LECTURE NOTES

§ 1 | A Different Construction of $Z(f)$

Definition. Define the *Haar basis* $\{h_{n,k}\}_{n,k\in\mathbb{N}}$ for $L^2([0,1])$ by $h_{0,0}\equiv 1$ and

$$h_{n,k}(t) \coloneqq \begin{cases} 2^{(n-1)/2} & \text{if } 2^{-n}(2^k)\le t<2^{-n}(2k+1)\,, \\ -2^{(n-1)/2} & \text{if } 2^{-n}(2k+1)\le t<2^{-n}(2k+2)\,, \\ 0 & \text{otherwise} \end{cases}$$

if $n\geq 1$ and $0\le k\le 2^{n-1}$ (otherwise set $h_{n,k}\equiv 0$).

It is easily verified that $\{h_{n,k}\}$ is an orthonormal set in $L^2([0,1])$, ie,

$$\int_0^1 h_{n,k}(t)h_{m,\ell}(t)\,dt = \begin{cases} 1 & \text{if } n=m \text{ and } k=\ell, \\ 0 & \text{otherwise} \end{cases}$$

Now suppose $\{Z_{n,k}\}$ is a collection of independent $N(0,1)$ variables and $f\in L^2([0,1])$ expands as

$$f(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{2^{n-1}} a_{n,k}h_{n,k}(t)\,, \quad \text{where} \quad a_{n,k}=\int_0^1 f(t)h_{n,k}(t)\,dt\,.$$

The random variable $Z(f)$ defined by

$$Z(f)\coloneqq\sum_{n=0}^{\infty}\sum_{k=0}^{2^{n-1}} a_{n,k}Z_{n,k}$$

gives a Gaussian process $\{Z(f)\}$ for which

$$\operatorname{cov}[Z(f),Z(g)]=\sum_{n=0}^{\infty}\sum_{k=0}^{2^{n-1}}\sum_{m=0}^{\infty}\sum_{\ell=0}^{2^{m-1}} a_{n,k}b_{m,\ell}\operatorname{cov}[Z_{n,k},Z_{m,k}]=\sum_{n=0}^{\infty}\sum_{k=0}^{2^{n-1}} a_{n,k}b_{n,k}\,.$$

In particular we have

$$\operatorname{var}[Z(f)]=\sum_{n=0}^{\infty}\sum_{k=0}^{2^{n-1}} a_{n,k}^2=\int_0^1|f(t)|^2\,dt\,,$$

where the latter equality is given by Parseval's identity (see page 85 of Rudin's *Real & Complex Analysis*). In the following example we see that Brownian motion can again be constructed by applying Z to a particular collection of functions.

1.1 Example (*Lévy's Construction*) Setting $B_t=Z(\mathbf{1}_{[0,t)})$ for $t\in[0,1]$, we have

$$a_{n,k}=\int_0^1 mathbf1_{[0,t)}h_{n,k}(s)\,ds=\int_0^t h_{n,k}(s)\,ds$$

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and

$$B_t = \sum_{n=0}^{\infty} \sum_{k=0}^{2^{n-1}} Z_{n,k} \int_0^t h_{n,k}(s)\,ds.$$

Since Z is linear (with respect to f), for $s \leq t$ we have

$$\operatorname{cov}[B_t - B_s] = \operatorname{cov}[Z(\mathbf{1}_{[0,t)}) - Z(\mathbf{1}_{[0,s)})] = \operatorname{cov}[Z(\mathbf{1}_{[s,t)})] = \|\mathbf{1}_{[s,t)}\|_2^2 = t - s\,.$$

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§ 2 | The Brownian Bridge

Let $U_t = (B_t \mid B_1 = 0)$ for $0 \leq t \leq 1$. Since $B_1 = Z_{0,0}$, we have

$$U_t = (B_t \mid Z_{0,0} = 0) = \sum_{n=1}^{\infty} \sum_{k=1}^{2^{n-1}} Z_{n,k} \int_0^t h_{n,k}(s)\,ds = B_t - tB_t.$$