Topology Vol. 32, No. 2, pp. 325–328, 1993.
Printed in Great Britain

0040-9383/93 $6.00 + 0.00

ON FUNDAMENTAL THEOREMS OF ALGEBRAIC *K*-THEORY

RANDY MCCARTHY†

(*Received* 8 *January* 1992)

INTRODUCTION

GRAYSON [2] and Staffeldt [6] have given simplified proofs of some of the foundational theorems of algebraic K-theory [5]. Especially in Staffeldt's treatment these theorems are shown to be consequences, via standard homotopy theoretical methods, of the so-called *additivity theorem* which therefore assumes the role of a basic result in algebraic K-theory.

There are two formulations of the additivity theorem, one in terms of original *Q-construction* of Quillen and another in terms of the somewhat more general *S.-construction* of Waldhausen. Up till now the only available proof of the additivity theorem in the latter context [8] has been rather complicated.

On the other hand, work of the author not directly related to K-theory (i.e. Hochschild homology and cyclic homology, cf. [4]) involved a result formally similar to, and in fact motivated by, the additivity theorem. In retrospect it turns out that the proof of that result, when suitably modified, will give a new, and simpler, proof of the additivity theorem (in the general form). The purpose of the present paper is to give a short and self-contained account of that proof.

Notation. We will use without further comment the notion of a *category with cofibrations* (cf. the first few lines after the introduction of [8]) as well as the definition of the *S.-construction* (cf. the first two pages of Section 1.3 of [8]). We recall some notation, introduced in [1], which is extremely convenient for the present purposes. If X is a simplicial set, we let XR and XL denote the bisimplicial sets $XR([m],[n]) = X([n])$ and $XL([m],[n]) = X([m])$ (with trivial simplicial maps in the first and second variables respectively).

Let $\mathscr{C}$ and $\mathscr{D}$ be categories with cofibrations, $F: \mathscr{C} \to \mathscr{D}$ an exact functor, and let $S.F$: $S.\mathscr{C} \to S.\mathscr{D}$ denote the simplicial functor induced by F. For $m, n \in \mathrm{N}$, we let the diagram $(*)$

$$\left(\frac{0 = C_0 \rightarrowtail C_1 \rightarrowtail \dots \rightarrowtail C_m}{0 = D_0 \rightarrowtail D_1 \rightarrowtail \dots \rightarrowtail D_m \rightarrowtail E_0 \rightarrowtail E_1 \rightarrowtail \dots \rightarrowtail E_n}\right)$$

denote the following information (suppressing the chosen quotients):

$$(0 = D_0 \rightarrowtail D_1 \rightarrowtail \dots \rightarrowtail D_m \rightarrowtail E_0 \rightarrowtail \dots \rightarrowtail E_n) \in S_{m+n+1}\mathscr{D}$$
$$(0 = C_0 \rightarrowtail C_1 \rightarrowtail \dots \rightarrowtail C_m) \in S_m\mathscr{C}$$

†This research was partially funded by the Institut de Recherche Mathématique Avancée in Strasbourg, France during the spring of 1988.

plus the identity in $S_m\mathscr{D}$:

$$\begin{pmatrix} 0 = & F(C_0) \rightarrowtail & F(C_1) \rightarrowtail & \dots \rightarrowtail & F(C_m) \\ & \| & \| & & \| \\ 0 = & D_0 \rightarrowtail & D_1 \rightarrowtail & \dots \rightarrowtail & D_m \end{pmatrix}$$

Definition (following [1]): We let $S.F|\mathscr{D}$ denote the following bisimplicial set:

$$(S.F|\mathscr{D})([m],[n]) = \{\textit{diagrams of the type } (*) \textit{ above}\}$$

We let π: $S.F|\mathscr{D} \to S.\mathscr{C}L$ denote the natural projection of bisimplicial sets and we let ρ denote the natural projection of bisimplicial sets ρ: $S.F|\mathscr{D} \to S.\mathscr{D}R$ defined by sending the diagram $(*)$ to $(0 = E_0/E_0 \rightarrowtail E_1/E_0 \rightarrowtail \dots \rightarrowtail E_n/E_0)$. The following is essentially theorem A′ of [1] which is one possible reformulation of theorem A from [5] in the setting of simplicial sets.

PROPOSITION. *The following are equivalent*:

(a) *The simplicial map $S.F: S.\mathscr{C} \to S.D$ is a homotopy equivalence*
(b) *The bisimplicial map ρ: $(S.F|\mathscr{D}) \to (S.\mathscr{D}R)$ is a homotopy equivalence*

Proof. There is a commutative diagram of bisimplicial sets:

$$\begin{array}{ccccc} S.\mathscr{D}R & \longleftarrow & S.F|\mathscr{D} & \xrightarrow{\simeq} & S.\mathscr{C}L \\ \| & & \downarrow F & & \downarrow F \\ S.\mathscr{D}R & \xleftarrow{\simeq} & S.id_{\mathscr{D}}|\mathscr{D} & \xrightarrow{\simeq} & S.\mathscr{D}L \end{array}$$

One can show the arrows marked $\simeq$ are homotopy equivalences by first fixing a simplicial direction and recalling the fact that the nerve of a category with either an initial or final object is contractible and then applying the *realization lemma* (see for example Lemma 5.1 of [7]).

For each $n \in \mathbf{N}$, we define the simplicial map E_n from $S.F|\mathscr{D}(-,[n])$ to itself by sending $(*)$ to

$$\left(\frac{0 = 0 = \dots = 0}{0 = 0 = \dots = 0} = E_0/E_0 \rightarrowtail E_1/E_0 \rightarrowtail \dots \rightarrowtail E_n/E_0\right)$$

COROLLARY. *If the simplicial maps E_n are homotopy equivalences for all $n \in \mathbf{N}$ then the simplicial map $S.F: S.\mathscr{C} \to S.\mathscr{D}$ is a homotopy equivalence.*

Proof. For $n \in \mathbf{N}$, the simplicial map $\rho(-,[n])$ from $S.F|\mathscr{D}(-,[n])$ to $S.\mathscr{D}R(-,[n])$ is split by a simplicial map I_n such that $I_n \circ \rho(-,[n])$ is E_n. Part (b) of the previous proposition is now satisfied by the realization lemma.

For $\mathscr{C}$ a category with cofibrations, we let $E(\mathscr{C})$ denote the category with objects the cofibration sequences $A \rightarrowtail C \twoheadrightarrow B$ in $\mathscr{C}$. This is naturally a category with cofibrations. The following proposition is one of several equivalent formulations of the additivity theorem (see Proposition 1.3.2 of [8]).

ADDITIVITY THEOREM ([5] Section 3 and [8] Section 1.4). *The exact functor F: $E(\mathscr{C}) \to \mathscr{C} \times \mathscr{C}$ defined by sending $(A \rightarrowtail C \twoheadrightarrow B)$ to (A, B) induces a homotopy equivalence $S.F$: $S.E(\mathscr{C}) \to S.(\mathscr{C}) \times S.(\mathscr{C})$.*

Proof. We will show that in this situation the above map E_n is a homotopy equivalence for all $n \in \mathbb{N}$. Define the simplicial map Γ from $S.F|\mathscr{C}^2(-,[n])$ to itself by taking an arbitrary simplex $e \in S.F|\mathscr{C}^2([m],[n])$ like

$$\left[\begin{array}{l}
\begin{array}{cccccccc}
0 = & A_0 & \rightarrowtail & A_1 & \rightarrowtail & \dots & \rightarrowtail & A_m \\
& \downarrow & & \downarrow & & & & \downarrow \\
0 = & C_0 & \rightarrowtail & C_1 & \rightarrowtail & \dots & \rightarrowtail & C_m \\
& \downarrow & & \downarrow & & & & \downarrow \\
0 = & B_0 & \rightarrowtail & B_1 & \rightarrowtail & \dots & \rightarrowtail & B_m
\end{array} \\
\hline
0 = A_0 \rightarrowtail A_1 \rightarrowtail \dots \rightarrowtail A_m \rightarrowtail S_0 \rightarrowtail S_1 \rightarrowtail \dots \rightarrowtail S_n \\
0 = B_0 \rightarrowtail B_1 \rightarrowtail \dots \rightarrowtail B_m \rightarrowtail T_0 \rightarrowtail T_1 \rightarrowtail \dots \rightarrowtail T_n
\end{array}\right]$$

and setting $\Gamma(e)$ to be:

$$\left[\begin{array}{l}
\begin{array}{cccccccc}
0 = & 0 & = & 0 & = & \dots & = & 0 \\
& \downarrow & & \downarrow & & & & \downarrow \\
0 = & B_0 & \rightarrowtail & B_1 & \rightarrowtail & \dots & \rightarrowtail & B_m \\
& \| & & \| & & & & \| \\
0 = & B_0 & \rightarrowtail & B_1 & \rightarrowtail & \dots & \rightarrowtail & B_m
\end{array} \\
\hline
0 = 0 = 0 = \dots = 0 = S_0/S_0 \rightarrowtail S_1/S_0 \rightarrowtail \dots \rightarrowtail S_n/S_0 \\
0 = B_0 \rightarrowtail B_1 \rightarrowtail \dots \rightarrowtail B_m \rightarrowtail T_0 \rightarrowtail T_1 \rightarrowtail \dots \rightarrowtail T_n
\end{array}\right]$$

Let X be the subspace of $S.F|\mathscr{C}^2(-,[n])$ determined by elements e such that all the A_i's are 0. Thus, Γ is a retraction of $S.F|\mathscr{C}^2(-,[n])$ to X. The result will follow from:

(1) Γ is homotopic to the identity

(2) $E_n|_X$ is homotopic to the identity of X

The homotopy for (2) follows from the fact that X behaves simplicially as the nerve of a category with a final object. We motivate the homotopy for (1) by noting that the lower row of A_i's and S_j's also behaves simplicially like the nerve of a category with a final object. One then needs to check that the usual contracting homotopy for this setting can actually be lifted to the entire diagram by the use of push-outs.

The homotopy for (1) needs a more detailed description. We use the notation of a simplicial homotopy as found in section 5 of [3]. Let Δ denote the category of non empty finite ordered sets and order preserving set maps with one object $[n] = \{0 < 1 < \dots < n\}$ for each $n \in \mathbb{N}$. A simplicial homotopy h between the simplicial maps f and g from X to Y is a simplicial map h from $X \times Hom_\Delta(-,[1])$ to Y such that

$$h(x \times \delta_0 \sigma_{q-1} \circ \dots \circ \sigma_0) = f(x)$$

$$h(x \times \delta_1 \sigma_{q-1} \circ \dots \circ \sigma_0) = g(x)$$

Such a homotopy h is determined by a sequence of functions $h_i : X_q \to Y_{q+1}$ $(0 \le i \le q)$ satisfying combinatorial relations to the face and degeneracy maps of X and Y. We recall that h is recovered from the $\{h_i\}$ by the prescription

$$h(x \times \sigma_{q-1} \circ \dots \sigma_i \circ \dots \circ \sigma_0) = d_{i+1} h_i(x)$$

Another equivalent definition of simplicial homotopies (see for example [8], p. 335) is as a natural transformation from X^* to Y^* where X^* is defined to be the functor from $\Delta/[1]$ (the category of objects over [1]; the objects are the maps $[n] \to [1]$) to sets produced from X by $(\Delta/[1])^{op} \to \Delta^{op} \xrightarrow{X} Sets$. The translation from the previous definition to this can be obtained by defining the natural transformation h^* by $h^*(\eta) = h(- \times \eta)$. Now, for the current situation, we will simply define the h_i's and leave it to the interested reader to check that these do assemble to provide a simplicial homotopy.

The homotopy for (1) is given by the simplicial homotopy h defined by taking $e \in S.F|\mathscr{C}^2([m], [n])$, setting $X_j = C_j \coprod_{A_j} S_0$, and letting $h_i(e)$ be (we give only the left-hand side of the diagram, the rest is clear):

$$\left[\begin{array}{ccccccccccccccc}
0 = A_0 & \rightarrowtail & A_1 & \rightarrowtail & \cdots & \rightarrowtail & A_i & \rightarrowtail & S_0 & = & S_0 & = & \cdots & = & S_0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 = C_0 & \rightarrowtail & C_1 & \rightarrowtail & \cdots & \rightarrowtail & C_i & \rightarrowtail & X_i & \rightarrowtail & X_{i+1} & \rightarrowtail & \cdots & \rightarrowtail & X_m \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 = B_0 & \rightarrowtail & B_1 & \rightarrowtail & \cdots & \rightarrowtail & B_i & = & B_i & \rightarrowtail & B_{i+1} & \rightarrowtail & \cdots & \rightarrowtail & B_m \\
\hline
0 = A_0 & \rightarrowtail & A_1 & \rightarrowtail & \cdots & \rightarrowtail & A_i & \rightarrowtail & S_0 & = & S_0 & = & \cdots & = & S_0 \\
0 = B_0 & \rightarrowtail & B_1 & \rightarrowtail & \cdots & \rightarrowtail & B_i & = & B_i & \rightarrowtail & B_{i+1} & \rightarrowtail & \cdots & \rightarrowtail & B_m
\end{array}\right]$$

For $i = 0, 1, \ldots, m$ the crucial row of the diagram (the A_i row) is given by

$$\begin{array}{l}
h_0 : 0 = A_0 \rightarrowtail S_0 = S_0 = \ldots \\
h_1 : 0 = A_0 \rightarrowtail A_1 \rightarrowtail S_0 = \ldots \\
\vdots \quad \vdots \\
h_m : 0 = A_0 \rightarrowtail A_1 \rightarrowtail \ldots \rightarrowtail S_0 = S_0 \rightarrowtail \ldots
\end{array}$$

This completes the description of the homotopies and thus the argument.

REFERENCES

1. H. Gillet and D. Grayson: On the loop space of the Q-construction, *Illinois J. Math.* **31** (1987), 574–597.
2. D. Grayson: Exact sequences in algebraic K-theory, *Illinois J. Math.* **31** (1987), 598–617.
3. P. May: *Simplicial objects in algebraic topology*, D. Van Nostrand (1967).
4. R. McCarthy: *Cyclic homology of an exact category*, Thesis at Cornell University (1990).
5. D. G. Quillen: Higher algebraic K-theory I, *Springer Lecture Notes in Math.* **341** (1973), 85–147.
6. R. Staffeldt: On fundamental theorems of algebraic K-theory, *K-theory* **1** (1989), 511–532.
7. F. Waldhausen: Algebraic K-theory of generalized free products, *Ann. Math.* **108** (1978), 135–256.
8. F. Waldhausen: Algebriac K-theory of spaces, *Springer Lecture Notes in Math.* **1126** (1985), 318–419.

Department of Mathematics
Brown University
Providence, RI 02912
U.S.A.