Ex 30. Uniqueness of minimizers (1)

$f: \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \longrightarrow [0,\infty[$

usual assumptions and additionally

strictly convex in A but not in u, i.e.

$\forall u_0, u_1 \in \mathbb{R}^m, A_0, A_1 \in \mathbb{R}^{m\times d}, \lambda \in [0,1]$, a.e. $x \in \Omega$

$$f(x, u_\lambda, A_\lambda) \le (1-\lambda) f(x, u_0, A_0) + \lambda f(x, u_1, A_1)$$

where $u_\lambda := (1-\lambda)u_0 + \lambda u_1$, $A_\lambda := (1-\lambda)A_0 + \lambda A_1$

inequality is strict if $A_0 \neq A_1$, $\lambda \in]0,1[$

a) We show that minimizer $u_* \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^m)$ is unique.

Let $u_1, u_2 \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^m)$ be two minimizers with $u_1 \neq u_2$.

Step 1. First note that

$$\mathcal{L}^d(\{x \in \Omega \mid \nabla u_1(x) \neq \nabla u_2(x)\}) > 0$$

otherwise $u_1 - u_2 \equiv \text{constant}$ a.e. in Ω however on Γ_D, the Dirichlet boundary condition implies $u_1 = u_2$. $\lightning$ $\square$

Step 2. Observing that $f(x, u_i, \nabla u_i) < \infty$ ②

a.e. in Ω has to hold for $i = 1, 2$, we conclude that by the strict convexity of f w.r.t. A $u_\theta = (1-\theta)u_1 + \theta u_2$ is such that

$$f(x, u_\theta, \nabla u_\theta) < (1-\theta) f(x, u_1, \nabla u_1) + \theta f(x, u_2, \nabla u_2)$$

on ~~a~~ the non-negligible set $\{\nabla u_1 \neq \nabla u_2\}$.

Thus, integrating over Ω gives a contradiction

b) • $f_1(x, u, A) = \frac{1}{p}|A|^p$ (no u-dependence)

• $f_2(x, u, A) = \underbrace{f_1(x, A)}_{\substack{\text{strictly} \\ \text{convex}}} + \underbrace{f_2(x, u)}_{\substack{\text{non-strictly} \\ \text{convex}}}$

• $f_3(u, A) = \frac{1}{2}(1 + |u|)|A|^2$

Ex 31 $f : \mathbb{R}^n \to \mathbb{R}$ convex

a) f is locally bounded

Let $Q_R = [-R, R]^n$ denote cube for $R > 0$

denote by $v^{(i)}$, $i = 1, \dots, 2^n$ the vertices of Q_R

i.e. $v_j^{(i)} \in \{\pm R\}$ for $j = 1, \dots n$

For every $v \in Q_R$ there exists $\lambda^{(i)} \geq 0$ such that $\sum_{i=1}^{2^n} \lambda^{(i)} = 1$ and $v = \sum_{i=2}^{2^n} \lambda^{(i)} v^{(i)}$

Jensen's inequality (comp. Ex. 14, Sh. 4) gives

$$f(v) \leq \sum_{i=1}^{2^n} \lambda^{(i)} f(v^{(i)}) \leq \overbrace{\max_{i=1,\dots,2^n} \{f(v^{(i)})\}}^{=: M_R}$$

$\Rightarrow$ f is bounded from above by M_R

On the other hand, for $v \in Q_R \Rightarrow -v \in Q_R$

$0 = \frac{1}{2} v + \frac{1}{2}(-v) \in Q_R$. Thus, via convexity

$$f(0) \leq \frac{1}{2} f(v) + \frac{1}{2} f(-v)$$

$$\Rightarrow f(v) \geq 2 f(0) - f(-v) \geq 2 f(0) - M_R =: m_R$$

$\Rightarrow f$ is bounded from below by m_R on Q_R (+)

b) We show that f is locally Lipschitz continuous, $\forall v_1, v_2 \in B_r(0)$, $0 < r < R$

$$|f(v_1) - f(v_2)| \leq \underbrace{\frac{\max_{|v| \leq R} f(v) - \min_{|v| \leq R} f(v)}{R - r}}_{\Rightarrow \, Lip_R \geq 0} |v_1 - v_2|$$

Fix $v_1, v_2 \in B_r(0)$ and set $\alpha = |v_1 - v_2| > 0$

and $\tilde{v} = v_2 + \frac{R-r}{\alpha}(v_2 - v_1)$

$\Rightarrow |\tilde{v}| \leq r + \frac{R-r}{\alpha} \alpha = R \Rightarrow \tilde{v} \in Q_R$

we have

$$\left(1 + \frac{R-r}{\alpha}\right) v_2 = \tilde{v} + \frac{R-r}{\alpha} v_1$$

$$\Rightarrow v_2 = \frac{1}{1 + \frac{R-r}{\alpha}} \tilde{v} + \frac{R-r}{\alpha} \frac{1}{1 + \frac{R-r}{\alpha}} v_1$$

$$= \frac{\alpha}{\alpha + R - r} \tilde{v} + \frac{R-r}{\cancel{\alpha}} \frac{\cancel{\alpha}}{R - r + \alpha} v_1$$

$$= \lambda \tilde{v} + (1 - \lambda) v_1, \quad \lambda \in [0,1]$$

$$\overset{\text{convex}}{\Rightarrow} f(v_2) - f(v_1) \le \lambda \left(f(\tilde{v}) - f(v_1) \right)$$

$$\le \underbrace{\frac{\alpha}{\alpha + R - r}} \left(\max_{|v| \le R} f(v) - \min_{|v| \le r \le R} f(v) \right)$$

$$\le \frac{\alpha}{R-r} \quad (\text{recall that } \alpha = |v_1 - v_2|)$$

The same holds with the roles of v_1 and v_2 swapped. Thus, the claim follows.

c) We assume now $f \in C^1(\mathbb{R}^n)$ and $\exists C > 0,\ p \in]1, \infty[\ \forall v \in \mathbb{R}^n : |f(v)| \le C(1 + |v|^p)$

We take the inequality in b) and estimate for $v_1 \neq v_2$, $v_1, v_2 \in B_r(0)$

$$\frac{|f(v_1) - f(v_2)|}{|v_1 - v_2|} \le \frac{2C(1 + R^p)}{R - r}$$

where we used that

$$\left| \max_{v \in B_R(0)} f(v) - \min_{v \in B_R(0)} f(v) \right| \le 2 \max_{v \in B_R(0)} |f(v)|$$

$$\le 2C(1 + R^p)$$

We can choose $R = 2(1 + |v_1| + |v_2|)$ and $r = |v_1| + |v_2|$. Clearly, $0 < r < R$ and $v_1, v_2 \in B_r(0)$.

Then, we get

$$\frac{|f(v_1) - f(v_2)|}{|v_1 - v_2|} \leq \frac{\tilde{C}(1 + |v_1|^p + |v_2|^p)}{2 + |v_1| + |v_2|}$$

Letting $v_2 \to v_1$ in $\mathbb{R}^n$, we get

$$|\partial_v f(v_1)| \leq \frac{\hat{C}(1 + |v_1|^p)}{1 + |v_1|}$$

$$\leq C(1 + |v_1|^{p-1})$$

$\Rightarrow$ growth condition for $\partial_v f$ is automatically satisfied in convex case, if f satisfies a growth condition.

This is important for the existence of the weak Euler-Lagrange eqns.

Ex 32 Consider $f : \mathbb{R}^d \longrightarrow \mathbb{R}$

Assume that $I(u) = \int_\Omega f(Du)\,dx$

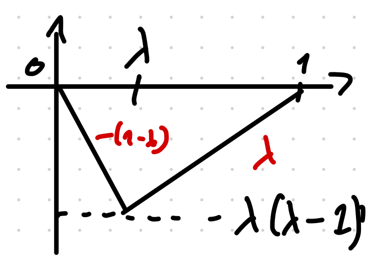
is weak sequentially lower semicontinuous on $W^{1,p}(\Omega)$ (scalarvalued functions)

We show that f is convex.

Let us consider the sequence

$$u_k(x) := \big(\lambda A + (1-\lambda)B\big) \cdot x$$
$$+ \frac{1}{k} \tilde{\chi}_\lambda\big(k x \cdot (B-A)\big)$$

where
- $\lambda \in]0,1[$
- $A, B \in \mathbb{R}^d$
- $\tilde{\chi}_\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ is 1-periodic extension of



$$\chi_\lambda(t) = \begin{cases} -(1-\lambda)t & \text{if } t \in [0,\lambda[\\ \lambda(t-1) & \text{if } t \in [\lambda, 1] \end{cases}$$

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$u_k \in W^{1,\infty}(\Omega)$

- $u_k \longrightarrow u$ in $L^\infty(\Omega)$ where

$$u(x) = (\lambda A + (1-\lambda)B)\cdot x$$

linear

- $\partial_{x_i} u_k(x) = \lambda A_i + (1-\lambda)B_i + \chi_\lambda'(kx\cdot(B-A))(B_i - A_i)$

we have $\chi_\lambda'(t) = \begin{cases} -(1-\lambda) & t\in[0,\lambda[\\ \lambda & t\in]\lambda,1] \end{cases}$

Ex. 13, Sh. 3

$$\Rightarrow \partial_{x_i} u_k \rightharpoonup \lambda A_i + (1-\lambda)B_i + (B_i - A_i)\cdot\underbrace{\{\lambda(1-\lambda) - \lambda(1-\lambda)\}}_{=0}$$

$$= \lambda A_i + (1-\lambda)B_i = \partial_{x_i} u$$

conclude $u_k \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega)$

The assumption (I w.l.s.c.) gives

$$\liminf_{k\to\infty} I(u_k) \geq I(u) = \int_\Omega f(\nabla u)\,dx$$
$$= |\Omega| f(\lambda A + (1-\lambda)B)$$

We compute

$$\int_\Omega f(\nabla u_k)\,dx = \int_{\Omega\cap U_k} f(\lambda A + (1-\lambda)B - (1-\lambda)(B-A))$$

$$+ \int_{\Omega\cap V_k} f(\lambda A + (1-\lambda)B + \lambda(B-A))$$

where $U_k = \{x \mid kx\cdot(B-A) \in \bigcup_{m\in\mathbb{Z}} (m+[0,\lambda[)\}$

$V_k = \{x \mid kx\cdot(B-A) \in \bigcup_{m\in\mathbb{Z}} (m+[\lambda,1])\}$

$$\int_\Omega f(\nabla u_k)\,dx = \int_{\Omega\cap U_k} f(A)\,dx + \int_{\Omega\cap V_k} f(B)\,dx$$

$$= \int_\Omega \chi_{U_k} f(A) + \chi_{V_k} f(B)\,dx$$

$$\chi_{U_k} \rightharpoonup \lambda$$

$$\chi_{V_k} \rightharpoonup (1-\lambda)$$

Ex. 13

Thus, $\liminf\limits_{k\to\infty} I(u_k) = |\Omega| \{ \lambda f(A) + (1-\lambda) f(B) \}$

We conclude

$$|\Omega| \{ \lambda f(A) + (1-\lambda) f(B) \} = \liminf_{k\to\infty} I(u_k)$$

$$\geq I(u) = |\Omega| f(\lambda A + (1-\lambda) B).$$

$\square$