INTRO TO DE RHAM COHOMOLOGY

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ABSTRACT. Some notes I wrote a while back to introduce myself to de Rham cohomology. I'd recommend these to anyone who already knows basic algebraic topology.

CONTENTS

1. The de Rham Cohomology

1.1. **The Exact Sequence of Smooth Functions of $\mathbb{R}^3$.**

In calculus, one learns that smooth functions $f : \mathbb{R}^3 \to \mathbb{R}$ and $F : \mathbb{R}^3 \to \mathbb{R}^3$ have the property that $\mathrm{div}(\mathrm{curl}(F)) = 0$ and $\mathrm{curl}(\mathrm{grad}(f)) = \mathbf{0}$. One also learns that if $\mathrm{curl}F = \mathbf{0}$ then F is the divergence of a smooth scalar function, and possibly that if $\mathrm{div}F = 0$ then F is the curl of a smooth vector function. Implicitly developed is an exact sequence, and a highly important one at that. It is the one below.

$$\begin{matrix} C^\infty \text{ scalar} \\ \text{functions} \\ \mathbb{R}^3 \to \mathbb{R} \end{matrix} \xrightarrow{\text{grad}} \begin{matrix} C^\infty \text{ vector} \\ \text{functions} \\ \mathbb{R}^3 \to \mathbb{R}^3 \end{matrix} \xrightarrow{\text{curl}} \begin{matrix} C^\infty \text{ vector} \\ \text{functions} \\ \mathbb{R}^3 \to \mathbb{R}^3 \end{matrix} \xrightarrow{\text{div}} \begin{matrix} C^\infty \text{ scalar} \\ \text{functions} \\ \mathbb{R}^3 \to \mathbb{R} \end{matrix}$$

We may pass between this sequence and the cochain of differential forms on $\mathbb{R}^3$ by the following identifications. For $f : \mathbb{R}^3 \to \mathbb{R}$, we may leave it as is, in which case it is already a 0-form, or we may identify it with a 3-form via $f \longleftrightarrow f\ dx \wedge dy \wedge dz$. For $F = (P, Q, R) : \mathbb{R}^3 \to \mathbb{R}^3$, we may identify it with a 1-form or a 2-form via

$$(P, Q, R) \longleftrightarrow P\ dx + Q\ dy + R\ dz,$$
$$(P, Q, R) \longleftrightarrow P\ dy \wedge dz - Q\ dx \wedge dz + R\ dx \wedge dy.$$

These four identifications are vector space isomorphisms between the spaces in the exact sequence above and the spaces of differential forms, as in the following diagram. Furthermore, the isomorphisms commute with exterior derivation, i.e. the diagram is commutative.

$$\begin{tikzcd}
\begin{matrix} C^\infty \text{ scalar} \\ \text{functions} \\ \mathbb{R}^3 \to \mathbb{R} \end{matrix} \arrow[r, "\text{grad}"] \arrow[d, leftrightarrow, "\cong"'] & \begin{matrix} C^\infty \text{ vector} \\ \text{functions} \\ \mathbb{R}^3 \to \mathbb{R}^3 \end{matrix} \arrow[r, "\text{curl}"] \arrow[d, leftrightarrow, "\cong"'] & \begin{matrix} C^\infty \text{ vector} \\ \text{functions} \\ \mathbb{R}^3 \to \mathbb{R}^3 \end{matrix} \arrow[r, "\text{div}"] \arrow[d, leftrightarrow, "\cong"] & \begin{matrix} C^\infty \text{ scalar} \\ \text{functions} \\ \mathbb{R}^3 \to \mathbb{R} \end{matrix} \arrow[d, leftrightarrow, "\cong"] \\
\Omega^0(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"'] & \Omega^3(\mathbb{R}^3)
\end{tikzcd}$$

With this framework, the classical divergence, gradient, and curl of functions of $\mathbb{R}^3$ are special cases of the exterior derivative d. Exact forms correspond bijectively with functions whose divergence or curl is zero, and closed forms with functions which are the gradient or curl of another.

1.2. **The Poincaré Lemma.**

A natural next step is to ask how the situation changes if we change the domain of our forms to an open subset U of $\mathbb{R}^n$. It turns out that the properties of the associated *cochain complex of forms*

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \longrightarrow \cdots$$

Vary depending on the topological structure of U. In 1887, Henri Poincaré proved the first version of a now-classical result about this cochain complex, in particular when it is exact. (In other words, when closed forms on U are always exact.)

Lemma 1.1 (The Poincaré Lemma). *Every closed form on $\mathbb{R}^n$ is an exact form. Equivalently, the cochain complex of forms associated to $\mathbb{R}^n$ is exact.*

We may restate this lemma for a slightly broader family of sets, which is the version some sources canonize. It is not what Poincaré originally proved—but with some extra machinery, the two turn out to be essentially equivalent. It is stated in terms of *contractible* sets.

Definition 1.2. Suppose U is open in $\mathbb{R}^n$. We say U is *contractible* if there is a smooth map $H : U \times [0,1] \to U$ such that $H(u,0) = u$ and $H(u,1)$ is constant.

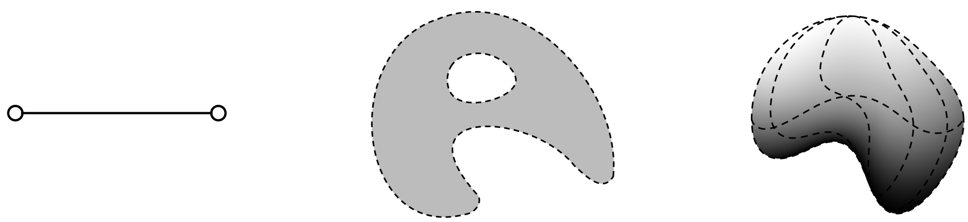

FIGURE 1. A contractible set in $\mathbb{R}$, a non-contractible set in $\mathbb{R}^2$, and a contractible set in $\mathbb{R}^3$.

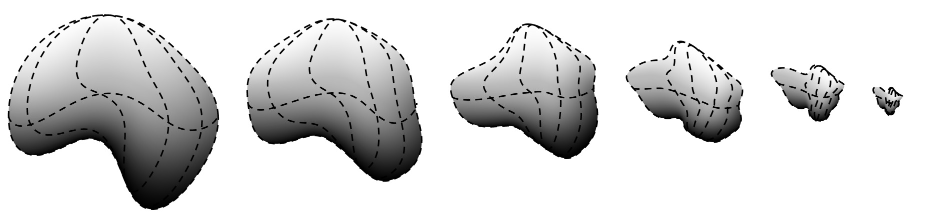

FIGURE 2. A visualization of a contraction of the third contractible set in Figure 2. One should think of these as the images of the original set under H as the second parameter varies.

Corollary 1.3 (Poincaré Lemma For Open Sets)**.** *Every closed form on a contractible open subset of $\mathbb{R}^n$ is an exact form. Equivalently, the cochain complex of forms associated to a contractible set is exact.*

Some go even further and state Poincaré's Lemma for contractible manifolds. We delay this and omit any proofs of the other statements, for the same reason: it can all be combed into a more fertile, cohesive program once we have some language of de Rham cohomology.

That being said, we remark that Poincaré's Lemma can be proven *without* the aforementioned machinery, at least a version of it. Namely, one can prove the lemma for open balls in $\mathbb{R}^n$ using only the Lie derivative, the fundamental theorem of calculus, and the Cartan formula. With a little more work and no new ideas, it can even be proven for *star-convex* sets, which are open sets with a point satisfying a convexity condition.

1.3. **The de Rham Cohomology.** We now consider domains with non-exact cochain complexes of forms. First, we work out an example on such a domain.

> **Example 1.4.** Consider the 1-form ω on the punctured plane $\mathbb{R}^2 - \mathbf{0}$ defined by
> $$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}.$$
> One finds that $d\omega = 0$, so ω is closed. Is ω exact? Converting to polar coordinates, we compute
> $$\int_{S^1} \omega = \int_{S^1} \frac{r\cos\theta \cdot d(r\sin\theta) - r\sin\theta \cdot d(r\cos\theta)}{r^2} = \int_{S^1} d\theta = \pm 2\pi.$$
> Stokes' theorem implies that if ω were exact then its integral around S^1 would be zero. It is not, so ω is not exact. Of course, in this computation we found that $\omega = d\theta$; the reason this does not mean ω is exact is because θ is not continuous on all of $\mathbb{R}^2 - \mathbf{0}$.

The exterior derivative d satisfies $d \circ d = 0$, hence exact forms are necessarily closed regardless of domain. However, we have just seen that some domains have closed forms which are not exact. Said differently, the cochain of forms associated to a domain may fail to be exact. The *de Rham cohomology* of the domain measures how the cochain differs from the underlying exact one.

Definition 1.5. We define the k-th **de Rham cohomology** of M to be the quotient vector space
$$H^k(M) = \{\text{closed } k\text{-forms}\}/\{\text{exact } k\text{-forms}\}.$$
Elements of $H^k(M)$ are equivalence classes of closed forms, called their *cohomology classes*. Two closed forms which determine the same cohomology class are *cohomologous*. By definition, two forms are cohomologous if and only if they differ by an exact form. We may refer to the de Rham cohomology of M as simply the cohomology of M, if it is not ambiguous in context.

Definition 1.6. For an open set U of $\mathbb{R}^n$, we have been calling the sequence
$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \longrightarrow \cdots$$
The cochain complex of forms associated to U. In fact, it already has a name: the **de Rham complex** of U, or more generally, of a smooth manifold M. We may refer to this complex by $\Omega^\bullet(M)$.

We now have two basic questions: what is the zero-th cohomology of a manifold and where is the cohomology nontrivial?

Proposition 1.7. *Suppose M has r connected components, then its zero-th de Rham cohomology is $H^0(M) = \mathbb{R}^r$.*

Proof. Since there are no nonzero exact 0-forms, $H^0(M) = \{\text{closed 0-forms}\}$. Supposing f is a closed 0-form, we have that on any chart $(U, x_1, \ldots, x_n)$,
$$df = \sum \frac{\partial f}{\partial x_i} dx_i.$$

Since $df = 0$, it must be that the $\partial f/\partial x_i$ vanish on M. Thus, f is locally constant. On a given connected component, f must then be constant, because it is smooth. There are r connected components, hence f may be specified by an r-tuple of real numbers, the value it takes on each component. This is what we wanted to show. $\square$

Proposition 1.8. *For M an n-manifold, $H^k(M) = 0$ for $k > n$.*

Proof. For $k > n$, the only k-form on M is the zero form. $\square$

We next ask for the cohomology of a manifold that is the smooth image of another. Recall that for $F : N \to M$ a smooth map of manifolds, we have the *pullback* $F^* : \Omega^*(M) \to \Omega^*(N)$ of forms.

Proposition 1.9. *The pullback commutes with exterior derivation.*

Proof. We are to show that for $\omega \in \Omega^k(M)$, $F^*(d\omega) = d(F^*\omega)$. The $k = 0$ case is immediate. For $k > 0$, let $p \in N$ and let $(V, y_1, \ldots, y_m)$ be a chart about $F(p)$. Then about $F(p)$, $\omega = \sum a_I dy_I$. We compute

$$\begin{aligned} F^*\omega &= \sum (F^* a_I)\, F^* dy_I, \\ &= \sum (a_I \circ F)\, dF_I, \\ \implies d(F^*\omega) &= \sum d(a_I \circ F) \wedge dF_I. \end{aligned}$$

On the other hand, $F^*(d\omega) = \sum F^* da_I \wedge F^* dy_I = \sum d(F^* a_I) \wedge dF_I = \sum d(a_I \circ F) \wedge dF_I$. We conclude the desired equality. $\square$

Proposition 1.10. *The pullback sends closed forms to closed forms and exact forms to exact forms.*

Thus, we may speak of the linear *pullback map in cohomology* induced by F^* (or F), described by

$$F^\# : H^k(M) \to H^k(N),\ [\omega] \mapsto [F^*(\omega)].$$

Often, $F^\#$ is also denoted F^*, just as the pullback. We may make this convention, of course clear in context.

Corollary 1.11. *Evident from the preceding discussion, given a smooth map of manifolds $F : N \to M$, the following diagram is commutative.*

$$\begin{tikzcd} H^0(M) \arrow[r, "d"] \arrow[d, "F^\#"'] & H^1(M) \arrow[r, "d"] \arrow[d, "F^\#"'] & H^2(M) \arrow[r] \arrow[d, "F^\#"'] & \cdots \\ H^0(N) \arrow[r, "d"'] & H^1(N) \arrow[r, "d"'] & H^2(N) \arrow[r] & \cdots \end{tikzcd}$$

Here, $d : H^k \to H^{k+1}$ is the map induced by $d : \Omega^k \to \Omega^{k+1}$ defined by $\omega \mapsto [d\omega]$.

Before seeing some computations, the last thing we introduce is the algebra structure of $\Omega^\bullet(M)$. With the wedge product, $\Omega^\bullet(M)$ has a product structure. This induces a product structure on cohomology, whose properties are given by the following proposition.

Proposition 1.12. *We define a wedge of cohomology classes of forms to be the cohomology class of their wedge. In symbols, for $\omega \in H^k(M)$ and $\tau \in H^\ell(M)$,*

$$[\omega] \wedge [\tau] = [\omega \wedge \tau] \in H^{k+\ell}(M).$$

This wedge has three important properties.

(1) The wedge product $\omega \wedge \tau$ of two forms is closed.
(2) The cohomology class $[\omega \wedge \tau]$ is independent of the representative of $[\tau]$.
(3) The cohomology class $[\omega \wedge \tau]$ is independent of the representative of $[\omega]$.

Definition 1.13. For M a smooth manifold, we define the *cohomology ring* of M

$$H^*(M) \stackrel{\text{def}}{=} \bigoplus_{k=0}^{\infty} H^k(M).$$

An element α of $H^*(M)$ can be written either as a tuple $(\alpha_0, \alpha_1, \dots)$ or as a sum $\alpha_0 + \alpha_1 + \dots$, where α_i is a cohomology class in $H^i(M)$. The latter is highly preferred in this context. The cohomology ring $H^*(M)$ is truthfully a ring, whose multiplication is the wedge product. One adds and multiplies its elements just like polynomials.

Since the wedge product sends k, l-forms to a $(k+l)$-form, $H^*(M)$ is *graded.* Since $[\omega] \wedge [\ell] = [(-1)^{kl}\tau \wedge \omega]$, $H^*(M)$ is also *anticommutative.* Along with the fact that $H^*(M)$ is a vector space makes $H^*(M)$ into an anticommutative graded $\mathbb{R}$-algebra.

1.4. **Some de Rham Cohomologies.** Now we compute the cohomology of some spaces. By all means, our current methods for computing cohomology are primitive. The machinery we are postponing will allow us to compute the cohomology for much more complicated spaces.

> **Example 1.14** (Cohomology of $\mathbb{R}^1$)**.** By Propositions 1.7 and 1.8, $H^0(\mathbb{R}^1) = \mathbb{R}$ and is zero elsewhere.

> **Example 1.15** (Cohomology of $\mathbb{R}^3$)**.** By Propositions 1.7 and 1.8, $H^0(\mathbb{R}^3) = \mathbb{R}$ and $H^k(\mathbb{R}^3) = 0$ for $k > 3$. As for the other cases, recall from the previous section that the we may pass to the de Rham complex of $\mathbb{R}^3$ via isomorphisms from an exact sequence of smooth functions of $\mathbb{R}^3$ which commute with d. This is sufficient to conclude that the de Rham complex itself is exact. This implies that the remaining cohomologies are trivial, i.e. are zero-dimensional. (Actually, we are overlooking the most difficult part of this argument—it is quite nontrivial that a smooth, zero-divergence vector field is the curl of another.)

> **Example 1.16** (Cohomology of S^1)**.** By Propositions 1.7 and 1.8, $H^0(S^1) = \mathbb{R}$ and $H^k(S^1) = 0$ for $k > 1$. We are left to compute $H^1(S^1)$.
>
> Firstly, since S^1 is one-dimensional, every 1-form on S^1 is closed. So we must determine the space of nonexact forms.
>
> Let $F : [0, 2\pi] \to S^1$ be the parametrization of S^1 defined by $t \mapsto (\cos t, \sin t)$, and let $\omega = -y\,dx + x\,dy$ be the nowhere-vanishing 1-form we considered earlier. Since $F^*\omega = dt$, we have
>
> $$\int_{S^1} \omega = \int_0^{2\pi} dt = 2\pi.$$

Stokes' theorem implies exact 1-forms on S^1 are in the kernel of the integration map φ, defined by $\tau \mapsto \int_{S^1} \tau$. Now write $h : \mathbb{R} \to S^1$ to denote the map $t \mapsto (cos\, t, \sin\, t)$. Suppose $\alpha = f\omega$ is a 1-form on S^1 in the kernel of φ; then $\bar{f} = h^* f = f \circ h$ is periodic of period 2π. One computes

$$\int_0^{2\pi} \bar{f}(t)\, dt = 0,$$

And a technical lemma asserts that this implies $\bar{f}\, dt = d\bar{g}$ for some smooth, 2π-periodic $\bar{g}$, which satisfies $d\bar{g} = \bar{f}(t)\, dt$. Since smooth functions on S^1 correspond to smooth periodic functions of $\mathbb{R}$ via pullback, we can write $\bar{g} = h^* g$ for a smooth function $g : S^1 \to \mathbb{R}$. On one hand, we now have $d\bar{g} = h^*(dg)$. On the other, $\bar{f}(t)\, dt = h^* \alpha$. Since h^* is injective, we conclude $\alpha = dg$. Thus, the kernel of φ is precisely the space of exact forms on S^1, so that it induces an isomorphism from $H^1(S^1)$ to $\mathbb{R}$.

2. Interlude One

Our goal now is to develop techniques for computing the cohomology of a manifold. For we have seen that even in the simplest of cases, i.e. $H^1(S^1)$, direct computation is unattractive. The natural direction to proceed at this point is toward the *homotopy axiom for de Rham cohomology* and the *Mayer-Vietoris sequence*. In this section, we will lay down some groundwork so that the debut of these tools is C^∞ and painless.

The section is organized as follows. We will first develop the basic concepts of category theory, namely a *category* and *functor*. Next, we will do the same for *homological algebra*, namely developing *cochain complexes* and cohomology in general. Crucially, we will meet the *long exact sequence in cohomology*, whose existence is the conclusion of the *zig-zag lemma*. Lastly, we will recall some *topics involving the word homotopy*, proving that *homotopic smooth maps between manifolds induce the same map between de Rham cohomologies*. (We avoid the term 'homotopy theory,' as we are not studying homotopy groups, the overwhelming focus of homotopy theory.)

Continually, we will move between abstracted concepts and their restrictions to the subject at hand.

2.1. Some Category Theory.

Algebraic topology concerns mappings from topology to algebra. Category theory gives us a language to express this.

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Definition 2.1. A **category** $\mathcal{C}$ consists of the following:

(1) A set of objects $\mathrm{Ob}(\mathcal{C})$.
(2) For each pair $A, B \in \mathrm{Ob}(\mathcal{C})$, a set of morphisms $\mathrm{Mor}_{\mathcal{C}}(A, B)$ from A to B, such that for any $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(B, C)$, there exists a map $g \circ f \in \mathrm{Mor}_{\mathcal{C}}(A, C)$ called the *composite*,

Subject to the following two rules:

(1) (Identity axiom) For each A, there is a morphism $\mathrm{id}_A \in \mathrm{Mor}_{\mathcal{C}}(A, A)$ such that for any $x \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ and $y \in \mathrm{Mor}_{\mathcal{C}}(B, A)$,
$$x \circ \mathrm{id}_A = x \text{ and } \mathrm{id}_A \circ y = y,$$
(2) (Associative axiom) And for any $a \in \mathrm{Mor}_{\mathcal{C}}(A, B)$, $b \in \mathrm{Mor}_{\mathcal{C}}(B, C)$, and $c \in \mathrm{Mor}_{\mathcal{C}}(C, D)$,
$$c \circ (b \circ a) = (c \circ b) \circ a.$$

It is essential that one sees examples of categories.

Examples 2.2 (Categories)**.**

(1) The collection of sets together with the functions between them forms a category.
(2) The collection of groups together with the homomorphisms between them forms a category.
(3) The collection of topological spaces together with the continuous maps between them forms the *continuous category*.
(4) The collection of smooth manifolds together with the smooth maps between them forms the *smooth category*.
(5) A *pointed manifold* is a pair (M, q) consisting of a manifold and a point, the *basepoint*, in it. The collection of pointed manifolds together with the smooth maps taking one basepoint to the other

form the *category of pointed manifolds.* A similar category is the *category of pointed topological spaces.*

Definition 2.3. A **covariant functor** $\mathcal{F}$ from one category $\mathcal{C}$ to another $\mathcal{D}$ is a map that associates

(1) To each object A of $\mathcal{C}$ an object $\mathcal{F}(A)$ of $\mathcal{D}$, and
(2) To each morphism $f : A \to B$ a morphism $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$,

Satisfying the *functorial properties* $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$ and $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$.

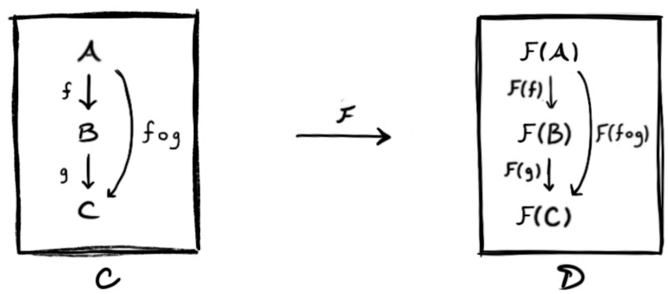

FIGURE 3. A schematic to convey how a covariant functor $\mathcal{F}$ from one category $\mathcal{C}$ to another $\mathcal{D}$ acts on its objects and the morphisms between them.

Example 2.4 (Tangent space as a functor)**.** When we construct the tangent space to a manifold, we specify the manifold N and a point p on it; i.e. we require a pointed manifold. The result is a vector space, T_pN. To each smooth map $f : (N, p) \to (M, f(p))$, we associate the differential $f_{*,p} : T_pN \to T_{f(p)}M$. The differential of the identity map on N is the identity map, and the chain rule

$$(g \circ f)_{*,p} = g_{*,f(q)} \circ f_{*,p}$$

Is the second functorial property. So, the tangent space construction is a functor from smooth pointed manifolds to vector spaces.

Example 2.5 (π_1 as a functor)**.** The fundamental group of a pointed topological space (X, x_0), denoted $\pi_1(X, x_0)$, is the homotopy class of continuous images of $[0, 1]$ in X based at x_0 (i.e. which start and end at x_0), given a group structure by the operation of path concatenation. For a continuous map of pointed topological spaces $f : (X, x_0) \to (Y, y_0)$, there is an induced homomorphism of fundamental groups sending each homotopy class of loops to the (unique) homotopy class of the image of their representatives under f. This is usually denoted f_*. However, it may equally be written $\pi_1(f)$ since π_1 is well-known to satisfy the functorial properties and hence is a functor, one from the category of pointed topological spaces to the category of groups.

Definition 2.6. Two objects A, B are **isomorphic** if there are morphisms $f : A \to B$ and $g : B \to A$ satisfying $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. These f, g are called *isomorphisms.*

Proposition 2.7. *A functor takes isomorphisms to isomorphisms.*

Proof. Suppose $f : A \to B$, $g : B \to A$ are isomorphisms whose compositions are id. Then $\mathcal{F}(f \circ g) = \mathcal{F}(\mathrm{id}_B)$. By the definition of a functor, we have $\mathcal{F}(f) \circ \mathcal{F}(g) = \mathrm{id}_{\mathcal{F}(B)}$. Likewise, $\mathcal{F}(g) \circ \mathcal{F}(f) = \mathrm{id}_{\mathcal{F}(A)}$. Thus $\mathcal{F}(f), \mathcal{F}(g)$ are isomorphisms. $\square$

Definition 2.8. A **contravariant functor** $\mathcal{F}$ from one category $\mathcal{C}$ to another $\mathcal{D}$ is a map whose properties are identical to that of a covariant functor, except $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$ rather than $\mathcal{F}(f) \circ \mathcal{F}(g)$.

In the following example, we will see a particularly relevant contravariant functor.

> **Example 2.9** (Pullback as a contravariant functor)**.** A smooth map of manifolds $F : N \to M$ induces the pullback $F^* : \Omega^\bullet(M) \to \Omega^\bullet(N)$. If $F : N \to N$ is the identity, F^* is the identity on the differential forms of N. This is the first functorial property. Pullback also has the second contravariant functorial property, and so is a contravariant functor.
>
> We have also seen in the previous section that the smooth map $F : N \to M$ induces the linear pullback map in cohomology, $F^\# : H^k(M) \to H^k(N)$, defined by $F^\#([\omega]) = [F^*(\omega)]$. This definition in terms of * (pullback), which we know to have the contravariant functorial properties, gives for free that $^\#$ (pullback in cohomology) has the contravariant functorial properties. Thus, pullback in cohomology is a contravariant functor from the category of smooth manifolds with smooth maps (the smooth category) to the category of vector spaces with linear maps.
>
> In fact, we have seen that an $\mathbb{R}$-algebra can be formed as the direct sum of all the cohomologies of a manifold, in which case $F^\# : \Omega^*(M) \to \Omega^*(N)$ is a linear map of $\mathbb{R}$-algebras (still vector spaces, but a 'smaller category' of them.) With this perspective, $\Omega^\bullet$ is a contravariant functor, one from the category of smooth manifolds and smooth maps between them to the category of commutative differential graded algebras and their homomorphisms. In fact, it is the unique such functor that is the pullback of functions on $\Omega^0(\mathbb{R}^n)$.

Corollary 2.10. *By the discussion in the previous example, for $F : N \to M$ a diffeomorphism of manifolds, the pullback $F^\# : H^k(M) \to H^k(N)$ is a vector space isomorphism. We conclude that de Rham cohomology is diffeomorphism-invariant.*

2.2. **Cochain Complexes.**

Definition 2.11. A **cochain complex** C is a collection of vector spaces $\{C^i\}_\mathbb{Z}$ such that there are linear maps $d_k : C^k \to C^{k+1}$

$$\cdots C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \longrightarrow \cdots$$

Satisfying $d_k \circ d_{k-1} = 0$. We often abbreviate d_k to d and call it the *differential.*

Definition 2.12. A sequence of homomorphisms of vector spaces (or groups, or modules. . .) $A \xrightarrow{f} B \xrightarrow{g} C$ Is called **exact** at B if $\operatorname{Im} f = \operatorname{Ker} g$. An exact sequence is one exact at every term. A *short exact* sequence is a five-term sequence with 0 at each end.

Proposition 2.13. *Certain exact sequences have important properties.*

(1) $0 \to A \xrightarrow{g} B$ *is exact iff* g *is injective.*

(2) $A \xrightarrow{f} B \to 0$ *is exact iff* f *is surjective.*

(3) $0 \to A \xrightarrow{f} B \to 0$ *is exact iff* f *is an isomorphism.*

(4) $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is exact iff* f *is injective,* g *is surjective, and* $\operatorname{Ker} g = \operatorname{Im} f$. *In this case,* g *induces an isomorphism* $C \cong B/\operatorname{Im} f$.

One should feel that we have just redefined, in slightly more general terms, phenomena we have seen before. We do the same now for cohomology.

Definition 2.14. Since $\operatorname{Im} d_{k-1} = \operatorname{Ker} d_k$, we may form the quotient vector space

$$H^k(C) = \frac{\operatorname{Ker} d_k}{\operatorname{Im} d_{k-1}},$$

Called the k-th **cohomology vector space** of C. Elements of C^k are *degree k-cochains* or *k-cochains* of C. The k-cochains in $\operatorname{Ker} d_k$ is called a *k-cocyle*, and likewise a *k-coboundary* if it lies in the image of d_{k-1}. The equivalence class of a k-cocycle c, denoted $[c]$, is its *cohomology class.* We may denote the set of cocycles by $Z^k(C)$, and the set of coboundaries by $B^k(C)$.

Definition 2.15. Suppose A, B are two cochain complexes with differentials d and d', respectively. A **cochain map** φ is a collection of maps $\varphi_k : A_k \to B_k$ that commute with d and d', meaning $d' \circ \varphi_k = \varphi_{k+1} \circ d$. In other words, the following diagram is commutative.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{k-1} \arrow[r, "d"] \arrow[d, "\varphi^{k-1}"'] & A^k \arrow[r, "d"] \arrow[d, "\varphi^k"'] & A^{k+1} \arrow[r] \arrow[d, "\varphi^{k+1}"'] & \cdots \\
\cdots \arrow[r] & B^{k-1} \arrow[r, "d'"'] & B^k \arrow[r, "d'"'] & B^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

Whenever we speak of a cochain map $\varphi : A \to B$, we may also consider the naturally induced map $\varphi^* : H^k(A) \to H^k(B)$ defined by $[c] \mapsto [\varphi(c)]$.

Example 2.16 (Cohomology as a functor)**.** The collection of cochain complexes together with the cochain maps between them form a category. For a complex C, we associate to it the complex of its cohomology spaces $H^*(C)$. To a cochain map $f : C \to D$, we associate to it a cochain map $H^*(f) : H^*(C) \to H^*(D)$. We have that $H^*(gf) = H^*(g)H^*(f)$ and $H^*(\mathrm{id}_C) = \mathrm{id}_{H^*(C)}$, hence H^* is a covariant functor, the *cohomology functor.*

Example 2.17 (Pullback as a cochain map)**.** A smooth map of manifolds $F : N \to M$ induces the pullback $F^* : \Omega^\bullet(M) \to \Omega^\bullet(N)$. We have proven that F^* commutes with d. So, F^* is a cochain map. We have also seen that F^* induces the pullback in cohomology $F^\# : H^*(M) \to H^*(N)$ defined by $[\omega] \mapsto [F^*(\omega)]$, and this commutes with the $d : H^k \to H^{k+1}$ naturally induced by the differential d. This $F^\#$ is the map in de Rham cohomology naturally induced by F^*, the pullback in forms.

Next, given a sequence of cochains and cochain maps between them, we construct an important homomorphism, the *connecting homomorphism.*

Definition 2.18. Suppose A, B, C are complexes such that $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is *short exact* in the sense that in each dimension k, the sequence $0 \to A^k \xrightarrow{i_k} B^k \xrightarrow{j_k} C^k \to 0$ is exact.

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
 & \arrow[u] & \arrow[u] & \arrow[u] & \\
0 \arrow[r] & A^{k+1} \arrow[r, "i"] \arrow[u] & B^{k+1} \arrow[r, "j"] \arrow[u] & C^{k+1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A^k \arrow[r, "i"] \arrow[u, "d_A"] & B^k \arrow[r, "j"] \arrow[u, "d_B"] & C^k \arrow[r] \arrow[u, "d_C"] & 0 \\
0 \arrow[r] & A^{k-1} \arrow[r, "i"] \arrow[u, "d_A"] & B^{k-1} \arrow[r, "j"] \arrow[u, "d_B"] & C^{k-1} \arrow[r] \arrow[u, "d_C"] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

Given a short exact sequence of complexes as at right, we will construct a certain homomorphism $d^* : H^k(C) \to H^{k+1}(A)$. Begin with $[c] \in H^k(C)$. By exactness, j is onto, hence $c = j(b)$ for some $b \in B^k$. Then $d_B b \in B^{k+1}$. In fact, $d_B b$ is in the kernel of j, since $(j \circ d_B)b = (d_C \circ j)b$ by commutativity, which equals $d_C c$, which equals 0 as c is a cocycle. Then by exactness, $d_B b = i(a)$ for some $a \in A^{k+1}$.

We take $d^*([c]) = [a]$. This $d^* : H^k(C) \to H^{k+1}(A)$ is a homomorphism. Well-definedness follows essentially from the fact that for any $c + d_C e \in C^k$, which are all the representatives of $[c]$, the exact part $d_C e$ is killed by a second application of d_C. We call d^* the **connecting homomorphism**.

Lemma 2.19 (Zig-zag lemma)**.** *Suppose A, B, C are cochain complexes such that $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is exact. Then there is a long exact sequence in cohomology*

$$
\begin{tikzcd}
H^{k+1}(A) \arrow[r, "i^*"] & \cdots & \\
H^k(A) \arrow[r, "i^*"] & H^k(B) \arrow[r, "j^*"'] & H^k(C) \arrow[llu, "d^*"'] \\
 & \cdots \arrow[r, "j_*"'] & H^{k-1}(C) \arrow[llu, "d^*"']
\end{tikzcd}
$$

Here, i^* and j^* are the maps induced by i and j (equivalently, the images of i and j under the cohomology functor) and d^* is the connecting homomorphism.

To prove this lemma, one must demonstrate exactness at A, B, and C in every dimension. The technique used is called *diagram chasing* in homological algebra.

I've got a funny i0xd about topology.
u/lector57

2.3. **Not Homotopy Theory.**

Definition 2.20. Two continuous maps $f, g : X \to Y$ between topological spaces are called **homotopic** if there exists a *homotopy* $H : X \times [0, 1] \to Y$ such that H is continuous, $H(x, 0) = f(x)$, and $H(x, 1) = g(x)$. If in addition X and Y are smooth manifolds and H is smooth, we say H is a *smooth homotopy* of f and g, which are in this case **smoothly homotopic**.

Definition 2.21. Suppose $F, G : M \to N$ are smooth maps of manifolds. A collection of linear maps $h : \Omega^k(N) \to \Omega^{k-1}(M)$ such that

$$d(h\omega) + h(d\omega) = G^*\omega - F^*\omega$$

Is called a **homotopy operator** between F^* and G^*, a case of the more general *cochain homotopy.*

A homotopy operator is useful because if ω is closed, then $(d(h\omega) + h(d\omega))^{\#} = [d(h\omega) + h(d\omega)] = [d(h\omega)] = 0$, hence $F^{\#} = G^{\#}$. Thus, proving two smooth maps induce the same map between cohomologies amounts to proving a homotopy operator exists between their pullbacks. We will use this to show that (*not necessarily smoothly*) homotopic smooth maps induce the same map on cohomologies. The homotopy axiom for de Rham cohomology is essentially a simpler instance of this, so we prove it along with the homotopy axiom itself in the next section.

We will need two lemmas, the first of which is a special case of the endgame result. A proof is omitted, for it uses several tools (the Lie derivative, Cartan's formula, the interior product) not developed here. A proof can be found in [Lee]. We will also need a significant theorem regarding the approximation of continuous maps between manifolds by smooth ones.

Lemma 2.22. *Suppose M is a smooth manifold, with or without boundary. Let $i_t : M \to M \times [0,1]$ be the map $i_t(p) = (p, t)$. Then there exists a homotopy operator between their pullbacks $i_0^*, i_1^* : \Omega^\bullet(M \times [0,1]) \to \Omega^\bullet(M)$.*

Theorem 2.23 (Whitney's Approximation theorem)**.** *Suppose $F, G : M \to N$ are homotopic smooth maps between smooth manifolds with or without boundary, then F and G are smoothly homotopic.*

Lemma 2.24. *Suppose $F, G : M \to N$ are homotopic smooth maps between smooth manifolds, with or without boundary. Then the induced maps $F^{\#}, G^{\#} : H^k(N) \to H^k(M)$ are the same map in every dimension k.*

Proof. By Lemma 2.22, the maps in cohomology $i_0^{\#}, i_1^{\#} : H^*(M \times [0,1]) \to H^*(M)$ are the same. By Whitney's approximation theorem, there is a smooth homotopy $H : M \times [0,1] \to N$ between F and G. We may write $F = H \circ i_0$ and $G = H \circ i_1$, from which we have

$$F^{\#} = (H \circ i_0)^{\#} = i_0^{\#} \circ H^{\#} = i_1^{\#} \circ H^{\#} = (H \circ i_1)^{\#} = G^{\#}.$$

□

3. The Homotopy Axiom

In this section, we will prove that de Rham cohomology is homotopy-invariant. This will follow as what is essentially a special case of the result that homotopic smooth maps between smooth manifolds induce the same map on de Rham cohomologies. This homotopy invariance is surprising, for the de Rham groups of a manifold depend on its smooth forms, so we should have little expectation that a non-smooth homotopy (which may change the smooth structure) would leave these groups unchanged.

Another conundrum is more pedantic: why do we refer to the homotopy axiom, a result we shall prove, as an axiom? This naming is in consultation with the broader study of (co)homologies in general. The *homotopy axiom* is one of the five listed by Eilenburg and Steenrod in 1952 in their definition of a *(co)homology theory* as part of their successful attempt to axiomatize, largely by the use of category theory (which they also established), the many variations of (co)homology popping up at the time. The *homotopy axiom for de Rham cohomology* we are concened with here is the homotopy axiom posited in the definition of a (co)homology theory.

Theorem 3.1 (Homotopy Axiom for de Rham Cohomology)**.** *If M and N are homotopy equivalent smooth manifolds with or without boundary, then $H^k(M) \cong H^k(N)$ in every dimension k. In other words, de Rham cohomology is a homotopy invariant.*

Proof. Let $F : M \to N$ be a homotopy equivalence with homotopy inverse $G : N \to M$. By Theorem 2.23, there exist smooth maps $\tilde{F} : M \to N$ and $\tilde{G} : N \to M$ homotopic to F and G, respectively. We have that $\tilde{F} \circ \tilde{G} \simeq F \circ G \simeq \mathrm{id}_N$ and likewise $\tilde{G} \circ \tilde{F} \simeq \mathrm{id}_M$, thus $\tilde{F}$ and $\tilde{G}$ are homotopy inverses of each other.

By Lemma 2.24, $\tilde{F}^{\#} \circ \tilde{G}^{\#} = (\tilde{G} \circ \tilde{F})^{\#}$. By the above, we know this to be equal to $(\mathrm{id}_M)^{\#} = \mathrm{id}_{H^*(M)}$. Likewise, $\tilde{G}^{\#} \circ \tilde{F}^{\#} = \mathrm{id}_{H^*(N)}$. Thus, $\tilde{F}^{\#} : H^*(N) \to H^*(M)$ is an isomorphism. □

Corollary 3.2. *Since a homeomorphism is a homotopy equivalence, de Rham cohomology is a topological invariant.*

3.1. Applications.

As promised, we can now obtain the Poincaré lemma with ease, for any *contractible* (i.e. homotopy equivalent to a point) manifold.

Lemma 3.3 (Poincaré Lemma)**.** *Suppose M is a contractible manifold, then $H^0(M) = \mathbb{R}$ and $H^k(M) = 0$ for $k > 0$.*

Proof. Since M is contractible, by Corollary 3.2 its cohomologies are that of a point. One easily finds that $H^0(\{pt\}) = \mathbb{R}$ and 0 otherwise. □

The homotopy axiom is useful for simplifying the computation of a complicated space's cohomology to that of a more familiar one. In general, though, we still need more machinery to explicitly compute the cohomologies of these familiar spaces. In the next section, we develop the Mayer-Vietoris sequence, which aids us in doing exactly that.

> **Example 3.4** (Cohomology of $\mathbb{R}^n$)**.** The n-dimensional Euclidean space $\mathbb{R}^n$ is contractible, hence it has trivial cohomology in all dimensions.

> **Example 3.5** (Cohomology of $\mathbb{R}^n - \mathbf{0}$)**.** The punctured space $\mathbb{R}^n - \mathbf{0}$ deformation retracts onto S^{n-1}, hence $H^k(\mathbb{R}^n - \mathbf{0}) = H^k(S^{n-1})$.

Example 3.6 (Cohomology of a Möbius strip)**.** The Möbius strip admits a retract onto its core circle by collapsing its edge. This core circle is homotopy equivalent to S^1, hence the cohomology of the Möbius strip is that of S^1.

4. The Mayer-Vietoris Sequence

The Mayer-Vietoris sequence is a powerful tool for computing cohomology groups, relating a space's cohomology to those of an open cover's constituents. Beginning with a manifold M and an open cover $\{U, V\}$, consider the diagram of inclusions and the diagram of their induced maps under the pullback functor $\Omega^\bullet$:

$$\begin{tikzcd}
 & U \arrow[dl, hook', "\iota_U"'] & \\
M & & U \cap V \arrow[ul, hook', "j_U"'] \arrow[dl, hook, "j_V"] \\
 & V \arrow[ul, hook, "\iota_V"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \Omega^\bullet(U) \arrow[dr, "j_U^*"] & \\
\Omega^\bullet(M) \arrow[ur, "\iota_U^*"] \arrow[dr, "\iota_V^*"'] & & \Omega^\bullet(U \cap V) \\
 & \Omega^\bullet(V) \arrow[ur, "j_V^*"'] &
\end{tikzcd}$$

Here, ι_U, ι_V are the inclusions of U and V into M, respectively, and likewise for j_U, j_V. The induced pullbacks are simply the maps restricting domain. For instance, $i_U^* \omega = \omega|_U$. Now, we may assemble a sequence of de Rham complexes

$$\begin{array}{ccccccccc}
0 \longrightarrow & \Omega^\bullet(M) & \longrightarrow & \Omega^\bullet(U) \oplus \Omega^\bullet(V) & \longrightarrow & \Omega^\bullet(U \cap V) & \longrightarrow 0 \\
& \sigma & \longmapsto & (\iota_U^* \sigma, \iota_V^* \sigma) & & & \\
& & & (\omega, \tau) & \longmapsto & j_V^* \tau - j_U^* \omega &
\end{array}$$

Proposition 4.1. *The above sequence of de Rham complexes is exact.*

Proof. Exactness at $\Omega^\bullet(M)$ is immediate. Exactness at $\Omega^\bullet(U) \oplus \Omega^\bullet(V)$ is similarly straightforward.

For exactness to hold at $\Omega^\bullet(U \cap V)$, we must check that the difference map is surjective. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to $\{U, V\}$. For $\omega \in \Omega^\bullet(U \cap V)$, we define

$$\omega_U = \begin{cases} \rho_V(x)\omega_x & \text{if } x \in U \cap V, \\ 0 & \text{if } x \in U - (U \cap V). \end{cases}$$

Likewise for ω_V. Now, ω_U is defined (in fact, smooth) on all of U, and likewise for ω_V. On $U \cap V$, $(-\omega_U, \omega_V)$ restricts to $(-\rho_V \omega, \rho_U \omega)$. Then under the difference map, $(-\omega_U, \omega_V) \in \Omega^k(U) \oplus \Omega^k(V)$ maps to $\rho_V \omega - (-\rho_U \omega) = \omega$. Thus, the difference map is surjective, and the sequence is exact. $\square$

Definition 4.2. Recall the zig-zag lemma (Lemma 2.19), which constructs a long exact sequence in cohomology from a short exact sequence of chain complexes. Applying this to the exact sequence of complexes just discussed, we get a long exact sequence, the **Mayer-Vietoris sequence**

$$\begin{tikzcd}
H^{k+1}(M) \arrow[r, "\iota^*"] & \cdots & \\
H^k(M) \arrow[r, "\iota^*"] & H^k(U) \oplus H^k(V) \arrow[r, "j^*"'] & H^k(U \cap V) \arrow[ull, "d^*"] \\
 & \cdots \arrow[r, "j_*"'] & H^{k-1}(U \cap V) \arrow[ull, "d^*"]
\end{tikzcd}$$

In this sequence, ι_* and j_* are the maps induced by the inclusion and difference maps, respectively. They are given by

$$\iota^*[\sigma] = ([\sigma|_U], [\sigma|_V]),$$
$$j^*([\tau], [\omega]) = [\tau|_{U\cap V} - \omega|_{U\cap V}].$$

Next, to describe d^* we recall the diagram chase by which it was constructed, in this context of forms. Given $[\omega] \in H^{p-1}(U \cap V)$, we arrive at $d^*[\omega] \in H^p(M)$ as follows:

(1) By exactness, there is a $\theta \in \Omega^\bullet(U) \oplus \Omega^\bullet(V)$ such that $j\theta = \omega$, namely $\theta = (-\omega_V, \omega_U)$.
(2) The form $d\theta = (-d\omega_V, d\omega_U)$ maps to zero under j since $jd\theta = dj\theta = 0$. By exactness, there is an $\alpha \in \Omega^\bullet(M)$ such that $\iota\alpha = (\iota_V\alpha, \iota_U\alpha) = d\theta$. In other words, $-d\omega_V$ and $d\omega_U$ patch together into a higher-order form α defined on all of M.
(3) Take $d^*[\omega] = [\alpha] \in H^{p+1}(M)$. As discussed earlier, this is well-defined.

4.1. **Applications.**

5. Interlude Two

A virtue of (co)homology theories is that certain (co)homologies will agree on certain classes of spaces. This can be practically useful, or at least illuminating, as some (co)homologies lend themselves to computation or geometrization more readily than others. In the next two sections, we will develop some (co)homology theories closely related to de Rham cohomology:

(1) *Singular homology*, which should be a familiar face to the reader with some background in algebraic topology. It is defined for general topological spaces. Alongside it, we will develop its smooth variant, defined for smooth manifolds. Then, for both homologies we will introduce their dual cohomology theories.
(2) *Ĉech cohomology*, which the reader is not expected to have seen before. In general, Ĉech cohomology may refer to a number of things; a modern and efficient notion is phrased in the language of sheaf theory and category theory, in which Ĉech cohomology is more like an algorithm to compute *sheaf cohomology* (which is a very general class of cohomologies parametrized by *(pre)sheaves*) than an actual cohomology theory. For our first encounter with Ĉech cohomology, however, we will limit our perspective to only intersect the essential portions of sheaf and category theory. In particular, we fix the presheaf we are working over. This restriction is motivated by the belief that an all-at-once approach would greatly obscure the purpose of the theory in the first place.

We shall see that all these cohomologies (singular, smooth singular, and Ĉech 'with values in the constant presheaf') agree on smooth manifolds. That H^*_{dR} and H^*_{sng} agree is *de Rham's theorem.* This result expresses a fundamental relationship between the analytic properties (e.g. solutions to differential equations $d\omega = \sigma$) and topological properties (e.g. twists and holes) of smooth manifolds. The fact that H^*_{dR} and Ĉech cohomology agree is sometimes also called de Rham's theorem, or its modern version, or its sheaf-theoretical version.

5.1. **The Five-Lemma.** We will need a result from homological algebra in the next section, called the *five-lemma.* For its proof, we will use two lemmas, the *four-lemmas.* All three statements refer to the following commutative diagram, whose rows are exact and whose objects lie in the same abelian category (e.g. vector spaces over a field, abelian groups, ...).

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "l"] & B \arrow[r, "g"] \arrow[d, "m"] & C \arrow[r, "h"] \arrow[d, "n"] & D \arrow[r, "j"] \arrow[d, "p"] & E \arrow[d, "q"] \\
A' \arrow[r, "r"'] & B' \arrow[r, "s"'] & C' \arrow[r, "t"'] & D' \arrow[r, "u"'] & E'
\end{tikzcd}$$

Lemma 5.1 (Four-lemma one)**.**

Lemma 5.2 (Four-lemma two)**.**

Lemma 5.3 (Five-lemma)**.**

5.2. **Axioms for (Co)homology Theories.** Soon, we will be handling a number of things carrying the word '(co)homology.' In this subsection, we will make precise what a *(co)homology theory* is. One might ask whether this should be done after these new (co)homologies have been developed. The answer is 'perhaps,' but we point out that (i) we are supposing some exposure to (co)homology already, particularly singular homology, and (ii) the Čech cohomology we will see is not truly a cohomology theory. Also, we will be using this axiomatic framework to avoid direct and highly-technical proofs, namely the original proof of de Rham's theorem, the convenience of which cannot be understated.

The basic motivation for these axioms is to succinctly capture the properties of various (co)homologies. Hereafter, a pair (X, A) refers to a topological space X and a subspace A, possibly empty.

Definition 5.4 (Homology theory)**.** A *homology theory* consists of:

(1) A sequence of functors H_n from the category of pairs of spaces to the category of abelian groups, indexed by $\mathbb{Z}$, and
(2) A sequence of natural transformations $\partial : H_n(X, A) \to H_{n-1}(A, \varnothing)$,

Satisfying the following axioms.

Dimension: If X is a point, then $H_0(X)$ is nonzero only for $n = 0$.

Exactness: The following sequence is exact, where unlabeled arrows are induced by the inclusions $A \hookrightarrow X$ and $(X, \varnothing) \hookrightarrow (X, A)$:

$$\cdots \longrightarrow H_k(A) \longrightarrow H_k(X) \longrightarrow H_k(X, A) \xrightarrow{\partial} H_{k-1}(A) \longrightarrow \cdots$$

Excision: If $(X; A, B)$ is such that A, B are subspaces of X whose interiors together cover X, then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism

$$H_*(A, A \cap B) \longrightarrow H_*(X, B).$$

Additivity: If (X, A) is the disjoint union of a set of pairs (X_i, A_i), then their inclusions into (X, A) induce an isomorphism

$$\bigoplus_i H_i(X_i, A_i) \longrightarrow H(X, A).$$

Homotopy: If $f, g : (X, A) \to (Y, B)$ are homotopic, then their images under H_n (denoted f_*, g_*) are identical as homomorphisms from $H_n(X, A)$ to $H_n(Y, B)$.

Definition 5.5 (Cohomology theory)**.** A *cohomology theory* consists of:

(1) A sequence of contravariant functors H^n from the category of pairs of spaces to the category of abelian groups, indexed by $\mathbb{Z}$, and
(2) A sequence of natural transformations $\delta : H^n(A, \varnothing) \to H^{n+1}(X, A)$,

Satisfying the following axioms.

Dimension: If X is a point, then $H^0(X)$ is nonzero only for $n = 0$.

Exactness: The following sequence is exact, where unlabeled arrows are induced by the inclusions $A \hookrightarrow X$ and $(X, \varnothing) \hookrightarrow (X, A)$:

$$\cdots \longrightarrow H^k(X, A) \longrightarrow H^k(X) \longrightarrow H^k(A) \xrightarrow{\delta} H^{k+1}(X, A) \longrightarrow \cdots$$

Excision: If $(X; A, B)$ is such that A, B are subspaces of X whose interiors together cover X, then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism

$$H^*(X, B) \longrightarrow H^*(A, A \cap B).$$

Additivity: If (X, A) is the disjoint union of a set of pairs (X_i, A_i), then their inclusions into (X, A) induce an isomorphism

$$H(X, A) \longrightarrow \bigotimes_i H_i(X_i, A_i)$$

Homotopy: If $f, g : (X, A) \to (Y, B)$ are homotopic, then their images under H^n (denoted f^*, g^*) are identical as homomorphisms from $H^n(Y, B)$ to $H^n(X, A)$.

Definition 5.6. For a (co)homology theory, $H_0(\{\text{pt}\}) = G$ is called the *coefficient group* of the theory.

Cohomology theories are dual to homology theories. This duality is clear directly from their definitions, i.e. as functors vs. contravariant functors. In general, homology and cohomology do not carry exceptionally different information. However, cohomology has a naturally richer structure as a result of contravariance; we refer the reader to [Hatcher] for more on this.

5.3. **Mayer-Vietoris Sequences.** For our purposes, the most important fact about (co)homology theories is that they come with Mayer-Vietoris sequences. Said sequences between homology and cohomology theories are strikingly similar, down to proofs and construction. For this reason, conciseness, and because we will be mostly concerned with cohomology, we will develop only the Mayer-Vietoris sequence for cohomology theories.

We start with two preliminaries.

Lemma 5.7. *Suppose (X, A, B) is such that B is a subspace of A a subspace of X. Then the following sequence is exact:*

$$\cdots \longrightarrow H^q(X, A) \xrightarrow{j^*} H^q(X, B) \xrightarrow{i^*} H^q(A, B) \xrightarrow{\delta} H^{q+1}(X, A) \longrightarrow \cdots$$

Where $j : (X, B) \hookrightarrow (X, A)$ and $i : (A, B) \hookrightarrow (X, B)$ are the inclusions, and δ is the composition

$$H^k(A, B) \longrightarrow H^q(A) \xrightarrow{\delta} H^q(X, A).$$

Lemma 5.8. *Suppose $(X; A, B)$ is such that A, B are subspaces of X whose interiors cover X. Let $C = A \cap B$. Then the inclusions $(A, C) \hookrightarrow (X, C)$ and $(B, C) \hookrightarrow (X, C)$ induce an isomorphism*

$$H^*(X, C) \xrightarrow{\cong} H^*(A, C) \oplus H^*(B, C).$$

Theorem 5.9 (Mayer-Vietoris sequence)**.** *Let $(X; A, B)$ and C be as in the preceding lemma. Then the following sequence is exact:*

$$\cdots \longrightarrow H^k(X) \xrightarrow{\phi^*} H^k(A) \oplus H^k(B) \xrightarrow{\psi^*} H^k(C) \xrightarrow{\Delta^*} H^{k+1}(X) \longrightarrow \cdots$$

Where, if $i : C \hookrightarrow A$, $j : C \hookrightarrow B$, $q : A \hookrightarrow X$, and $\ell : B \hookrightarrow X$ are the inclusions, then

$$\phi^*(\chi) = (q^*(\chi), \ell^*(\chi)), \quad \textit{and} \quad \psi^*(\alpha, \beta) = i^*(\alpha) - j^*(\beta),$$

And Δ^* *is the composition*

$$H^k(C) \longrightarrow H^k(A, C) \xrightarrow[\text{EXCISION}]{\cong} H^k(X, B) \xrightarrow{\delta} H^{k+1}(X).$$

6. Singular (Co)homology

6.1. **Singular Homology.** We are assuming some background in algebraic topology, so our recollection of singular homology will be brief.

Definition 6.1. We denote by Δ_k the **standard k-simplex** in $\mathbb{R}^{k+1}$, defined to be the convex hull of the standard basis vectors in $\mathbb{R}^{k+1}$.

Definition 6.2. Suppose X is a topological space. A **singular k-simplex in X** is a continuous map $\sigma : \Delta_k \to X$. We denote by $C_k(X)$ the free abelian group generated by the singular k-simplices in X, whose elements are called **singular k-chains**. In symbols,

$$C_k(X) = \mathrm{FAb}\,\{(\sigma : \Delta_k \to X) : \sigma \text{ is continuous}\}.$$

Definition 6.3. We denote by $\partial_k : C_k(X) \to C_{k-1}(X)$ the **boundary map** defined by

$$\partial_k \sigma = \sum_{i=0}^{k} (-1)^i \sigma|_{e_0,\dots\hat{e_i},\dots,e_k}$$

where $\sigma|_{e_0,\dots\hat{e_i},\dots,e_k}$ is the restriction of σ to the $(k-1)$-simplex whose vertices are $e_0, \dots e_k$ with e_i omitted.

Definition 6.4. The maps ∂ satisfy $\partial_k \circ \partial_{k+1} = 0$, hence for a space X we have its **singular chain complex**

$$C_0(X) \xleftarrow{\partial} C_1(X) \xleftarrow{\partial} C_2(X) \leftarrow \cdots$$

Analogous to cohomology, we define the n-th **singular homology group** of X to be quotient

$$H_n^{\text{sing}}(X) \overset{\text{def}}{=} \frac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}}.$$

This defines a homology theory.

6.2. **Smooth Singular Homology.** We will eventually establish an isomorphism between de Rham cohomology and singular cohomology by integrating forms over singular chains. However, singular chains in general are only continuous, whereas integration requires a pullback with respect to a smooth map. This turns out to be a nonissue, in the sense that it is 'good enough' to look only at smooth simplices. This is the *smooth equivalence*. To make this precise, let us quickly develop *smooth singular homology.*

Suppose U is a subset of $\mathbb{R}^n$. A continuous map $f : U \to M$ is *smooth on $V \subseteq U$* (possibly a point) if it extends to a smooth map in an open neighborhood of V in $\mathbb{R}^n$. A **smooth k-simplex** is simply a smooth map $\sigma : \Delta_k \to M$. We define **smooth singular k-chains** and **smooth singular homology groups** $H_n^\infty(M)$ of M for smooth k-simplices exactly as we did singular k-chains and singular homology for arbitrary k-simplices.

It turns out that smooth singular homology is isomorphic to the usual singular homology, making the smoothness requirement for integration a nonissue. We prove this nontrivial fact in the next subsection.

6.3. **The Smooth Equivalence.** The free abelian group generated by smooth k-simplices $C_k^\infty(M)$ is a subgroup of $C_k(M)$. The inclusion $\iota : C_k^\infty(M) \hookrightarrow C_k(M)$ commutes with ∂, i.e. it is a chain map, so it naturally induces a well-defined map $\iota_* : H_k^\infty(M) \to H_k(M)$ given by $[\sigma] \mapsto [\iota\sigma]$.

Theorem 6.5 (Smooth equivalence)**.** *The map $\iota_* : H_k^\infty(M) \to H_k(M)$ is an isomorphism.*

The proof of Theorem 6.5 involves some lengthy technical adventures, so we will settle for a proof sketch. A thorough proof can be found in [Lee].

Proof sketch. We summarize the proof in steps.

(1) We will need two lemmas, the first being the one below. Here, a *boundary face* of a simplex is a subsimplex generated by all but one of the vertices from the original simplex.

  Lemma 6.6. *Suppose $f : \Delta_k \to M$ is continuous, and furthermore smooth when restricted to each boundary face of Δ_k. Then f is smooth as a map on the entirely of $\partial\Delta_k$.*

  The proof of this lemma begins with the observation that an arbitrary point x of $\partial\Delta_k$ lies in at most k of its faces, say j. If $\partial_i\Delta_k$ enumerates the j faces containing x, our hypothesis is that there are open sets $U_i \subseteq \partial_i\Delta_k$ so that f extends to a smooth $\tilde{f}_i : U_i \to M$. Rearranging the simplex diffeomorphically if needed, one constructs inductively (on j) a smooth map $\tilde{f} : \bigcup U_i \to M$ that coincides with f on $\partial_i\Delta_k$, defined in terms of $\tilde{f}_0, \tilde{f}_j$. An observation useful to this proof is that by appealing to locality, possibly after replacing U by a subneighorhood, we may regard f as a map to a coordinate patch about $f(x)$ which can be identified with $\mathbb{R}^m$.

(2) Next, we prove a second lemma.

  Lemma 6.7. *For each singular k-simplex $\sigma : \Delta_k \to M$, there exists a continuous map $H_\sigma : \Delta_k \times [0,1] \to M$ such that the following properties hold:*

  I H_σ is a homotopy from $\sigma(x) = H_\sigma(x,0)$ to a smooth k-simplex $\tilde{\sigma}(x) = H_\sigma(x,1)$.
  II For each inclusion of a boundary face $F_{i,k} : \Delta_{k-1} \cong \partial_i\Delta_k \hookrightarrow \Delta_k$, $H_{\sigma\circ F_{i,k}}(x,t) = H_\sigma(F_{i,k}(x),t)$, where $(x,t) \in \Delta_{k-1} \times [0,1]$.
  III If σ is a smooth k-simplex, then H_σ is the constant homotopy.

  One proceeds by constructing the homotopies H_σ by induction on k. The base case is trivial, as the arbitrary 0-simplex is already smooth. For higher k, if σ is smooth, then the constant homotopy is straightfowardly checked to satisfy I-III. For non-smooth σ, we consider the subset

$$S \overset{\text{def}}{=} (\Delta_k \times \{0\}) \cup (\partial\delta_k \times [0,1])\Delta_k \times [0,1]$$

  and a map $H_0 : S \to M$ defined by

$$H_0(x,t) \overset{\text{def}}{=} \begin{cases} \sigma(x) & x \leq x \in \Delta_k,\ t = 0, \\ H_{\sigma\circ F_{i,k}}(F_{i,k}^{-1}(x).t) & x \in \partial_i\Delta_k,\ t \in [0,1]. \end{cases}$$

  One verifies that H_0 is continuous on overlapping sections, hence H_0 is continuous as a function of S by the gluing lemma. Since S is a retraction of $\Delta_k \times [0,1]$ (for example, by radial projection), H_0 extends to a continuous map $H : \Delta_k \times [0,1] \to M$ by precomposing it with the retraction. This H is a homotopy between σ and some other singular simplex $\sigma'(x) = H(x,1)$ which satisfies II by construction.

Next, we must extend H to be a homotopy between σ and a smooth simplex σ'', since σ' is not necessarily smooth. One checks that σ' is continuous on each of $\partial_i \Delta_k \times \{1\}$ so that Lemma 6.7 applies. Thus, H_0 is smooth as a function of $\Delta_k \times \{1\}$. Denote the continuous extension of σ' by $\sigma'' : U \to M$, where U is a neighborhood of σ'. By Theorem 2.23, we can extend σ'' to a smooth map by a new homotopy K such that $K(x,t) = \sigma''(x)$ whenever x is in $\partial \Delta_k$. This K restricts to a homotopy G between σ' to a smooth singular simplex $\tilde{\sigma}(x) \stackrel{\text{def}}{=} K(x,1)$. Finally, one patches together H and K appropriately and verifies that the new map G is a homotopy from σ to $\tilde{\sigma}$, and that it satisfies properties I and II.

(3) Finally

□

6.4. **Singular Cohomology.** To state de Rham's theorem, it will be convenient to work with cohomology vector spaces rather than homology groups. We will dualize (smooth) singular homology to obtain (smooth) singular cohomology. We remark that by the *universal coefficient theorem*, these singular cohomology theories are no richer than their duals in homology.

We begin by defining dual vector spaces $C^k(M;\mathbb{R}) \stackrel{\text{def}}{=} \mathrm{Hom}(C_k(M);\mathbb{R})$, the space of *singular k-cochains with real coefficients,* and the homomorphisms d_k as the duals to ∂_k, so that the d_k are maps $C^k(M) \to C^{k+1}(M)$. These C^k together with d_k form a cochain complex, whose cohomology spaces, the *singular cohomology* of M, we denote $H^k_(M;\mathbb{R})$.

Categorically, we have applied the contravariant functor $\mathrm{Hom}(-,\mathbb{R})$ to the category of chain complexes whose morphisms are chain maps. Denoting by A, B chain complexes, chain maps $F : A \to B$ become cochain maps $F^* : \mathrm{Hom}(B,\mathbb{R})$ to $\mathrm{Hom}(A,\mathbb{R})$.

By similarly dualizing via $\mathrm{Hom}(-,\mathbb{R})$, from chain complexes of smooth singular simplices we obtain *smooth singular cohomology.* We denote the k-th smooth singular cohomology of M by $H^k_\infty(M)$.

6.5. **de Rham's Theorem.** We have shown that singular and smooth singular homology agree, so their dual cohomologies are the same. What's left is to show that these cohomologies agree with de Rham cohomology on smooth manifolds. For this, the isomorphism $H^n(M) \cong H^n_\infty(M)$ will be paramount, since it allows us to work entirely with chains over which we may take integrals.

Definition 6.8. Suppose ω is a k-form defined on M and $\sigma = \sum a_i \sigma_i$ is a smooth k-chain in M, i.e. $\sigma \in C^\infty_k(M)$. Then we define the *integral of ω over σ*

$$\int_\sigma \omega \stackrel{\text{def}}{=} \sum \left(a_i \int_{\Delta_k} \sigma_i^* \omega \right).$$

Theorem 6.9 (Stokes' Theorem on chains)**.** *Suppose M is a smooth manifold over which a smooth $(k-1)$-form ω is defined, and in which σ is a smooth k-chain.*

Then

$$\int_{\partial\sigma} \omega = \int_{\sigma} d\omega.$$

Proof. The proof is a straightforward application of linearity and Stokes' theorem, with a change of variables. □

Definition 6.10. We denote by Ψ the *de Rham homomorphism*, defined by

$$\begin{aligned} \Psi : \Omega^k(M) &\longrightarrow C^k_\infty(M;\mathbb{R}) \\ \omega &\longmapsto \left(\sigma \mapsto \int_\sigma \omega\right) \end{aligned}$$

The de Rham homomorphism is a cochain map (one can check), i.e. $\Psi \circ d = d \circ \Psi$, so it naturally induces a map $\Psi^* : H^k_{\mathrm{dR}}(M) \to H^k_\infty(M)$ defined by $[\omega] \to [\Psi\omega]$. This map Ψ^* commutes with pullbacks in cohomology, as well as the connecting homomorphism induced by a short exact sequence of complexes. That Ψ^* is in fact an isomorphism will be the *piéce de résistance* of this section.

One can prove that Ψ^* is an isomorphism directly, as de Rham originally did in 1931. We will not. The proof we give follows [Bredon] and enjoys the categorical/homological foresight we now have of algebraic topology.

We need a lemma regarding statements defined on open subsets of manifolds.

Lemma 6.11. *Let M be a smooth n-manifold. Suppose that $P(U)$ is a statement about open subsets of M, satisfying three properties:*

(1) $P(U)$ is true for U diffeomorphic to a convex, open subset of $\mathbb{R}^n$;
(2) $P(U), P(V), P(U \cap V) \implies P(U \cup V)$; and
(3) $\{U_\alpha\}$ disjoint and $P(U_\alpha)$ for all $\alpha \implies P(\bigcup U_\alpha)$.

Then $P(M)$ is true.

Proof. First, we will prove this for M diffeomorphic to an open subset of $\mathbb{R}^n$. In this case, we will regard M as an open subset of $\mathbb{R}^n$ without loss of generality.

We would like to work over a finite union of convex open subsets. To ensure we can do this, we need a proper map $f : M \to [0,\infty)$, i.e. a map for which preimages of compact sets are compact. We construct one such map like so: covering M with a coutable collection of open sets with compact closure, take a partition of unity $\{\phi_i\}$ subordinate to this cover and let $f(x) = \sum \phi_i f_i(x)$.

Now define $A_n = f^{-1}([n, n+1])$. Since f is proper, A_n is compact. Now take a covering U_n of A_n that is the finite union of convex open sets contained in $f^{-1}([n - \frac{1}{2}, n + \frac{3}{2})$. It follows that the U_{even} are pairwise disjoint, as are the U_{odd}.

Since the U_n are finite unions of convex open sets, by (1) and (3) we have that $P(U_n)$ is true for all n. By (3), $P(\bigcup_{\text{even}} U_i)$ and $P(\bigcup_{\text{odd}} U_i)$ are true. Now, $(\bigcup_{\text{even}} U_i) \cap (\bigcup_{\text{odd}} U_i) = \bigcup (U_{2i} \cap U_{2i+1})$, and the latter is a disjoint union of sets which are finite unions of convex open sets, from which it follows that $P((\bigcup_{\text{even}} U_i) \cap (\bigcup_{\text{odd}} U_i))$ is true. By (2), $P(M)$ is true.

So we have proven that $P(U)$ is true when U is diffeomorphic to an open subset of $\mathbb{R}^n$. The result in general follows by substituting, in (1) and the proof, the words 'open subset of $\mathbb{R}^n$' for 'convex open subset of $\mathbb{R}^n$.' □

Theorem 6.12 (de Rham's Theorem)**.** *The homomorphism $\Psi^* : H^k_{\mathrm{dR}}(M) \to H^k(M)$ is an isomorphism for smooth manifolds M.*

Proof. We will show that the statement $P(U)$ = '$\Psi^* : H^k_{\mathrm{dR}}(U) \to H^k_\infty(U)$ is an isomorphism for all k' satisfies the properties listed in the previous lemma. This is sufficient by the smooth equivalence.

(1) If U is contractible, the homotopy axiom asserts U has the cohomology of a point. Thus, for $k > 0$, Ψ^* is a map from 0 to 0 and so is trivially an isomorphism. For $k = 0$, Ψ^* is a linear map from $\mathbb{R}$ to $\mathbb{R}$. In this case, it is an isomorphism unless it is the zero map; and it is not so, since $\Psi^*[1] = [(\sigma \mapsto \int_\sigma 1)]$ is not zero in general.

(2) We said earlier that Ψ^* commutes with pullbacks in cohomology and the connecting homomorphism.

$\square$

7. Ĉech Cohomology