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Theorem: (i) X reflexive Banach space

(ii) $M \subset X$ weak sequentially closed

(iii) $I : X \to \mathbb{R}_\infty$ weak seq. lsc

(iv) $I|_M$ is coercive (on M!)

$\Rightarrow \exists\, u_* \in M$ such that $I(u_*) = \inf_{\tilde{u} \in M} I(\tilde{u})$.

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subset M$ such that

$$I(u_n) \to \alpha := \inf_{\tilde{u} \in M} I(\tilde{u}).$$

We assume that $I(u_n) \le C_1 < \infty$ uniformly in n.

Since I is coercive on M, we get that $\|u_n\|_X \le C_1 < \infty$. By the reflexivity of X, we can extract a subsequence $n_k \xrightarrow{k\to\infty} \infty$ such that $u_{n_k} \rightharpoonup u_* \in X$.

Since M is weak seq. closed, we get $u_* \in M$ and the weak lower semicontinuity yields

$$I(u_*) \le \liminf_{k \to \infty} I(u_{n_k}) = \lim_{n \to \infty} I(u_n) = \alpha$$

Thus, $u_* \in M$ is a minimizer of I over M.

(34) Poincaré inequality. $\Omega \subset \mathbb{R}^d$ bounded, open Lipschitz

In lecture: $\forall u \in W_0^{1,p}(\Omega)$: $\int_\Omega |u|^p dx \leq C_\Omega^p \int_\Omega |\nabla u|^p dx$

where $C_\Omega := \operatorname{diam} \Omega$

$\boxed{1 < p < \infty \text{ not in exercise}}$

a) We define $\left.\begin{array}{l} I_p(u) = \int_\Omega |\nabla u|^p dx \\ J_p(u) = \int_\Omega |u|^p dx \end{array}\right\}$ on $W_0^{1,p}(\Omega)$

We consider $M_1 := \{ u \in W_0^{1,p}(\Omega) \mid J_p(u) = 1 \}$

Theorem 3.60 gives the existence of $u_* \in M_1$ such that $\inf_{\tilde{u} \in M_1} I_p(\tilde{u}) = I_p(u_*) =: (C_{opt})^{-p}$

and $\lambda_p \in \mathbb{R}$ such that $DI_p(u_*) = \lambda_p DJ_p(u_*)$.

This means that

$$\forall v \in W_0^{1,p}(\Omega) : p \int_\Omega |\nabla u_*|^{p-2} \nabla u \cdot \nabla v \, dx = \lambda_p \, p \int_\Omega |u_*|^{p-2} u_* v \, dx$$

$\boxed{\lambda_p > 0 !}$

choosing $v = u_*$ gives $\int_\Omega |\nabla u_*|^p dx = \lambda_p \int_\Omega |u_*|^p dx$

i.e. $I_p(u_*) = \lambda_p J_p(u_*) = \lambda_p \cdot 1$

In particular, we get $\lambda_p = C_{opt}^{-p}$ ③

Let us now consider $\tilde{u} \in W_0^{1,p}(\Omega) \setminus \{0\}$.

we get
$$\frac{\int_\Omega |\nabla \tilde{u}|^p dx}{\int_\Omega |\tilde{u}|^p dx} = \frac{I_p(\tilde{u})}{J_p(\tilde{u})} = I_p(\hat{u})$$
where $\hat{u} = \frac{\tilde{u}}{\|\tilde{u}\|_{L^p}} \in M_1$

$$\Rightarrow \int_\Omega |\nabla \tilde{u}|^p dx \geq \underbrace{I_p(u_*)}_{(C_{opt})^{-p}} \int_\Omega |\tilde{u}|^p dx$$

$$\forall\, \tilde{u} \in W_0^{1,p}(\Omega): \int_\Omega |\tilde{u}|^p dx \leq C_{opt}^p \int_\Omega |\nabla \tilde{u}|^p dx$$

Claim: We have $C_{opt} \leq C_\Omega$.

Indeed, we have

$$1 = \int_\Omega |u_*|^p dx = C_{opt}^p \int_\Omega |\nabla u_*|^p dx \geq \left(\frac{C_{opt}}{C_\Omega}\right)^p \int_\Omega |u_*|^p dx = \left(\frac{C_{opt}}{C_\Omega}\right)^p$$

thus $\frac{C_{opt}}{C_\Omega} \leq 1$ $\square$

C_{opt} is the optimal Poincaré constant!
The Poincaré inequality is sharp for u_*.

b) $p = 2$, $\Omega =]0, \ell[\Rightarrow C_\Omega = \ell > 0$ (4)

We show that optimal Poincaré constant
$C_{opt} = \frac{1}{\sqrt{\lambda_2}} < C_\Omega$

We solve the Euler-Lagrange equations explicitly:

$$(EL) \quad \begin{cases} -u_*'' = \lambda_2 u_* \text{ in } \Omega \\ u_*(0) = u_*(\ell) = 0 \\ \int_0^\ell u_*^2 \, dx = 1 \quad , \quad \lambda_2 \in \mathbb{R} \end{cases}$$

We know that $u_* \in W_0^{1,2}(\Omega)$ and λ_2 exist.

<u>(1) Assume that $\lambda_2 > 0$:</u>

$$\Rightarrow u_*(x) = C_1 \cos(\sqrt{\lambda_2}\, x) + C_2 \sin(\sqrt{\lambda_2}\, x)$$

such that $u_*(0) = 0 \Rightarrow C_1 = 0$

$u_*(\ell) = 0 \Rightarrow \boxed{\sqrt{\lambda_n}\, \ell = n\pi}$ for some $n \in \mathbb{Z}$

$\lambda_n = \left(\frac{n\pi}{\ell}\right)^2$

We still need to solve constraint: $\int_0^\ell u_*^2 \, dx \stackrel{!}{=} 1$

$$\Rightarrow 1 = C_2^2 \int_0^\ell \sin\left(\frac{n\pi}{\ell} x\right)^2 dx$$

$$= C_2^2 \frac{\ell}{n\pi} \int_0^{n\pi} \sin(z)^2 dz = C_2^2 \frac{\ell}{n\pi} \left(\frac{1}{2} z - \frac{1}{2} \sin z \cos z \right) \Big|_0^{n\pi}$$

$$= C_2^2 \frac{\ell}{\cancel{n\pi}} \left(\frac{1}{2} \cancel{n\pi} \right)$$

$$\Rightarrow \quad C_2^2 = \frac{2}{\ell}$$

$$u_{\lambda_n,\pm}(x) = \pm\sqrt{\frac{2}{\ell}}\,\sin\left(\frac{n\pi}{\ell}x\right)$$

is solution of Euler-Lagrange equations

Plugging in I_2 yields

$$I_2(u_{\lambda_n,\pm}) = \int_0^\ell \frac{2}{\ell}\cos\left(\frac{n\pi}{\ell}\right)^2\left(\frac{n\pi}{\ell}\right)^2 dx$$

$$= \frac{2}{\ell}\left(\frac{n\pi}{\ell}\right)^2\frac{\ell}{n\pi}\int_0^{n\pi}\cos(z)^2\,dz$$

$$= 2\,\frac{n\pi}{\ell^2}\left(\frac{1}{2}n\pi\right) = \left(\frac{n\pi}{\ell}\right)^2 = \lambda_n$$

Hence, we have $\forall n\in\mathbb{N}: \; I_2(u_{\lambda_n,\pm}) = \lambda_n = \left(\frac{n\pi}{\ell}\right)^2$

and $\quad \inf\limits_{u\in M_1} I_2(u) = \lambda_1 = \frac{\pi^2}{\ell^2}$

2. Assume that $\lambda = 0$

$$\Rightarrow \quad u_*(x) = C_1 + C_2 x$$

boundary conditions imply $u_* \equiv 0 \notin M_1$ ↯

3. Assume that $\lambda < 0$

$$\Rightarrow \quad u_*(x) = C_1\cosh(\sqrt{-\lambda}\,x) + C_2\sinh(\sqrt{-\lambda}\,x)$$

$u_*(0) = C_1 \quad \Rightarrow \quad C_1 = 0$

$u_*(\ell) = C_2\sinh(\sqrt{-\lambda}\,\ell) \Rightarrow \quad C_2 = 0$ ↯

Only first case is relevant.

$\Rightarrow$ optimal Poincaré constant satisfies $C_{opt} = \frac{1}{\sqrt{\lambda_1}} = \frac{\ell}{\pi} < \ell = C_\Omega$

$\square$

35 Gâteaux and Fréchet differentials

$X = L^2(\Omega; \mathbb{R}^m)$, $\quad I(u) = \int_\Omega g(u(x))\,dx$

with $g \in C^1(\mathbb{R}^m)$ satisfying

$(*)$ $\exists C > 0\ \forall a, b \in \mathbb{R}^m : |g(a+b) - g(a)| \leq C(1+|a|)|b|$

a) Gâteaux differentiability

$$\frac{I(u+tv) - I(u)}{t} = \int_\Omega \frac{g(u+tv) - g(u)}{t}\,dx$$

$$\overset{(*)}{\leq} \frac{C}{t} \int_\Omega (1+|u|)t|v|\,dx$$

$$= C \int_\Omega (1+|u|)|v|\,dx \quad \forall v \in L^2(\Omega; \mathbb{R}^m)$$

Let $h_t(x) = \dfrac{g(u(x) + tv(x)) - g(u(x))}{t}$.

The estimate above shows that $h_t \in L^1$ has majorant $H(x) = C(1+|u(x)|)|v(x)| \in L^1$. Since $g \in C^1$, we have a.e. in Ω $\quad h_t(x) \longrightarrow h_0(x) = \partial_u g(u(x)) \cdot v(x)$. (7)

Thus, Lebesgue's theorem on dominated convergence can be applied and we conclude

$$D^G I(u)[v] = \lim_{t\to 0} \int_\Omega h_t(x)\,dx = \int_\Omega \lim_{t\to 0} h_t(x)\,dx$$

$$= \int_\Omega h_0(x)\,dx = \int_\Omega \partial_u g(u)\cdot v\,dx$$

$\square$

b) Fréchet differentiability

Assume additionally that

(**) $\exists\, C_2 > 0,\ \theta \in]0,1]$ such that

$$|\partial_u g(a) - \partial_u g(b)| \leq C_2 (1 + |a| + |b|)^{1-\theta} |a-b|^\theta$$

for all $a, b \in \mathbb{R}^m$.

We have to show that, if $v \to u$ in L^2, then

$$D^G I(v) = \partial_u g(v) \longrightarrow \partial_u g(u) = DI(u)$$

in $(L^2(\Omega;\mathbb{R}^m))' \cong L^2(\Omega;\mathbb{R}^m)$.

We consider

$$|D^G I(u)[w] - D^G I(v)[w]| \leq \int |\partial_u g(u) - \partial_u g(v)||w|\,dx$$

$$\overset{(**)}{\leq} C_2 \int_\Omega (1+|u|+|v|)^{1-\theta} |u-v|^\theta |w|\,dx$$

If $\theta = 1$ $\Rightarrow$ Hölder estimate

$$\le C_2 \|w\|_{L^2} \|u-v\|_{L^2}$$

$\Rightarrow$ Fréchet differentiable

If $\theta \in]0,1[$ we apply Hölder inequality with three functions

$$\int_\Omega |f \cdot g \cdot h| \, dx \le \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}$$

if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

For $p, q, r \in [1, \infty]$

thus,

$$|D^G I(u)[w] - D^G I(v)[w]|$$

$$\le C_1 \left(\int_\Omega (1+|u|+|v|)^2 \right)^{\frac{1-\lambda}{2}} \left(\int_\Omega |u-v|^2 dx \right)^{\frac{\lambda}{2}} \left(\int_\Omega |w|^2 dx \right)^{\frac{1}{2}}$$

where $p = \frac{2}{1-\lambda}$, $q = \frac{2}{\lambda}$, $r = 2$

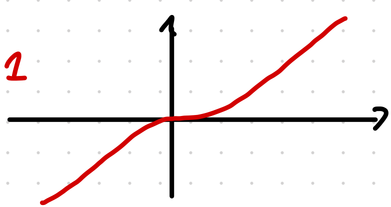
$$\frac{1-\lambda}{2} + \frac{\lambda}{2} + \frac{1}{2} = 1$$

$$\le C \left(\|u\|_{L^2}^2 + \|v\|_{L^2}^2 \right)^{\frac{1-\lambda}{2}} \|u-v\|_{L^2}^{\lambda} \|w\|_{L^2}$$

$\Rightarrow$ Fréchet differentiable

c) $X = L^1(\Omega)$, $\Omega =]0, \ell[$, $m = 1$

$$g(a) = \frac{a^3}{1 + a^2}$$



We show that $I(u) = \int_0^\ell g(u)\,dx$ is (9)

Gâteaux-differentiable at $u \equiv 0$ but <u>not</u> Fréchet differentiable.

1. Gâteaux differentiability

Note:
$$\partial_u g(a) = \frac{3a^2}{1+a^2} - \frac{2a^4}{(1+a^2)^2} = \frac{3a^2(1+a^2) - 2a^4}{(1+a^2)^2}$$
$$= \frac{3a^2 + a^4}{(1+a^2)^2}$$

$$\frac{I(0+tv) - I(0)}{t} = \frac{1}{t} I(tv) = \int_\Omega \frac{1}{t} \frac{t^3 v^3}{1+t^2v^2}\,dx$$
$$= \int_\Omega \frac{t^2 v^3}{1+t^2v^2}\,dx$$

We have that
$$\frac{t^2v^3}{1+t^2v^2} = m(t,x)\,v(x) \quad \text{with } m(t,x) = \frac{t^2 v(x)^2}{1+t^2 v(x)^2}$$

Note that $0 \leq m(t,x) \leq 1$, hence $v(x)$ is an integrable majorant. Moreover, for $t \to 0$ we have that $m(t,x) \to 0$.
Lebesgue's theorem allows us to pass to the limit to get $\frac{1}{t} I(tv) \to 0 = DI(0)[v]$

$\Rightarrow$ Gâteaux-differentiable in $u \equiv 0$.

Fréchet - differentiability

If I was Fréchet-differentiable, we would have $D^G I(0) = D^F I(0)$. Consider $v_n(x) = \chi_{]0, 1/n[}(x)$ such that $\|v_n\|_{L^1} = \frac{1}{n} \to 0$ for $n \to \infty$, and $I(v_n) = \frac{1}{2n} = \frac{1}{2}\|v_n\|_{L^1}$.

Thus,

$$\frac{|I(0+v_n) - I(0) - D^G I(0)[v_n]|}{\|v_n\|} = \frac{\left(\frac{1}{2n}\right)}{\left(\frac{1}{n}\right)} = \frac{1}{2} \not\to 0$$

no Fréchet-differentiability

$\square$

REMINDER

$I : X \to \mathbb{R}$ is called

- Gâteaux-diff'able in $u \in X$ if Operator $T_u \in Lin(X, \mathbb{R})$ (linear + continuous) exists such that

$$\lim_{t \to 0} \frac{1}{t}\left(I(u+tv) - I(u)\right) = T_u v$$

$$DI(u)[v] := T_u v$$

- Fréchet diff'able if

$$\lim_{\|v\|_X \to 0} \frac{| I(u+v) - I(u) - T_u v |}{\|v\|_X} \to 0$$ (11)

"uniform in v"

If $I : X \to \mathbb{R}$ is Gâteaux-diff'able and $u \mapsto DI(u)$ is continuous (w.r.t. (X, X') topology)

then, I is also Fréchet diff'able.

Proof. Mean value theorem gives

$$| I(u+v) - I(u) - D^G I(u)[v] | \le \| D^G I(u + \vartheta v) - D^G I(u) \|_{X'} \|v\|_X$$

for some $\vartheta \in (0,1)$.

$$(f(\vartheta) = I(u + \vartheta v) - D^G I(u)[\vartheta v], \Rightarrow f \in C^1$$

$$f'(\vartheta) = D^G I(u + \vartheta v) - D^G I(u)$$

$$\exists \vartheta \in (0,1) : | f(1) - f(0) | \le | f'(\vartheta) | .$$

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X reflexive Banach space

$K \subset X$ closed convex

$I \in C^1(X)$ (Fréchet) + convex

a) Let us assume that $u_* \in K$ is minimizer of I over K.

Consider $v \in K$, $\vartheta \in (0,1)$

$$\Rightarrow \quad v_\vartheta = (1-\vartheta)u_* + \vartheta v \in K \quad (K \text{ convex})$$

$$\Rightarrow 0 \leq \frac{I(u_* + \vartheta(v-u_*)) - I(u_*)}{\vartheta} \xrightarrow{\vartheta \to 0} D^F I(u_*)[v-u_*]$$

$$\Rightarrow \quad D^F I(u_*)[v-u_*] \geq 0$$

Let us now assume that $D^F I(u_*)[v-u_*] \geq 0$

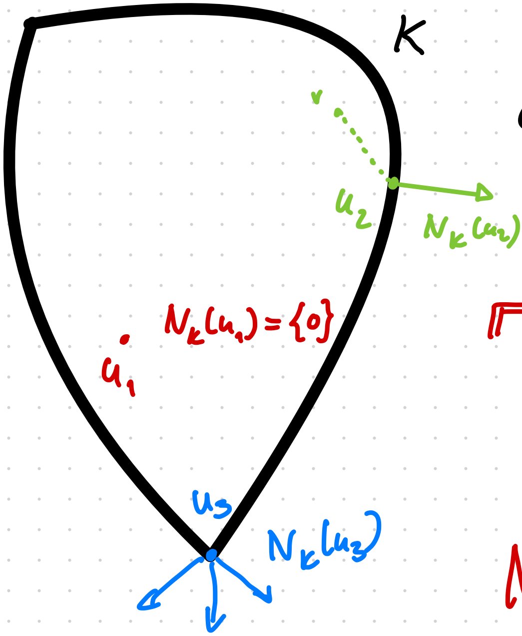
Since I is convex and Fréchet-diff'able we have

$$I(v) \geq I(u_*) + \underbrace{D^F I(u_*)[v-u_*]}_{\geq 0} \geq 0 \quad \forall v \in K$$

$\Rightarrow$ u_* is minimizer in K.

The equivalence $DI(u_*)[v-u_*] \geq 0 \iff -DI(u_*) \in N_K(u_*)$ is trivial $\square$

$N_K(u)$ is a cone, i.e. if $\ell \in N_K(u)$ then $\alpha \ell \in N_K(u)$ for all $\alpha \geq 0$



claim: $N_K(u) = 0$ if $u \in int(K)$

wrong sign in Definition of normal cone on sheet

$$N_K(u) = \{\ell \in X' \mid \ell(v-u) \leq 0\}$$

b) $$\chi_K(u) = \begin{cases} 0 & \text{if } u \in K \\ \infty & \text{otherwise} \end{cases}$$

convexity follows from that of K

$$0 \leq \chi_K((1-\lambda)u_0 + \lambda u_1) \leq (1-\lambda)\chi_K(u_0) + \lambda \chi_K(u_1)$$
$$= \begin{cases} 0 & \text{if } u_0 \text{ and } u_1 \text{ in } K \\ +\infty & \text{otherwise} \end{cases}$$

χ_K is lower semicontinuous since K is closed:

If $\{u_n\} \subset X$ is such that $u_n \to u$ in X

For $\alpha := \liminf_{n\to\infty} \chi_K(u_n)$, we either have $\alpha = +\infty$ or $\alpha = 0$. In the first case, there is nothing to show. The second case means that there exists a subsequence $n_\ell \to \infty$ such that $u_{n_\ell} \in K$. However, since $u_{n_\ell} \to u$ and K is closed, we get $u \in K$ and $\alpha = \chi_K(u)$.

The subdifferential of χ_K is given via

$$\xi \in \partial \chi_K(u) \iff \forall v \in X: \chi_K(v) \geq \chi_K(u) + \langle \xi, v-u \rangle$$

If $v \notin K$, then $\chi_K(v) = +\infty$. Hence, we can restrict to $v \in K$. If $u \notin K$ then no $\xi \in \partial \chi_K(u)$ can exist. For $u \in K$, we get

$$\begin{aligned} \xi \in \partial \chi_K(u) &\iff 0 \geq \langle \xi, v-u \rangle \quad \forall v \in K \\ &\iff \xi \in N_K(u) \end{aligned}$$

c) trivial

Application: Let $X = H_0^1(\Omega)$ and fix $\psi \in H_0^1(\Omega)$ to define $K_\psi = \{u \in X \mid u \geq \psi \text{ a.e. in } \Omega\}$

K_ψ is convex and closed

We consider $I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - f(x)u\,dx$ on K_ψ

$f \in L^2(\Omega)$, $\Rightarrow$ I is coercive, convex, Fréchet-diff'able

$\Rightarrow$ Minimizer satisfies $DI(u_*)[v-u_*] \geq 0 \;\; \forall v \in K_\psi$

This means

$$\int_\Omega \nabla u_* \cdot \nabla(v-u_*)\,dx \geq \int_\Omega f(v-u_*)\,dx$$

We define $\Omega_+ := \{x \in \Omega \mid u_*(x) > \psi(x)\}$

Set $v = u_* + hw$ with $w \in C_c^\infty(\Omega)$, $h \in \mathbb{R}$

such that $|h|$ is small and $\operatorname{supp} w \subset \Omega_+$

We get $v \in K_\psi$ and

$$h\int_\Omega \nabla u_* \cdot \nabla w\,dx - h\int_\Omega f w\,dx \geq 0$$

Since $h \in \mathbb{R}$ we can replace it with $-h$ to get

$$\int_{\Omega_+} \nabla u_* \cdot \nabla w - f w\,dx = 0 \qquad \forall w \in C_c^\infty(\Omega)$$
s.t. $\operatorname{supp} w \subset \Omega_+$

If $u_* \in H^2(\Omega)$ we get $-\Delta u = f$ in Ω_+

analogously we show $-\Delta u \geq f$ in $\Omega \setminus \Omega_+$