Exercise 22 (simpler)

At first, consider $u \in C_c^1(\mathbb{R}^d)$ and let $x, y \in \mathbb{R}^d$. The fundamental theorem of calculus implies

$$u(x) = u(y) - \int_0^1 \frac{d}{ds}\left[u(x + s(y-x)) \right] ds$$
$$= u(y) - \int_0^1 \nabla u(x + s(y-x)) \cdot (y-x)\, ds .$$

Integration with respect to $y \in B_1(x)$ leads to

$$u(x) = \frac{1}{|B_1(x)|}\left[\int_{B_1(x)} u(y)\, dy - \iint_{B_1(x)}\!\!\int_0^1 \nabla u(x + s(y-x)) \cdot (y-x)\, ds\, dy \right]$$

For the first integral, we use Hölder's inequality ($\frac{1}{p} + \frac{1}{p'} = 1$) to estimate

$$\left| \int_{B_1(x)} u(y) dy \right| \leq \int_{B_1(x)} |u(y)| \cdot 1\, dy \leq \left(\int_{B_1(x)} |u(y)|^p dy \right)^{\frac{1}{p}} \left(\int_{B_1(x)} 1\, dy \right)^{\frac{1}{p'}} = C \|u\|_{L^p(\Omega)}.$$

For the second integral, we use Fubini's theorem and the transformation theorem to obtain

$$\left| \int_{B_1(x)} \int_0^1 \nabla u(x + s(y-x)) \cdot (x-y)\, ds\, dy \right|$$
$$\leq \int_0^1 \int_{B_1(x)} |\nabla u(x + s(y-x))|\, |x-y|\, dy\, ds$$
$$\underset{\substack{z = s(y-x) \\ dz = s^d dy}}{=} \int_0^1 \int_{B_s(0)} |\nabla u(x+z)|\, s^{-d}\, dz\, ds$$
$$\underset{\text{Hölder}}{\leq} \int_0^1 \left(\int_{B_s(0)} |\nabla u(x+z)|^p dz \right)^{\frac{1}{p}} \left(\int_{B_s(0)} 1\, dz \right)^{\frac{1}{p'}} s^{-d}\, ds$$
$$\leq \int_0^1 s^{\frac{d}{p'} - d}\, ds \quad \|\nabla u\|_{L^p(\mathbb{R}^d)}$$
$$= \int_0^1 s^{-\frac{d}{p}} ds \quad \|\nabla u\|_{L^p(\mathbb{R}^d)}$$

Since $d < p$ by assumption, we have $-\frac{d}{p} > -1$, and the remaining integral is finite.

Combining both estimates, we arrive at

$$|u(x)| \leq C\left(\|u\|_{L^p(\mathbb{R}^d)} + \|\nabla u\|_{L^p(\mathbb{R}^d)}\right),$$

and thus

$$\|u\|_{L^\infty} \leq C\,\|u\|_{W^{1,p}(\mathbb{R}^d)}$$

for all $u \in C_c^1(\mathbb{R}^d)$.

By a standard density argument, this holds for all $u \in W^{1,p}(\mathbb{R}^d)$.

"Standard density argument"

Let $u \in W^{1,p}(\Omega)$. Then there exists a sequence $(\varphi_n) \subset C_c^1(\bar\Omega)$ such that $\varphi_n \to u$ in $W^{1,p}(\Omega)$. Since $\varphi_n \in C_c^1(\bar\Omega)$ and $\varphi_n - \varphi_m \in C_c^1(\bar\Omega)$, we have seen above that

$$\|\varphi_n\|_{L^\infty(\Omega)} \leq C\|\varphi_n\|_{W^{1,p}(\Omega)}, \qquad \|\varphi_n - \varphi_m\|_{L^\infty} \leq C\|\varphi_n - \varphi_m\|_{W^{1,p}(\Omega)}.$$

By assumption, (φ_n) is a Cauchy sequence in $W^{1,p}(\Omega)$. Then (φ_n) is a Cauchy sequence in $L^\infty(\Omega)$ by the second estimate. By completeness of $L^\infty(\Omega)$, we thus have $\varphi_n \to v$ in $L^\infty(\Omega)$ for some $v \in L^\infty(\Omega)$. Taking a subsequence that converges a.e. in Ω, we see that $u = v$ a.e. Now a passage to the limit in the first inequality yields $\|u\|_{L^\infty(\Omega)} \leq C\,\|u\|_{W^{1,p}(\Omega)}$.

Exercise 24

a) Let $(\varphi_n) \subset C^1(\bar{\Omega})$ with $\varphi_n|_{\Gamma_D} = 0$ and such that $\varphi_n \to u$ in $W^{1,p}(\Omega)$.

Let $x_0 \in \Gamma_D$ and let $j \in \{1, \dots, d\}$ such that $n_j(x_0) \neq 0$, w.l.o.g. $n_j(x_0) > 0$.

Since $n: \partial\Omega \to \mathbb{R}^d$ is continuous, there exists $R > 0$ such that $B_R(x_0) \subset U$ and $n_j(x) > 0$ for all $x \in B_R(x_0) \cap \partial\Omega = B_R(x_0) \cap \Gamma_D$.

Now let $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\psi \geq 0$, $\operatorname{supp} \psi \subset B_R(x_0)$ and $\psi(x) = 1$ for $x \in B_r(x_0)$ for some $r \in (0, R)$. Then $\varphi_n(x)\psi(x) = 0$ for all $x \in \partial\Omega$, so that

$$0 = \int_{\partial\Omega} \varphi_n \psi n_j \, da = \int_\Omega \partial_j(\varphi_n \psi) \, dx = \int_\Omega \partial_j \varphi_n \psi + \varphi_n \partial_j \psi \, dx$$

Passing to the limit $n \to \infty$, we obtain

$$0 = \int_\Omega \partial_j u \psi + u \partial_j \psi \, dx = 0 + \int_\Omega c \, \partial_j \psi \, dx = c \int_{\partial\Omega} \psi n_j \, da \geq c \int_{B_r(x_0) \cap \Gamma_D} n_j \, da .$$

Since $\int_{B_r(x_0) \cap \Gamma_D} n_j \, da > 0$, we conclude $c = 0$.

b) Assume that the estimate is wrong. Then for every $n \in \mathbb{N}$ there exists $u_n \in W^{1,p}_{\Gamma_D}(\Omega)$ such that

$$\|u_n\|_{L^p(\Omega)} > n \|\nabla u_n\|_{L^p(\Omega)} .$$

After a renormalization, we may assume that $\|u_n\|_{L^p(\Omega)} = 1$, so that

$$1 > n \|\nabla u_n\|_{L^p(\Omega)}$$

for all $n \in \mathbb{N}$. In particular, the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$. By reflexivity of $W^{1,p}(\Omega)$, there exists a weakly convergent subsequence $(u_{n_k})_{k \in \mathbb{N}}$ with weak limit $u \in W^{1,p}_{\Gamma_D}(\Omega)$. Rellich's compactness theorem yields strong convergence $u_{n_k} \to u$ in $L^p(\Omega)$. Since $\|u_n\|_{L^p(\Omega)} = 1$, we also have $\|u\|_{L^p(\Omega)} = 1$.

Moreover, from weak convergence $\nabla u_{n_k} \rightharpoonup \nabla u$ in $L^p(\Omega)$ we conclude

$$\|\nabla u\|_{L^p(\Omega)} \leq \liminf_{k\to\infty} \|\nabla u_{n_k}\|_{L^p(\Omega)} \leq \lim_{k\to\infty} \frac{1}{n_k} = 0,$$

which shows $\nabla u = 0$ and thus $u \equiv c$ a.e. for some $c \in \mathbb{R}$ by Exercise 23. As we show below, $u \in W^{1,p}_{\Gamma_D}(\Omega)$ now implies $c = 0$, i.e., $u = 0$ a.e. But this contradicts $\|u\|_{L^p(\Omega)} = 1$. We thus conclude the asserted inequality.