Assignment #3

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Question 1

Let $A = \{1, 2, 3, 4, 5\}$ and let $B = \{6, 7, 8, 9\}$.
Prove or disprove the following statements.

a. For all functions $f : A \to B$, there exists a function $g : B \to A$ so that $g \circ f = I_A$

Solution. This statement is false. The negation of the statement is: There exists a function $f : A \to B$ so that for all functions $g : B \to A$, $g \circ f \neq I_A$.

Proof. Let $f = \{(1, 6), (2, 6), (3, 6), (4, 6)\}$. Suppose g is a function $g : B \to A$. Argue by contradiction: suppose $g \circ f = I_A$. Then, $g(f(1)) = g(6) = 1$ and $g(f(2)) = g(6) = 2$, so g must have elements $(6, 1)$ and $(6, 2)$. Then, g is not a function, since 6 does not have a unique element. But g is a function, this is a contradiction, so the original claim cannot be true. Therefore, $g \circ f \neq I_A$.

b. For all functions $f : A \to B$, there exists a function $g : B \to A$ so that $f \circ g = I_B$

Solution. This statement is false. The negation of the statement is: There exists a function $f : A \to B$ so that for all functions $g : B \to A$, $f \circ g \neq I_B$.

Proof. Let $f = \{(1,6),(2,6),(3,6),(4,6)\}$. Suppose g is a function $g : B \to A$. Argue by contradiction: suppose $f \circ g = I_B$. Then, $f(g(7)) = 7$, but $7 \notin Im(f)$, so $f(g(7)) \neq 7$. This is a contradiction, so the original claim cannot be true. Therefore, $f \circ g \neq I_B$.

Therefore, $g \circ f \neq I_A$.

c. There exist functions $f : A \to B$ and $g : B \to A$ so that $g \circ f = I_A$

Solution. This statement is false. The negation of the statement is: For all functions $f : A \to B$ and $g : B \to A$, $g \circ f \neq I_A$.

Proof. Suppose functions $f : A \to B$, $g : B \to A$. Argue by contradiction: suppose $g \circ f = I_A$. Then, $g(f(x)) = x$ for all $x \in A$.

d. There exist functions $f : A \to B$ and $g : B \to A$ so that $f \circ g = I_B$

Solution. This statement is true.

Proof. Let function $f : A \to B$ be defined by

$$f = \{(1,6),(2,7),(3,8),(4,9),(5,9)\}$$

and let the function $g : B \to A$ be defined by

$$g = \{(6,1),(7,2),(8,3),(9,4)\}$$

Then

$$f \circ g = \{(6,6),(7,7),(8,8),(9,9)\} = I_B$$

Where $f : A \to B$ and $g : B \to A$ so that $f \circ g = I_B$.

Question 2

Prove or disprove each of the following statements.

a. For every nonempty set A, if f is a function from A to A so that $f \circ f = I_A$, then f is one-to-one and onto.

Solution. This statement is true.

Proof. Suppose A is a nonempty set. Suppose that $f : A \to A$ and $f \circ f = I_A$. Then, for all $x \in A$, $f(f(x)) = x$. Suppose we have elements $a, b \in A$ such that $f(a) = f(b)$. Then, $f(f(a)) = f(f(b))$, and therefore $a = b$, so f must be one-to-one. Now suppose $c \in A$. Let $d \in A$, $d = f(c)$. Then, $f(d) = f(f(c)) = c$, so f must be onto.

b. For every nonempty set A, if f is a one-to-one and onto function from A to A, then $f \circ f = I_A$.

Solution. This statement is false. The negation of the statement is: There exists nonempty set A where function $f : A \to A$ is one-to-one and onto, but $f \circ f \neq I_A$.

Proof. Let $A = \{1, 2, 3\}$. Now, let $f = (1, 2), (2, 3), (3, 1)$. Observe that the function is one-to-one and onto. But, $f \circ f = \{(1, 3), (2, 1), (3, 2)\} \neq \{(1, 1), (2, 2), (3, 3)\} = I_A$.

c. There exists a function f from A to A where $A = \{1, 2, 3, 4\}$ so that $f \circ f = I_A$ and $f(x) \neq x$ for all $x \in A$.

Solution. This statement is true.

Proof. Let $f = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$. Then

$$f \circ f = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\} = I_A$$

And $1 \neq 4, 2 \neq 3, 3 \neq 2, 4 \neq 1$, so $\forall x \in A, f(x) \neq x$.

d. There exists a function f from A to A where $A = \{1, 2, 3, 4, 5\}$ so that $f \circ f = I_A$ and $f(x) \neq x$ for all $x \in A$.

Solution. This statement is false. The negation of the statement is: For all functions f from A to A where $A = \{1, 2, 3, 4, 5\}$, $f \circ f \neq I_A$ or $f(x) = x$.

Proof.

e. For parts (c) and (d), if such a function exists, count the number of such functions. Give a detailed recipe and simplify your answer to a number.

Solution. One recipe for such a function in part (c) is:

(a) Choose a pair of numbers (such that f(a) = b and f(b) = a) ($\binom{4}{2}$ ways).

(b) Choose the other pair of numbers (1 way). THIS DOESNT INCLUDE IDENTIYTY

So there are

$$\begin{aligned}\binom{4}{2} &= \frac{4!}{2!(4-2)!} \\ &= \frac{4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)(2 \cdot 1)} \\ &= 3 \cdot 2 \\ &= 6\end{aligned}$$

such functions.

Question 3

Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = \lfloor 2x \rfloor - x$, for all $x \in \mathbb{R}$.

a. Prove that for all $x \in \mathbb{R}$, $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

Proof. Suppose $x \in \mathbb{R}$. By definition of floor, $\lfloor x \rfloor \leq x < \lfloor x + 1 \rfloor$.

b. Prove that f is one-to-one.

c. Prove that f is onto.