SMM 641
Revenue Management and Pricing

Week 8:
Price Optimization with Consumer Choice
Peak Period Pricing

Price Differentiation with Self Selection

- Customers choose from available product versions according to
  - their willingness to pay for the different versions of the product
  - the prices of the different versions of the product
- The goal of the seller is to set the prices to
  - encourage the buyers with higher willingness to pay to choose the higher priced versions
  - capturing the buyers with lower willingness to pay through less expensive versions.

Price Differentiation with Self Selection

- A commonly used notion for modeling such consumer choice is the consumer surplus, which is defined as follows:

Surplus = Customer's willingness to pay for a product
- Product's price

- The customer is then assumed to pick whichever choice gives her the maximum surplus.

Price Differentiation with Self Selection Example

- Assume that a firm has two versions of the same product and has two market segments of equal size with different WTP. Below data shows each segment's WTP for these two versions:

	Low quality product	High quality product
Segment 1	$30	$100
Segment 2	$20	$50

- If firm offers only the low quality product, what price should it set to maximize its revenue?
  - $20; both segments would buy because their willingness-to-pay is greater than or equal to $20, resulting in a revenue of $40.
  - (If pick $30, only Segment 1 buys, generating a revenue of $30.)
  - Segment 1 would be happy because they will be left with (30-20=$10) surplus.

Price Differentiation with Self Selection Example

- Assume that a firm has two versions of the same product and has two market segments of equal size with different WTP. Below data shows each segment's WTP for these two versions:

	Low quality product	High quality product
Segment I	$30	$100
Segment 2	$20	$50

- If firm offers only the high quality product, what price should it set to maximize its revenue?
  - $50 or $100; at $50, both segments would buy because their willingness-to-pay is greater than or equal to $50, resulting in a revenue of $100. (If pick $100, only Segment 1 buys, again generating a revenue of $100.)
  - At price $50, Segment 1 would be happy because they will be left with (100-50=$50) surplus.

Price Differentiation with Self Selection Example

- Assume that a firm has two versions of the same product and has two market segments of equal size with different WTP. Below data shows each segment's WTP for these two versions:

	Low quality product	High quality product
Segment I	$30	$100
Segment 2	$20	$50

- If the firm wants to introduce *both products*, but target Segment 1 with high-quality and segment 2 with low quality product, then at what price those products should be offered?
- In this case, both products will be in the market and customers in both segments will self-select based on which option gives them the maximum surplus. (Note that we assume firm cannot verify which segment a customer belongs.)

Price Differentiation with Self Selection Example

- Assume that a firm has two versions of the same product and has two market segments of equal size with different WTP. Below data shows each segment's WTP for these two versions:

	Low quality product	High quality product
Segment I	$30	$100
Segment 2	$20	$50

- Segment 2 can be targeted with low quality product only if the low quality product is sold at $20.
- High-Segment would have a surplus of $10 (30-20) if they buy this low quality product. This means, we need give the high-segment back this $10 surplus if we want to make them buy the high-quality product.
- Hence the price of the high-quality product should be $100-$10=$90.
- So, when prices are set as ($20,$90), Segment 2 will buy the low quality product at $20, Segment 1will buy the high-quality product at $90.

Price Differentiation through Bundling

- A bundle is a collection of products or services that are sold together as a package.
  - Pure Bundling: Products are available only in a bundle and not individually.
  - Mixed bundle: Products are available individually as well as in a bundle.
- Bundling can be used for price differentiation

Price Differentiation through Bundling

- Consider the WTP of four buyers for two products.

	Coldplay Concert	Ariana Grande Concert
A	$80	$40
B	$40	$80
C	$90	$10
D	$10	$90

- Three ways the tickets can be sold
  - Price and sell each ticket as an individual item
  - Price and sell the tickets only as a bundle of two (pure bundling)
  - Price and sell the tickets individually and as a bundle (mixed bundling)
- Which option generates the highest revenue?

Price Differentiation through Bundling

- Consider the WTP of four buyers for two products.

	Coldplay Concert	Ariana Grande Concert
A	$80	$40
B	$40	$80
C	$90	$10
D	$10	$90

- Pricing and selling each ticket as an individual item:
  - Consider the Coldplay concert ticket: If priced at
    $10, demand is 4, revenue is $40.
    $40, demand is 3, revenue is $120.
    $80, demand is 2, revenue is $160.
    $90, demand is 1, revenue is $90.
  - Price at $80. Similarly, price Ariana Grande Concert also at $80.
  - Total revenue is $320.

Price Differentiation through Bundling

- Consider the WTP of four buyers for two products.

	Coldplay Concert	Ariana Grande Concert	Bundle
A	$80	$40	$120
B	$40	$80	$120
C	$90	$10	$100
D	$10	$90	$100

- Pricing and selling each ticket as a pure bundle:
  - Assume the WTP for the bundle is the sum of the WTP for its components. If price bundle at
    $100, demand is 4, revenue is $400.
    $120, demand is 2, revenue is $240.
  - Price bundle at $100.
  - Total revenue is $400, higher than selling individually.

Price Differentiation through Bundling

- Consider the WTP of four buyers for two products.

	Coldplay Concert	Ariana Grande Concert	Bundle
A	$80	$40	$120
B	$40	$80	$120
C	$90	$10	$100
D	$10	$90	$100

- Pricing and selling each ticket as a mixed bundle:
  - Consider the following prices, bundle is priced at $120 and individual concerts are priced at $90.
    A and B purchase the bundle
    C and D purchase individual concerts. (C: Coldplay, D: Ariana Grande)
  - Total revenue is $240 + $180 = $420 , higher than either selling individually or a pure bundle.

Parametric vs Non-parametric estimates

- Parametric estimate
  - Assume a demand model which is defined by a modest number of parameters
  - Estimate the parameters from data
  - Example:
    - A demand function is linear with intercept a and slope b (d(p) = a - bp)
    - Demand is Normally distributed with mean m, standard deviations s
  - Advantages: Concise description; can extrapolate beyond observed history; may help smooth noisy data
  - Disadvantages: Makes assumptions on form of response that may not be valid
- Nonparametric estimate
  - Use the raw data directly to estimate demand without making assumptions on the functional form of the relationships
  - Examples:
    - Empirical histogram of demand volume
    - Empirical histogram of willingness to pays
  - Advantages: No assumptions; uses values actually observed
  - Disadvantages: Subject to "noise" in data; cannot extrapolate beyond observed history

Example: New York Health Club Part A

- Please see the case document, data file `NYHCSurvey.csv` and the R Supplement.

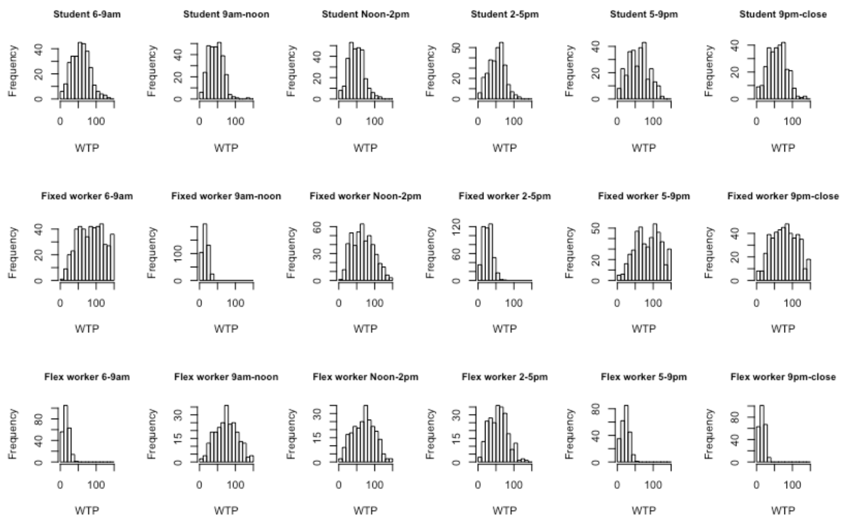
```
# Read the Survey Data
nyhc = read.csv("NYHCSurvey.csv",header=T)
# Row count
N=nrow(nyhc)
# Print top few rows of data
head(nyhc)
```

Example: New York Health Club Part A

- Please see the case document, data file `NYHCSurvey.csv` and the R Supplement.
- Snippet of WTP survey data of clients across time slots:

	Client	Type	TimeSlot1	TimeSlot2	TimeSlot3	TimeSlot4	TimeSlot5	TimeSlot6
1	1	1	18	50	41	76	69	41
2	2	1	66	14	86	62	71	46
3	3	1	60	43	43	26	91	58
4	4	1	59	48	62	15	91	15
5	5	1	21	24	59	45	81	131
6	6	1	66	69	29	80	37	79
7	7	1	126	41	61	97	111	56
8	8	1	72	71	40	11	78	116
9	9	1	49	64	54	26	49	107
10	10	1	17	21	41	54	76	27

Histograms for Willingness to Pay across different segments



Example: New York Health Club Part A.1

- Suppose that NYHC offered a flat rate for personal training for all members and all time periods.
- What price would you recommend?

New York Health Club Part A.1 – Single Price

- First compute the maximum willingness to pay for each client across all the time slots:

```
# Compute maximum willingness to pay for each client across time slots and
enter it as a new column
for (i in 1:N){
    nyhc$maxWTP[i]=max(nyhc[i,3:8])
}
# The above can also be achieved without a for loop by typing
# nyhc$maxWTP=apply(nyhc[,3:8],1,max)

#Displaying the first ten rows of data including the maxWTP for each client
nyhc[1:10,]
```

New York Health Club Part A.1 – Single Price

- First compute the maximum willingness to pay for each client across all the time slots:

	Client	Type	TimeSlot1	TimeSlot2	TimeSlot3	TimeSlot4	TimeSlot5	TimeSlot6	maxWTP
1	1	1	18	50	41	76	69	41	76
2	2	1	66	14	86	62	71	46	86
3	3	1	60	43	43	26	91	58	91
4	4	1	59	48	62	15	91	15	91
5	5	1	21	24	59	45	81	131	131
6	6	1	66	69	29	80	37	79	80
7	7	1	126	41	61	97	111	56	126
8	8	1	72	71	40	11	78	116	116
9	9	1	49	64	54	26	49	107	107
10	10	1	17	21	41	54	76	27	76

New York Health Club Part A.1 – Single Price

- Then, for each price point (i.e., from 1 to the maximum of all WTPs), determine how many clients would purchase a session.

```
# The maximium WTP in data, we can use this as the upper bound for our price
search.
# No need to consider a price if noone can afford it.
maxprice=max(nyhc$maxWTP)

# Defining empty array variables we will be introducing
demand=rep(NA,maxprice)
revenue=rep(NA,maxprice)
```

New York Health Club Part A.1 – Single Price

- Then, for each price point (i.e., from 1 to the maximum of all WTPs), determine how many clients would purchase a session.

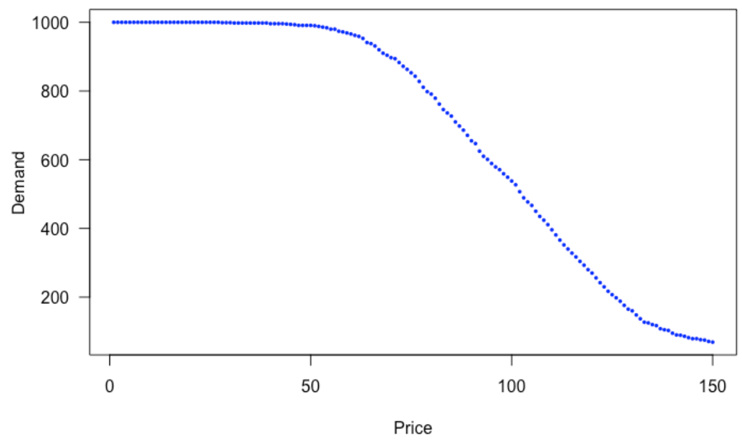
```
# Find how many people buy at each price level
for (p in 1:maxprice){
    # For any price level, say for p=75,
    # nyhc$maxWTP>=75 assigns a true/false value for each element in nyhc max WTP column
    # (note there are N=1000 elements (rows), one for each client)
    # the value is TRUE if for that particular client has a maxWTP greater than or equal to 75
    # That means the client will buy if their maxWTP is greater than or equal to.
    # The value is FALSE if client's maxWTP is less than 75.
    # The sum() function adds up all the values.  It treats TRUEs as 1, and FALSEs as 0.
    # So if for a price, say 75, there are 853 TRUEs, the demand at that price level is 853.
    demand[p]=sum(nyhc$maxWTP>=p)
    revenue[p]=p*demand[p]
}
```

```
# Identifying the Best Price
revenueBest=max(revenue)
priceBest=which(revenue == revenueBest)
print(paste("If a single price is to be charged across all time slots, the optimal price is:",p
riceBest))
```

```
"If a single price is to be charged across all time slots, the optimal price is: 76"
```

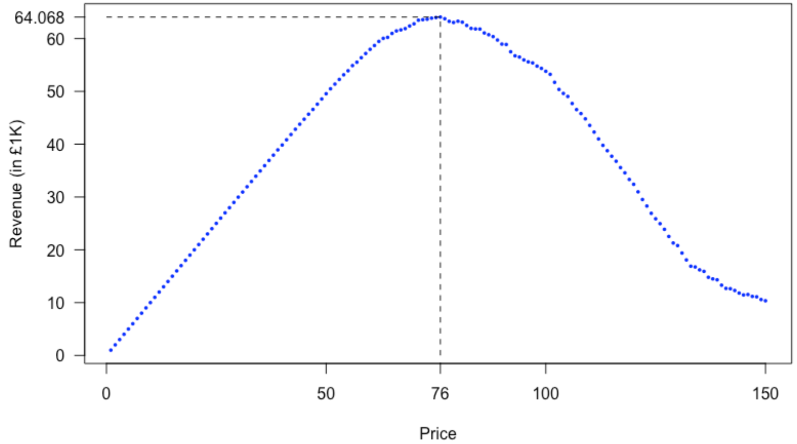
New York Health Club Part A.1 – Single Price

- Plotting Demand vs Price:



New York Health Club Part A.1 – Single Price

- Plotting Revenue vs Price:



New York Health Club Part A.2 (Peak Period Pricing)

- Suppose now that the gym's strategy is to offer personal training sessions at two price levels:
  - all times other than during TimeSlot5 (the 5–9 pm slot) will be priced at $65 per hour, while
  - sessions in TimeSlot5 will be priced at some other level, which is to be determined.
- Under that assumption, construct a linear demand model for the aggregate potential demand from all member segments for personal training sessions between 5pm and 9pm.
- Use this model to determine a peak period price for 5pm-9pm that will maximize the revenue from all time slots.

New York Health Club Part A.2 (Peak Period Pricing)

Tips:

- Members decide which product to buy by first calculating the net utility of a personal training session (i.e., their willingness-to-pay minus the actual cost of the session) at various time slots.
- They then select the option that maximizes net utility; if all options result in negative utility for a particular member, then that member does not buy any sessions at all.
- By fixing the price at $65 per hour for all time periods except 5pm – 9pm, you can compute the net utilities for all these products for each member who participated in the survey. You can then vary the price for the 5pm – 9pm slot and, for each possible choice, count the number of members who would choose a session in the 5pm – 9pm period.
- You should then try to approximate this demand function by a linear relation of the form d(p) = D – b*p, where d(p) is the number of members who purchase a session during the 5pm – 9pm slot at price *p*, D is the maximum potential demand when *p* = 0, and *b* is the price sensitivity parameter that explains how demand drops as price increases. The goal is to find D and b.
- Repeat the estimation procedure for nonpeak period demand as a function of the peak period price.

New York Health Club Part A.2 (Peak Period Pricing)

- Step 1: Max Surplus from NonPeak

```
# We will refer to all time slots except time slot 5 as NonPeak.
# Similarly timeslot 5 that is between 5-9pm is referred to as Peak

# Price for all time slots (i.e., nonpeak) except for time slot 5 (5-9pm) (peak)
basePrice=65
```

```
# For each client we will obtain their maximum WTP and
# maximum Surplus among the Nonpeak time slots
# Columns 3 to 6 and Column 8 are the NonPeak time slots.

maxWTPNonPeak<-rep(0,N)
maxsurplusNonPeak<-rep(0,N)

for (i in 1:N){
    maxWTPNonPeak[i]=max(nyhc[i,c(3:6,8)])
    maxsurplusNonPeak[i]=max(nyhc[i,c(3:6,8)]-basePrice)

    # We can also generate new column(s) and add this information to our daya nyhc:
    nyhc$maxWTPNonPeak[i]=max(nyhc[i,c(3:6,8)])
    nyhc$maxsurplusNonPeak[i]=max(nyhc[i,c(3:6,8)]-basePrice)
}
```

New York Health Club Part A.2 (Peak Period Pricing)

- Step 1: Max Surplus from NonPeak

	Client	Type	TimeSlot1	TimeSlot2	TimeSlot3	TimeSlot4	TimeSlot5	TimeSlot6	maxWTP	maxWTPNonPeak	maxsurplusNonPeak
1	1	1	18	50	41	76	69	41	76	76	11
2	2	1	66	14	86	62	71	46	86	86	21
3	3	1	60	43	43	26	91	58	91	60	-5
4	4	1	59	48	62	15	91	15	91	62	-3
5	5	1	21	24	59	45	81	131	131	131	66
6	6	1	66	69	29	80	37	79	80	80	15
7	7	1	126	41	61	97	111	56	126	126	61
8	8	1	72	71	40	11	78	116	116	116	51
9	9	1	49	64	54	26	49	107	107	107	42
10	10	1	17	21	41	54	76	27	76	54	-11

New York Health Club Part A.2 (Peak Period Pricing)

- Step 2: Surplus from Peak across various Peak Period Prices

```
# For each possible price point:
# and for all clients at the particular price point currently in consideration:

# Compare a client's surplus from NonPeak and Peak

# If a client's surplus from NonPeak is greater than their surplus for Peak
# and if the client's surplus from NonPeak is greater than 0,
# That client will purchase NonPeak.

# If a client's surplus from Peak is greater than their surplus for NonPeak
# and if the client's surplus from Peak is greater than 0,
# That client will purchase Peak.

# If both surpluses are less than 0, the customer will not buy.
```

New York Health Club Part A.2 (Peak Period Pricing)

- Step 2: Surplus from Peak across various Peak Period Prices

```
# Let's first compute clients' surpluses for Peak across all possible Peak Price choices
# There are 1000 clients and 150 possible price choices
# So we will create a matrix of dimension: 1000 rows (for each client) and 150 Columns
# where for example, the element [10,50] is Client 10's surplus from Peak if Peak Price is 50.

surplusPeak<-matrix(0,N,maxprice)

for (p in 1:maxprice){
    for (i in 1:N){
        surplusPeak[i,p]=nyhc[i,7]-p
    }
}
```

New York Health Club Part A.2 (Peak Period Pricing)

- Step 2: Surplus from Peak across various Peak Period Prices

	p=65	p=70	p=75	p=80	p=85	p=90	p=95	p=100
[1,]	4	-1	-6	-11	-16	-21	-26	-31
[2,]	6	1	-4	-9	-14	-19	-24	-29
[3,]	26	21	16	11	6	1	-4	-9
[4,]	26	21	16	11	6	1	-4	-9
[5,]	16	11	6	1	-4	-9	-14	-19
[6,]	-28	-33	-38	-43	-48	-53	-58	-63
[7,]	46	41	36	31	26	21	16	11
[8,]	13	8	3	-2	-7	-12	-17	-22
[9,]	-16	-21	-26	-31	-36	-41	-46	-51
[10,]	11	6	1	-4	-9	-14	-19	-24

New York Health Club Part A.2 (Peak Period Pricing)

- Step 3: Identify which Time Slot each Consumer will choose and aggregate total Peak and NonPeak Demand at each Peak Period Price level.

```
# Next, let's compare each client's surplus from NonPeak and Peak for each Peak price point p
# And for each of these price pints p's, we will count how many clients will buy NonPeak and
# how many clients will buy Peak.

# surplusPeak[,p] corresponds to the surpluses across all clients at a particular price p
# (maxsurplusNonPeak>surplusPeak[,p]) returns an array of TRUEs and FALSEs  each element
# corresponding to each Client.
# If for a Client, their maximum surplus from Non Peak exceeds their surplus from Peak
# Then they will prefer NonPeak, indicated by TRUE.  Otherwise for that client we have FALSE.
# The Client also needs that their maximum surplus from Non Peak is >= 0 to buy.
# If everything is negative, they cannot be forced to buy.
# The (maxsurplusNonPeak>=0) returns TRUE if their maximum surplus from Non Peak is >= 0
# and returns FALSE otherwise.
# The multiplication of these two logical variables is TRUE only if both are TRUE.
# The sum function adds ip all TRUE values (treating them as 1s)
# and hence gives total NonPeak demand at price p.

# The logic for demandPeak[p] is the same.
```

New York Health Club Part A.2 (Peak Period Pricing)

- Step 3: Identify which Time Slot each Consumer will choose and aggregate total Peak and NonPeak Demand at each Peak Period Price level.

```
for (p in 1:maxprice){
  demandNonPeak[p]=sum((maxsurplusNonPeak>surplusPeak[,p])*(maxsurplusNonPeak>=0))
  demandPeak[p]=sum((surplusPeak[,p]>=maxsurplusNonPeak)*(surplusPeak[,p]>=0))
  revenue[p]=basePrice*demandNonPeak[p]+p*demandPeak[p]
}
```

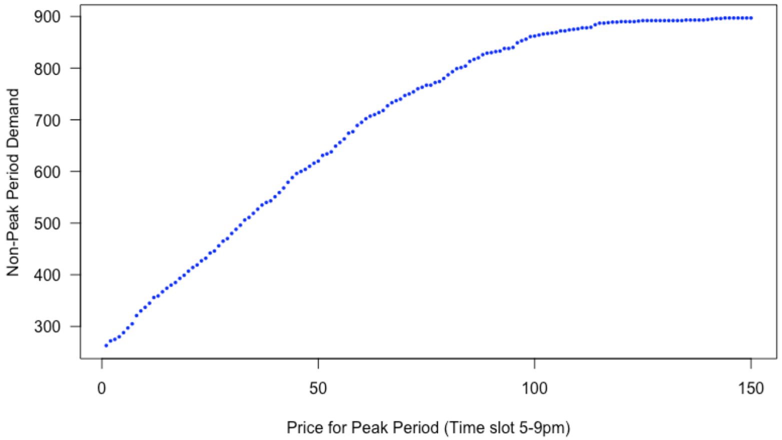
```
# Plotting NonPeak Demand vs Peak Period Price
xaxis=1:maxprice
plot(xaxis,demandNonPeak,pch = 16, type="s",col="blue", las=1, xaxt="n",
     xlab="Price for Peak Period (Time slot 5-9pm)",ylab="Non-Peak Period Demand")
xticks <- seq(0, maxprice, by=50)
axis(side = 1, at = xticks)
```

```
# Plotting Peak Demand vs Peak Period Price
xaxis=1:maxprice
plot(xaxis,demandPeak,pch = 16, type="s",col="blue", las=1, xaxt="n",
     xlab="Price for Peak Period (Time slot 5-9pm)",ylab="Peak Period Demand")
xticks <- seq(0, maxprice, by=50)
axis(side = 1, at = xticks)
```

New York Health Club Part A.2 (Peak Period Pricing)

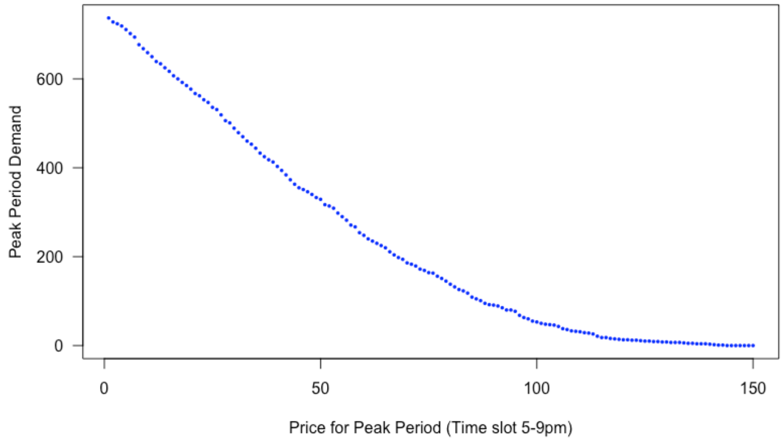
- Step 3: Identify which Time Slot each Consumer will choose and aggregate total Peak and NonPeak Demand at each Peak Period Price level.

Non-Peak Period Demand vs Peak Period Price (base p=65)



New York Health Club Part A.2 (Peak Period Pricing)

- Peak Period Demand vs Peak Period Price (base p=65)

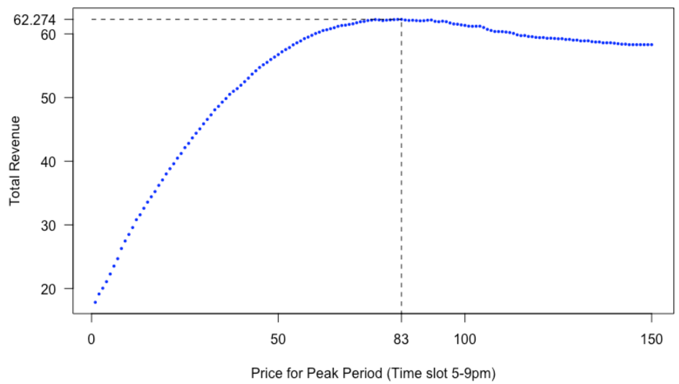


New York Health Club Part A.2 (Peak Period Pricing)

```
# Plotting Revenue vs Peak Period Price
xaxis=1:maxprice
plot(xaxis,revenue/1000,pch = 16, type="s",col="blue", las=1, xaxt="n",
     xlab="Price for Peak Period (Time slot 5-9pm)",ylab="Total Revenue")
xticks <- seq(0, maxprice, by=50)
axis(side = 1, at = xticks)
revenueBest=max(revenue[basePrice:maxprice])
priceBest=which(revenue == revenueBest)
axis(side = 1, at = priceBest)
lines(c(priceBest,priceBest),c(0, revenueBest/1000),lty=2)
axis(side = 2, at = round(revenueBest/1000,3),las=1)
lines(c(0,priceBest),c(revenueBest/1000, revenueBest/1000),lty=2)
```

New York Health Club Part A.2 (Peak Period Pricing)

- Based on this nonparametric demand model, optimal peak period price is 83 (when base price is 65). (Note that this revenue is less than the previous single price model, so perhaps base price should not have been set at 65. We will study setting both prices shortly.



New York Health Club Part A.2 (Peak Period Pricing)

- Now, let's instead fit a (parametric) linear demand model:

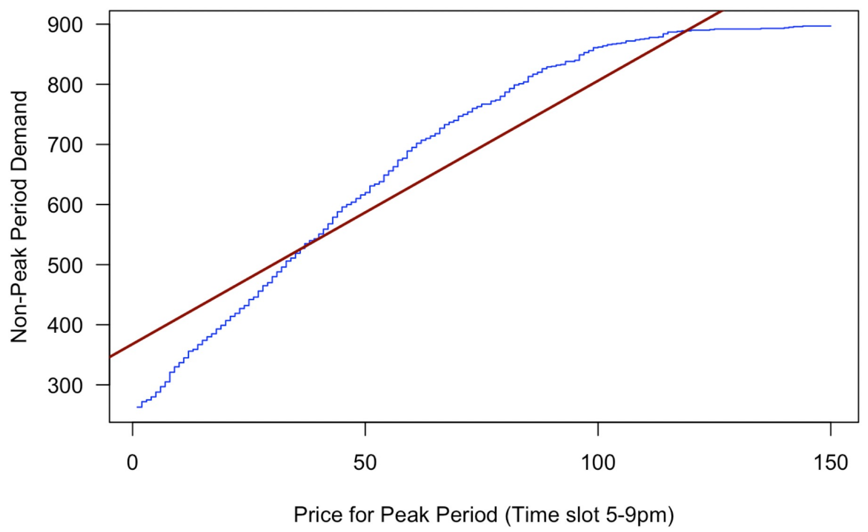
```
# STEP 3: Fitting a linear demand model

PeakPrice=1:maxprice

# Fitting a Linear Demand Model for Non-Peak Period Demand
fitNonPeak <-lm(demandNonPeak ~ PeakPrice)
InterceptNonPeak=coef(fitNonPeak)[1]
CoefPriceNonPeak=coef(fitNonPeak)[2]
```

```
plot(xaxis,demandNonPeak,pch = 16, type="s",col="blue", las=1, xaxt="n",
     xlab="Price for Peak Period (Time slot 5-9pm)",ylab="Non-Peak Period Demand")
xticks <- seq(0, maxprice, by=50)
axis(side = 1, at = xticks)
abline(lm(demandNonPeak ~ PeakPrice), lwd=2, col="darkred")
```

New York Health Club Part A.2 (Peak Period Pricing)

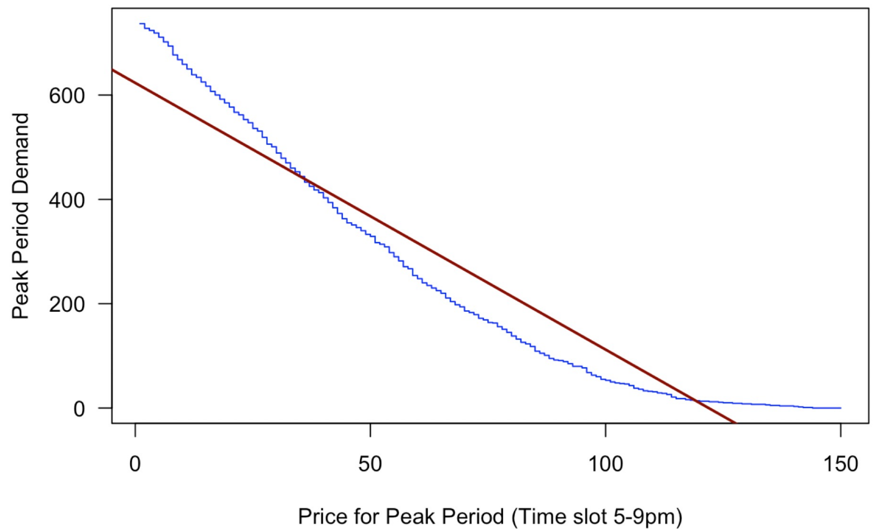


New York Health Club Part A.2 (Peak Period Pricing)

```
# Fitting a Linear Demand Model for Peak Period Demand
fitPeak <-lm(demandPeak ~ PeakPrice)
InterceptPeak=coef(fitPeak)[1]
CoefPricePeak=coef(fitPeak)[2]


plot(xaxis,demandPeak,pch = 16, type="s",col="blue", las=1, xaxt="n",
     xlab="Price for Peak Period (Time slot 5-9pm)",ylab="Peak Period Demand")
xticks <- seq(0, maxprice, by=50)
axis(side = 1, at = xticks)
abline(lm(demandPeak ~ PeakPrice), lwd=2, col="darkred")
```

New York Health Club Part A.2 (Peak Period Pricing)



New York Health Club Part A.2 (Peak Period Pricing)

```r
# To display the regression results in a Latex table that can be easily
included in a report written in Latex
library(stargazer)
stargazer(fitNonPeak,fitPeak, type="text")
```

```r
# stargazer(fitNonPeak,fitPeak, type="latex", out="Outputexample.tex")
```

New York Health Club Part A.2 (Peak Period Pricing)

- Now, let's instead fit a (parametric) linear demand model:

Table 1:

	Dependent variable:	
	demandNonPeak	demandPeak
	(1)	(2)
Prices	4.381***	−5.114***
	(0.115)	(0.127)
Constant	367.910***	623.440***
	(10.008)	(11.060)
Observations	150	150
R^2	0.908	0.916
Adjusted R^2	0.907	0.916
Residual Std. Error (df = 148)	60.978	67.387
F Statistic (df = 1; 148)	1,452.017***	1,619.896***
Note:	*p<0.1; **p<0.05; ***p<0.01	

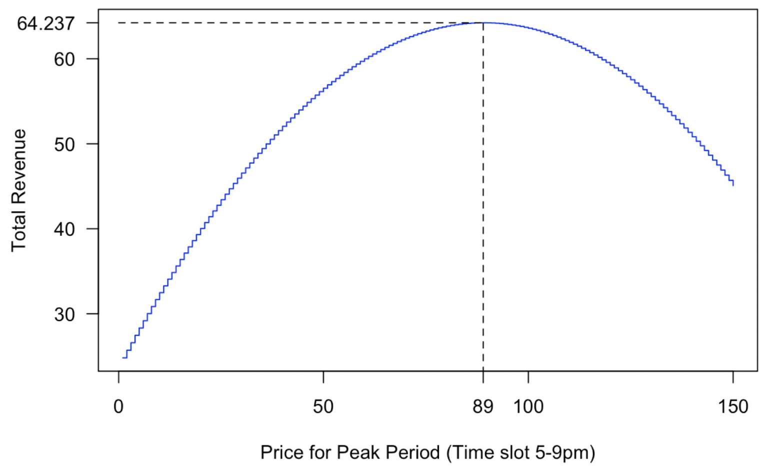
New York Health Club Part A.2 (Peak Period Pricing)

```r
# Maximizing Revenue From the Linear Demand Model
demandNonPeakLinear<-rep(0,maxprice)
demandPeakLinear<-rep(0,maxprice)
revenueLinear<-rep(0,maxprice)

for (p in 1:maxprice){
  demandNonPeakLinear[p]=InterceptNonPeak+CoefPriceNonPeak*p
  demandPeakLinear[p]=InterceptPeak+CoefPricePeak*p
  revenueLinear[p]=basePrice*demandNonPeakLinear[p]+p*demandPeakLinear[p]
}

revenueLinearBest=max(revenueLinear)
priceBestLinear=which(revenueLinear == revenueLinearBest)
```

New York Health Club Part A.2 (Peak Period Pricing)



Examples of Other Price Response Functions

- Linear Price Response Function:
  - The general formula is given by *d(p)=D-mp*, where *D*>0 is the demand at zero price, and *m* is the slope indicating price sensitivity.

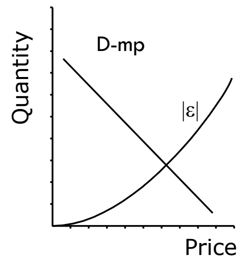
- Constant Elasticity (Iso-Elastic) Price Response Function
  - The constant elasticity price response function has the property that the ratio of the percentage change in demand to the percentage change in price is constant across all price points.
  - The general formula is given by $d(p)=Dp^{-\varepsilon}$ where ε denotes elasticity of demand.

- Exponential (Log-Linear) Price Response Function
  - The general formula is given by $d(p)=De^{-\varepsilon p}$

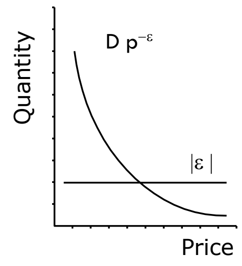
Examples of Other Price Response Functions

- The differences between these models is how the "price elasticity of demand" behaves.
- Elasticity: a measure of the percentage change in demand in response to a 1% change in price.

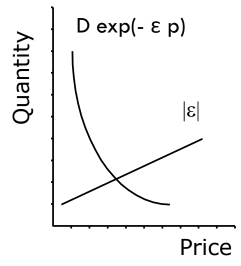
$$Elasticity = \frac{p \cdot d'(p)}{d(p)}$$



Linear



Iso-elastic



Log-linear

(Aside: Fitting a Constant Elasticity Price Response Function)

- Fitting a Constant Elasticity (Iso-elastic) Price Response Function
  - $d(p)=Dp^{-\varepsilon}$

- Fitting Technique: Nonlinear Regression
  - Transform the variables and use Linear Regression!
  - *Ln*(quantity) = a + b *Ln*(price)

  - *Ln*(.) is the natural logarithm, i.e., logarithm with base e, e=2.71828

  - Once *a* and *b* are identified we can solve for *D* and ε.

  - Note *a* = *Ln(D)* and *b* = -ε
  - Hence, $D = e^{a}$ and ε = -*b*

(Aside: Fitting a Log-Linear Model)

- Fitting a Log-Linear Price Response Function
  - $d(p)=De^{-\varepsilon p}$

- Fitting Technique: Nonlinear Regression
  - Transform only the quantity variable
  - *Ln*(quantity) = a + b price

  - Once *a* and *b* are identified we can similarly solve for *D* and ε.

  - Note *a* = *Ln(D)* and *b* = -ε
  - Hence, $D = e^{a}$ and ε = -*b*

New York Health Club Part A.3 (Selecting Both Prices)

- Calculate Nonpeak and Peak Demand for various price combinations

```
# We would like to create new data that has four columns,
# the first two columns will be the prices for Nonpeak and Peak time slots
# The remaining columns will count how many people buy NonPeak and Peak at
that price combination.
# Lets search prices from 25 to 125 with an increment of 5.
# This gives us 21 price points for basePrice (i.e., 25, 30, ..., 125) and
similarly
# it will lead to 21 price points for Peak price (i.e., 25, 30, ..., 125).
# We will not require that Peak Price exceeds basePrice now because our go
al at this stage
# is to model the demand in terms of prices.
# The total number of price combinations is 21*21=441
# We will keep track of this through an index variable called index.
```

New York Health Club Part A.3 (Selecting Both Prices)

- Calculate Nonpeak and Peak Demand for various price combinations

```
index=1
for (basePrice in seq(from = 50, to = 100, by = 5)){
  for (peakPrice in seq(from = 50, to = 100, by = 5)){
    for (i in 1:N){
      surplusNonPeak[i]=max(nyhc[i,c(3:6,8)]-basePrice)
      surplusPeak[i]=nyhc[i,7]-peakPrice
    }
    demandNonPeak[index]=sum((surplusNonPeak>surplusPeak)*(surplusNonPeak>=0))
    demandPeak[index]=sum((surplusPeak>=surplusNonPeak)*(surplusPeak>=0))
    index=index+1
  }
}
```

New York Health Club Part A.3 (Selecting Both Prices)

- Calculate Nonpeak and Peak Demand for various price combinations

```
# Create a data table which we will use to run the two regressions:
newdata<-data.frame(matrix(nrow=121,ncol = 5))
colnames(newdata)=c("index","basePrice","peakPrice","NonPeakDemand", "PeakDemand")
index=1
for (basePrice in seq(from = 50, to = 100, by = 5)){
  for (peakPrice in seq(from = 50, to = 100, by = 5)){
    newdata[index,1]=index
    newdata[index,2]=basePrice
    newdata[index,3]=peakPrice
    newdata[index,4]=demandNonPeak[index]
    newdata[index,5]=demandPeak[index]
    index=index+1
  }
}
```

New York Health Club Part A.3 (Selecting Both Prices)

- Calculate Nonpeak and Peak Demand for various price combinations

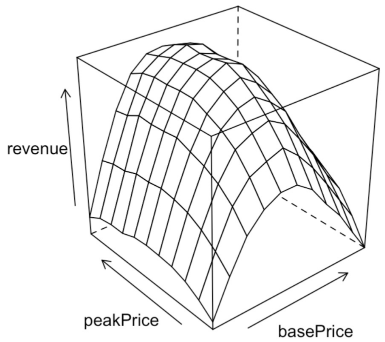
basePrice	peakPrice	NonPeakDemand	PeakDemand
50	50	765	226
50	55	798	193
50	60	822	169
50	65	847	143
50	70	873	116
50	75	893	96
50	80	907	80
50	85	931	56
50	90	938	49
50	95	948	39
50	100	963	23
55	50	729	251
55	55	754	226
55	60	787	193
55	65	811	168
55	70	835	142
100	100	390	148

New York Health Club Part A.3 (Selecting Both Prices)

- Visualizing Revenue as a function of NonPeak and Peak prices

```r
# Visualizing Revenue as a Function of Base and Peak Price
newdata$revenue=newdata$basePrice*newdata$NonPeakDemand+ne
wdata$peakPrice*newdata$PeakDemand

library(lattice)
wireframe(revenue ~ basePrice * peakPrice, data=newdata)
```



New York Health Club Part A.3 (Selecting Both Prices)

- Constructing Linear Demand models for NonPeak and Peak time slots as a function of NonPeak and Peak prices

```r
# Regression for the dependent variable NonPeakDemand

fit2NonPeak <-lm(NonPeakDemand ~ basePrice+peakPrice, data=newdata)
```

```r
# Regression for the dependent variable NonPeakDemand

fit2Peak <-lm(PeakDemand ~ basePrice+peakPrice, data=newdata)
```

```r
stargazer(fit2NonPeak,fit2Peak, type="text")
```

New York Health Club Part A.3 (Selecting Both Prices)

- Regression Models with Two Prices

Table 1:

	Dependent variable:	
	NonPeakDemand	PeakDemand
	(1)	(2)
basePrice	−11.854***	4.168***
	(0.117)	(0.119)
peakPrice	4.059***	−5.949***
	(0.117)	(0.119)
Constant	1,202.843***	348.190***
	(12.517)	(12.771)
Observations	121	121
R^2	0.990	0.969
Adjusted R^2	0.990	0.969
Residual Std. Error (df = 118)	20.300	20.712
F Statistic (df = 2; 118)	5,761.901***	1,860.117***
Note:	*p<0.1; **p<0.05; ***p<0.01	

New York Health Club Part A.3 (Selecting Both Prices)

- Regression Models with Two Prices

```r
a1=coef(fit2NonPeak)[1]
b11=coef(fit2NonPeak)[2]
b12=coef(fit2NonPeak)[3]
```

1202.843
-11.85405
4.058678

```r
a2=coef(fit2Peak)[1]
b21=coef(fit2Peak)[2]
b22=coef(fit2Peak)[3]
```

348.1901
4.167934
-5.94876

New York Health Club Part A.3 (Selecting Both Prices)

- Optimization Model with Two Prices

```r
# Finding optimal revenue by optimization
library("nloptr")

# Differentiated Prices

eval_f <- function(x){
  basePrice=x[1]
  peakPrice=x[2]
  NonPeakDemand=max(0,a1+b11*basePrice+b12*peakPrice)
  PeakDemand=max(0,a2+b21*basePrice+b22*peakPrice)
  revenue=basePrice*NonPeakDemand+peakPrice*PeakDemand
  objfunction=-revenue
  return(objfunction)
}

eval_g_ineq <- function(x) {
  basePrice=x[1]
  peakPrice=x[2]
  NonPeakDemand=max(0,a1+b11*basePrice+b12*peakPrice)
  PeakDemand=max(0,a2+b21*basePrice+b22*peakPrice)
  constraint <- c(-NonPeakDemand,
                  -PeakDemand,
                  x[1]-x[2])
  return(constraint)
}
```

New York Health Club Part A.3 (Selecting Both Prices)

- Optimization Model with Two Prices

```r
# initial values
x0 <- c(70,90)
# lower and upper bounds of control
lb <- c(50,50)
ub <- c(100,100)
opts <- list( "algorithm" = "NLOPT_LN_COBYLA",
              "xtol_rel"  = 1.0e-9,
              "maxeval"   = 1000)
result <- nloptr(x0=x0,eval_f=eval_f,lb=lb,ub=ub,
                 eval_g_ineq=eval_g_ineq,opts=opts)
# print(result)

priceOpt<-result$solution
RevenueOpt<- -result$objective
```

New York Health Club Part A.3 (Selecting Both Prices)

- Optimization Model with Two Prices

```r
print(paste("Optimal Base Price:",priceOpt[1]))
```

```
## [1] "Optimal Base Price: 80.1121177652542"
```

```r
print(paste("Optimal Peak Price:",priceOpt[2]))
```

```
## [1] "Optimal Peak Price: 84.6597709195169"
```

```r
print(paste("Optimal Revenue:",RevenueOpt))
```

```
## [1] "Optimal Revenue: 62919.9944037216"
```