Exercise Sheet 11

discussed on January 31, 2023

Exercise 37. Quasi-convexity. The definition of quasi-convexity for a continuous function $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ reads as follows:

$$\forall\, A \in \mathbb{R}^{m\times d}\ \forall\, \phi \in \mathrm{C}_\mathrm{c}^\infty(B_1(0);\mathbb{R}^m): \quad \int_{B_1(0)} f(A{+}\nabla\phi(x))\,\mathrm{d}x \geq \int_{B_1(0)} f(A)\,\mathrm{d}x,$$

where $B_1(0)$ is the open unit ball in $\mathbb{R}^d$ centered at 0.

(a) Show that in the definition of quasi-convexity any open bounded domain $\Omega \subset \mathbb{R}^d$ can be used without changing the definition.

(b) Show that the much larger set $\mathrm{W}_0^{1,\infty}(\Omega;\mathbb{R}^m)$ of test functions can be used instead of $\mathrm{C}_\mathrm{c}^\infty(\Omega;\mathbb{R}^m)$.

(c) Considering $\Omega = Q =]0,1[^d \subset \mathbb{R}^d$, we may look at periodic functions $\psi \in \mathrm{C}_\mathrm{per}^\infty(\mathbb{R}^d;\mathbb{R}^m)$, i.e. $\psi \in \mathrm{C}^\infty(\mathbb{R}^d;\mathbb{R}^m)$ with $\psi(m{+}y) = \psi(y)$ for $m \in \mathbb{Z}^d$ and $y \in \mathbb{R}^d$. Show that quasi-convexity is equivalent to

$$\forall\, A \in \mathbb{R}^{m\times d}\ \forall\, \psi \in \mathrm{C}_\mathrm{per}^\infty(\mathbb{R}^d;\mathbb{R}^m): \quad \int_Q f(A{+}\nabla\psi(x))\,\mathrm{d}x \geq f(A).$$

Exercise 38. Quasi-convexity implies rank-one convexity. Consider the periodic function $h : \mathbb{R} \to \mathbb{R}$ with period 1, $h(t) = 1{-}\theta$ for $t \in [0,\theta[$ and $h(t) = -\theta$ for $t \in [\theta,1[$ and its primitive function $H(t) = \int_0^t h(r)\,\mathrm{d}r$.

(a) For $a \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^d$ define the sequence $u_k(x) = \frac{1}{k}H(k\eta{\cdot}x)a$. Show that $u_k \rightharpoonup 0$ in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$.

(b) For a function $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ and fixed $A \in \mathbb{R}^{m\times d}$ calculate $\lim\limits_{k\to\infty} \int_\Omega f\big(A{+}\nabla u_k(x)\big)\,\mathrm{d}x$.

(c) Show for continuous $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ that quasi-convexity implies rank-one convexity.

Exercise 39. Counterexample concerning Reshetnyak's theorem. Take $m = d = p = 2$ and $\Omega =]-1,1[^2$ and the sequence

$$u^k(x_1,x_2) = \frac{1}{\sqrt{k}}(1{-}|x_2|)^k\big(\sin(kx_1),\cos(kx_1)\big).$$

(a) Show that $u^k \rightharpoonup 0$ in $\mathrm{H}^1(\Omega;\mathbb{R}^2)$.

(b) Prove that $\int_\Omega \det(\nabla u^k)\varphi\,\mathrm{d}x \to 0$ for all $\varphi \in \mathrm{C}_\mathrm{c}(\Omega)$.

(c) Show that $\det(\nabla u^k)$ does not converge weakly to 0 in $\mathrm{L}^1(\Omega)$.
Hint: Consider suitable $\varphi \in \mathrm{L}^\infty(\Omega)$ in (b).

(37) $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ continuous

and quasi-convex, i.e.

$QC(B_1(0))$ $$\int_{B_1(0)} f(A + \nabla\phi)\,dx \geq vol(B_1(0))\, f(A) \quad \forall \phi \in C_c^\infty(B_1(0), \mathbb{R}^m)$$

a) We show that $QC(B_1(0))$ is equivalent to

$QC(D)$ $$\int_D f(A+\nabla\psi)\,dy \geq vol(D)\, f(A) \quad \forall \psi \in C_c^\infty(D, \mathbb{R}^m)$$

where $D \subset \mathbb{R}^d$ is open and bounded.

We assume that $QC(B_1(0))$ holds and show that f also satisfies $QC(D)$.

Choose $R>0$ sufficiently large s.t.

$D \subseteq Q_R :=]-R, R[^n$

For $\psi \in C_c^\infty(D; \mathbb{R}^m)$ define

$$\tilde{\psi}(y) := \begin{cases} \psi(y) & \text{if } x \in D \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow \quad \tilde{\phi} \in C_c^\infty(Q_R; \mathbb{R}^m)$$

Fix $k \in \mathbb{N}$ such that $Q_{R/k} \subset\subset B_1(0)$

define $\phi \in C_c^\infty(B_1(0); \mathbb{R}^m)$ via

$$\phi(x) := \begin{cases} \frac{1}{k}\tilde{\psi}(kx) & x \in Q_{R/k}, \\ 0 & \text{else}. \end{cases}$$

We get $\nabla_x \phi = \nabla_y \tilde{\psi}$

$$\int_D f(A + \nabla_y \psi)\,dy = \int_{Q_R} f(A + \nabla_y \tilde{\psi})\,dy - \int_{Q_R \setminus D} f(A)\,dy$$

$$= \int_{Q_{R/k}} k^d f(A + \nabla_y \tilde{\psi}(kx))\,dx - \big((2R)^d - \mathrm{vol}(D)\big) * f(A)$$

$$= \int_{B_1(0)} k^d f(A + \nabla_x \phi)\,dx - \Big\{ k^d \mathrm{vol}((B_1(0)) - \cancel{(2R)^d} + \cancel{(2R)^d} - \mathrm{vol}(D) \Big\} f(A)$$

$$\overset{\text{Assumption}}{\geq} k^d \operatorname{vol}(B_1(0)) f(A)$$

$$- \left\{ k^d \operatorname{vol}(B_1(0)) - \operatorname{vol}(D) \right\} f(A)$$

$$= \operatorname{vol}(D) f(A).$$

The reverse direction is anologous. $\square$

b) We need to show that if f satisfies $QC(B_1(0))$ for $\phi \in C_c^\infty(B_1(0); \mathbb{R}^m)$, then it satisfies the same estimate for $\psi \in W_0^{1,\infty}(B_1(0); \mathbb{R}^m)$.

Note that $C_c^\infty(B_1(0); \mathbb{R}^m) \subset W_0^{1,\infty}(B_1(0); \mathbb{R}^m)$.

Thus, the reverse implication is trivial.

Let us fix $\psi \in W_0^{1,\infty}(B_1(0); \mathbb{R}^m)$.

We find a sequence $\psi_n \in C_c^\infty(B_1(0); \mathbb{R}^m)$ such that $\|\psi_n\|_{W^{1,\infty}} \leq C_\psi$ (uniformly) and $\nabla \psi_n \longrightarrow \nabla \psi$ in $L^1(B_1(0); \mathbb{R}^m)$.

(not that trivial)

Thus, we have $f(A+\nabla\varphi_n) \to f(A+\nabla\varphi)$ a.e. in $B_1(0)$, since f is continuous on $\mathbb{R}^{m\times d}$, there exists $M>0$ such that

$|f(\hat{A})| \leq M$ for all $\hat{A}$ with $|\hat{A}| \leq |A| + C_\varphi$

$\Rightarrow$ Lebesgue's theorem on dominated convergence gives

$$\mathrm{vol}(B_1(0))\, f(A) \leq \lim_{n\to\infty} \int_{B_1(0)} f(A+\nabla\varphi_n)\,dx = \int_{B_1(0)} f(A+\nabla\varphi)\,dx$$

$\square$

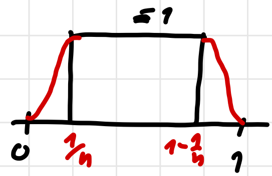
c) $Q =]0,1[^d$, $C^\infty_{per}(Q;\mathbb{R}^m)$

Clearly, if $\phi \in C^\infty_c(Q;\mathbb{R}^m)$, we can extend ϕ to a periodic function, why?

Assume $QC(Q)$ holds for $\phi \in C^\infty_c(Q;\mathbb{R}^m)$.

Let $\hat{\phi}$ denote a periodic function in $C^\infty_{per}(Q;\mathbb{R}^m)$. We take $\eta_n \in C^\infty_c(Q;\mathbb{R}^m)$ such that $\eta_n \equiv 1$ on $[\frac{1}{n}, 1-\frac{1}{n}]^d$ and

η_n vanishes in a neighborhood of the boundary and $|\nabla \eta_n| \le C_1 n$

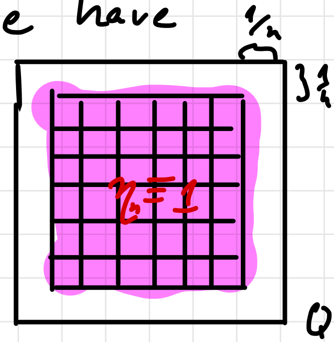


and $0 \le \eta_n \le 1$

We define $\psi_n(x) = \eta_n(x)\,\frac{1}{n}\,\phi(nx)$

$\Rightarrow$ $\psi_n \in C_c^\infty(Q;\mathbb{R}^m)$ and we have

$$f(A) \le \int_Q f(A + \nabla \psi_n)\,dx$$



we need to show the right-hand side converges to $\int_Q f(A + \nabla \phi)\,dx$

We compute

$$\boxed{\nabla_x\big(\phi(\overbrace{nx}^{y})\big) = n\,\nabla_y\,\phi(\overbrace{nx}^{y})}$$

$$\nabla_x \psi_n = \eta_n(x)\,\nabla_y \phi(n\,x) + \frac{1}{n}\,\phi(n\,x)\,\nabla_x \eta_n(x)$$

In particular, we have

$$\|\nabla_x \psi_n - \nabla_y \phi(n\cdot)\|_{L^\infty}$$
$$= \|(\eta_n - 1)\nabla_y \phi(n\cdot) + \phi(n\cdot)\,\frac{1}{n}\nabla_x \eta_n\|_{L^\infty}$$

$$\leq C_2 \|\phi\|_{W^{1,\infty}}$$

Since f is continuous, it is bounded on compact subsets $K \subset \mathbb{R}^{m \times d}$, thus

$$\| f(A + \nabla \psi_n) - f(A + \nabla_y \phi(n \cdot)) \|_{L^\infty} \leq C_3$$

However, we have

$$\mathrm{vol}(Q)\, f(A) \leq \int_Q f(A + \nabla \psi_n)\, dx$$

$$= \int_Q f(A + \nabla_y \phi(n \cdot))\, dx$$

$$+ \int_Q | f(A + \nabla \psi_n) - f(A + \nabla_y \phi(n \cdot)) |\, dx$$

Since $\nabla \psi_n = \nabla_y \phi(n \cdot)$ on $Q \setminus Q_n$, we get for the last term

$$\int_Q | f(A + \nabla \psi_n) - f(A + \nabla_y \phi(n \cdot)) | \leq \int_{Q \setminus Q_n} C_3\, dx$$

$$= C_3 \operatorname{vol}(Q \setminus Q_n)$$

$$\longrightarrow 0$$

Finally, we have due to the periodicity of ϕ and a change of variables

$$\int_Q f(A + D_y \phi(n\cdot))\,dx = \sum_{j=1}^{n^d} \int_{Q_n^j} f(A + D_y \phi(n\cdot))\,dx$$

$y = nx$

$dy = n^d dx$

$$= \frac{\cancel{n^d}}{\cancel{n^d}} \int_Q f(A + D_y \phi)\,dy \,.$$

Here Q_n^j are cubes wide side length $1/n$ and volumes $1/n^d$.

$\square$

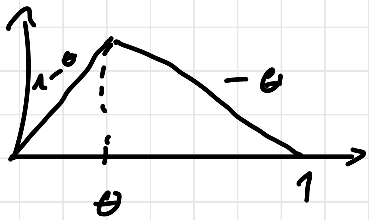
(38)

$$h(t) = \begin{cases} 1-\theta & \text{if } t \in [0,\theta[\\ -\theta & \text{if } t \in [t,1] \end{cases}$$

$\theta \in]0,1[$

periodically extended to $\mathbb{R}$

$$H(t) = \int_0^t h(r)\,dr$$



a) Fix $a \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^d$ define

$$u_k(x) = \frac{1}{k} H(k\eta \cdot x)\, a$$

$$\Rightarrow u_k \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^m)$$

$$\nabla u_k(x) = \frac{1}{\cancel{k}} H'(k\eta \cdot x)\cancel{k}\, a \otimes \eta$$

$(a \otimes \eta) \in \mathbb{R}^{m\times d}$, $(a\otimes\eta)_{ij} = a_i \eta_j$ $\quad i = 1,\dots,m$, $j = 1,\dots,d$

$\shortparallel$

$a\eta^T$

$$\begin{pmatrix} a_1\eta_1 & \cdots & a_1\eta_d \\ \vdots & \ddots & \vdots \\ a_m\eta_1 & \cdots & a_m\eta_d \end{pmatrix}$$ has rank one!

More precisely, we have

$$\nabla u_k(x) = \begin{cases} (1-\theta)\, a \otimes \eta & \text{if } k\eta\cdot x \in z+[0,\theta[\\ & \text{for some } z\in\mathbb{Z} \\ -\theta\, a\otimes\eta & \text{if } k\eta\cdot x \in z+[\theta,1] \\ & \text{for some } z\in\mathbb{Z} \end{cases}$$

Basic idea if $A, B \in \mathbb{R}^{m\times d}$ are rank-1-connected, i.e., $\operatorname{rank}(A-B)=1$ then $\exists\, a\in\mathbb{R}^m, \eta\in\mathbb{R}^d$ $A-B = a\otimes\eta$

$$\Rightarrow \theta A + (1-\theta)B + \nabla u_k(x)$$

$$= \theta A + (1-\theta) B + \begin{cases} (1-\theta)(A-B) & \text{if } k\eta\cdot x \in \dots \\ \theta(B-A) & \text{if } k\eta\cdot x\in\dots \end{cases}$$

$$= \begin{cases} A & \text{if } kx\cdot\eta \in \dots \\ B & \text{if } kx\cdot\eta\in\dots \end{cases}$$

Clearly, we have that $|u_k| \leq \frac{|a|}{k} \to 0$

i.e. $u_k \longrightarrow 0$ in $L^\infty(\Omega;\mathbb{R}^m)$

and as before, we have that

$$\nabla u_k \longrightarrow (1-\theta)\, a\otimes\eta \cdot \theta - \theta\, a\otimes\eta \cdot (1-\theta)$$
$$= 0 \qquad \checkmark$$

see previous exercises
(Sheet 9, Exercise 32
Sheet 4, Exercise 13)

As before, we compute

$$\int_\Omega f(\tilde{A} + \nabla u_k(x))\,dx$$
$$= \int_\Omega \underbrace{f(\tilde{A} + h(kx\cdot\eta)\, a\otimes\eta)}_{\text{oscillating in direction } \eta\in\mathbb{R}^d}\,dx$$
$$\longrightarrow \int_\Omega \theta f(\tilde{A} + (1-\theta)\, a\otimes\eta) + (1-\theta) f(\tilde{A} - \theta\, a\otimes\eta)\,dx$$

c) We now show that quasi-convexity implies rank-1-convexity.
We use the characterization of quasi-convexity in Exercise 37 (c) via periodic function.

Note that we can assume w.l.o.g. that $\eta = e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^d$ (why?).

We then take $Q = [0,1]^d$. Since u_k is periodic, we get

$$\int_Q f(\hat{A} + Du_k)\,dx \geq f(\hat{A})$$

$(\mathrm{vol}(Q) = 1)$

with A, B as above

Choosing $\hat{A} = \theta A + (1-\theta)B$ yields

$$\lim_{k\to\infty} \int_Q f(\hat{A} + Du_k)\,dx = \int_Q \theta f(A) + (1-\theta) f(B)\,dx$$

$$= \theta f(A) + (1-\theta) f(B)$$

$$\geq f(\theta A + (1-\theta) B) .$$

$\square$