LECTURE NOTES

§ 1 | A Gaussian Process on $L^2(\mathbb{R})$

We will construct a process on $L^2(\mathbb{R})$ by taking a limit of a simpler process on the space of sequences

$$\ell^2(\mathbb{R}) \coloneqq \left\{ (a_k)_{k\in\mathbb{Z}} \colon \sum_{k\in\mathbb{Z}} |a_k|^2 < \infty,\, a_k \in \mathbb{R} \right\}.$$

Given a set of independent random variables $\{X_k\}_{k\in\mathbb{Z}}$ satisfying $X_k \sim N(0,1)$ (think of each X_k as "noise" at the integer k), define $Z(a) \coloneqq \sum_{k\in\mathbb{Z}} a_k X_k$ for $a = (a_k) \in \ell^2(\mathbb{R})$. The collection $\{Z(a)\}_{a\in\ell^2(\mathbb{R})}$ is a Gaussian process that is centered (ie, $\mathbb{E}[Z(a)] = 0$) and satisfies

$$\operatorname{var}[Z(a)] = \operatorname{cov}\left[\sum_{k\in\mathbb{Z}} a_k X_k, \sum_{l\in\mathbb{Z}} a_l X_l\right] = \sum_{k,l\in\mathbb{Z}} a_k a_l \operatorname{cov}[X_k, X_l] = \sum_{k\in\mathbb{Z}} a_k^2$$

for each $a \in \ell^2(\mathbb{R})$.

Now recall $L^2(\mathbb{R}) \coloneqq \{f \colon \mathbb{R} \to \mathbb{R} \colon \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty\}$. Given $f \in L^2(\mathbb{R})$, define a sequence of random variables $(Z^{(N)}(f))_{N\in\mathbb{N}}$ by

$$Z^{(N)}(f) \coloneqq \frac{1}{\sqrt{N}} \sum_{k\in\mathbb{Z}} f\left(\frac{k}{N}\right) X_k.$$

It can be shown that $Z^{(N)}(f)$ converges in distribution to a random variable which we will call $Z(f)$. It can be also shown that the collection $\{Z(f)\}_{f\in L^2(\mathbb{R})}$ is a centered Gaussian process satisfying $\operatorname{cov}[Z(f), Z(g)] = \int_{-\infty}^{\infty} f(x)g(x)\, dx$. It turns out that these properties (centered Gaussian and satisfying the above covariance formula) characterize $\{Z(f)\}_{f\in L^2(\mathbb{R})}$, so we might as well define it this way:

> Definition. The collection $\{Z(f)\}_{f\in L^2(\mathbb{R})}$ is **the centered Gaussian process on** $L^2(\mathbb{R})$ satisfying
>
> $$\operatorname{cov}[Z(f), Z(g)] \coloneqq \int_{-\infty}^{\infty} f(x)g(x)\, dx\,.$$

Note that $Z(af + bg) = aZ(f) + bZ(g)$ for all $a, b \in \mathbb{R}$ and all $f, g \in L^2(\mathbb{R})$, so $f \mapsto Z(f)$ defines a linear map from $L^2(\mathbb{R})$ to the space of random variables. The random variable $Z(f)$ is called the *stochastic integral* of f, denoted

$$Z(f) =\colon \int_{-\infty}^{\infty} f(t)\, dW_t\,,$$

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and the integrator dW_t is interpreted as "white noise" that weights the value of $f(t)$. Assuming it is known that $\mathrm{cov}[dW_s, dW_t] = \delta_s(t)\, ds\, dt$, where δ_s is the Dirac δ-functional centered at s, we can recover the covariance formula using stochastic integral notation:

$$\begin{aligned}\mathrm{cov}[Z(f), Z(g)] &= \mathrm{cov}\left[\int_{-\infty}^{\infty} f(s)\, dW_s, \int_{-\infty}^{\infty} g(t)\, dW_t\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s)g(t)\ \mathrm{cov}[dW_s, dW_t] = \int_{-\infty}^{\infty} f(t)g(t)\, dt.\end{aligned}$$

The following examples show that some stochastic processes can be defined as stochastic integrals.

1.1 Example (*Brownian motion*) If we define $B_t := Z(\mathbf{1}_{[0,t)}) = \int_0^t dW_s$, then the process $\{B_t\}_{t\in[0,\infty)}$ is a Brownian motion. Indeed,

(i) B_t is Gaussian,

(ii) $\mathrm{cov}[B_s, B_t] = \int_{-\infty}^{\infty} \mathbf{1}_{[0,s)}(u)\mathbf{1}_{[0,t)}(u)\, du = \int_0^{\min\{s,t\}} du = \min\{s,t\}$ (and hence $\mathrm{var}[B_t] = t$),

(iii) if $u < v \leq s < t$ then

$$\mathrm{cov}[B_t - B_s, B_v - B_u] = \int_{-\infty}^{\infty} \mathbf{1}_{[s,t)}(x)\mathbf{1}_{[u,v)}(x)\, dx = \int_{-\infty}^{\infty} \mathbf{1}_{[s,t)\cap[u,v)}(x)\, dx = 0\,.$$

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1.2 Example (*Energy*) Define a process on $\{Y_t\}_{t\in\mathbb{R}}$ by

$$Y_t := Z\left(e^{-(t-s)}\mathbf{1}_{(-\infty,t]}(s)\right) = \int_{-\infty}^{\infty} e^{-(t-s)}\, dW_s\,.$$

Then

(i) Y_t is Gaussian,

(ii) for $s \leq t$ we have

$$\mathrm{cov}[Y_s, Y_t] = \int_{-\infty}^{\infty} e^{-(s-u)}\mathbf{1}_{(-\infty,s]}(u)^2 e^{-(t-u)}\mathbf{1}_{(-\infty,t]}(u)^2\, du = e^{-s-t}\int_{-\infty}^{s} e^{2u}\, du = \frac{e^{-(t-s)}}{2}\,,$$

and hence $\mathrm{var}[Y_t] = \frac{1}{2}$.

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