Here is something ordinary. Let a finite group G act on a k-vector space V. We denote by V^G and V_G the invariants and coinvariants of this action, respectively, and manifestly have a map $V^G \to V_G$. We also have a *norm map* $V_G \to V^G$ given by $[v] \mapsto \sum_g gv$. The composite is multiplication by $|G|$. If $k = \mathbb{Q}$ or $\mathbb{C}$, then this is an isomorphism, and the norm map exhibits a "canonical duality" $V_G \cong V^G$. If the characteristic is nonzero, this generally does not occur.

Here is something surprising. Morava K-theories let us recreate this situation in "intermediary characteristic." Fix a prime p and finite group G and consider a $K(n)$-local spectrum X with a G-action. We again have a canonical norm map $X_G \to X^G$, and a 1996 result of Hovey, Sadofsky, and Greenlees says it is an isomorphism. Rognes obtained related results in 2005, but these lacked good context. In 2013, Hopkins and Lurie fit all this into a very broad framework. They say that rather than a $K(n)$-local space X with a G-action, i.e. a map $BG \to Sp_{K(n)}$, we should consider *any* map $Y \to Sp_{K(n)}$ such that Y has finite and finitely-many homotopy groups. They exhibit a "generalized norm map" and show it is *still* an isomorphism

$$\operatorname{colim}(Y \to Sp_{K(n)}) \xrightarrow{\sim} \lim(Y \to Sp_{K(n)}).$$

The surprise here is twofold. First, the ordinary result depended upon multiplication by $|G|$ being invertible and thus the zero characteristic, yet its analogue in "intermediary characteristic" still holds. Second, the analogue holds much more generally than you would probably guess given the ordinary result.

This intiated a general study of (1) the behaviour of canonical "norm maps" between colimit and limits, and (2) categories C such that for every space Y with finiteness conditions on $\pi_n Y$ and functor $Y \to C$, this norm map exists and is an isomorphism, which we interpret as a very general form of "duality" (e.g., I think this is how Lurie formulates "nonabelian Poincaré duality"). *Ambidexterity* is about when these canonical maps in (1) are isomorphisms. A category C is called *m-semiadditive* when, as in (2), we have limit-colimit duality for every m-finite space Y and map $Y \to C$. Examples abound in chromatic, K-theory, ... The above discussion says that $Sp_{K(n)}$ is ∞-semiadditive. It turns out that $Sp_{T(n)}$ is too.

This formulation (Hopkins and Lurie's) of ambidexterity and semiaddivity is centered around these *canonical norm maps*, which are complicated. I can give a clearer explanation. This starts from something familiar and primitive: summation. (In fact, you can start farther back, with the notion of zero.)

Here's how this goes. **In ordinary category theory**, C is called *semiadditive* if we can coherently "sum" pairs of objects, which means it is pointed and has finite biproducts. Semiadditivity is fundamentally related to the notion of abelian monoids: a category C with coproducts is semiadditive if and only if the forgetful $CMon(C) \to C$ is an equivalence, and there is a "free commutative monoid on one object" property somewhere in there.

A remark that will have ramifications: Γ-spaces are one way to access infinite loop spaces, and Γ-sets are one way to access ordinary commutative monoids. This picture sets up a preliminary fact: commutative monoids are precisely product-preserving functors $Span(Fin) \to Set$.

Now we ask: given an ∞-category C with coproducts, can we make a similar statement? The answer is yes, C is 0-semiadditive (pointed with biproducts) if and only if C is equivalent to the category of its coherent abelian monoids, i.e. Γ-objects, i.e. functors $Span(Fin) \to C$. We learn that if X, Y are objects of C, then $Map(X, Y)$ inherits a canonical $\mathbb{E}_\infty$-space structure.

It is natural to then ask about m-semiadditive categories for $m > 0$. **It turns out that everything in the above paragraph is secretly striated in m.** That is, we should think of an $\mathbb{E}_\infty$-space as a 0-(commutative monoid), which is part of a notion of m-commutative monoids we define by

$$CMon_m(C) := Fun^\times(Span(m\text{-finite spaces}), C).$$

Analogous to before, C is m-semiadditive if and only if $CMon_m(C) \to C$ is an equivalence. And if C is m-semiadditive, then it is naturally enriched in $CMon_m(Spaces)$. Recall that this is a concrete statement: it means that the mapping spaces between (say) $K(n)$-local spectra are not only $\mathbb{E}_\infty$-spaces, but in fact something much more structured having strong duality properties.

It remains to ask whether any of this matters. Should we really think of $\mathbb{E}_\infty$-spaces as 0-monoids? Probably not in general, but semiadditivity appears fundamental: the control and conceptual benefit we gain by studying this finer structure has helped us to study known things. It appears in BSY's proof of redshift. Ishan tells me it was important in disproving the telescope conjecture. And a host of people are using it to import arithmetic phenomena to chromatic homotopy theory: roots of unity, cyclotomic extensions, Fourier transforms, Grothendieck-Witt theory, ... which has supposedly had good conceptual and calculational consequences.

Oh, I forgot to say: the idea to define "higher monoids" and use them to reinterpret semiaddivity is thanks to Yonatan Harpaz. He did this in 2018. His was the first paper I set out to understand. I think this REU paper would be a great service—this story has not been properly told (not even in Yonatan's paper). And in addition, I could briefly cover Γ-sets/spaces/... , which I also think would be a nice resource to have out there.