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Matthew Niemiro

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I January

> Prior to the advent of the brain, there was no color and no sound in the universe, nor was there any flavor or aroma...before brains the universe was also free of pain and anxiety.
>
> *Roger Sperry*

New year. And a new notebook. Last one was getting tedious to compile, and it was about time I tweak the TEXanyway. I should start hearing back from graduate schools soon. I am pretty excited to start grad school, and now that the anxiety of applications is gone, I can feel and and appreciate how much time six years (the duration I plan to be in graduate school) is to do things. I have some math-related new year's resolutions: learn some combinatorics, GAP 4, probability theory and ergodic theory, and number theory. Maybe special mention to Cohen-Lenstra statistics. This list reflects personal interests I feel I have neglected. Bonus points if I can make things weird (i.e., categorical). This notebook remains a documentation of what I'm studying in higher category theory.

I.1 (1/1) Cardinality

A "set of all sets" is set-theoretically impossible, although there is a proper class T of such things. A group is a set, hence *the class of groups of order* 2 is at least the size of T since each set $S \in T$ gives rise to a group of order 2, namely $\{S, S\}$ with whatever[1] group structure you'd like. In particular, there is a *proper* class of groups of order 2. The lesson: for many purposes, we should count up to isomorphism.

Let X be a finite set and $\sim$ an equivalence relation. The quotient $X/\sim$ is finite, hence its cardinality is well-defined and equals the number of equivalence classes, which we can count "over X" via the easy formula

$$|X/\sim| = \sum_{x \in X} \frac{1}{|[x]|}.$$

Notice this formula does not use the relation $\sim$ in any meaningful way. In certain situations, we may not like this.

Observation I.1. Let X be a finite set and $\sim$ an equivalence relation. *A priori*, the natural distribution on a finite set X is the uniform one. You can also put the uniform distribution on $X/\sim$, but it is more natural that the likelihood of obtaining $[x]$ is induced by the likelihood of obtaining any $y \in [x]$. These quantities should be inversely proportional(?), hence $\mathbb{E}([x]) = \sum(|[x]|)^{-1}$.) Then another notion of "size" for $X/\sim$ is perhaps the sum $\sum_{[x]} |[x]|^{-1}$.

Example I.1. Finite group $G = \mathbb{Z}/2\mathbb{Z}$ acting on the set

$$X := \{1, 2, 3, 4, 5\}$$

as the cycle (15)(24). Then X/G has three points, those being the orbits $\{1, 5\}$, $\{2, 4\}$, and $\{3\}$. Both X and X/G are finite sets. If we concede that the natural dsitribution on a finite set is the uniform one, however, then the induced distribution on X/G is *not* the natural one. See, the induced distribution weighs the listed orbits $2/5, 2/5$, and $1/5$, respectively, but the uniform weighs them each $1/3$.

Our use of probability here is not essential. It just gives a natural language to talk about the "size" of objects—bigger objects should take up more of the distribution.

Remark I.1. Let G be any finite group acting on a finite set X. If the action is free, then there is no issue like the above, for freeness is equivalent to trivial stabilizers, which implies that the orbits of G have uniform size, namely the orbits are of size $|G|$.

[1] Which ever. Of the two.

Remark I.2. The category of finite sets is the "categorification" of the natural numbers. We can divide two natural numbers—how to divide two finite sets S and T? If T is a finite group acting on S, we may consider the quotient S/T. By the previous remark, if this action is free, then $|S/T| = |S|/|T|$. Hence, quotienting sets by finite free actions is a "categorification" of division. However, this process only produces finite sets (categorified naturals, even though we should get rationals), and furthermore does *not* work for non-free actions (e.g., in the example, $|S/T| = 3$ but $|S| = 5$ and $|T| = 3$). These issues are quite related—to address the latter, we will need a new type of object, for which we will need a new notion of cardinality, from which we get the rationals.

The problem is that $3 \in X$ has more automorphisms (a smaller orbit), so it "appears larger" in X/G than it "really is" because the set X/G does not see automorphisms. This suggests that the "size" of an element should be inversely proportional to its number of automorphisms. We therefore should replace sets with objects that carry automorphism data (groupoids) and extend the notion of cardinality to account for that data.

Definition I.1. Let X be an ordinary groupoid. Define its homotopy cardinality as the sum

$$|X| = \sum_{[x]\in\pi_0 X} \frac{1}{|\mathrm{Aut}(x)|}.$$

Example I.2. Let $G = \mathbb{Z}/2\mathbb{Z}$ act on a five-element X as the cycle $\sigma = (15)(24)$ again. Consider the associated action groupoid $X//G$ [nLab]. It has $\mathrm{Ob}(X//G) = X$ and a morphism $g : x \to y$ if and only if $\exists g : gx = y$. In this case, we can compute its homotopy cardinality as

$$|X| = \frac{1}{|\mathrm{Aut}(1)|} + \frac{1}{|\mathrm{Aut}(2)|} + \frac{1}{|\mathrm{Aut}(3)|} = \frac{1}{|\{\sigma^2 = \mathrm{id}\}|} + \frac{1}{|\{\sigma^2 = \mathrm{id}\}|} + \frac{1}{|\{\mathrm{id}, \sigma\}|} = 5/2.$$

This works. In the above example, it weights elements in the quotient correctly, namely prescribing 3 a weight of $1/2$. More generally, given $G \curvearrowright X$, we have $|X//G| = |X|/|G|$ as desired. One can crank out more exotic examples. You may ponder the cardinality of the core of your favorite category. For example, $|\mathsf{FinSet}^{\cong}| = e$.

Homotopy cardinality is homotopy invariant. Furthermore, it has the essential properties of ordinary cardinality: it is additive and multiplicative over products and coproducts, respectively. How to extend this notion to ∞-groupoids? Let me just give you the definition: for an ∞-groupoid X, we define its homotopy cardinality as the sum

$$|X| := \sum_{[x]\in\pi_0 X} \prod_{n=1}^{\infty} |\pi_n(X,x)|^{(-1)^k} = \sum_{[x]\in\pi_0 X} \frac{|\pi_2(X,x)| \cdot |\pi_4(X,x)| \cdots}{|\pi_1(X,x)| \cdot |\pi_3(X,x)| \cdots}.$$

This satisfies a more general multiplicativity property: given a fibration $F \to E \to B$ over a connected base B, the homotopy long exact sequence yields $|E| = |F||B|$. (This is more general because fiber bundles are "twisted" cartesian products.) This tells us, for instance, that

$$|BG| = \frac{1}{|G|}.$$

What's with the alternation in the product? Crudely, it is a manifestation of an iterated inclusion-exclusion argument. I think of this informally: the cardinality of the underlying set of X is the number of its points. But homotopy theory says to replace sets with 0-groupoids, in which case the cardinality is the number o f connected components. For simplicity, suppose X is connected; then its underlying 1-groupoid has one point up to isomorphism, but that point may have automorphisms, accounted for by $\pi_1 X$, and we know that cardinality should be inversely proportional to this. But wait—those automor mtomorphisms "smaller." But wait—the automorphisms of automorphisms have automorphisms, accounted for by $\pi_3 X$, which by parallel reasoning make $|X|$ *smaller*. So on and so forth *ad infinitum*.

Remark I.3. We are implicitly assuming the defining series for $|X|$ converges. Call spaces for which this is the case tame.

Recall that a space X is called n-finite if $\pi_{k>n}X = 0$ and $\pi_{k\leq n}X$ is finite, and that X is called π-finite if it is n-finite for some n. It is clear that π-finite spaces are tame. Thus, π-finite spaces seem like a good category of spaces in which to think about homotopy cardinality.

Definition I.2. Write $\mathsf{S}_{\mathrm{fin}}$ for the ∞-category generated by a point under finite colimits. We also consider $\mathsf{S}_{n-\mathrm{fin}}$ and $\mathsf{S}_{\pi\text{-fin}}$, the full subcategories of n- and π-finite spaces.

Homotopy cardinality defines a functor[2] $\mathsf{S}_{\pi\text{-fin}} \to \mathbb{Q}_{\geq 0}$. It is the unique extension of the cardinality of finite sets that is homotopy invariant, additive w.r.t. disjoint unions, and multiplicative w.r.t. fibrations. See the answer to my MO question.

I.2 (1/4) Colimit completions and filtering classes

Here's a story I really like. Consider an ordinary category C. Its presheaf category $\mathrm{PShv}(\mathsf{C}) := \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ is the "free completion at colimits" of C in the following sense.

Theorem I.1. The Yoneda embedding $\text{よ} : \mathsf{C} \to \mathrm{PShv}(\mathsf{C})$ is a free cocompletion of C. That is, $\mathrm{PShv}(\mathsf{C})$ has small colimits, and if D admits small colimits, then restriction along よ defines a natural equivalence

$$\text{よ}^* : \mathrm{Fun}^{cocts}(\mathrm{PShv}(\mathsf{C}), \mathsf{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathsf{C}, \mathsf{D}).$$

This is standard. I think it also characterizes よ (and hence $\mathrm{PShv}(\mathsf{C})$) by the usual "thing satisfying universal property is unique" argument. Here is another characterization.

Definition I.3. Consider a presheaf $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. Its category of elements $\mathrm{el}\, F$ has (objects) transformations $\text{よ}c \to F$ and (arrows) transformations $\text{よ}c \to \text{よ}d$ such that the evident triangle commutes.

Proposition I.1 (Density)**.** There is a canonical map $\mathrm{el}\, F \to \mathsf{C}$. Its colimit canonically presents F, i.e.,

$$\mathrm{colim}(\mathrm{el}\, F \to \mathsf{C} \to \mathrm{PShv}(\mathsf{C})) \cong F.$$

The density theorem canonically associates to each presheaf $F : * \to \mathrm{PShv}(\mathsf{C})$ a diagram of representables $\mathrm{el}\, F \to \mathrm{PShv}(\mathsf{C})$, which we equivalently regard as its underlying diagram $\mathrm{el}\, F \to \mathsf{C}$. I think you can upgrade this to say "the category $\mathrm{PShv}(\mathsf{C})$ is *equivalent* to the category of diagrams in C."

We have roughly provided three equivalent definitions of $\mathrm{PShv}(\mathsf{C})$: it is (I) the functor category $\mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$, (II) the category of diagrams in C,[3] and (III) the free colimit completion of C. The equivalence (I) $\to$ (III) associates to F its diagram $\mathrm{el}\, F \to \mathsf{C}$. The equivalence (III) $\to$ (I) takes the colimit of the diagram. Either can be shown to fit the description (II).

Remark I.4. For a presheaf F, its category of elements can also be defined as the pullback below. The map $\mathrm{el}\, F \to \mathsf{C}$ is again evident. Perhaps because of this definition, we sometimes write $\mathsf{C}/F := \mathrm{el}\, F$.

$$\begin{tikzcd}
\mathrm{el}\, F \arrow[r, dashed] \arrow[d, dashed, "\pi_F"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{PShv}(\mathsf{C})_{/F} \arrow[d] \\
\mathsf{C} \arrow[r, "\text{よ}"] & \mathrm{PShv}(\mathsf{C})
\end{tikzcd}$$

Remark I.5. The category of elements C/F can also be defined as the *comma category* $* \downarrow F$. I honestly do not know about comma categories and am not sure I need to know about them right now.

Often, we want to think about the completion C at a *certain class* of colimits—say, filtered or sifted colimits. We described the colimit completion $\mathrm{PShv}(\mathsf{C})$ in three ways (I), (II), and (III) above; these suggest three ways to complete at a *chosen class* of colimits. Namely, the completion of C at a "nice" colimits should be...

[2] I don't think this is actually a functor. It is a function from the set of equivalence classes of π-finite Kan complexes to $\mathbb{Q}_{\geq 0}$.

[3] The description (III) needs more attention than we have provided. First of all, you want *small* diagrams. Then you must define morphisms. This requires some care. Will I get around to typing this out?

(I) A subcategory of "nice" presheaves in $\mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$;

(II) A subcategory of "nice" diagrams in C; and

(III) The "free nice colimit completion" of C (in the sense of a universal property).

All three models are useful. For example, *filtered colimits* are easiest to define as colimits over *filtered diagrams*, and this diagrammatic definition suggests we define the ind-completion $\mathrm{Ind}(\mathsf{C})$[4] as the category of filtered diagrams in C, in the likeness of (II).[5] (They are equivalently the diagrams whose colimits in Set commute with finite limits.) Meanwhile, *sifted colimits* are not so easily described by the shape[6] of their diagram; we say a diagram is *sifted* if its colimits commute with finite products (in Set). This suggests a definition of the sifted completion in the likeness of (I): when C has finite (co?)products, I think you can define its sifted completion as the full subcategory of presheaves commuting with finite products. (You can also define it as the subcategory of sifted diagrams in C, but I'm trying to make the point that (I) and (II) are both natural.)

Remark I.6 (Further reading)**.** I just said lots of stuff, but maybe left out lots of stuff. A little while ago, I gave a detailed account of the above picture in Lecture 6 of our condensed seminar, notes here.

I have not really thought about colimit completions for ∞-categories before. But, it seems the ordinary picture persists verbatim. Let me spell out an extended example, the motivating example in Charles' paper [Rez22], which is what made me start thinking about colimit completions again.

Example I.3 (Completion at κ-filtered colimits)**.** For a regular cardinal κ, an ∞-category C is called κ-filtered if every κ-small simplical set diagram $K \to \mathsf{C}$ admits an extension $K^{\triangleright} \to \mathsf{C}$ [Lur08, 5.3.1.7]. Parallel to the ordinary case, κ-filteredness is equivalent to commutation with all small limits in Space [Lur08, 5.3.3.3]. It turns out that κ-smallness is a generally well-behaved property (e.g., it is preserved by categorical equivalences), enough so that we may *complete* C *at* κ*-small colimits* to obtain $\mathrm{Ind}_\kappa(\mathsf{C})$. Also parallel to the ordinary picture, $\mathrm{Ind}_\kappa(\mathsf{C})$ admits models in the likeness of (I), (II), and (III) above:

(I) In terms of presheaves, $\mathrm{Ind}_\kappa(\mathsf{C})$ is the full subcategory of $\mathrm{PShv}(\mathsf{C})$ spanned by filtered colimits of representables [Lur08, 5.3.5.4]. Furthermore, if C admits small colimits, then Ind_κ can be more concretely described as the full subcategory spanned by preseheaves preserving finite limits.

(II) In terms of point categories, $\mathrm{Ind}_\kappa(\mathsf{C})$ consists of diagrams $\mathsf{J} \to \mathsf{C}$ such that J is a κ-filtered simplicial set(?)

(III) In terms of its universal property, $\mathrm{Ind}_\kappa(\mathsf{C})$ admits κ-filtered colimits, Yoneda factors as よ $: \mathsf{C} \to \mathrm{Ind}_\kappa(\mathsf{C})$, and if D admits κ-filtered colimits, then restriction along よ defines an equivalence

$$\mathrm{Fun}_\kappa(\mathrm{PShv}(\mathsf{C}), \mathsf{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathsf{C}, \mathsf{D}).$$

In other words, if D admits κ-filtered colimits, then each arrow $\mathsf{C} \to \mathsf{D}$ admits an essentially unique extension to $\mathrm{Ind}_\kappa(\mathsf{C})$ [Lur08, 5.3.5.10].

Remark I.7. Lurie's verbatim definition [Lur08, 5.3.5.1] is "$\mathrm{Ind}_\kappa(\mathsf{C})$ is the full subcategory of $\mathrm{PShv}(\mathsf{C})$ spanned by presheaves F which classify right fibrations $\mathsf{C}/F \to \mathsf{C}$ such that C/F is κ-filtered." I *feel* like this is just (I) and (II) above at the same time. Lurie basiclally says in [Lur08, 5.3] that he exhibits model (II), but he does not seem to exhibit that explicitly. But I actually do not know anything about (un)straightening, so I cannot really navigate.

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[4] Filtered colimits used to be called inductive limits, so we call the completion of C at filtered colimits its "ind-completion" or "indization."

[5] Then you can recharacterize it in the likeness of (I) and (III). For (I), it turns out $\mathrm{Ind}(\mathsf{C})$ is the full subcategory of $\mathrm{PShv}(\mathsf{C})$ spanned by filtered colimits of representable presheaves. If C is finitely cocomplete, so that C^{op} has finite limits, then we can be more explicit: $\mathrm{Ind}(\mathsf{C})$ is precisely those presheaves commuting with finite colimits. For (III), take the universal property of $\mathrm{PShv}(\mathsf{C})$ but replace "cocontinuous" with "preserves filtered colimits."

[6] I think there is such a description, but it is not nice.

Remark I.8. In the ordinary case, it is standard to characterize (say) ind-objects by their point category, i.e., to say that an ind-object is a filtered diagram in C, equivalently a presheaf $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ whose category of elements is filtered (and cofinal) [KS06, 3.3.13, 6.1.5].

I regard Lurie's definition of $\mathrm{Ind}_\kappa(\mathsf{C})$ as an amalgam of models (I) and (II). These are "concrete" models which one shows have the relevant universal property. Charles' paper [Rez22] considers the general question: for which classes of ∞-categories $\mathcal{F}$ do these "concrete" models still work (i.e., actually have the relevant universal property)? More precisely, which classes of small ∞-categories $\mathcal{F}$ are such that for every ∞-category C, the category $\{X \in \mathrm{PShv}(\mathsf{C}) : \mathsf{C}/X \in \mathcal{F}\}$ has the "free $\mathcal{F}$-colimit completion" property?

For this purpose, we make two definitions. First, we define $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) := \{X \in \mathrm{PShv}(\mathsf{C}) : \mathsf{C}/X \in \mathcal{F}\}$. Second, we define $\mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ as the *minimal full subcategory of PShv*(C) *generated by representables under* $\mathcal{F}$*-colimits,* to mean the smallest full subcategory containing $よ(\mathsf{C}) \subseteq \mathrm{PShv}(\mathsf{C})$ and closed under $\mathcal{F}$-colimits. Yoneda factors as $よ : \mathsf{C} \to \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$, and this functor is a *free* $\mathcal{F}$*-colimit completion* [Lur08, 5.3.6.2]. Then we seek to compare $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ and $\mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$.

This is not complicated. First, note that $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \subseteq \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ since any presheaf $X \in \mathrm{PShv}(\mathsf{C})$ is a colimit of $\mathsf{C}/X \to \mathsf{C} \to \mathrm{PShv}(\mathsf{C})$, thus X is an $\mathcal{F}$-colimit of representables as soon as $\mathsf{C}/X \in \mathcal{F}$. Now, if we can show that $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \subseteq \mathrm{PShv}(\mathsf{C})$ contains $よ\mathsf{C}$ and is stable under $\mathcal{F}$-colimits, then we get the reverse inclusion are conclude

$$\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) = \mathrm{PShv}_{\mathcal{F}}(\mathsf{C}).$$

We consider sufficient conditions for $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ to obtain these properties.

- For which $\mathcal{F}$ does $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ always contain the representables? For an object c, we have $\mathsf{C}/よc \cong \mathsf{C}_{/c}$. Hence, you could ask that $\mathcal{F}$ contains all ∞-categories with a terminal object. Alternatively(?): there c is terminal, so $よc$ is a terminal presheaf. By definition, we in fact have $よc \in \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$. Hence, if $\mathcal{F}$ contains D whenever $\mathrm{PShv}_{\mathcal{F}}(\mathsf{D})$ contains a terminal presheaf, then we would have $よc \in \mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$.

- I have nothing to say about when $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ is stable under $\mathcal{F}$-colimits. See the proof of Prop. 4.2 in Charles paper [Rez22].

We say that $\mathcal{F}$ is filtering if it contains D whenever $\mathrm{PShv}_{\mathcal{F}}(\mathsf{D})$ contains a terminal presheaf. We seek this property (rather than just ask that $\mathcal{F}$ contain D whenever D has a terminal object) because filtration turns out to be closely related to our problem:

Proposition I.2. Suppose that a family of ∞-categories $\mathcal{F}$ is such that for any C, the full subcategory inclusion $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \subseteq \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ is an isomorphism. Then $\mathcal{F}$ is filtering.

Proof. If $1 \in \mathrm{PShv}(\mathsf{D})$ denotes a terminal presheaf, then $\mathsf{D}/1 \cong \mathsf{D}$. Therefore[7] $\mathsf{D}/1 \cong \mathsf{D}$ lies in $\mathrm{PShv}_{\mathcal{F}}(\mathsf{D})$ and thus $\mathrm{Ind}_{\mathcal{F}}(\mathsf{D})$ by assumption. This exactly says that $\mathcal{F}$ is filtering. □

To summarize, we explained that if $\mathcal{F}$ is filtering, then $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \subseteq \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ is an equivalence, and we proved the converse. Hence, given a class of ∞-categories $\mathcal{F}$, the "diagrammatic" model for the $\mathcal{F}$-colimit completion $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ (presheaves whose point category belong to $\mathcal{F}$) is an actual $\mathcal{F}$-colimit completion (in the sense of possessing the relevant universal property, equivalently $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \hookrightarrow \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ being an equality) if and only if $\mathcal{F}$ is filtering. I wonder what the diagrammatic model is good for in general. Charles also studies the notion of filtering classes, which I have not read yet. May also be interesting to look at [Du23].

Remark I.9. In light of this writing, my mind has changed to consider there as being only *two* truly distinct models: a *diagrammatic* model and a *universal property* model. Maybe it's not right to even call the latter a model, since it's really what we're trying *to model*, but that seems to be the language I've found most efficient.

Remark I.10. Someone should rename *filtering classes*. But I am not sure what a good name would be. I do not like *filtering* because we already have *filtered/filtering/filtrant* colimits, which are a very specific example of filtering classes. Something to reflect that $\mathcal{F}$-diagrams are special. Diagrammatic? Graphical? Figurative? "Graphical classes" has a nice ring to it. Or just call them "graphics." Or "complete graphics."

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[7] Hmm. Why does $\mathsf{D}/1$ lie in $\mathrm{PShv}_{\mathcal{F}}(\mathsf{D})$? Does

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