Exercise 20

$C_p = \{u \in L^p(\Omega;\mathbb{R}^m) \mid u(x) \in C \text{ for a.a. } x \in \Omega\}$

a) Let $C \subset \mathbb{R}^m$ be convex. Then $C_p \subset L^p(\Omega;\mathbb{R}^m)$ is convex. By Mazur's lemma, it suffices to show that C_p is strongly closed.
To show this, let $(u_n) \subset C_p$ be a sequence that converges (strongly) to $u \in L^p(\Omega;\mathbb{R}^m)$. Then there is a subsequence (u_{n_k}) such that $u_{n_k}(x) \to u(x)$ for a.a. $x \in \Omega$ as $k \to \infty$. Since $u_{n_k}(x) \in C$ for a.a. $x \in \Omega$ and all $k \in \mathbb{N}$ and since C is closed, this implies $u(x) \in C$ for a.a. $x \in \Omega$, so that $u \in C_p$. Therefore, C_p is closed.

b) Let C_p be weakly closed. Consider $a, b \in C$ and $\lambda \in (0,1)$.
To show that $(1-\lambda)a + \lambda b \in C$, we construct a suitable rapidly oscillating sequence (cf. Exercise 13 and Exercise 16).
First define a function $g: [0,1] \to [0,1]$, $g(t) = \begin{cases} 1, & t \in [0,\lambda), \\ 0, & t \in [\lambda, 1]. \end{cases}$
Extend g to $\mathbb{R}$ periodically $(g(t+1) = g(t))$ and define

$$g_n: \mathbb{R} \to [0,1], \quad g_n(t) := g(nt).$$

Then $g_n \rightharpoonup g_{av} := \int_0^1 g(t)\,dt = \lambda$.
Let $D := (t_0,t_1) \times \tilde{D} \subset \Omega$ with $\tilde{D} \subset \mathbb{R}^{n-1}$ open, and set

$$u_n(x) = a + \chi_D(x)\, g_n(x_1)\,(b-a).$$

Then $u_n \in C_p$ since $u_n(x) \in \{a,b\}$ and Ω is bounded.
Moreover, for any test function $v \in L^{p'}(\Omega)$ we have

$$\int_\Omega u_n(x)\cdot v(x)\,dx = \int_\Omega a\, v(x)\,dx + \int_{\tilde{D}} \underbrace{\int_{t_0}^{t_1} g_n(x_1)\, v(x_1,x')\,dx_1}_{\to\, g_{av} \int_{t_0}^{t_1} v(x_1,x')\,dx_1} (b-a)\,dx'$$

$$\xrightarrow{n\to\infty} \int_\Omega a\, v(x)\,dx + \int_D \lambda(b-a)\, v(x)\,dx$$

Hence $u_n \rightharpoonup u$, where $u(x) = \begin{cases} a, & x \in \Omega \setminus D, \\ a + \lambda(b-a), & x \in D. \end{cases}$
Since C_p is weakly closed, we have $u \in C_p$, so that $a + \lambda(b-a) \in C$.

Exercise 21

Let $u \in L^p(\Omega)$ be fixed.

At first, consider $g : \Omega \to \mathbb{R}^n$ such that $g(x) \in \partial f(u(x))$ for a.a. $x \in \Omega$.

For $v \in L^p(\Omega; \mathbb{R}^n)$, we then have

$$I(v) - I(u) = \int_\Omega f(v(x)) - f(u(x))\,dx \geq \int_\Omega g(x)\cdot(v(x) - u(x))\,dx$$

Identifying $(L^p(\Omega;\mathbb{R}^n))'$ with $L^{p'}(\Omega;\mathbb{R}^n)$ for $\frac{1}{p} + \frac{1}{p'} = 1$, we thus obtain

$$\{ g \in L^{p'}(\Omega;\mathbb{R}^n) \mid g(x) \in \partial f(u(x)) \text{ for a.a. } x \in \Omega \} \subset \partial I(u).$$

To prove the converse inclusion, use the above isomorphism to identify elements of $\partial I(u)$ with functions $g \in L^{p'}(\Omega;\mathbb{R}^n)$.

Then $g \in \partial I(u)$ if

$$\forall v \in L^p(\Omega;\mathbb{R}^n): \quad I(v) \geq I(u) + \int_\Omega g(x)\cdot(v(x) - u(x))\,dx \quad .$$

Now fix $x_0 \in \Omega$ and let

$$v(x) := \begin{cases} u(x), & x \in \Omega \setminus B_R(x_0), \\ b, & x \in B_R(x_0), \end{cases}$$

for $R > 0$ sufficiently small and $b \in \mathbb{R}^m$ arbitrary. Then

$$\int_{B_R(x_0)} [f(b) - f(u(x))]\,dx \geq \int_{B_R(x_0)} g(x)\cdot(b - u(x))\,dx$$

$$(\Rightarrow) \quad f(b) - \fint_{B_R(x_0)} f(u(x))\,dx \geq \left[\fint_{B_R(x_0)} g(x)\,dx\right]\cdot b - \fint_{B_R(x_0)} g(x)\cdot u(x)\,dx,$$

where $\displaystyle\fint_{B_R(x_0)} h(x)\,dx := \frac{1}{\mathrm{vol}(B_R(x_0))}\int h(x)\,dx$.

Passing to the limit $R \to 0$, and using Lebesgue's differentiation theorem, we obtain

$$f(b) \geq f(u(x_0)) + g(x_0) \cdot [b - u(x_0)] \qquad \text{for a.a. } x_0 \in \Omega.$$

Note: The set where the integrals may not converge only depends on $f \circ u$, g and $g \circ u$, but it is independent of b!

Since this holds for all $b \in \mathbb{R}^m$, we conclude $g(x_0) \in \partial f(u(x_0))$ for a.a. $x_0 \in \Omega$. This yields the converse inclusion, and we obtain

$$\partial I(u) = \{ g \in L^{p'}(\Omega; \mathbb{R}^m) : g(x) \in \partial f(x) \quad \text{for a.a. } x \in \Omega \}.$$

More direct (but wrong!) argument avoiding Lebesgue's differentiation theorem:

As above, we conclude for any open set $D \subset \Omega$ the inequality

$$\int_D [f(b) - f(u(x))] \, dx \geq \int_D g(x) \cdot (b - u(x)) \, dx$$

for all $b \in \mathbb{R}^m$. Since D is arbitrary, this yields the pointwise inequality almost everywhere, i.e.,

$$\forall b \in \mathbb{R}^m: \text{ for a.a. } x \in \Omega: \quad f(b) - f(u(x)) \geq g(x) \cdot (b - u(x))$$

But to conclude $g(x) \in \partial f(u(x))$ for a.a. $x \in \Omega$, we need

$$\text{for a.a. } x \in \Omega: \ \forall b \in \mathbb{R}^m: \quad f(b) - f(u(x)) \geq g(x) \cdot (b - u(x)).$$

This is not equivalent!

The statement only follows if we can choose the exceptional set (i.e., those $x \in \Omega$, where the inequality may not hold) independently of $b \in \mathbb{R}^m$, which is why we argue via Lebesgue's differentiation theorem.