$\boxed{15}$ We consider $\Omega =]\alpha, \beta[$ and

$f \in C^1([\alpha,\beta] \times \mathbb{R}^m \times \mathbb{R}^m)$

on $C^1([\alpha,\beta]; \mathbb{R}^m)$, we consider the functional

$$I(u) = \int_\alpha^\beta f(x, u(x), u'(x))\,dx$$

We want to show the implication

$$u_* \text{ strong local minimizer}$$
$$\Downarrow$$
$$E(x, u_*(x), u_*'(x), B) \geq 0 \quad \forall B \in \mathbb{R}^m \quad (*)$$

where

$$E(x,u,A,B) = f(x,u,A+B) - f(x,u,A) - \partial_A f(x,u,A)\cdot B$$

From the lecture we know that quasi-convexity of $\tilde{f}(A) = f(x, u_*(x), A)$ in $A = u_*'(x)$

is necessary for strong local minimizer.

Thus, if we can show that (*) is equivalent to quasiconvexity, we have proven the claim.

Note: In higher dimensions, this equivalence is wrong! It holds only for $\Omega \subset \mathbb{R}^1$!

We prove:

$$\boxed{A \mapsto \tilde{f}(A) \text{ quasiconvex in } A_0} \iff \boxed{\tilde{E}(A_0,B) \geq 0 \;\; \forall B \in \mathbb{R}^m}$$

with $\tilde{E}(A,B) = \tilde{f}(A+B) - \tilde{f}(A) - \partial_A \tilde{f}(A)\cdot B$

First prove $\square \Leftarrow \square$, i.e. $\tilde{E}(A_0,B) \geq 0 \; \forall B$ holds. Pick $v \in C_0^1([-1,1];\mathbb{R}^m)$, then

$$\int_{-1}^{+1} \tilde{f}(A_0+v'(y)) - \tilde{f}(A_0)\, dy = \int_{-1}^{+1} \overbrace{\tilde{E}(A_0,v'(y))}^{\geq 0} dy + \int_{-1}^{+1} \partial_A \tilde{f}(A_0)\cdot v'(y)\, dy$$

$$\overset{\square}{\geq} \; 0 + \int_{-1}^{+1} \partial_A \tilde{f}(A_0) \cdot v'(y)\, dy$$

$$\overset{\text{(why?)}}{=} \; \partial_A \tilde{f}(A_0) \cdot \underbrace{\int_{-1}^{+1} v'(y)\, dy}_{=0 \;\; \text{since } v(-1) = v(+1) = 0} = 0$$

$$\Rightarrow \; \int_{-1}^{+1} \tilde{f}(A_0 + v'(y))\, dy \geq 2\, \tilde{f}(A_0)$$

$$\forall\, v \in C_0^1([-1,1];\mathbb{R}^m)$$

i.e. $\tilde{f}$ is quasiconvex in A_0

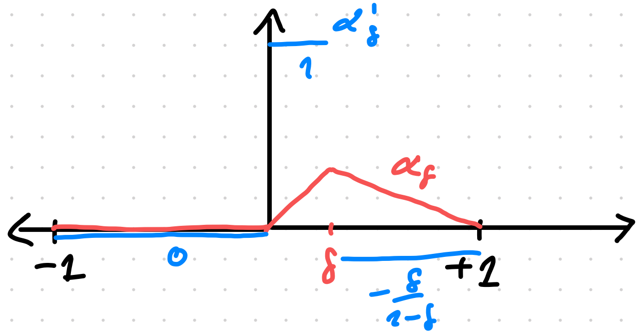
Now, we show $\square \Rightarrow \square$

$\delta > 0$

Let $\tilde{v}_\delta \in PC_0^1([-1,1];\mathbb{R}^m)$ defined via

$$\tilde{v}_\delta(y) = \alpha_\delta(y)\, B \quad \text{with } B \in \mathbb{R}^m \text{ and}$$

$$\alpha_\delta(y) = \begin{cases} 0 & y \in [-1,0) \\ y & y \in [0,\delta) \\ \frac{\delta}{1-\delta}(1-y) & y \in [\delta, 1] \end{cases}$$



Since $\tilde{f}$ is quasiconvex in A_0 we have

$$0 \le \int_{-1}^{+1} \frac{1}{\delta}\left[\tilde{f}(A_0+\alpha_\delta' B) - \tilde{f}(A_0)\right] dy \underbrace{- \frac{1}{\delta}\int_{-1}^{+1} \partial_A \tilde{f}(A_0)\cdot v_\delta'(y)\,dy}_{=0}$$

due to boundary conditions

$$= \frac{1}{\delta}\left(\int_{-1}^{0} 0\,dy + \int_{0}^{\delta}\left[\tilde{f}(A_0+1\cdot B) - \tilde{f}(A_0) - \partial_A \tilde{f}(A_0)B\right] dy \right.$$
$$\left. + \int_{\delta}^{1}\left[\tilde{f}\left(A_0 - \frac{\delta}{1-\delta}B\right) - \tilde{f}(A_0) + \partial_A \tilde{f}(A_0)\frac{\delta}{1-\delta}B\right] dy\right)$$

$$= \frac{1}{\delta}\left(0 + \delta \tilde{E}(A_0, B) + (1-\delta)\left(\tilde{f}\left(A_0 - \frac{\delta}{1-\delta}B\right) - \tilde{f}(A_0) + \partial_A \tilde{f}(A_0)\frac{\delta}{1-\delta}B\right)\right.$$

Note that for $h = \frac{\delta}{1-\delta} \xrightarrow{\delta \to 0} 0$ we get

$$\frac{1}{h}\left(\tilde{f}(A_0 - hB) - \tilde{f}(A_0)\right) + \partial_A \tilde{f}(A_0)B \longrightarrow 0$$

Thus, we conclude that $\tilde{E}(A_0, B) \geq 0$.

$\square$

$\boxed{16}$ Let $f \in C^0(\bar{\Omega} \times \mathbb{R}^m)$ with

$\exists\, C > 0,\ p \in [1, \infty[,\ h \in L^1(\Omega)\ \forall (x,u) \in \Omega \times \mathbb{R}^m$

$$|f(x,u)| \leq C(h(x) + |u|^p)$$

$$I : \begin{cases} L^p(\Omega;\mathbb{R}^m) & \longrightarrow \quad \mathbb{R} \\ u & \longmapsto \quad \int_\Omega f(x,u(x))\,dx \end{cases}$$

Clearly, for $u \in L^p(\Omega;\mathbb{R}^m)$ we have

$$I(u) \leq C\left(\int_\Omega h(x)\,dx + \|u\|_{L^p}^p\right) < \infty$$

Thus I is finite on $L^p(\Omega;\mathbb{R}^m)$

a) Let $u_n \in L^p(\Omega;\mathbb{R}^m)$ be such that $u_n \to u$ in $L^p(\Omega;\mathbb{R}^m)$.

We can extract subsequence $n_k \xrightarrow{k\to\infty} \infty$ such that $|u_{n_k}(x)| \leq g$ for a $g \in L^p(\Omega)$ and $u_{n_k}(x) \to u(x)$ for almost every $x \in \Omega$.

Since f is continuous, we get

$\tilde{f}_k(x) := f(x, u_{n_k}(x)) \longrightarrow f(x, u(x))$ for a.e. x

$\tilde{f}_k$ has majorant $C(h + |g|^p)$ (why?)

By Lebesgue's dominated convergence theorem we conclude $I(u_{n_k}) = \int_\Omega \tilde{f}_k \, dx \to I(u)$

Limit $I(u)$ does not depend on subsequence, this whole sequence u_n satisfies

$I(u_n) \to I(u)$

$\square$

b) If $g \in L^\infty(\mathbb{R})$ periodic with period 1, i.e. $g(t+1) = g(t)$ $\forall t \in \mathbb{R}$.

we get for $g_k(t) = g(kt)$ for $k \in \mathbb{N}$, from

Exercise 13: $g_k \xrightarrow{*} g_{av} := \int_0^1 g(s)\,ds$.

Fix $\theta \in [0,1]$ and consider

$g(t) = 0$ for $t \in [0, \theta[$ and

$g(t) = 1$ for $t \in [\theta, 1[$

periodically extended to $\mathbb{R}$.

Clearly $g_{av} = 1 - \theta$

Fix $w_0, w_1 \in \mathbb{R}^n$ and arbitrary set $B \subset \Omega$
(measurable!)

and define

$$u_k(x) = \chi_B(x)\left(w_1 + g_k(x_1)\cdot(w_0 - w_1)\right)$$

i.e. in B u_k oscillates between w_1 and w_0 (how does θ come into play?)

Moreover, we get

$$u_k \xrightarrow{*} \chi_B\Big(\underbrace{w_1 + g_{av}(w_0 - w_1)}^{= w_1 + (1-\theta)(w_0 - w_1) =: w_\theta}\Big) \text{ in } L^\infty$$

Assuming that I is weakly continuous

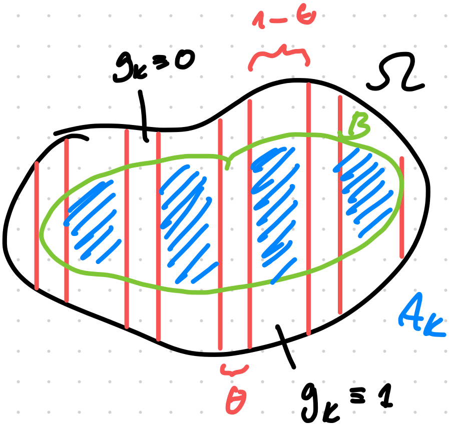
$$I(u_k) \to I(u) = \int_B f(x, w_\theta)\,dx + \int_{\Omega \setminus B} f(x,0)\,dx$$

On the other hand defining

$$A_k := \{ x \in \Omega \mid u_k(x) = w_0 \}$$

we see that $\chi_{A_k} = \chi_B \cdot g_k(x_1) \rightharpoonup (1-\theta)\chi_B$

why



we have

$$I(u_k) = \int_B \Big(f(x, w_0) \underbrace{\chi_{A_k}}_{\rightharpoonup (1-\theta)} + f(x, w_1) \underbrace{(1 - \chi_{A_k})}_{\rightharpoonup \theta} \Big) dx$$

$$+ \int_{\Omega \setminus B} f(x, 0)\, dx$$

$$\to \int_B f(x, w_0)(1-\theta) + f(x, w_1)\theta \, dx$$

$$+ \int_{\Omega \setminus B} f(x, 0)\, dx$$

Thus, for an arbitrary subset $B \subset \Omega$ we have

$$\int_B f(x, (1-\theta) w_0 + \theta w_1)\, dx = \int_B f(x, w_0)(1-\theta) + f(x, w_1)\theta \, dx$$

$\Rightarrow$ we have almost everywhere

$$f(x, (1-\theta) w_0 + \theta w_1) = (1-\theta) f(x, w_0) + \theta f(x, w_1)$$

$\Rightarrow$ f is affine in u

$\Rightarrow$ $f(x, u) = b(x) \cdot u + a(x)$

$\square$