Poisson–Gamma Aggregate Loss: Exam Template

1. Model Specification

- Claim frequency:
$$N \sim \text{Poisson}(\lambda).$$
- Claim severities (independent of N):
$$X_i \sim \text{Gamma}(\alpha, \beta) \quad \text{(rate parameterisation)}.$$
- Aggregate loss:
$$S = \sum_{i=1}^{N} X_i, \qquad S = 0 \text{ if } N = 0.$$
- Assumptions:
  - Independence between N and $\{X_i\}$.
  - Identically distributed severities.
  - Parameters λ, α, β are given or estimated.

2. Monte Carlo Method (Describe This Exactly)

- Choose M simulations (e.g., $M = 50{,}000$ or $100{,}000$ for stability).
- For each simulation m:
$$N_m \leftarrow \texttt{rpois}(1, \lambda)$$
$$S_m = \begin{cases} \sum_{i=1}^{N_m} X_{mi}, & N_m > 0, \\ 0, & N_m = 0, \end{cases}$$
where
$$X_{mi} \leftarrow \texttt{rgamma}(1, \alpha, \beta).$$
- Store all results in a vector:
$$S = (S_1, \dots, S_M).$$
- Large M ensures accurate tail estimation for the 99.5th percentile.

3. Empirical (Non-Parametric) Statistics

Report:

- $\text{mean}(S)$,
- $\text{median}(S)$,
- $\text{sd}(S)$,
- sample skewness:
$$\text{skew}(S) = \frac{E[(S - \bar{S})^3]}{\text{sd}(S)^3},$$
- empirical quantiles:
$$q_p = \text{quantile}(S, p), \qquad p \in \{0.50, 0.75, 0.90, 0.95, 0.99, 0.995\}.$$

Key Interpretation Points

- mean > median $\Rightarrow$ right skew (expected for Gamma).
- skewness > 0 indicates heavy right tail.
- 99.5th percentile is used as a 1-in-200 solvency measure.
- If the distribution is very skewed, normal approximations become unreliable.

4. Analytic (Parametric) Moments

For $X \sim \text{Gamma}(\alpha, \beta)$ (rate parametrisation):

$$E[X] = \frac{\alpha}{\beta}, \qquad \text{Var}(X) = \frac{\alpha}{\beta^2}.$$

For $S = \sum_{i=1}^{N} X_i$, $N \sim \text{Poisson}(\lambda)$:

$$E[S] = \lambda E[X] = \frac{\lambda\alpha}{\beta},$$

$$\text{Var}(S) = \lambda\left(\text{Var}(X) + (E[X])^2\right) = \lambda\left(\frac{\alpha}{\beta^2} + \frac{\alpha^2}{\beta^2}\right).$$

Normal-approximate 99.5% quantile:

$$q_{0.995}^{\text{Normal}} = E[S] + z_{0.995}\sqrt{\text{Var}(S)}, \qquad z_{0.995} = 2.5758.$$

Interpretation

- The analytic mean and variance should closely match Monte Carlo estimates.
- For right-skewed distributions, the Normal approximation *underestimates* the 99.5th percentile.
- Therefore, solvency assessment should use the empirical (simulation-based) quantile.

5. Parametric vs Non-Parametric Comparison

Checklist of Comments

- Agreement of mean: should be close (LLN).
- Agreement of variance: should also be close.
- Tail Behaviour:
  - Simulation captures skewness.
  - Normal-based approximation does not.
- Recommendation:
  - Use empirical 99.5% quantile for capital/reserves.
  - Normal approach acceptable only for light-tailed portfolios.

6. Visualisation (If Required)

- Histogram of S with vertical line at 99.5th percentile.
- Interpretation to include:
  - Long right tail.
  - Concentration of mass near the median.
  - Extreme events drive capital requirements.

7. Final Written Answer Structure

Use This for Full Marks

1. Clearly state Poisson–Gamma assumptions and independence.
2. Describe Monte Carlo procedure step-by-step.
3. Present table of empirical statistics:

   $$\text{mean, median, sd, skewness, quantiles (esp. 99.5\%).}$$

4. Compute analytic mean and variance and display alongside MC values.
5. Compare analytic vs MC values; interpret differences.
6. Comment explicitly on:
   - Heavy-tailed behaviour,
   - Skewness,
   - Reliability of Normal approximation,
   - Why empirical 99.5% is preferred.
7. Conclude with a robust statement about solvency/risk (e.g. Monte Carlo estimate should be used as the capital requirement).

8. One-Sentence Conclusions to Choose From

- "The aggregate loss distribution is right-skewed, so the simulation-based 99.5th percentile is the appropriate solvency measure."
- "The Normal approximation underestimates tail risk due to positive skewness."
- "Monte Carlo results and analytic moments agree for mean/variance but diverge in the tail, confirming the need for a non-parametric tail estimate."