# Title tbd

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## 0.1 December 2022

### 0.1.1 (11/28) Ok, let's give this a try

I want to read *Higher Topos Theory* (HTT). That book is $> 700$ pages, and fairly dense ones, so it's a bit of a project. On top of that, HTT isn't really a self-contained read (and that's sort of the nature of the subject). So I'll be drawing on lots of additional material. The subject seems well worth learning, even unavoidable at times (e.g., in the areas I am interested in), so this should be a productive little activity.

...

I need to cover some ground before actually opening HTT. First I need to think about *simplicial sets*. Their role in higher category theory is ubiquitous. They give us a combinatorial model for the homotopy theory of spaces, and also a model for $\infty$-categories. (On the list of things to do is make precise sense of those statements.) Some references are [Rie], [Mat], [Fri08], and `kerodon.net`.

**Definition 0.1.1.** Denote by $\mathbf{\Delta}$ the *simplex category*, defined to have...
- As objects, the ordered set $[n] := \{0 < 1 < \cdots < n\}$ for each $n \geq 0$; and
- As maps, the weakly order-preserving set maps.

**Definition 0.1.2.** A *simplicial set* is a contravariant functor $\mathbf{\Delta} \to \mathsf{Set}$. The *category of simplicial sets*, denoted $\mathsf{sSet}$, is the functor category $\mathsf{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set})$.

**Notation 0.1.1.** Let $X : \mathbf{\Delta}^{op} \to \mathsf{Set}$ be a simplicial set. We may write it $X_\bullet$, and denote by $X_n$ the set $X([n])$. We call the elements of $X_n$ the *$n$-simplices* of $X$.

**Notation 0.1.2.** We may write $\langle f_0, \ldots, f_n \rangle$ to denote the function $[n] \to [m]$ given by $n \mapsto f_n$.

Simplical sets are not just simplices. They carry additional structure, that arising from morphisms in $\mathbf{\Delta}$. We can give a simple description of $\mathbf{\Delta}$. This in turn gives some intuition for what a simplicial set "is."

**Proposition 0.1.3** (The structure of $\mathbf{\Delta}$). *For each $n \geq 0$ and $0 \leq i \leq n$, define the*

$$i\text{-th face map } d^i : [n-1] \to [n] \text{ as } \langle 0, \ldots, \hat{i}, \ldots n \rangle, \text{ and the}$$
$$i\text{-th degeneracy map } s^i : [n+1] \to [n] \text{ as } \langle 0, \ldots, i, i, \ldots, n \rangle.$$

*Every morphism in $\mathbf{\Delta}$ may be written as a composition of face and degeneracy maps. (Also, the face/degeneracy maps satisfy various relations, the simplicial identities; in fact $\mathbf{\Delta}$ is the category generated by those maps, subject to these identities.)*

Thus a simplicial set $X_\bullet$ can be described as a collection of sets $X_n$ ($n$-cells) together with face and degeneracy maps which satisfy the "simplicial identities." I should write more about how this notion arises from topology, in particular the singular complex. That in turn would be a good time to relate all this back to topology (nerves, geometric realization, ...) which is important.

### 0.1.2 (12/1) Why simplicial sets, simplicial complexes

I had stuff written here. But it was incomplete, and the "story" here is an aside I want to write about a bit more carefully at some point. I'm leaving this day blank for the time being.

### 0.1.3 (12/4) Basic structure in $\mathsf{sSet}$

We need to make some terminology regarding simplicial sets.

**Definition 0.1.4.** Let $X_\bullet, Y_\bullet$ be simplicial sets. We say $Y_\bullet$ is a *simplicial subset* of $X_\bullet$ if $Y_n \subseteq X_n$ and $Xf|_{Y_n} = Yf$ for every $n \geq 0$ and simplicial operator $f$. In other words, the action of operators on $Y$ is the restriction of their action on $X$.

**Definition 0.1.5.** The *standard $n$-simplex* $\Delta^n$ is the simplicial set represented by $[n]$, i.e. $\Delta^n := \mathrm{Hom}_{\mathbf{\Delta}}(-, [n])$.

**Proposition 0.1.6.** *Let $X_\bullet$ be a simplicial set. Yoneda's lemma provides a bijection* $\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, X_\bullet) \cong X_n$. *Under this bijection, each n-cell $a \in X_n$ corresponds to a map $f_a : \Delta^n \to X_\bullet$ satisfying $f_a(\mathrm{id}_{[n]}) = a$.*

**Definition 0.1.7.** Let $X_\bullet$ be a simplicial set. By the above, we may identify its $n$-cells with maps $\Delta^n \to X_\bullet$. Call a cell $a \in X_n$ *degenerate* if it factors as $\Delta^n \to \Delta^m \to X_\bullet$ for some $m < n$. (See [Lur22, Tag 0011] for equivalent conditions.)

**Proposition 0.1.8.** *The standard simplex $\Delta^n$ has a unique non-degenerate n-simplex, that arising from* $\mathrm{id}_{[n]}$. *We may call this the generator of $\Delta^n$.*

**Definition 0.1.9** (Boundary of $\Delta^n$)**.** Define a simplicial subset $\partial\Delta^n$, the *boundary of $\Delta^n$*, by

$$(\partial\Delta^n)_k := \{\text{non-surjective maps } [k] \to [n]\} \subseteq \mathrm{Hom}_{\Delta^{op}}([k], [n]).$$

**Definition 0.1.10** (Horns in $\Delta^n$)**.** For $0 \le i \le n$, define a simplicial subset $\Lambda^n_i$, the *i-th horn in $\Delta^n$*, by

$$(\Lambda^n_i)_k := \{f \in \mathrm{Hom}_{\Delta^{op}}([k], [n])) : f([k]) \neq [n] \cup \{i\}\}.$$

A horn $\Lambda^n_i$ is called *outer* if $i \neq 0, n$ and *inner* otherwise.

**Remark 0.1.11.** The face maps $d^i : [n-1] \to [n]$ induce maps $d^i : \Delta^{n-1} \to \Delta^n$ via post-composition. Now, consider an $n$-cell $a \in X_n$ and its representation $a : \Delta^n \to X_\bullet$. We have that $d_i(a) \in X_{n-1}$ is represented by $ad^i$.

### 0.1.4 (12/6) Colimits in/functors out of sSet

Today I want to understand part of Akhil's notes, about functors out of sSet. This is closely related to understanding colimits in sSet, by general theory for presheaf categories. So we also want to understand colimits in sSet. (And we should want to understand these regardless.) Let's go over this.

Here's a standard structure result for presheaf categories.

**Proposition 0.1.12.** *If a category* $\mathsf{C}$ *is small, then every presheaf on* $\mathsf{C}$ *is canonically the colimit of representable presheaves. In particular, every simplicial set is canonically the colimit of standard simplices.*

*Proof.* This is written out in Akhil's notes. I'll give the idea. Also see [Lur22, Remark 00X5].
Consider a presheaf $F : \mathsf{C}^{op} \to \mathsf{Set}$. We associate to $F$ the category $\mathsf{D}_F$ with

- Objects: morphisms from represented presheaves to $F$, i.e. arrows $[-, X] \to F$; and
- Morphisms: morphisms between represented presheaves such that the obvious triangle commutes.

There is a functor $\phi_F : \mathsf{D}_F \to \mathsf{PShv}(\mathsf{C})$ which sends objects $[-, X] \to F$ to $[-, X]$. By construction, for each object $c \in \mathsf{D}_F$, there is a morphism $\phi_F(c) \to F$, and the diagram described by $\phi_F$ together with these morphisms commutes. We therefore have a distinguished morphism

$$\lim_{\mathsf{D}_F} \phi_F \to F.$$

This map turns out to be an isomorphism. □

Hereafter, denote by $\overline{\mathsf{C}}$ the category of presheaves on $\mathsf{C}$.

Suppose $\mathsf{D}$ is cocomplete. We want to understand functors $\overline{F} : \overline{\mathsf{C}} \to \mathsf{D}$. The previous proposition says that objects in $\overline{\mathsf{C}}$ are colimits of representables. So, if $\overline{F}$ preserves colimits, then $\overline{F}$ is determined by $\overline{F}|_{\mathsf{C}}$, i.e. what it does to $\mathsf{C}$ (embedded via Yoneda). We've described an injection of sets

$$\mathrm{Fun}'(\overline{\mathsf{C}}, \mathsf{D}) \hookrightarrow \mathrm{Fun}(\mathsf{C}, \mathsf{D}). \tag{13}$$

Here, $\mathrm{Fun}'$ denotes the set of colimit-preserving functors.

Conversely, suppose given a functor $F : \mathsf{C} \to \mathsf{D}$. Does it extend along the Yoneda embedding to a functor $\overline{F} : \overline{\mathsf{F}} \to \mathsf{D}$? We can do something here, let me write it out:

1. (1) As above, for each presheaf $G : \mathsf{C}^{op} \to \mathsf{Set}$, consider it as a colimit of $\phi_G : \mathsf{D}_G \to \mathsf{C}$. (We can do this because it lands in represented functors.)
2. (2) This is 'functorial' in the following sense: a morphism $G \to H$ induces a functor $\mathsf{D}_G \to \mathsf{D}_H$ such that the obvious triangle commutes.

(3) Define a functor $\overline{F} : \overline{\mathsf{C}} \to \mathsf{D}$ by

$$\overline{F}(G) := \operatorname*{colim}_{\mathsf{D}_G} F \circ \phi_G.$$

This is a functor because of (2).

This functor $\overline{F}$ really extends $F$, i.e. the obvious diagram commutes. For suppose $G = [-, c]$; then $\mathsf{D}_G$ has a final object $[-, c] \to [-, c]$, therefore $\overline{F}(G) = \operatorname{colim}_{\mathsf{D}_G} F \circ \phi_G = F(G)$.

**Proposition 0.1.14.** *Suppose given a functor $F : \mathsf{C} \to \mathsf{D}$ to a cocomplete category. Then the associated functor $\overline{F} : \overline{\mathsf{C}} \to \mathsf{D}$ constructed above preserves all colimits. In fact, $\overline{F}$ is a left adjoint. The right adjoint to $\overline{F}$ is the functor defined by*

$$\mathsf{D} \ni d \mapsto (c \mapsto \operatorname{Hom}_{\mathsf{D}}(Fc, d)) \in \overline{\mathsf{C}}.$$

**Proposition 0.1.15.** *Suppose given a functor $F : \mathsf{C} \to \mathsf{D}$ to a cocomplete category. Then the mapping $F \mapsto \overline{F}$ describes a bijection of sets*

$$\operatorname{Fun}(\mathsf{C}, \mathsf{D}) \xrightarrow{\sim} \operatorname{LeftAdjoints}(\overline{\mathsf{C}}, \mathsf{D}).$$

The proofs are short and formal.

**Corollary 0.1.16.** *If a functor $\overline{F} : \overline{\mathsf{C}}^{op} \to \mathsf{Set}$ takes colimits to limits, then it is representable.*

*Proof.* Suppose as given $\overline{F}$. By the above, it is left adjoint to some $\overline{G} : \mathsf{Set}^{op} \to \mathsf{C}$. Define $f := \overline{G}(\{pt\})$, the image of the terminal object in $\mathsf{Set}^{op}$. I claim that $f$ represents $F$. (Insert short, formal proof; it's in Akhil's notes.) □

### 0.1.5 (12/23) The singular complex and geometric realization

Finals are over and I've had some time to wind down at home. Last time I worked through part of Akhil's notes about functors out of $\mathsf{sSet}$. (Emily Riehl talks about something similar in her notes, but I have not gotten through those, so let me say nothing about that right now.)

Next I want to relate $\mathsf{Top}$, $\mathsf{Cat}$, and $\mathsf{sSet}$. This is the backdrop for the idea that higher categories "bridge" topology/homotopy theory and ordinary categories. Today I'll go over the relation of $\mathsf{sSet}$ to $\mathsf{Top}$, by which I mean the adjunction

$$\text{the geometric realization functor} \dashv \text{the total singular complex functor.}$$

There are a few ways to introduce this adjunction. Lurie, Charles, and Akhil each do it differently. As a matter of taste, I prefer Akhil's approach. (Possibly related: [Lur22, Tag 002D].) Lurie's approach has some important ideas behind it too, I think, but that is overruled because I am feeling sleepy today.

**Definition 0.1.17.** Define a functor $|-| : \mathbf{\Delta} \to \mathsf{Top}$ as follows.

- Each object $[n]$ is sent to the *topological $n$-simplex* $\Delta^n_{top} \subseteq \mathbb{R}^{n+1}$, defined as those $(t_0, \dots, t_n)$ satisfying $t_i \geq 0$ and $\sum t_i = 1$ and given the subspace topology.
- Each morphism $f : [m] \to [n]$ is sent to the map

$$(t_0, \dots, t_n) \mapsto (u_j), \quad u_j = \sum_{i : f(i) = j} t_i.$$

**Definition 0.1.18** (Geometric realization)**.** Since $\mathsf{Top}$ is cocomplete, according to Proposition 0.1.14 and 0.1.15 the functor of Definition 0.1.17 extends uniquely to a left adjoint $|-| : \mathsf{sSet} \to \mathsf{Top}$. We call this *geometric realization*.

In fact, as in Proposition 0.1.15, we know the right adjoint to geometric realization. It sends a space $X$ to the simplicial set $[n] \mapsto \operatorname{Hom}_{\mathsf{Top}}(|[n]| = \Delta^n_{top}, X)$. This is an important construction I maybe should have defined earlier.

**Definition 0.1.19** (Singular complex)**.** Let $X$ be a space. Denote by $\operatorname{Sing}(X)_\bullet$ the simplicial set given as follows.

- The $n$-cells are the continuous maps $\Delta^n_{top} \to X$, and
- Each simplicial operator $f : [m] \to [n]$ acts by precomposing with the continuous map

$$\Delta^m_{top} \to \Delta^n_{top}, \quad (t_j) \mapsto (u_j = \sum_{f(i)=j} t_i).$$

We call $\operatorname{Sing}(X)_\bullet$ the *singular complex* of $X$. We define a functor $\operatorname{Sing}(-)_\bullet : \mathsf{Top} \to \mathsf{sSet}$ in the obvious way.

**Proposition 0.1.20.** *Prior discussion tells us that geometric realization* $|-| : \mathsf{sSet} \to \mathsf{Top}$ *is left adjoint to the singular complex functor.*

**Proposition 0.1.21.** *Since geometric realization is a left adjoint, it commutes with colimits. Furthermore, geometric realization commutes with finite limits of compactly generated spaces.*

We will see later that this adjunction is homtopically well-behaved.

I next want to describe geometric realization. We already have the general construction laid out for us by Proposition 0.1.14 and the preceding discussion. Given $X_\bullet$, we will form the *category of simplices*, also called its *category of elements*, whose elements are the morphisms $\Delta^n \to X_\bullet$ (i.e., the cells of $X_\bullet$), and we take the colimit of $|-|$ restricted to this subcategory. (This is not circular since we are really applying the "baby" geometric realization to the simplex category, Yoneda embedded.)

**Definition 0.1.22.** Given $X_\bullet \in \mathsf{sSet}$, its *category of simplices* or *category of elements* has as objects all morphisms $\Delta^n \to X_\bullet$ for every $n$, and morphisms the maps $\Delta^m \to \Delta^n$ making the obvious diagram commute. We write this category $\mathsf{el}(X)$.

This category of elements/simplices $\mathsf{el}(X)$ is precisely the category $\mathsf{D}_X$ described on (12/6) with $\mathsf{C} = \mathbf{\Delta}$. (Lurie writes this $\mathbf{\Delta}_X$.) Also as noted there, there is a natural functor $\phi_X : \mathsf{el}(X) \to \mathsf{sSet}$. Geometric realization is by definition the colimit

$$|X| \cong \operatorname*{colim}_{\mathsf{el}(X)} |-| \circ \phi_X.$$

Here, we are thinking of the "baby" geometric realization defined only on $\mathbf{\Delta}$.

**Remark 0.1.23.** Much of what we have done the last two days—extension along Yoneda, constructing an adjoint, the use of the category of elements—can be understood and applied quite generally. As to understanding or explaining this, I'm not sure where to start. I think Riehl gives a good first look in [Rie, §4]. These things also come up in Subsection 01Q7. It might be some time before I really get into this. Understanding the "Grothendieck construction" might be a good first step. Also, Kan extensions. Also, consult Cisinski.

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