Exercise 39

Let $m=d=p=2$ and $\Omega =]-1,1[^2$. Define

$$u^k(x_1,x_2) = \frac{1}{\sqrt{k}}\,(1-|x_2|)^k\,(\sin(kx_1), \cos(kx_1)).$$

a) To show $u^k \rightharpoonup 0$ in $H^1(\Omega;\mathbb{R}^2)$, it suffices to show

(i) $u^k \rightharpoonup 0$ in $L^2(\Omega;\mathbb{R}^2)$, and

(ii) $\|\nabla u^k\|_{L^2(\Omega)} \leq C$ for some $C>0$.

Indeed, this implies that (u^k) is bounded in $H^1(\Omega;\mathbb{R}^2)$. By reflexivity, every subsequence has a weakly convergent subsequence with limit $v \in H^1(\Omega;\mathbb{R}^2)$. Due to the continuous embedding $H^1(\Omega;\mathbb{R}^2) \hookrightarrow L^2(\Omega;\mathbb{R}^2)$, statement (i) implies $v=0$. Therefore, $u^k \rightharpoonup 0$ in $H^1(\Omega;\mathbb{R}^2)$.

To show (i), we observe

$$\|u^k\|_{L^2(\Omega)} = \left(\int_{-1}^{1}\int_{-1}^{1} \frac{1}{k}\underbrace{(1-|x_2|)^{2k}}_{\leq 1}\,dx_2\,dx_1\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{k}}\left(\int_{-1}^{1}\int_{-1}^{1} 1\,dx_2\,dx_1\right)^{\frac{1}{2}} \leq \frac{2}{\sqrt{k}} \to 0$$

as $k\to\infty$. Therefore, we even have $u^k \to 0$ in $L^2(\Omega;\mathbb{R}^2)$, which implies (i).

The first-order derivatives of u^k are given by

$$\partial_1 u(x_1,x_2) = \sqrt{k}\,(1-|x_2|)^k\,(\cos(kx_1), -\sin(kx_1)),$$

$$\partial_2 u(x_1,x_2) = \sqrt{k}\,(1-|x_2|)^{k-1}\left(-\frac{x_2}{|x_2|}\right)(\sin(kx_1), \cos(kx_1)).$$

We thus obtain

$$|\partial_1 u(x_1,x_2)| \leq \sqrt{k}\,(1-|x_2|)^k \leq \sqrt{k}\,(1-|x_2|)^{k-1},$$

$$|\partial_2 u(x_1,x_2)| \leq \sqrt{k}\,(1-|x_2|)^{k-1},$$

which yields

$$\int_\Omega |\nabla u(x)|^2\,dx \leq k\int_{-1}^{1}\int_{-1}^{1}(1-|x_2|)^{2k-2}\,dx_2\,dx_1 = 4k\int_0^1 (1-x_2)^{2k-2}\,dx_2 = \frac{4k}{2k-1} \leq 4$$

This shows (ii). In total, we conclude $u^k \rightharpoonup 0$ in $H^1(\Omega;\mathbb{R}^2)$.

b) We have

$$\det(\nabla u^k) = k(1-|x_2|)^{2k-1}\left(-\frac{x_2}{|x_2|}\right)\cdot 1 = F_k'(x_2)$$

for the function $F_k(y) = \frac{1}{2}(1-|z|)^{2k}$.

For smooth test functions $\psi \in C_c^\infty(\Omega)$, integration by parts thus yields

$$\int_\Omega \det(\nabla u^k)\psi \, dx = \int_{-1}^{1}\int_{-1}^{1} F_k'(x_2)\,\psi(x_1,x_2)\, dx_2\, dx_1$$

$$= -2\int_{-1}^{1} F_k(x_2)\,\partial_{x_2}\psi(x_1,x_2)\, dx_2$$

Since $0 \le F_k \le \frac{1}{2}$, we can use dominated convergence to pass to the limit $k\to\infty$ and to obtain

$$\lim_{k\to\infty}\int_\Omega \det(\nabla u^k)\psi\, dx = -2\int_{-1}^{1} 0\cdot\partial_{x_2}\psi(x_1,x_2)\, dx_2 = 0.$$

For $\varphi \in C_c(\Omega)$ and $\varepsilon > 0$, we can find $\psi \in C_c^\infty(\Omega)$ such that $\|\varphi - \psi\|_{L^\infty(\Omega)} < \varepsilon$.

Then we have

$$\left|\int_\Omega \det(\nabla u^k)\varphi\, dx\right| \le \int_\Omega |\det(\nabla u^k)|\, dx\, \|\varphi-\psi\|_{L^\infty(\Omega)} + \left|\int_\Omega \det(\nabla u^k)\psi\, dx\right|$$

$$\le \varepsilon\, 4\int_0^1 F_k'(x_2)\, dx_2 + \left|\int_\Omega \det(\nabla u^k)\psi\, dx\right|$$

$$\le 2\varepsilon + \left|\int_\Omega \det(\nabla u^k)\psi\, dx\right| < 4\varepsilon$$

if we choose k sufficiently large. This shows $\lim_{k\to\infty}\int \det(\nabla u^k)\varphi\, dx = 0$.

c) To show that $\det(\nabla u^k)$ does not converge weakly to 0 in $L^1(\Omega)$, we consider the test function $\varphi(x_1,x_2) = 1_{(0,1)}(x_2)$. Then

$$\int_\Omega \det(\nabla u^k)\varphi\, dx = \int_{-1}^{1}\int_{-1}^{1} F_k'(x_2)\, 1_{(0,1)}(x_2)\, dx_2\, dx_1$$

$$= 2\int_0^1 F_k'(x_2)\, dx_2 = 2\left(F_k(1) - F_k(0)\right) = 0 - 1 = -1$$

for all $k\in\mathbb{N}$. Therefore, $\det(\nabla u^k)$ does not converge weakly in $L^1(\Omega)$.