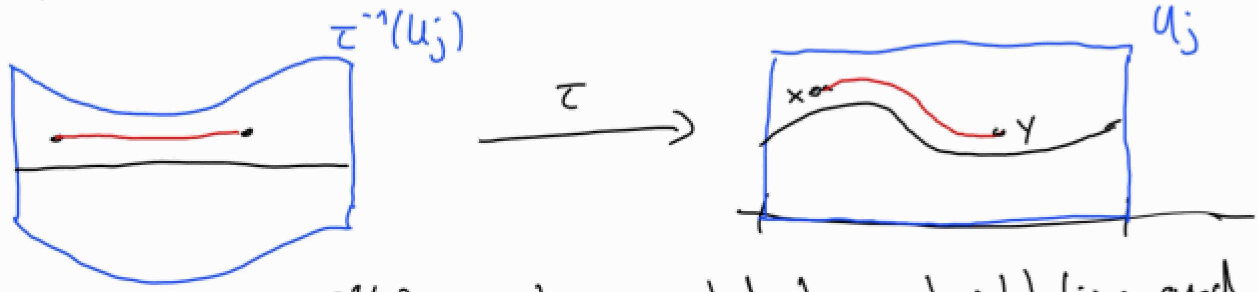
Let $z_j \in U_j \cap \Omega$ for $j = 1, \dots, k$ and let $z_{k+1}, \dots, z_\ell$ be such that $U_j := B_\varrho(z_j) \subset \Omega$ for $j = k+1, \dots, \ell$ and $\Omega \subset \bigcup_{j=1}^{\ell} U_j$, where $\varrho > 0$ is a fixed radius.




• For $x, y \in U_j$ with $j \geq k+1$, we let $\gamma(t) = (1-t)x + ty$. Then

$$L(\gamma) = \int_0^1 |\gamma'(t)|\, dt = |x - y|.$$

• For $x, y \in U_j$ with $j \leq k$, we use the Lipschitz map $\tau_j$ for $U_j$:




Then $\tau^{-1}(x)$ and $\tau^{-1}(y)$ can be connected by a straight line and we define

$$\gamma(t) = \tau\left( (1-t)\tau^{-1}(x) + t\tau^{-1}(y) \right).$$

Since $\tau$ is Lipschitz continuous (with constant $C_\tau$), we obtain

$$L(\gamma) \leq C_\tau \, |\tau^{-1}(x) - \tau^{-1}(y)| \leq C_\tau C_{\tau^{-1}} |x - y|.$$

- For $x \in U_i$, $y \in U_j$ with $i \neq j$, we have $|x-y| > c$ for some constant only depending on the covering.

Since $\Omega$ is open and connected, there exists a path $\gamma_{ij}$ from $z_i$ to $z_j$.

As in the previous cases, we further find paths connecting $x$ with $z_i$ and $y$ with $z_j$ and with lengths bounded by $C|x-z_i|$ and $C|y-z_j|$, respectively. Then the length of the connecting path $\gamma$ can be estimated as

$$\begin{aligned} L(\gamma) &\leq C|x-z_i| + L(\gamma_{ij}) + C|y-z_j| \\ &\leq 2C \operatorname{diam}(\Omega) + L(\gamma_{ij}) \leq C_{ij} \end{aligned}$$

for an absolute constant $C_{ij}$ only depending on $i$ and $j$. Hence,

$$L(\gamma) \leq C_{ij} \, c^{-1} |x-y|,$$

which also shows the estimate in this case.

## Exercise 27

a) If $g$ is convex, the difference quotients of $g$ are increasing:

Let $x<z$ and $y\in\mathbb{R}$. We show $$\frac{g(x)-g(y)}{x-y} \le \frac{g(z)-g(y)}{z-y}. \qquad (*)$$

i) If $x<y<z$, then $y=(1-\theta)x+\theta z$ for some $\theta\in(0,1)$, so that
$$\frac{g(x)-g(y)}{x-y} = \frac{g(y)-g(x)}{y-x} \le \frac{(1-\theta)g(x)+\theta g(z)-g(x)}{(1-\theta)x+\theta z-x} = \frac{\theta(g(z)-g(x))}{\theta(z-x)} = \frac{g(z)-g(x)}{z-x}$$
and
$$\frac{g(z)-g(y)}{z-y} \ge \frac{g(z)-((1-\theta)g(x)+\theta g(z))}{z-[(1-\theta)x+\theta z]} = \frac{(1-\theta)(g(z)-g(x))}{(1-\theta)(z-x)} = \frac{g(z)-g(x)}{z-x}.$$

ii) If $x<z<y$, then $z=(1-\theta)x+\theta y$ for some $\theta\in(0,1)$, so that
$$\frac{g(z)-g(y)}{z-y} = \frac{g(y)-g(z)}{y-z} \ge \frac{(1-\theta)g(x)+\theta g(y)-g(y)}{(1-\theta)x+\theta y-y} = \frac{(1-\theta)(g(x)-g(y))}{(1-\theta)(x-y)} = \frac{g(x)-g(y)}{x-y}.$$

iii) If $y<x<z$, then $x=(1-\theta)y+\theta z$ for some $\theta\in(0,1)$, so that
$$\frac{g(x)-g(y)}{x-y} \le \frac{(1-\theta)g(y)+\theta g(z)-g(y)}{(1-\theta)y+\theta z-y} = \frac{\theta(g(z)-g(y))}{\theta(z-y)} = \frac{g(z)-g(y)}{z-y}.$$

This shows $(*)$ in all three cases.

To show the claimed estimate, assume $x<y$.

We now use $(*)$ with $z=y+|y|+1>y>x$ and the estimate of $g$ to obtain
$$\frac{g(x)-g(y)}{x-y} \le \frac{g(z)-g(y)}{z-y} \le \frac{M(1+|z|^p)+M(1+|y|^p)}{|y|+1} \le C_pM\frac{(1+|y|^p)}{1+|y|}$$
$$\le C_pM(1+|y|^{p-1}) \le C_pM(1+|x|^{p-1}+|y|^{p-1}).$$

Similarly, taking $w=x-|x|-1<x<y$, we conclude
$$\frac{g(x)-g(y)}{x-y} \ge \frac{g(x)-g(w)}{x-w} \ge -\frac{M(1+|x|^p)+M(1+|w|^p)}{|x|+1} \ge -\frac{C_pM(1+|x|^p)}{1+|x|}$$
$$\ge -C_pM(1+|x|^{p-1}) \ge -C_pM(1+|x|^{p-1}+|y|^{p-1}).$$

In total, this shows
$$|g(x)-g(y)| \le C_pM|x-y|(1+|x|^{p-1}+|y|^{p-1}).$$

b) If $f$ is rank-one convex, we repeat the previous argument for every component.

Let $A = (A_{ij})$, $B = (B_{ij}) \in \mathbb{R}^{m\times d}$, and let $E_{ij} = (\delta_{ij}) \in \mathbb{R}^{m\times d}$. Then

$$
\begin{aligned}
|f(A) - f(B)| \leq\ & |f(A) - f(A + (B_{11} - A_{11}) E_{11})| \\
& + |f(A + (B_{11} - A_{11}) E_{11}) - f(A + (B_{11} - A_{11}) E_{11} + (B_{12} - A_{12}) E_{12})| \\
& + \dots \\
& + |f(B - (B_{md} - A_{md}) E_{md}) - f(B)|.
\end{aligned}
$$

Every summand is of the form $|f(D + xE_{ij}) - f(D + yE_{ij})|$ for some matrix $D \in \mathbb{R}^{m\times d}$ and some $x, y \in \mathbb{R}$. We again use the monotonicity of difference quotients:

Let $w := x - |D + xE_{ij}| - 1 < x < y < z := y + |D + yE_{ij}| + 1$.

As above, $(*)$ implies

$$
\begin{aligned}
\frac{f(D + xE_{ij}) - f(D + yE_{ij})}{x - y} &\leq \frac{f(D + zE_{ij}) - f(D + yE_{ij})}{z - y} \\
&\leq \frac{M(1 + |D + (1+y)E_{ij} + |D + yE_{ij}|E_{ij}|^p) + M(1 + |D + yE_{ij}|^p)}{1 + |D + yE_{ij}|} \\
&\leq \frac{C(1 + |D + yE_{ij}|^p)}{1 + |D + yE_{ij}|^p} \leq C(1 + |D + yE_{ij}|^{p-1})
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{f(D + xE_{ij}) - f(D + yE_{ij})}{x - y} &\geq \frac{f(D + xE_{ij}) - f(D + wE_{ij})}{x - w} \\
&\geq \frac{-M(1 + |D + xE_{ij}|^p) - M(1 + |D + (x-1)E_{ij} - |D + E_{ij}|E_{ij}|^p)}{1 + |D + xE_{ij}|} \\
&\geq -\frac{C(1 + |D + xE_{ij}|^p)}{1 + |D + xE_{ij}|} \geq -C(1 + |D + xE_{ij}|^{p-1}).
\end{aligned}
$$

Therefore,

$$|f(D+xE_{ij})-f(D+yE_{ij})| \leq C|x-y|\left(1+|D+xE_{ij}|^{p-1}+|D+yE_{ij}|^{p-1}\right)$$

With this estimate, we obtain

$$\begin{aligned}
|f(A)-f(B)| &\leq C|A_{11}-B_{11}|\left(1+|A|^{p-1}+|A+(B_{11}-A_{11})E_{11}|^{p-1}\right) \\
&\quad + C|A_{12}-B_{12}|\left(1+|A+(B_{11}-A_{11})E_{11}|^{p-1}+|A+(B_{11}-A_{11})E_{11}+(B_{12}-A_{12})E_{12}|^{p-1}\right) \\
&\quad + \dots \\
&\quad + C|A_{md}-B_{md}|\left(1+|B-(B_{md}-A_{md})E_{md}|^{p-1}+|B|^{p-1}\right) \\
&\leq C|A-B|\left(1+|A|^{p-1}+|B|^{p-1}\right).
\end{aligned}$$