Recall a realization of the basic noise for Gaussian processes looked like that in Figure **??**. Now, arrows are either muted or (rarely) point up. See Figure **??**.

§ 1 | Motivation

Suppose in some space X we lay down a large number of LED lights, each with their own battery, with density given by a σ-finite measure μ. We do this in a way so that, for each region $A \subset X$, we put down about $M\mu(A)$ lights in that region, where M is some large number. Independently we turn on each light with probability M^{-1}, and leave off otherwise.

Date: 01 October 2018.

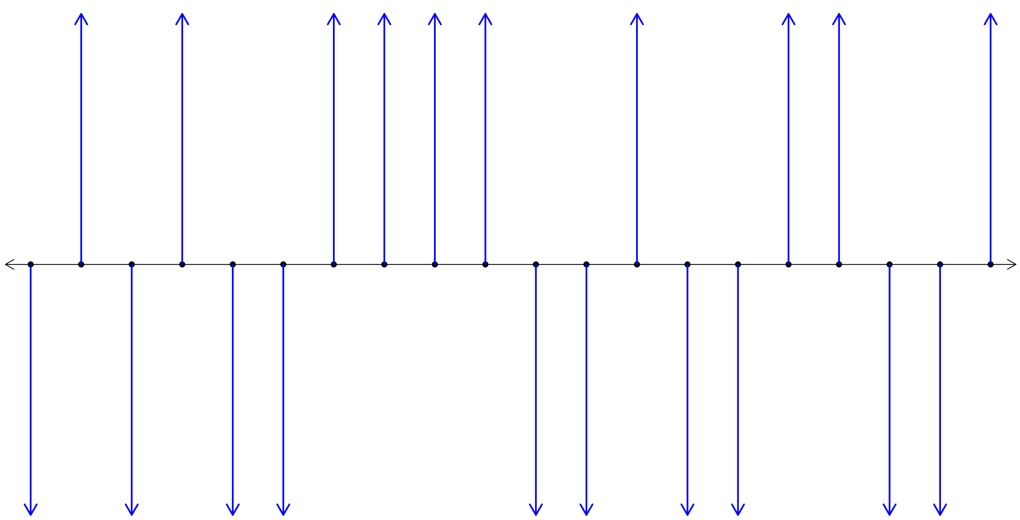

FIGURE 1. A realization of the basic noise used to construct a Gaussian process.

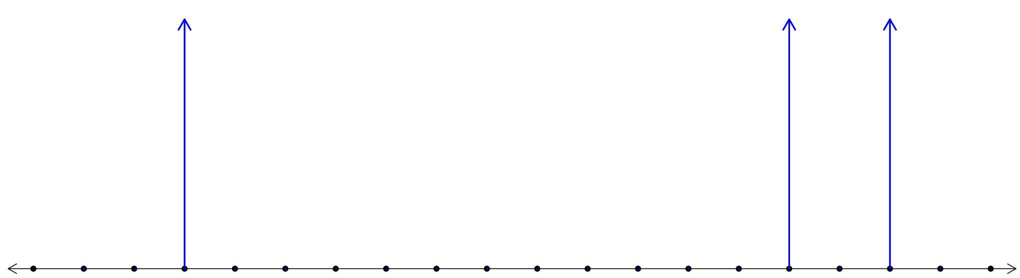

FIGURE 2. A realization of the basic noise used to construct a Poisson process.

We would like to answer the following question: how many lights in A are on? To that end, let $N(A)$ denote the number of lights on in A and compute

$$\mathbb{E}[N(A)] = \mathbb{E}\left[\sum_{\text{lights in } A} 1_{\{\text{light on}\}}\right] = \sum_{\text{lights in } A} \mathbb{P}\{\text{light is on}\} = M\mu(A)\left(\frac{1}{M}\right) = \mu(A). \tag{1}$$

Thus μ gives the expected density for the set of lights that are on in A. By construction, we know $N(A) \sim \text{Binom}\left(M\mu(A), M^{-1}\right)$, and hence the distribution of $N(A)$ is approximately $\text{Pois}(\mu(A))$. To see this, put $L = M\mu(A)$ and observe,

$$\begin{aligned}
\mathbb{P}\{N(A) = n\} &= \binom{L}{n}\left(\frac{1}{M}\right)^n\left(1 - \frac{1}{M}\right)^{L-n} && (2)\\
&= \frac{L(L-1)\cdots(L-n+1)}{n!M^n}\left(1 - \frac{1}{M}\right)^{L-n} && (3)\\
&\simeq \frac{1}{n!}\left(\frac{L}{M}\right)^n \exp\left(-\frac{L}{M}\right) + \mathcal{O}\left(\frac{1}{M}\right) && (4)\\
&\simeq \frac{\mu(A)^n}{n!}e^{-\mu(A)} && (5)
\end{aligned}$$

This motivates the following definition.

Definition. Let μ be a σ-finite measure on some space X. A *Poisson Point Process* (PPP) on X with *mean measure* (or, *intensity*) μ is a random point measure N such that:

(a) For any Borel set $A \subset X$, we have $N(A) \in \mathbb{Z}_{\geq 0}$ and $N(A) \sim \text{Pois}(\mu(A))$, i.e.

$$\mathbb{P}\{N(A) = n\} = \frac{\mu(A)^n}{n!}e^{-\mu(A)}. \tag{6}$$

(b) If A and B are disjoint Borel subsets of X, then $N(A)$ and $N(B)$ are independent random variables.

Recall a point measure is just a measure whose mass is atomic. That is, if $\{x_i\} \subset X$ then a point measure is of the form

$$\mu = \sum_i a_i \delta_{x_i} \tag{7}$$

where δ_x is the unit point mass at x.

§ 2 | PPP Properties

It is sometimes useful to think of a PPP as a random collection of points. With this in mind, we list some important properties of $N \sim \text{PPP}(\mu)$ on some space X:

- *Enumeration:* It is always possible to enumerate the points of N, i.e. there is a random collection of points $\{x_i\} \subset X$ such that

$$N = \sum_i \delta_{x_i}. \tag{8}$$

- *Mean measure:* If $f\colon X \to \mathbb{R}$ then

$$\mathbb{E}\left[\int f(x)\,dN(x)\right] = \int f(x)d\mu(x). \tag{9}$$

<u>Note:</u> This is a more general property of point processes, as any point process has a mean measure. To see (**??**) holds without needing N to be a *Poisson* point process, let f be a simple function, i.e.

$$f(x) = \sum_{i=1}^{n} f_i 1_{A_i}(x), \qquad \text{where} \qquad X = \bigcup_i A_i, \quad A_i \cap A_j = \varnothing \text{ for } i \neq j. \tag{10}$$

Then we compute
(11)

$$\mathbb{E}\left[\int_X f(x)\,dN(x)\right] = \mathbb{E}\left[\sum_i f_i N(A_i)\right] = \sum_i f_i \mathbb{E}\left[N(A_i)\right] = \sum_i f_i \mu(A_i) = \int_X f(x)\,d\mu(x).$$

This can then be extended to arbitrary measurable functions through the standard limiting procedure.

- *Thinning:* Independently discard each point of N with probability $1 - p(x)$ for a point at $x \in X$. The result is a $\mathrm{PPP}(\nu)$, where

$$\nu(A) = \int_A p(x)\,d\mu(x). \tag{12}$$

In other words, if $N = \sum_i \delta_{x_i}$ and $A_i = 1$ with probability $p(x_i)$ and $A_i = 0$ otherwise, then

$$\widetilde{N} = \sum_i A_i \delta_{x_i} \sim \mathrm{PPP}(\nu). \tag{13}$$

- *Additivity:* If $N_1 \sim \mathrm{PPP}(\mu_1)$ and $N_2 \sim \mathrm{PPP}(\mu_2)$ are independent on X, then $N_1 + N_2 \sim \mathrm{PPP}(\mu_1 + \mu_2)$. In particular, if $\mathbb{P}\left\{X = n\right\} = \frac{\lambda^n}{n!}e^{-\lambda}$ and $\mathbb{P}\left\{Y = n\right\} = \frac{\nu^n}{n!}e^{-\nu}$ are independent, then

$$\mathbb{P}\left\{X + Y = n\right\} = \frac{(\lambda+\nu)^n}{n!}e^{-(\lambda+\nu)}. \tag{14}$$

- *Labeling:* For each point in a PPP, associate an independent label from a space Y according to some probability distribution ν. Let $N = \sum_i \delta_{x_i}$ for $\{x_i\} \subset X$ and let $G_1, G_2, \ldots \in Y$ be iid with density ν. Then

$$\overline{N} := \sum_i \delta_{(x_i, G_i)} \sim \mathrm{PPP}(\mu \times \nu) \tag{15}$$

on $X \times Y$.

§ 3 | Examples

Henceforth, let λ denote Lebesgue measure.

3.1 Example Let $N \sim \mathrm{PPP}(\lambda)$ on $\mathbb{R}_{\geq 0}$, where λ is Lebesgue measure. As before, we think of the points of N as 'lights', here positioned on the positive reals.

(a) How far until the first light?
(b) Suppose each light is independently either red or green with probability $\frac{1}{2}$. How far until the first red light?

Solution. Let $N = \sum_i \delta_{x_i}$ and put $T = \min\{x_i\}$. Using (**??**) we compute

$$\mathbb{P}\{T > t\} = \mathbb{P}\{N([0,t]) = 0\} = e^{-t}. \tag{16}$$

This solves part (a). For the colorblind readers, this also solves part (b).

Now let $\{\tilde{x}_i\} \subset \{x_i\}$ be the (random) set of red lights and define $\widetilde{N} = \sum_i \delta_{\tilde{x}_i}$, the point process for the red lights from N. By the thinning property (**??**), $\widetilde{N} \sim \mathrm{PPP}\left(\frac{1}{2}\lambda\right)$. Similarly define $\widetilde{T} = \min\{\tilde{x}_i\}$ and observe

$$\mathbb{P}\left\{\widetilde{T} > t\right\} = \mathbb{P}\left\{\widetilde{N}([0,t]) = 0\right\} = e^{-t/2}, \tag{17}$$

thus (b) is solved. ♦

•

3.2 Example Rain falls for 10 minutes on a large patio at a rate of $\nu = 5000$ drops per minute per square meter. Each drop splatters to a random radius R that has an Exponential distribution, with mean 1cm, independently of the other drops. Assume the drops are 1mm thick and the set of locations of the raindrops is a PPP.

(a) What is the mean and variance of the total amount of water falling on a square with area $1\mathrm{m}^2$?
(b) A very small ant is running around the patio. See Figure **??**. What is the chance the ant gets hit?

Solution. Let $N = \sum_i \delta_{(x_i, y_i)}$ where (x_i, y_i) is the center of the ith drop. Take $N \sim \mathrm{PPP}(\nu\lambda)$ and let M denote the number of drops in $[0,1]^2$, so that $M = N([0,1]^2) \sim \mathrm{Pois}(\nu)$. Then the total volume V is

$$V = \sum_{i=1}^{M} \frac{\pi}{10^3} R_i^2 \tag{18}$$

where R_i is the radius of the ith drop. Note this is a sum of random variables where the number of terms is also a random variable. Thus we use Wald's equation (**??**) to obtain

$$\mathbb{E}[V] = \frac{\pi}{10^3}\mathbb{E}[M]\mathbb{E}[R_1^2] = \frac{\pi}{10^3} \cdot \nu \cdot \frac{2}{100^2} = \frac{2\pi}{10^7}\nu \tag{19}$$

The second step in (**??**) was obtained from the fact that an exponentially distributed random variable X with mean β^{-1} has higher moments given by

$$\mathbb{E}[X^n] = \frac{n!}{\beta^n}. \tag{20}$$

This is proved by an iterated application of integration by parts, and the result gives rise to

$$\mathrm{var}[X^n] = \mathbb{E}\left[X^{2n}\right] - \mathbb{E}[X^n]^2 = \frac{(2n)! - (n!)^2}{\beta^{2n}}. \tag{21}$$

The $n = 2$ case will turn out to be useful when computing the variance of V.

Indeed, to compute the variance we utilize the variance decomposition formula. Observe,

$$\operatorname{var}[V] = \mathbb{E}\left[\operatorname{var}[V \mid M]\right] + \operatorname{var}\left[\mathbb{E}[V \mid M]\right] \tag{22}$$

$$= \mathbb{E}\left[M\left(\frac{\pi}{10^3}\right)^2 \operatorname{var}(R^2)\right] + \operatorname{var}\left[M\left(\frac{\pi}{10^3}\right)\mathbb{E}[R^2]\right] \tag{23}$$

$$= \nu\left(\frac{\pi}{10^3}\right)^2\left(\frac{20}{100^4}\right) + \nu\left(\frac{\pi}{10^3}\right)^2\left(\frac{2}{100^2}\right)^2 \tag{24}$$

$$= \left(\frac{\pi}{10^3}\right)^2\left(\frac{24}{100^4}\right)\nu. \tag{25}$$

This solves part (a).

Now, part (b) can be solved by way of the labeling property. Here, we use the radius R_i of the ith drop to label the point (x_i, y_i). Recall the density of an Exponential random variable with mean 0.01 is $100\exp(-100r)\,dr$. So we define a measure μ on $X := \mathbb{R}^2 \times [0,\infty)$ by

$$\mu(A) = \int_A 100\nu\exp(-100r)\,dxdydr. \tag{26}$$

We think of X as the (closed) upper half plane in $\mathbb{R}^3$ where the third coordinate is a realization of R. By the labeling property (**??**), $\overline{N} := \sum_i \delta_{(x_i,y_i,R_i)} \sim \operatorname{PPP}(\mu)$ on X. For the ant to remain dry, any drop with radius r must land outside the circle of radius r centered at the ant. Viewed from the space X, we want to integrate over the cone with its tip at the ant, whose horizontal cross-section at height r is a circle of radius r. From this we compute

(27)

$$\mathbb{P}\left\{\text{ant is dry}\right\} = \mathbb{P}\left\{\overline{N}(A) = 0\right\} = \exp(-\mu(A)) = \exp\left(-100\pi\nu\int_0^\infty r^2e^{-100r}\,dr\right) = \exp\left(-\frac{\pi\nu}{5000}\right).$$

Plugging in the given value for ν yields $\mathbb{P}\left\{\text{ant is dry}\right\} = \exp(-\pi) \approx 0.0432$. The ant had better grab an umbrella! ♦

•

3.1 Wald's Equation

The following is the statement of Wald's equation, taken from Wikipedia [*].

3.3 Theorem (Wald's Equation) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued, independent and identically distributed random variables and let N be a nonnegative integer-valued random variable that is independent of the sequence $(X_n)_{n\in\mathbb{N}}$. Suppose that N and the X_n have finite expectations. Then

$$\mathbb{E}\left[\sum_{i=1}^N X_i\right] = \mathbb{E}\left[N\right]\mathbb{E}\left[X_1\right]. \tag{28}$$

[*] The proof is also on Wikipedia.

FIGURE 3. A realization of the ant from Example **??**. Looks like he had an umbrella after all.