Finding unknown heat transfer coefficients with 1-D IHCP solver based on the function specification method

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1 Acknowledgements

The method of solution for the inverse problem outlined below builds on Rauer et al. [2014] and Kumar [2004], which are based on the function specification method proposed by Beck et al. [1985].

2 Code

The MATLAB code used to implement this method is available at `https://github.com/mattlnhn/thermal-contact-resistance`.

3 Problem overview

The test section (Fig. 1) is split into three sections: the upstream section, the central section, and the downstream section. First, the heat flux at the boundaries of the upstream and downstream sections are estimated, assuming that conduction between the copper and Inconel is perfect. Then, using these estimated heat fluxes as boundary conditions on the central section, the heat transfer coefficients at the boundaries between the Inconel and H25 can be estimated.

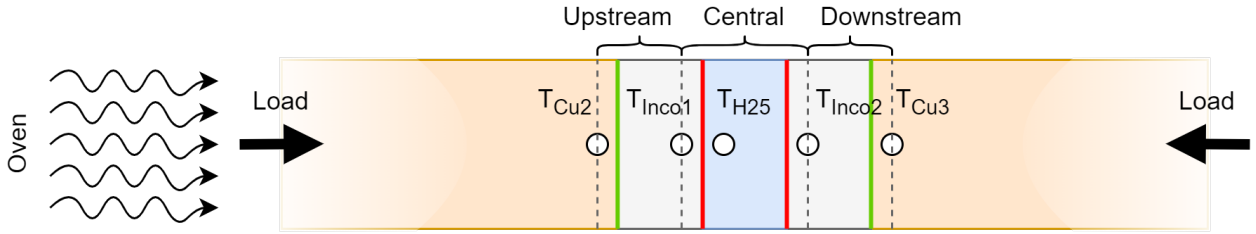

Figure 1: Illustration of the test section. Copper (orange), Inconel 718 (grey) and Haynes 25 (blue). Perfect conduction assumed across green boundaries, contact resistance on red boundaries. Thermocouple locations marked in white.

4 Method for the direct problem

The equation for transient 1-D conduction is

$$\frac{\partial T}{\partial t} = k\frac{\partial T}{\partial x}, \tag{1}$$

which for the general m^{th} node is discretised as

$$\frac{T_m^{i+1} - T_m^i}{\Delta t} = \frac{\alpha}{\Delta x^2}(T_{m-1}^i - 2T_m^i + T_{m+1}^i), \tag{2}$$

where i is the current time step and $\alpha = k/\rho c_p$ is the thermal diffusivity. For boundary nodes $m = 1$ and $m = N$ with a prescribed heat flux, this becomes

$$\frac{T_1^{i+1} - T_1^i}{\Delta t} = \frac{q_{in}}{\rho c_p \Delta x} + \frac{\alpha}{\Delta x^2}(T_2^i - T_1^i) \text{ and } \frac{T_N^{i+1} - T_N^i}{\Delta t} = -\frac{q_{out}}{\rho c_p \Delta x} + \frac{\alpha}{\Delta x^2}(T_{N-1}^i - T_N^i). \tag{3}$$

This is vectorised as

$$\mathbf{T}^{i+1} = \tau(\mathbf{A}\mathbf{T}^i + \mathbf{b}) + \mathbf{T}^i, \tag{4}$$

where

$$\tau = \frac{\alpha \Delta t}{\Delta x^2}, \mathbf{A} = \begin{bmatrix} -1 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & & & & & \\ & & & \ddots & & & & \\ & & & & -2 & 1 & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} q_{in}\Delta x/k \\ 0 \\ \vdots \\ 0 \\ -q_{out}\Delta x/k \end{bmatrix}. \tag{5}$$

To handle a change in material, constant τ becomes a diagonal matrix $\boldsymbol{\tau}$ to pass the appropriate thermal diffusivity to each material:

$$\boldsymbol{\tau} = \frac{\Delta t}{\Delta x^2} \left[\begin{array}{ccc|ccc} \alpha_1 & & & & & \\ & \ddots & & & & \\ & & \alpha_1 & & & \\ \hline & & & \alpha_2 & & \\ & & & & \ddots & \\ & & & & & \alpha_2 \end{array}\right]. \tag{6}$$

The entries of $\mathbf{A}$ corresponding to the boundary must also be adjusted. For perfect conduction,

$$\mathbf{A} = \left[\begin{array}{ccccc|ccccc} -1 & 1 & & & & & & & & \\ 1 & -2 & 1 & & & & & & & \\ & 1 & -2 & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & -(2-\gamma) & 2/(\kappa+1) & & & & \\ \hline & & & & 2\kappa/(\kappa+1) & -(2+\gamma) & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & -2 & 1 & \\ & & & & & & & 1 & -2 & 1 \\ & & & & & & & & 1 & -1 \end{array}\right] \tag{7}$$

where $\kappa = k_1/k_2$ and $\gamma = (\kappa - 1)/(\kappa + 1)$.

For a boundary with known heat transfer coefficient h,

$$\mathbf{A} = \left[\begin{array}{ccccc|ccccc} -1 & 1 & & & & & & & & \\ 1 & -2 & 1 & & & & & & & \\ & 1 & -2 & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & \xi_2 - 1 & -\xi_2 & & & & \\ \hline & & & & -\xi_1 & \xi_1 - 1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & -2 & 1 & \\ & & & & & & & 1 & -2 & 1 \\ & & & & & & & & 1 & -1 \end{array}\right] \tag{8}$$

where

$$\xi_1 = \frac{2\phi_1}{1-(1+\phi_1)(1+\phi_2)}, \xi_2 = \frac{2\phi_2}{1-(1+\phi_1)(1+\phi_2)}, \phi_1 = \frac{2k_1}{h\Delta x}, \phi_2 = \frac{2k_2}{h\Delta x}. \tag{9}$$

5 Method for the inverse problem

To estimate the k^{th} boundary heat flux on the upstream and downstream sections, a predictor-corrector method is implemented to minimise the objective function

$$E_k = \sum_{j=1}^{s}\sum_{i=1}^{r}\left[Y_{j,m+i-1} - T_{j,k,m+i-1}(q_{j,k,m+i-1})\right]^2, \tag{10}$$

where j is the sensor location (the position where a temperature measurement is taken), and s is the number of sensors; m is the current time step, i is the index for time steps, and r is the number of future time steps used in the calculation; Y is the measured temperature, and T is the calculated temperature, computed by the solver described above. An initial guess at the heat flux is provided to the direct solver to start the process, then an iterative process to minimise E_k allows $q_{k,m}$ to be found. The converged value from the previous step is then used as the initial estimate for the next.

The function specification method (Beck et al. [1985]) is used to simplify the problem. Here, it is assumed that the heat flux does not vary significantly over a short span of time, so the value is held constant over the next r steps:

$$q_{k,(m,\ldots,m+r-1)} \approx q_{k,m}. \tag{11}$$

This approximation is substituted into equation (10), which is differentiated to yield

$$\sum_{j=1}^{s}\sum_{i=1}^{r}\left[Y_{j,m+i-1} - T_{j,k,m+i-1}(q_{k,m})\right]^2 \frac{\partial T_{j,k,m+i-1}(q_{k,m})}{\partial q_{k,m}} = 0 \tag{12}$$

at minimum error. This is then solved using Newton's iterative method.

The sensitivity coefficients, which are the sensitivities of the calculated temperature to change in the k^{th} heat flux at the j^{th} sensor and $(m+i-1)^{th}$ time step, are defined as

$$X_{j,k,m+i-1} = \left.\frac{\partial T_{j,k,m+i-1}}{\partial q_{k,m}}\right|_{q_{k,m}}, \tag{13}$$

which is approximated by

$$X_{j,k,m+i-1} \approx \frac{T_{j,k,m+i-1}(q_{k,m}+\varepsilon q_{k,m}) - T_{j,k,m+i-1}(q_{k,m})}{\varepsilon q_{k,m}} \tag{14}$$

where ε is a small fraction.

The increment on the k^{th} heat flux for the p^{th} iteration of Newton's method is thus found to be

$$\Delta q_{k,m}^{p+1} = \frac{\sum_{j=1}^{s}\sum_{i=1}^{r}\left[Y_{j,m+i-1} - T_{j,k,m+i-1}(q_{k,m}^{p})\right] X_{j,k,m+i-1}(q_{k,m}^{p})}{\sum_{j=1}^{s}\sum_{i=1}^{r}\left[X_{j,k,m+i-1}(q_{k,m}^{p})\right]^{2}}, \tag{15}$$

and the updated heat flux is calculated as

$$q_{k,m}^{p+1} = q_{k,m}^{p} + \Delta q_{k,m}^{p+1}. \tag{16}$$

While the k^{th} sensitivities, heat flux increment, and heat flux are calculated, all remaining heat fluxes are held constant at their value from the previous time step (note that this differs from the 'serial' approach in Kumar [2004]). Iterations stop when one of several convergence criteria is met:

$$\frac{\Delta q_{k}^{p+1}}{q_{k}^{p+1}} < RTOLq \text{ for all } k \tag{17}$$

$$\frac{E_{k}^{p+1} - E_{k}^{p}}{E_{k}^{p}} < RTOLerror \text{ for all } k \tag{18}$$

$$p > p_{max} \tag{19}$$

where $RTOLq$ is tolerance to relative change in q, $RTOLerror$ is tolerance to relative change in error, and p_{max} is a maximum number of iterations.

An identical process is then used to extract the heat transfer coefficients h_k, with the previously estimated heat fluxes set as the boundary conditions, and temperatures and sensitivities calculated as a function of h_k.

The experimental data to be processed is not necessarily evenly spaced in time, and taken at intervals much greater than the minimum time step required for convergence of the direct solver. To solve this problem, the direct solver uses small constant time steps Δt, while the measured data points are indexed with integers, and the interval between them is variable. Starting at the m^{th} time step, the direct problem is solved in n small steps of Δt until the $(m+r-1)^{th}$ time step is reached. The values corresponding to the $(m, m+1, ..., m+r-1)^{th}$ steps and j^{th} sensors are extracted, and the remaining data discarded. This provides the calculated data used to find the sensitivity coefficients, the Newton method increments and the error terms. When the method converges, the direct solver is run a final time with the converged values from the current step to advance the initial temperature from the m^{th} step to the initial temperature of the $(m+1)^{th}$ step.

6 Flowchart

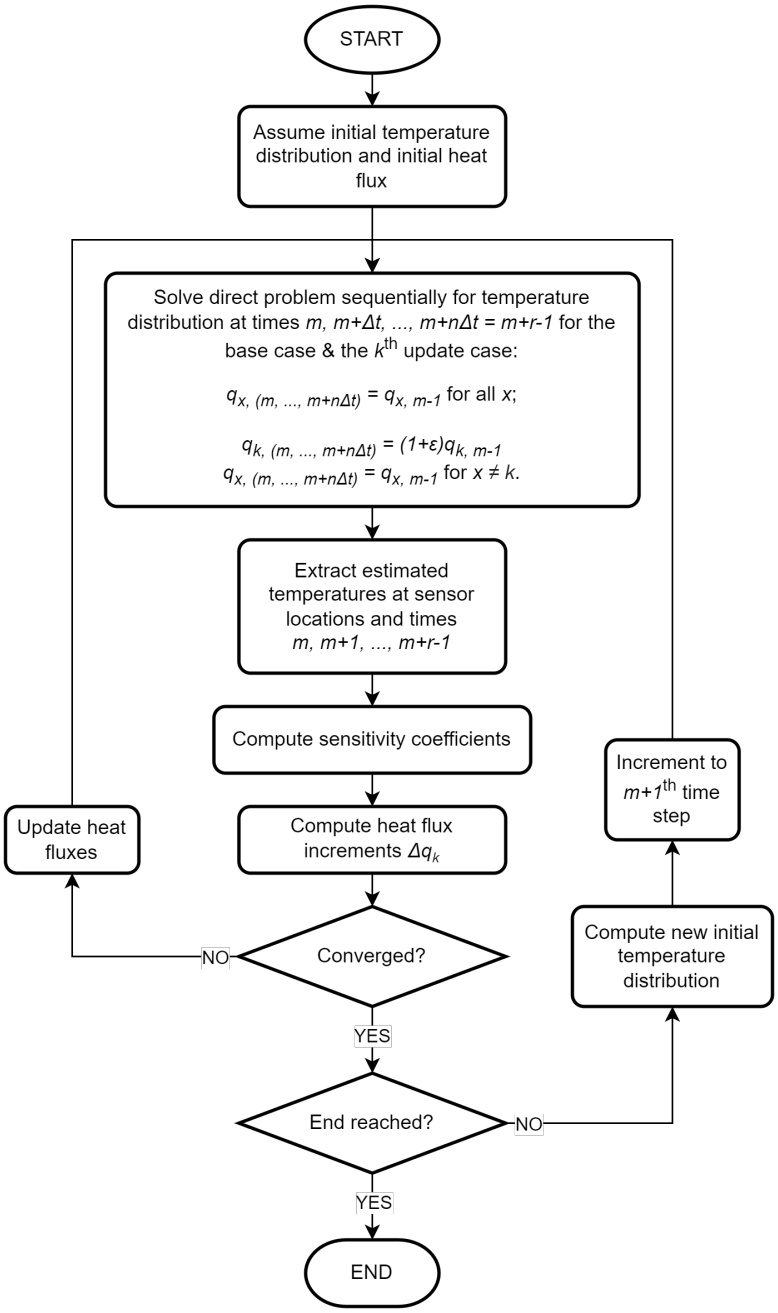

Figure 2: Process for determining unknown heat fluxes in upstream and downstream sections. Identical process followed to determine unknown heat transfer coefficients.

References

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