Ex 4. $I(x) = (1 - \|x\|_{\ell^2}^2)^2 + \sum_{k=1}^{\infty} \frac{1}{k} x_k^2$

a) <u>continuity</u>: Write $I = I_1 + I_2$

$I_1(x) = g(h(x))$ with $g(z) = (1 - z^2)^2$
$h(x) = \|x\|_{\ell^2}$

g continuous (polynome)
h continuous (norm)
$\Rightarrow I_1$ continuous

Consider $|I_2(x) - I_2(y)|$

$\leq \sum_{k=1}^{\infty} \frac{1}{k}(x_k^2 - y_k^2) = \sum_{k=1}^{\infty} \frac{1}{k}(x_k - y_k)(x_k + y_k)$

$\underset{\text{Hölder}}{\overset{1/k \leq 1}{\leq}} \|x+y\|_{\ell^2} \|x-y\|_{\ell^2}$

$\Rightarrow$ locally Lipschitz cont.

b) <u>positivity</u>

We obviously have $I(x) \geq 0$

I_1 and I_2 (see above) are nonnegative

Thus, $I(x) = 0$ if and only if

$$I_1(x) = 0 \quad \text{and} \quad I_2(x) = 0$$

Let $I_1(x) = 0 \implies \|x\|_{\ell^2} = 1$

$\implies \exists k_* \in \mathbb{N} : x_{k_*} \neq 0$

$\implies I_2(x) \geq \frac{1}{k_*} x_{k_*}^2 > 0$

$I_2(x) = 0 \implies x = 0$

$\implies I_1(x) = 1 > 0$

c) no minimizer exists.

We show $\inf_{\ell^2} I = 0$.

Consider sequence (of sequences)

$$x^{(\ell)} = e^{(\ell)} \text{ with } e^{(\ell)}_k = \begin{cases} 1 & \ell = k \\ 0 & \text{else} \end{cases}$$

Clearly $e^{(\ell)} \in \ell^2$ and $\|e^{(\ell)}\|_{\ell^2} = 1$

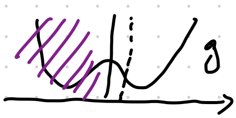
Thus, $I(x^{(\ell)}) = I(e^{(\ell)}) = 0 + \frac{1}{\ell} \xrightarrow{\ell \to \infty} 0$.

$$\Rightarrow \inf I = 0$$

If x_* was minimizer, we would have $I(x_*) = \inf I = 0$. However, due to part (b) that is not possible.

Interesting questions:

1. In which sense does $x^{(\ell)} = e^{(\ell)}$ converge? (weak sense)
2. Is it possible for an infimizing sequence to converge <u>strongly</u>? (no, see (a) and (b))
3. Is I convex? (no, $g(z) = (1 - z^2)^2$



defines double-well)

4. What happens if we restrict to bounded set $B_R(0) \subset \ell^2$? $B_R(0)$ is not compact w.r.t norm
I is not continuous w.r.t. "$\rightharpoonup$"

|Ana II: continuous function on compact set attains minimum!|

$\boxed{\text{Ex 5}}$

$$I : \begin{cases} \underbrace{\{ v \in C^1(\bar{\Omega}; \mathbb{R}^d) \mid v|_{\partial\Omega} = u_0|_{\partial\Omega} \}}_{=: \mathcal{M}} \to \mathbb{R} \\ u \mapsto \int \frac{\mu(x)}{2} |e(u)|^2 - f(x) \cdot u \, dx \end{cases}$$

$$e(u) = \frac{1}{2}(\nabla u + \nabla u^T) \in \mathbb{R}^{d \times d}_{sym}$$

$$A \in \mathbb{R}^{d \times d} : \quad |A|^2 = \sum_{i,j=1}^{d} A_{ij}^2, \quad A : B = \sum_{i,j=1}^{d} A_{ij} B_{ij} = \operatorname{tr}(A^T B)$$

$\mathcal{M}$ is an affine space, in particular,

$$u_1 \in \mathcal{M}, \; u_2 \in \mathcal{M} \not\Rightarrow u_1 + u_2 \in \mathcal{M}$$

except if $u_0 \equiv 0$.

$\mathcal{M} = u_0 + \mathcal{M}_0$ with

$$\mathcal{M}_0 = \{ v \in C^1(\bar{\Omega}; \mathbb{R}^d) \mid v|_{\partial\Omega} = 0 \}$$

$u \in \mathcal{M}, v \in \mathcal{M}_0 \Rightarrow u + v \in \mathcal{M}$ since

$$(u+v)|_{\partial\Omega} = u|_{\partial\Omega} + v|_{\partial\Omega} = u_0|_{\partial\Omega} + 0$$

Consider for $u \in \mathcal{M}$, $v \in \mathcal{M}_0$

$$\frac{1}{\varepsilon}\Big(I(\underbrace{u+\varepsilon v}_{\in \mathcal{M}}) - I(\underbrace{u}_{\in \mathcal{M}})\Big)$$

$$= \frac{1}{\varepsilon}\int_\Omega \frac{\mu(x)}{2}\left(|e(u+\varepsilon v)|^2 - |e(u)|^2\right) - f(x)\cdot(u+\varepsilon v - u)\,dx$$

$$\boxed{e(u+v) = e(u) + e(v)}$$

$$= \int_\Omega \frac{1}{\varepsilon}\frac{\mu(x)}{2}\left(|e(u)|^2 + \varepsilon^2 |e(v)|^2 + 2\varepsilon\, e(u):e(v) - |e(u)|^2\right) - f(x)v\,dx$$

$$= \int_\Omega \frac{\mu(x)}{2}\left(\varepsilon\, |e(v)|^2 + 2\, e(u):e(v)\right) - f(x)v\,dx$$

$$= \underbrace{\varepsilon J(v)}_{\xrightarrow{\varepsilon \to 0} 0} + \int_\Omega \mu(x)\, e(u):e(v) - f(x)v\,dx$$

$$\Rightarrow DI(u)[v] = \int_\Omega \mu(x)\, e(u) : e(v) - f(x) v \, dx$$

Observation: $\mathbb{R}^{d\times d}_{sym} \perp \mathbb{R}^{d\times d}_{skew}$

$$\mathbb{R}^{d\times d}_{sym} = \{A \in \mathbb{R}^{d\times d} \,/\, A^T = A\} \qquad \mathbb{R}^{d\times d}_{skew} = \{B \in \mathbb{R}^{d\times d} \mid B^T = -B\}$$

$$A : B = \sum_{i,j=1}^{d} A_{ij} B_{ji} \overset{B^T = -B}{=} -\sum_{i,j=1}^{d} A_{ij} B_{ji}$$

$$\overset{A^T = A}{=} -\sum_{i,j=1}^{d} A_{ji} B_{ji} = -A : B$$

$$\Rightarrow e(u) : e(v) = \underbrace{e(u)}_{\in \mathbb{R}^{d\times d}_{sym}} : \Big(\nabla v - \underbrace{\tfrac{1}{2}(\nabla v - \nabla v^T)}_{\in \mathbb{R}^{d\times d}_{skew}}\Big)$$

$$\Rightarrow DI(u)[v] = \int_\Omega \mu(x)\, e(u) : \nabla v - f \cdot v \, dx$$

$$\overset{\text{Integration by parts}}{=} \int_\Omega \Big(-\operatorname{div}(\mu(x)\, e(u)) - f\Big) v \, dx$$

[Maybe, discuss formulas for Integration-by-parts in vector-valued case]

$$+ \underbrace{\int_{\partial\Omega} \mu(x)\, e(u) \cdot \nu \, v \, da}_{=0}$$

Euler-Lagrange equation via fundamental theorem of calculus of variations

$$-\operatorname{div}(\mu(x)\, e(u)) = f \text{ in } \Omega$$

$$u = u_0 \text{ on } \partial\Omega$$

modifications $u = u_0$ on $\Gamma_{Dir} \subset \partial\Omega$

- "test functions" from which space?
- additional boundary contribution

$$\int_{\Gamma_{Neu}} \kappa u^2 \, da$$

and force $\int_{\Gamma_{Neu}} g \cdot u \, da$.

- Let's take

$$I(u) = \int_\Omega \Phi(x, e(u)) - f(x)\, u \, dx$$

with $\Phi \in C^2(\bar{\Omega} \times \mathbb{R}^{d\times d}_{sym})$

is $\frac{\partial \Phi}{\partial e}$ symmetric?

$\boxed{\text{open!!!}}$

Ex 6 $\Omega \subset \mathbb{R}^d$ (could be unbounded)

a) $a \in C^0(\Omega)$ s.t. $\forall \varphi \in C_c^\infty(\Omega)$

$$\int_\Omega a(x)\, \varphi(x)\, dx = 0$$

Show $a \equiv 0$ in Ω

Let's assume $a \not\equiv 0$. Then $x_0 \in \Omega$ exists with $a(x_0) = a_0 \neq 0$.

Set $\alpha(x) = a(x) \cdot a_0$ such that

$\alpha \in C^0(\Omega)$ and $\alpha(x_0) = a_0^2 > 0$

$\Rightarrow \exists \delta > 0$ s.t. $B_\delta(x_0) \subset \Omega$ (Ω open)

and $\forall x \in B_\delta(x_0)$ $\alpha(x) \geq \frac{1}{2}\alpha(x_0) > 0$

(continuity is important here)

Pick $\tilde{\psi} \in C^\infty(\Omega)$ s.t. $\boxed{\operatorname{sppt} \tilde{\psi} \subset \overline{B_\delta(x_0)}}$

and $\int_\Omega \tilde{\psi}\, dx = 1$, $\tilde{\psi} \geq 0$

(not important)

important

for example $\tilde{\psi}(x) = \frac{1}{\delta^d} \rho\left(\frac{x}{\delta}\right)$ with

$$\rho(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$\psi(x) = a(x_0)\tilde{\psi}(x)$

$$\Rightarrow \quad 0 = \int_\Omega a(x)\psi(x)\, dx = \int_{B_\delta(x_0)} \alpha(x)\tilde{\psi}(x)\, dx$$

$$\overset{\tilde{\psi}\geq 0}{\underset{\substack{\alpha(x)\geq\frac{1}{2}\alpha(x_0)\\ \text{on } B_\delta(x_0)}}{\geq}} \int_{B_\delta(x_0)} \frac{1}{2}\alpha(x_0)\,\tilde{\psi}(x)\,dx = \frac{1}{2}\alpha(x_0) > 0$$

$\lightning$

$\Rightarrow \quad a \equiv 0 \quad \text{in } \Omega.$

b) With Step (a) we know that $a \equiv 0$ by considering $\psi \in C_c^\infty(\Omega) \subset C^\infty(\bar{\Omega})$

Thus, we have $\displaystyle\int_{\partial\Omega} b(x)\,\psi(x)\,da = 0$

argue as before: Pick $x_1 \in \partial\Omega$ with $b(x_1) \neq 0 \Rightarrow \beta(x) = b(x_1)\cdot b(x)$

$\Rightarrow \exists\, \delta > 0\ \forall x \in B_\delta(x_1) \cap \partial\Omega : \beta(x) \geq \frac{1}{2}\beta(x_1) > 0$

Pick $\tilde{\psi}$ as above and set $\psi(x) = \beta(x)\tilde{\psi}(x)$

$$\Rightarrow \psi \in C^\infty(\bar{\Omega}) : 0 = \int_{\partial\Omega} b(x)\,\psi(x)\,da \geq \frac{1}{2}\beta(x_1)\int_{B_\delta(x_1)\cap\partial\Omega} \tilde{\psi}\,da > 0 \ \lightning$$

α, β are used to ensure positivity

Questions: What happens if $a \in L^1_{loc}(\Omega)$ only?

<u>Mollifier</u> $\eta \in C_c^\infty(\mathbb{R}^d)$

with $\int_{\mathbb{R}^d} \eta \, dx = 1$, $\eta \geq 0$

Example

normalization

$$\eta(x) = \begin{cases} c \exp\left(-\frac{1}{1-|x|^2}\right) & \text{for } x \in B_1(0) \\ 0 & \text{otherwise} \end{cases}$$

$$\eta_\delta(x) = \frac{1}{\delta^d} \eta\left(\frac{x}{\delta}\right)$$

we always assume $\operatorname{spt} \eta \subset B_1(0)$

extend a outside of Ω by 0.

$\Rightarrow \quad a_\delta(x) := (\eta_\delta * a)(x) = \int_{\mathbb{R}^d} \eta_\delta(x-y)\, a(y)\, dy$

well-defined, why?

on every $K \subset \Omega$ compact, we have $a_{\delta|_K} \to a_{|_K}$ in $L^1(K)$. Moreover

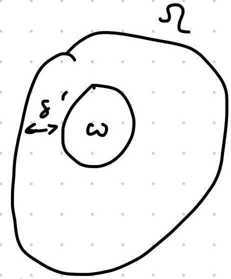
$a_\delta \in C^\infty(\Omega)$

Attention $\quad a_\delta \notin C_c^\infty(\Omega)$

Proof (Rindler's version is not very precise).

Let $\omega \subset \Omega$ open + bounded such that

$\operatorname{dist}(\omega, \partial\Omega) \geq \delta' > 0$



For any $\psi \in C_c^\infty(\omega)$ we have that

$$\psi_\delta = \psi * \eta_\delta \in C_c^\infty(\Omega) \text{ for } \delta < \delta'$$

Thus $0 = \int_\Omega a\, \psi_\delta\, dx = \int_\omega a_\delta\, \psi\, dx$

$\Rightarrow$ The result for continuous functions a gives $a_\delta = 0$ in ω.

Since ω is arbitrary (take for example suitable balls), we also get $a_\delta = 0$ in Ω.

The convergence $a_\delta \to a$ in $L^1(\bar{\omega})$ yields the existence of a subsequence $\delta_k \to 0$ for $k \to \infty$ such that

$a_{\delta_k} \to a$ <u>a.e.</u> in ω $\Rightarrow$ $a = 0$ in ω

By arbitraryness of ω we get $a = 0$ in Ω.