Exercise 11

$$f(x,u,A) = x^2 A^2 - x^4 u^2 + x^2$$

$$\partial_A f = 2x^2 A \quad , \qquad \partial_u f = -2x^4 u$$

$\Rightarrow$ Euler-Lagrange equations: $\quad -\left[2x^2 u'\right]' + 2x^4 u = 0$

$$\partial_A f(x,u,u')\, \nu(x) = 2x^2 u'(x)\, \nu(x) \underset{\partial\Omega = \{\pm 1\}}{=} 2(\pm 1)^2 u'(\pm 1) \underbrace{\nu(\pm 1)}_{= \pm 1} = 2u'(\pm 1)(\pm 1)$$

$\Rightarrow$ Neumann boundary conditions: $\quad u'(1) = u'(-1) = 0.$

Legendre-Hadamard in 1D $\Leftrightarrow$ $\partial_A^2 f(x,u,A) = 2x^2$ positive semi-definite

$$\Leftrightarrow \quad 2x^2 \geq 0.$$

Weierstraß condition: $f(x,u,A+\xi) - f(x,u,A) - \partial_A f(x,u,A)\xi$

$$= x^2(A+\xi)^2 - x^2 A^2 - 2x^2 A \xi = x^2 \xi^2 \geq 0 \quad \forall \xi \in \mathbb{R}.$$

Both are trivially satisfied.

Exercise 13

a) $$\int_{\mathbb{R}^d} p_\varepsilon(x)\phi(x)\,dx = \int_{\mathbb{R}^d} p\left(\tfrac{x}{\varepsilon}\right)\phi(x)\,dx \underset{y=\frac{x}{\varepsilon}}{=} \varepsilon^d \int_{\mathbb{R}^d} p(y)\,\phi(\varepsilon y)\,dy = \varepsilon^d \sum_{k\in\mathbb{Z}^d}\int_{k+[0,1]^d} p(y)\,\phi(\varepsilon y)\,dy$$

$$= \varepsilon^d \sum_{k\in\mathbb{Z}^d}\int_{[0,1]^d} p(y-k)\,\phi(\varepsilon(y-k))\,dy \overset{p \text{ periodic}}{=} \sum_{k\in\mathbb{Z}^d}\int_{[0,1]^d} p(y)\varepsilon^d\phi(\varepsilon(y-k))\,dy$$

Since ϕ has compact support, only finitely many of these integrals do not vanish, so that, this is actually a finite sum. Therefore,

$$\int_{\mathbb{R}^d} p_\varepsilon(x)\phi(x)\,dx = \int_{[0,1]^d} p(y) \underbrace{\sum_{k\in\mathbb{Z}^d}\varepsilon^d\phi(\varepsilon(y-k))}_{=:R_{\varepsilon,y}\phi}\,dy\,.$$

For any $y\in[0,1]^d$, the term $R_{\varepsilon,y}\phi$ is a Riemann sum for $\phi\in C_c^\infty(\mathbb{R}^d)$, so that

$$\lim_{\varepsilon\to 0} R_{\varepsilon,y}\phi = \int_{\mathbb{R}^d}\phi(x)\,dx.$$

To apply this in the previous integral, we want to use dominated convergence. To do so, let $M\in\mathbb{N}$ such that $\operatorname{supp}\phi\subset[-M,M]^d$. Then

$$R_{\varepsilon,y}\phi = \sum_{k\in\mathbb{Z}^d}\varepsilon^d\phi(\varepsilon(y-k)) \le \varepsilon^d\|\phi\|_{L^\infty}\sum_{k\in\mathbb{Z}^d}\mathbb{1}_{[-M,M]^d}(\varepsilon(y-k))$$
$$= \varepsilon^d\|\phi\|_{L^\infty}\sum_{k\in\mathbb{Z}^d}\mathbb{1}_{[-\frac{M}{\varepsilon},\frac{M}{\varepsilon}]^d}(y-k)$$
$$\le \varepsilon^d\|\phi\|_{L^\infty}\left(\frac{2M}{\varepsilon}+1\right)^d \le (2M+1)\|\phi\|_{L^\infty}$$

for $\varepsilon\le 1$ since

$$\mathbb{1}_{[-L,L]^d}(y-k)=1 \iff y-k\in[-L,L]^d \iff \forall j: k_j\in[y-L,y+L],$$

and $\#(\mathbb{Z}\cap[y-L,y+L])\le 2L+1$.

Therefore, the integrable function $y\mapsto p(y)(2M+1)\|\phi\|_{L^\infty}$ dominates the integrand, and by Lebesgue's theorem we conclude

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} p_\varepsilon(x)\phi(x)\,dx = \int_{[0,1]^d}\lim_{\varepsilon\to 0} p(y)\sum_{k\in\mathbb{Z}^d}\varepsilon^d\phi(\varepsilon(y-k))\,dy$$
$$= \int_{[0,1]^d} p(y)\int_{\mathbb{R}^d}\phi(x)\,dx\,dy = p_{av}\int_{\mathbb{R}^d}\phi(x)\,dx.$$

b) Now let $f \in L^1(\Omega)$. For $\phi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\int \rho_\varepsilon f \, dx - \int \rho_{av} f \, dx = \int \rho_\varepsilon (f - \phi) + \int (\rho_\varepsilon - \rho_{av}) \phi + \int \rho_{av} (\phi - f).$$

Let $\delta > 0$.

Since $C_c^\infty(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$, there exists $\phi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\|\phi - f\|_{L^1(\mathbb{R}^d)} < \frac{\delta}{3 \|\rho\|_{L^\infty}}.$$

By (a), we can choose $\varepsilon > 0$ so small that

$$\left| \int_{\mathbb{R}^d} \rho_\varepsilon \phi \, dx - \int \rho_{av} \phi \, dx \right| < \frac{\delta}{3}.$$

We obtain

$$\left| \int \rho_\varepsilon f \, dx - \int \rho_{av} f \, dx \right| \leq \|\rho_\varepsilon\|_{L^\infty} \frac{\delta}{3 \|\rho\|_{L^\infty}} + \frac{\delta}{3} + |\rho_{av}| \frac{\delta}{3 \|\rho\|_{L^\infty}}$$

$$\leq \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.$$

This shows

$$\int \rho f \, dx \to \int \rho_{av} f \, dx \quad \text{for all } f \in L^1(\mathbb{R}^d).$$