Exercise 41

Let $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ be given by $f(A) = (MA) : A$ for some $M \in \mathrm{Lin}(\mathbb{R}^{m\times d}; \mathbb{R}^{m\times d})$.

a) To show the equivalence

$$f \text{ convex} \iff \forall A \in \mathbb{R}^{m\times d} : f(A) \geq 0,$$

let $A, B \in \mathbb{R}^{m\times d}$ and $\lambda \in (0,1)$. Then

$$f((1-\lambda)A + \lambda B) = \left(M[(1-\lambda)A + \lambda B]\right) : [(1-\lambda)A + \lambda B]$$

$$= (1-\lambda)\,(MA) : [A + \lambda(B-A)] + \lambda\,(MB) : [B + (1-\lambda)(A-B)]$$

$$= (1-\lambda)\underbrace{(MA) : A}_{=f(A)} + (1-\lambda)\lambda\left[(MA) : (B-A) + (MB) : (A-B)\right] + \lambda \underbrace{(MB) : B}_{=f(B)}$$

We thus have

$$f((1-\lambda)A + \lambda B) \leq (1-\lambda) f(A) + \lambda f(B)$$

$$\iff \quad (MA) : (B-A) + (MB) : (A-B) \leq 0$$

$$\iff \quad 0 \leq (M(A-B)) : (A-B) = f(A-B)$$

This shows the assertion.

b) We want to show

$$f \text{ is polyconvex} \iff \exists \beta \in \mathbb{R}^{\tau_2(m,d)}\ \forall A \in \mathbb{R}^{m\times d} : f(A) \geq \langle \beta, T_2(A) \rangle.$$

„$\Leftarrow$" Define $\tilde f(A) := f(A) - \langle \beta, T_2(A) \rangle$. Since $\tilde f$ is quadratic and $\tilde f(A) \geq 0$, part a) implies that $\tilde f$ is convex. Then $f(A) = g(T(A))$ with the convex function g defined by

$$g(T(A)) = \tilde f(A) + \langle \beta, T_2(A) \rangle.$$

Hence, f is polyconvex.

„$\Rightarrow$" We have $f(A) = g(T(A))$ for a convex function $g: \mathbb{R}^{\tau(m,d)} \to \mathbb{R}_\infty$.

Since g is convex, there exists $\beta \in \mathbb{R}^{\tau(m,d)}$ such that

$$g(X) \geq g(0) + \langle \beta, X \rangle = \langle \beta, X \rangle$$

since $g(0) = f(0) = 0$. Here $\beta = (\beta_1, \beta_2, \dots, \beta_{\min\{m,d\}})$ with $\beta_j \in \mathbb{R}^{\tau_j(m,d)}$. For $\varepsilon > 0$, this implies

$$\varepsilon^2 f(A) = f(\varepsilon A) = g(T(\varepsilon A))$$

$$\geq \langle \beta, T(\varepsilon A) \rangle = \sum_{j=1}^{\min\{m,d\}} \langle \beta_j, T_j(\varepsilon A) \rangle = \sum_{j=1}^{\min\{m,d\}} \varepsilon^j \langle \beta_j, T_j(A) \rangle.$$

First, divide both sides by ε and pass to the limit $\varepsilon \to 0$ to obtain

$$0 \geq \langle \beta_1, T_1(A) \rangle = \langle \beta_1, A \rangle$$

Since $A \in \mathbb{R}^{m \times d}$ is arbitrary, this implies $\beta_1 = 0$.

Dividing the above inequality by ε^2 and passing to the limit $\varepsilon \to 0$, we thus obtain

$$f(A) \geq \langle \beta_2, T_2(A) \rangle.$$

c) To show that

$$f \text{ is rank-one-convex} \Leftrightarrow \forall \xi \in \mathbb{R}^m, \eta \in \mathbb{R}^d : f(\xi \otimes \eta) \geq 0,$$

we proceed in the same way as in a), with the restriction $\mathrm{rk}(A-B) = 1$.

This shows that f is rank-one-convex if and only if

$$0 \leq f(A-B)$$

for all $A, B \in \mathbb{R}^{m \times d}$ with $\mathrm{rk}(A-B) = 1$, i.e., with $A - B = \xi \otimes \eta$ for some $\xi \in \mathbb{R}^m$, $\eta \in \mathbb{R}^d$. This proves the claim.

Exercise 42

Consider $A_1, A_2, A_3 \in \mathbb{R}^{2\times 2}$ given by

$$A_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix},$$

and $f: \mathbb{R}^{2\times 2} \to \{0, \infty\}$ with

$$f(A) = \begin{cases} 0 & \text{if } A \in \{A_1, A_2, A_3\}, \\ \infty & \text{otherwise.} \end{cases}$$

To show that f is rank-one-convex, let $A, B \in \mathbb{R}^{2\times 2}$ with $\mathrm{rk}(A-B) = 1$ and $\lambda \in (0,1)$. We want to show

$$f((1-\lambda)A + \lambda B) \leq \lambda f(A) + (1-\lambda) f(B) \qquad (*)$$

We distinguish two cases:

i) $A \notin \{A_1, A_2, A_3\}$ or $B \notin \{A_1, A_2, A_3\}$. Then $f(A) = \infty$ or $f(B) = \infty$, and $(*)$ holds.

ii) $A = A_i$ and $B = A_j$ with $i, j \in \{1, 2, 3\}$.

We have
$$A_1 - A_2 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix},$$
$$A_1 - A_3 = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix},$$
$$A_2 - A_3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$
all of which have full rank. Therefore, $\mathrm{rk}(A-B) = 1$ if and only if $i = j$, that is $A = B$. Then $(*)$ is trivial.

In total, this shows that f is rank-one convex.

To show that f is not polyconvex, we assume the opposite. Then there exists a convex function $g: \mathbb{R}^5 \to \mathbb{R} \cup \{+\infty\}$ such that

$$f(A) = g(A, \det A).$$

To obtain a contradiction, consider a convex combination of A_1, A_2, A_3, namely

$$A_0 := \frac{1}{3}(A_1 + A_2 + A_3) = \frac{1}{3}\begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}.$$

Then $\det A_0 = 0$. Since also $\det A_1 = \det A_2 = \det A_3 = 0$, we have

$$\det A_0 = 0 = \frac{1}{3}\sum_{i=1}^{3} \det A_i .$$

This identity and the convexity of g imply

$$\begin{aligned} \infty = f(A_0) = g(A_0, \det A_0) &= g\left(\frac{1}{3}\sum_{i=1}^{3} A_i, \frac{1}{3}\sum_{i=1}^{3}\det A_i\right) \\ &\leq \frac{1}{3}\sum_{i=1}^{3} g(A_i, \det A_i) \\ &= \frac{1}{3}\sum_{i=1}^{3} f(A_i) = 0, \end{aligned}$$

which is a contradiction. Therefore, f is not polyconvex.