Title tbd

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0.1 December 2022

0.1.1 (11/28) Ok, let's give this a try

I want to read *Higher Topos Theory* (HTT). That book is > 700 pages, and fairly dense ones, so it's a bit of a project. On top of that, HTT isn't really a self-contained read (and that's sort of the nature of the subject). So I'll be drawing on lots of additional material. The subject seems well worth learning, even unavoidable at times (e.g., in the areas I am interested in), so this should be a productive little activity.

...

I need to cover some ground before actually opening HTT. First I need to think about *simplicial sets*. Their role in higher category theory is ubiquitous. They give us a combinatorial model for the homotopy theory of spaces, and also a model for ∞-categories. (On the list of things to do is make precise sense of those statements.) Some references are [Rie], [Mat], [Fri08], and `kerodon.net`.

Definition 0.1.1. Denote by $\mathbf{\Delta}$ the *simplex category*, defined to have...

- As objects, the ordered set $[n] := \{0 < 1 < \cdots < n\}$ for each $n \geq 0$; and
- As maps, the weakly order-preserving set maps.

Definition 0.1.2. A *simplicial set* is a contravariant functor $\mathbf{\Delta} \to \mathsf{Set}$. The *category of simplicial sets*, denoted sSet, is the functor category $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set})$.

Notation 0.1.1. Let $X : \mathbf{\Delta}^{op} \to \mathsf{Set}$ be a simplicial set. We may write it $X_\bullet$, and denote by X_n the set $X([n])$. We call the elements of X_n the *n-simplices* of X.

Notation 0.1.2. We may write $\langle f_0, \ldots, f_n\rangle$ to denote the function $[n] \to [m]$ given by $n \mapsto f_n$.

Simplical sets are not just simplices. They carry additional structure, that arising from morphisms in $\mathbf{\Delta}$. We can give a simple description of $\mathbf{\Delta}$. This in turn gives some intuition for what a simplicial set "is."

Proposition 0.1.3 (The structure of $\mathbf{\Delta}$)**.** *For each $n \geq 0$ and $0 \leq i \leq n$, define the*

$$i\text{-th face map } d^i : [n-1] \to [n] \text{ as } \langle 0, \ldots, \hat{i}, \ldots n\rangle, \text{ and the}$$
$$i\text{-th degeneracy map } s^i : [n+1] \to [n] \text{ as } \langle 0, \ldots, i, i, \ldots, n\rangle.$$

Every morphism in $\mathbf{\Delta}$ may be written as a composition of face and degeneracy maps. (Also, the face/degeneracy maps satisfy various relations, the simplicial identities; in fact $\mathbf{\Delta}$ is the category generated by those maps, subject to these identities.)

Thus a simplicial set $X_\bullet$ can be described as a collection of sets X_n (n-cells) together with face and degeneracy maps which satisfy the "simplicial identities." I should write more about how this notion arises from topology, in particular the singular complex. That in turn would be a good time to relate all this back to topology (nerves, geometric realization, ...) which is important.

0.1.2 (12/1) Why simplicial sets, simplicial complexes

I had stuff written here. But it was incomplete, and the "story" here is an aside I want to write about a bit more carefully at some point. I'm leaving this day blank for the time being.

0.1.3 (12/4) Basic structure in sSet

We need to make some terminology regarding / record examples of simplicial sets.

Definition 0.1.4. The *standard n-simplex* Δ^n is the simplicial set represented by $[n]$, i.e. $\Delta^n := \mathrm{Hom}_{\mathbf{\Delta}}(-, [n])$.

Definition 0.1.5. Let $X_\bullet, Y_\bullet$ be simplicial sets. We say $Y_\bullet$ is a *simplicial subset* of $X_\bullet$ if $Y_n \subseteq X_n$ and $Xf|_{Y_n} = Yf$ for every $n \geq 0$ and simplicial operator f. In other words, the action of operators on Y is the restriction of their action on X. In other words, $Y_\bullet$ is a subfunctor of $X_\bullet$.

Proposition 0.1.6. *Let $X_\bullet$ be a simplicial set. The Yoneda lemma asserts a bijection $\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, X_\bullet) \cong X_n$. Under this bijection, each n-cell $a \in X_n$ corresponds to a map $f_a : \Delta^n \to X_\bullet$ satisfying $f_a(\mathrm{id}_{[n]}) = a$.*

Definition 0.1.7. Let $X_\bullet$ be a simplicial set. By the above, we may identify its n-cells with maps $\Delta^n \to X_\bullet$. Call a cell $a \in X_n$ *degenerate* if it factors as $\Delta^n \to \Delta^m \to X_\bullet$ for some $m < n$. (See [Lur22, Tag 0011] for equivalent conditions.)

Proposition 0.1.8. *The standard simplex Δ^n has a unique non-degenerate n-simplex, that arising from $\mathrm{id}_{[n]}$. We may call this the generator of Δ^n.*

Definition 0.1.9 (Boundary of Δ^n)**.** Define a simplicial subset $\partial\Delta^n$, the *boundary of Δ^n*, by

$$(\partial\Delta^n)_k := \{\text{non-surjective maps } [k] \to [n]\} \subseteq \mathrm{Hom}_{\mathbf{\Delta}^{op}}([k], [n]).$$

Proposition 0.1.10. *The boundary of Δ^n is the maximal proper simplicial subset of Δ^n.*

Definition 0.1.11 (Horns in Δ^n)**.** For $0 \le i \le n$, define a simplicial subset Λ^n_i, the *i-th horn in Δ^n*, by

$$(\Lambda^n_i)_k := \{f \in \mathrm{Hom}_{\mathbf{\Delta}^{op}}([k], [n])) : f([k]) \cup \{i\} \neq [n]\}.$$

In other words, its cells are those maps "missing something besides i." A horn Λ^n_i is called *outer* if $i \neq 0, n$ and *inner* otherwise.

Any simplicial operator $f : [m] \to [n]$ factors through its image, i.e. we can uniquely write $f = f^{inj} f^{surj}$, a surjection followed by an injection. Furthermore, this is unique. We get the following.

Proposition 0.1.12. *Let $\sigma : \Delta^n \to X_\bullet$ be an n-cell of $X_\bullet$. Then σ factors uniquely as*

$$\Delta^n \xrightarrow{\alpha} \Delta^m \xrightarrow{\tau} X_\bullet,$$

Where α represents a surjection $[n] \to [m]$ and τ is not degenerate. Call m the dimension of the cell σ. (My notation, maybe poor, not that important.)

So, degenerate n-simplices are just non-degenerate simplices in a lower dimension (their "dimension"), trying to bite off more than they can chew.

Definition 0.1.13 (Skeleta)**.** Let $X_\bullet$ be a simplicial set. For $k \ge -1$, define a simplicial subset $\mathrm{sk}_k(X_\bullet)$, the *k-skeleton* of $X_\bullet$, by

$$(\mathrm{sk}_k(X_\bullet))_n := \{\, n\text{-simplices of } X_\bullet \text{ with dimension at most } k \,\}.$$

Remark 0.1.14. The face maps $d^i : [n-1] \to [n]$ induce maps $d^i : \Delta^{n-1} \to \Delta^n$ via post-composition. Now, consider an n-cell $a \in X_n$ and its representation $a : \Delta^n \to X_\bullet$. We have that $d_i(a) \in X_{n-1}$ is represented by ad^i.

0.1.4 (12/6) Colimits in/functors out of sSet

Today I want to understand part of Akhil's notes, about functors out of sSet. This is closely related to understanding colimits in sSet, by general theory for presheaf categories. So we also want to understand colimits in sSet. (And we should want to understand these regardless.) Let's go over this.

Here's a standard structure result for presheaf categories.

Proposition 0.1.15. *If a category C is small, then every presheaf on C is canonically the colimit of representable presheaves. In particular, every simplicial set is canonically the colimit of standard simplices.*

Proof. This is written out in Akhil's notes. I'll give the idea. Also see [Lur22, Remark 00X5].
Consider a presheaf $F : \mathsf{C}^{op} \to \mathsf{Set}$. We associate to F the category D_F with

- Objects: morphisms from represented presheaves to F, i.e. arrows $[-, X] \to F$; and
- Morphisms: morphisms between represented presheaves such that the obvious triangle commutes.

There is a functor $\phi_F : \mathsf{D}_F \to \mathsf{PShv(C)}$ which sends objects $[-, X] \to F$ to $[-, X]$. By construction, for each object $c \in \mathsf{D}_F$, there is a morphism $\phi_F(c) \to F$, and the diagram described by ϕ_F together with these morphisms commutes. We therefore have a distinguished morphism

$$\lim_{\mathsf{D}_F} \phi_F \to F.$$

This map turns out to be an isomorphism. □

Hereafter, denote by $\overline{\mathsf{C}}$ the category of presheaves on C.

Suppose D is cocomplete. We want to understand functors $\overline{F} : \overline{\mathsf{C}} \to \mathsf{D}$. The previous proposition says that objects in $\overline{\mathsf{C}}$ are colimits of representables. So, if $\overline{F}$ preserves colimits, then $\overline{F}$ is determined by $\overline{F}|_{\mathsf{C}}$, i.e. what it does to C (embedded via Yoneda). We've described an injection of sets

$$\mathrm{Fun}'(\overline{\mathsf{C}}, \mathsf{D}) \hookrightarrow \mathrm{Fun}(\mathsf{C}, \mathsf{D}). \tag{16}$$

Here, Fun' denotes the set of colimit-preserving functors.

Conversely, suppose given a functor $F : \mathsf{C} \to \mathsf{D}$. Does it extend along the Yoneda embedding to a functor $\overline{F} : \overline{\mathsf{F}} \to \mathsf{D}$? We can do something here, let me write it out:

(1) As above, for each presheaf $G : \mathsf{C}^{op} \to \mathsf{Set}$, consider it as a colimit of $\phi_G : \mathsf{D}_G \to \mathsf{C}$. (We can do this because it lands in represented functors.)

(2) This is 'functorial' in the following sense: a morphism $G \to H$ induces a functor $\mathsf{D}_G \to \mathsf{D}_H$ such that the obvious triangle commutes.

(3) Define a functor $\overline{F} : \overline{\mathsf{C}} \to \mathsf{D}$ by

$$\overline{F}(G) := \operatorname*{colim}_{\mathsf{D}_G} F \circ \phi_G.$$

This is a functor because of (2).

This functor $\overline{F}$ really extends F, i.e. the obvious diagram commutes. For suppose $G = [-, c]$; then D_G has a final object $[-, c] \to [-, c]$, therefore $\overline{F}(G) = \mathrm{colim}_{\mathsf{D}_G} F \circ \phi_G = F(G)$.

Proposition 0.1.17. *Suppose given a functor $F : \mathsf{C} \to \mathsf{D}$ to a cocomplete category. Then the associated functor $\overline{F} : \overline{\mathsf{C}} \to \mathsf{D}$ constructed above preserves all colimits. In fact, $\overline{F}$ is a left adjoint. The right adjoint to $\overline{F}$ is the functor defined by*

$$\mathsf{D} \ni d \mapsto (c \mapsto \mathrm{Hom}_{\mathsf{D}}(Fc, d)) \in \overline{\mathsf{C}}.$$

Proposition 0.1.18. *Suppose given a functor $F : \mathsf{C} \to \mathsf{D}$ to a cocomplete category. Then the mapping $F \mapsto \overline{F}$ describes a bijection of sets*

$$\mathrm{Fun}(\mathsf{C}, \mathsf{D}) \xrightarrow{\sim} \mathrm{LeftAdjoints}(\overline{\mathsf{C}}, \mathsf{D}).$$

The proofs are short and formal.

Corollary 0.1.19. *If a functor $\overline{F} : \overline{\mathsf{C}}^{op} \to \mathsf{Set}$ takes colimits to limits, then it is representable.*

Proof. Suppose as given $\overline{F}$. By the above, it is left adjoint to some $\overline{G} : \mathsf{Set}^{op} \to \mathsf{C}$. Define $f := \overline{G}(\{pt\})$, the image of the terminal object in Set^{op}. I claim that f represents F. (Insert short, formal proof; it's in Akhil's notes.) □

0.1.5 (12/23) The singular complex and geometric realization

Finals are over and I've had some time to wind down at home. Last time I worked through part of Akhil's notes about functors out of sSet. (Emily Riehl talks about something similar in her notes, but I have not gotten through those, so let me say nothing about that right now.)

Next I want to relate Top, Cat, and sSet. This is the backdrop for the idea that higher categories "bridge" topology/homotopy theory and ordinary categories. Today I'll go over the relation of sSet to Top, by which I mean the adjunction

the geometric realization functor $\dashv$ the total singular complex functor.

There are a few ways to introduce this adjunction. Lurie, Charles, and Akhil each do it differently. As a matter of taste, I prefer Akhil's approach. (Possibly related: [Lur22, Tag 002D].) Lurie's approach has some important ideas behind it too, I think, but that is overruled because I am feeling sleepy today.

Definition 0.1.20. Define a functor $|-|:\mathbf{\Delta}\to\mathsf{Top}$ as follows.

- Each object $[n]$ is sent to the *topological n-simplex* $\Delta^n_{top}\subseteq\mathbb{R}^{n+1}$, defined as those $(t_0,\dots,t_n)$ satisfying $t_i\geq 0$ and $\sum t_i=1$ and given the subspace topology.
- Each morphism $f:[m]\to[n]$ is sent to the map

$$(t_0,\dots,t_n)\mapsto(u_j),\quad u_j=\sum_{i:f(i)=j}t_i.$$

Definition 0.1.21 (Geometric realization)**.** Since Top is cocomplete, according to Proposition 0.1.17 and 0.1.18 the functor of Definition 0.1.20 extends uniquely to a left adjoint $|-|:\mathsf{sSet}\to\mathsf{Top}$. We call this *geometric realization*.

In fact, as in Proposition 0.1.18, we know the right adjoint to geometric realization. It sends a space X to the simplicial set $[n]\mapsto\mathrm{Hom}_{\mathsf{Top}}(|[n]|=\Delta^n_{top},X)$. This is an important construction I maybe should have defined earlier.

Definition 0.1.22 (Singular complex)**.** Let X be a space. Denote by $\mathrm{Sing}(X)_\bullet$ the simplicial set given as follows.

- The n-cells are the continuous maps $\Delta^n_{top}\to X$, and
- Each simplicial operator $f:[m]\to[n]$ acts by precomposing with the continuous map

$$\Delta^m_{top}\to\Delta^n_{top},\quad (t_j)\mapsto(u_j=\sum_{f(i)=j}t_i).$$

We call $\mathrm{Sing}(X)_\bullet$ the *singular complex* of X. We define a functor $\mathrm{Sing}(-)_\bullet:\mathsf{Top}\to\mathsf{sSet}$ in the obvious way.

Proposition 0.1.23. *Prior discussion tells us that geometric realization $|-|:\mathsf{sSet}\to\mathsf{Top}$ is left adjoint to the singular complex functor.*

Proposition 0.1.24. *Since geometric realization is a left adjoint, it commutes with colimits. Furthermore, geometric realization commutes with finite limits of compactly generated spaces.*

We will see later that this adjunction is homtopically well-behaved.

I next want to describe geometric realization. We already have the general construction laid out for us by Proposition 0.1.17 and the preceding discussion. Given $X_\bullet$, we will form the *category of simplices*, also called its *category of elements*, whose elements are the morphisms $\Delta^n\to X_\bullet$ (i.e., the cells of $X_\bullet$), and we take the colimit of $|-|$ restricted to this subcategory. (This is not circular since we are really applying the "baby" geometric realization to the simplex category, Yoneda embedded.)

Definition 0.1.25. Given $X_\bullet\in\mathsf{sSet}$, its *category of simplices* or *category of elements* has as objects all morphisms $\Delta^n\to X_\bullet$ for every n, and morphisms the maps $\Delta^m\to\Delta^n$ making the obvious diagram commute. We write this category $\mathsf{el}(X)$.

This category of elements/simplices $\mathsf{el}(X)$ is precisely the category D_X described on (12/6) with $\mathsf{C}=\mathbf{\Delta}$. (Lurie writes this $\mathbf{\Delta}_X$.) Also as noted there, there is a natural functor $\phi_X:\mathsf{el}(X)\to\mathsf{sSet}$. Geometric realization is by definition the colimit

$$|X|\cong\operatorname*{co\,lim}_{\mathsf{el}(X)}|-|\circ\phi_X.$$

Here, we are thinking of the "baby" geometric realization defined only on $\mathbf{\Delta}$.

Remark 0.1.26. General machinery gave us geometric realization. I think there are a few things worth saying about this, but I don't totally know what. I'll leave this remark here as a "to-do." Some possibly related keywords and references: "Grothendieck construction," "Kan extension," nLab, [Rie, §4], and Subsection 01Q7.

0.2 January 2023

0.2.1 (1/23) Plans have changed, nerves of categories

I've gone radio silent for a month. One big reason why is that I am busy this semester. Another is that some mutuals want to organize a reading group/seminar similar, but not identical to, what I've been trying to do here, and I may join them. Maybe the biggest difference is that they want to focus on Charles' quasicategory notes (under Charles' supervision).

This will probably mean repeating myself a bit while I change tracks to Charles' notes.

In any case, I want to talk about the nerve of a category. This is part of the basic "Spaces, categories, and simplicial sets" picture. In particular, the nerve of a category is a simplicial set encoding that category.

Definition 0.2.1. Let C be a category. Define a simplicial set NC, the *nerve* of C to have as cells $(NC)_n := \mathrm{Hom}_{\mathsf{Cat}}([n], C)$ and so that operators $f : [m] \to [n]$ act by precomposition. This defines the *nerve functor* $N : \mathsf{Cat} \to \mathsf{sSet}$.

Here's a feel for the structure of a nerve. The n-cells of $N\mathsf{C}$ may be canonically identified with the set of length n tuples of composable arrows in C. The 0-cells in particular may be identified with objects of C. An operator $f : [m] \to [n]$, or in Charles' notation $\langle f_1, \ldots, f_m \rangle$, acts by taking n-strings of composable arrows and "collapsing edges" by composing the arrows, those collapsed edges being determined by the f_i. (At least that's how I try to think about it. I think that's correct. UPDATE: Yes this is correct, see Charles' notes, Proposition 4.4.) For instance, $\langle 0, 2 \rangle^*$ takes a pair of composable arrows $(f, g) \in (NC)_2$ and sends them to their composite $gf \in (NC)_1$. See also Charles notes, p. 13.

Now we ask a nascent question: can we characterize nerves of categories?

Proposition 0.2.2. *Let X be a simplicial set. For $n \geq 2$, consider the function*

$$\phi_n : X_n \to \{(g_i) \in (X_1)^n : g_j\langle 1 \rangle = g_{j+1}\langle 0 \rangle \text{ for all } i\}$$

(The latter set being the collection of "n-paths" of 1-cells in X) Which acts by $a \mapsto (a\langle 0, 1 \rangle, \ldots, a\langle n-1, n \rangle)$. These ϕ_n are bijections for all $n \geq 2$ if and only if X is the nerve of a category.

Maybe a lazy way to digest this is that "a simplicial set is the nerve of a category iff, thinking of n-cells as length n strings of arrows, their 1-*dimensional structure* exactly reflects the structure that should arise from the existence and uniqueness of composites."

Proposition 0.2.3. *The nerve functor $N : \mathsf{Cat} \to \mathsf{sSet}$ is fully-faithful. That is, morphisms of nerves $NC \to ND$ correspond exactly to functors $C \to D$.*

0.2.2 (1/26) Spines

We characterized nerves as those simplicial sets whose n-cells were exactly determined by the collection of length n strings of "composable" 1-cells, in the obvious way. This captures the existence and uniqueness of composites for morphisms in a category. We can go about this characterization a bit more systematically.

Definition 0.2.4. Let $n \geq 0$. The *spine* of the standard n-simplex Δ^n is the simplicial subset defined by

$$(\mathrm{Spine}^n)_k := \{\langle f_0, \ldots, f_k \rangle : [k] \to [n] : f_k \leq f_0 + 1\} \subseteq \Delta^n_k.$$

Informally, the spine is the set of vertices of Δ^n together with the arrows between adjacent vertices (considered with their total ordering).

Proposition 0.2.5. *Let X be a simplicial set. For every $n \geq 0$, the map*

$$\mathrm{Hom}(\mathrm{Spine}^n, X) \to \{(a_i) \in (X_1)^n : a_i\langle 1 \rangle = a_{i+1}\langle 0 \rangle\} \tag{6}$$

Given by $f \mapsto (f\langle 0, 1 \rangle, \ldots, f\langle n-1, n \rangle)$ is a bijection.

Pictorially, I think this is obvious. Here's a clean proof.

Proof. One point we need: we previously talked about colimits in sSet. Or at least I intended to. Here's the main fact: *a colimit of simplicial sets X_α exists and has as its n-cells the colimit of the n-cells of the X_α.* This is true for presheaves in general; we say their (co)limits are "computed objectwise."

Another point we need: here's a definition. Suppose S is a totally ordered set. We denote by Δ^S the simplicial set having $(\Delta^S)_n := \{\text{order-preserving maps } [n] \to S\}$. If S is finite and nonempty, there is a unique isomorphism $\Delta^{|S|-1} \cong \Delta^S$. In the case that $S \subseteq [n]$, this is a good way to notate subcomplexes of Δ^n.

Here's a fact I won't prove: *given a subcomplex $K \subseteq \Delta^n$, writing A for the poset of $S \subseteq [n]$ such that $\Delta^S \subseteq K$, the canonical map $\operatorname{colim}_{S\in A} \Delta^S \to K$ is an isomorphism.*

Finally, our proposition: in the case that $K = \mathrm{Spine}^n$, the poset A consists of sets of the form $\{j\}$ and $\{j+1\}$, and we have that $\operatorname{colim}_{S\in A} \Delta^S \cong \mathrm{Spine}^n$. Now:

$$\mathrm{Hom}(\mathrm{Spine}^n, X) \cong \mathrm{Hom}(\operatorname*{colim}_{S\in A} \Delta^S, X) \cong \lim_A \mathrm{Hom}(\Delta^S, X).$$

See that the latter set is precisely the RHS of (6). □

Maybe the key observation is that Δ^n is "generated" precisely by the arrows of Spine^n. (Make this formal? Say this better? Well, this is how I think about it.)

Proposition 0.2.7. *A simplicial set X is the nerve of some category if and only if for each $n \geq 2$, every morphism $f : \mathrm{Spine}^n \to X$ extends uniquely along the inclusion $\mathrm{Spine}^n \hookrightarrow \Delta^n$.*

Proof. The unique extension condition is equivalent to bijectivity of the restriction $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\mathrm{Spine}^n, X)$. By Proposition 0.2.5, the latter set is isomorphic to

$$\{(a_i) \in (X_1)^n : a_i\langle 1\rangle = a_{i+1}\langle 0\rangle\}.$$

Then the desired result is immediate considering Proposition 0.3.34. □

0.2.3 (1/30) Inner Horns

Recall that for each n and $0 \leq i \leq n$ we defined the *i-th horn* $\Lambda^n_k \subseteq \Delta^n$ to have as k-cells those cells $f : [k] \to [n]$ of Δ^n which "miss something other than i," i.e. those satisfying $f([n]) \cup \{i\} \neq [n]$. If $j \neq 0, n$ then we called Λ^n_j an *inner horn*.

Drawing some pictures of horns and thinking of 1-cells as arrows, we think of inner horns as those collections of arrows that "should be composable." Similar to how we handled spines, we may think to characterize nerves as those simplicial sets whose inner horns have unique extensions. By our analogy comparing inner horns to composable arrows, this unique extension condition is analogous the existence and unqiueness of composites.

Theorem 0.2.8. *A simplicial set X is the nerve of some category if and only if for every $n \geq 2$ and inner horn Λ^n_j, every morphism $\Lambda^n_j \to X$ extends uniquely along $\Lambda^n_j \hookrightarrow \Delta^n$.*

Proof. Charles gives a full proof on p. 21 of his notes. The "only if" direction is not complicated. For the "if" direction:

- We can construct a category C whose nerve realizes X explicitly. The objects and morphisms are specified by X_0 and X_1.
- Existence *and* uniqueness of fillers are necessary for the existence and uniqueness of composites.
- Existence of a filler for Λ^3_1 or Λ^3_2 is necessary for the associative law to hold.
- Then one exhibits an isomorphism $X \to NC$.

□

0.3 February 2023

0.3.1 (2/4) Quasicategories

A *quasicategory* or *∞-category* is a simplicial set X such that every inner horn $\Lambda^n_j \to X$ has a filler (i.e. an extension along $\Lambda^n_j \hookrightarrow \Delta^n$.) We have shown that ordinary categories are precisely those quasicategories with *unique* horn extensions.

Definition 0.3.1. Some terminology. Let X, Y be quasicategories.

- *Objects* of $X := X_0$.
- *Morphisms* of $X := X_1$.
- *Identity morphism* of $x \in X_0 := x\langle 0, 0\rangle$.
- *Products of quasicategories* are just their products as simplicial sets.
- *Coroducts of quasicategories* are just their products as simplicial sets.
- A *morphism of quasicategories* $X \to Y$ is a map of simplicial sets.
- A *Natural transformation* $f_0 \implies f_1$ of functors $f_0, f_1 : X \to Y$ is a map of simplicial sets $\phi : X \times \Delta^1 \to Y$ such that $\phi|_{X\times\{i\}} = f_i$.

It's a fact to be proven that the (co)product of quasicategories is again a quasicategory. Here's more terminology.

Definition 0.3.2. Let X be a simplicial set. Let $\sim$ denote the equivalence relation on the set $\coprod X_n$ of cells of X generated by the relation which identifies a cell a with any other cell of the form af for some simplicial operator f. A *connected component* of X is an equivalence class of $\sim$. We write $\pi_0 X$ for the set of equivalence classes. A simplicial set is called *connected* if $\pi_0 X$ is a singleton.

Proposition 0.3.3. *Let X be a simplicial set and suppose x, y are cells in the same connected component of X, i.e. $x = yf$ for some $f : [m] \to [n]$. If $F : X \to Y$ is a map of simplicial sets, then $F(yf) = F(y)Y(Xf)$. The latter is $F(x)$ by hypothesis, thus $F(x) \sim F(y)$. So morphisms $F : X \to Y$ induce maps $\pi_0 X \to \pi_0 Y$ on connected components.*

Proposition 0.3.4. *The induced map $\pi_0(X \times Y) \to \pi_0(X) \times \pi_0(Y)$ is a bijection.*

0.3.2 (2/6) Sub, opposite quasicategories

Definition 0.3.5. Let C be a quasicategory. A subcomplex $C' \subseteq C$ is called a *subcategory* of C if for all $n \geq 2$ and $0 < k < n$, every $f : \Lambda^n_k \to C$ such that $f(\Lambda^n_k) \subseteq C'$ extends "into C'," i.e. extends to a map $f : \Delta^n \to C'$.

It is clear that subcategories of quasicategories are quasicategories.

Definition 0.3.6. Let C be a simplicial set. A simplicial subset $C' \subseteq C$ is called *full* if

- For every cell $\sigma : \Delta^n \to C$ such that for every $0 \leq i \leq n$ the vertex $\sigma(i) \in C$ belongs in C', the cell σ belongs in C'.

If C is a quasicategory and C' is a full subcomplex, then it is a quasicategory. In this case we say C' is a *full subcategory*.

Next we can define the opposite of a quasicategory. In ordinary categories, we do this by reversing composition. We can do something similar once we identify an involution on the simplex category $\mathbf{\Delta}$.

Definition 0.3.7. Define an involution functor $\mathrm{op} : \mathbf{\Delta} \to \mathbf{\Delta}$ as follows.

- It acts as the identity on objects.

- It sends the morphism $\langle f_0, \dots, f_m \rangle : [m] \to [n]$ to its "reverse" $\langle n - f_m, \dots, n - f_0 \rangle : [m] \to [n]$.

Definition 0.3.8. Let $X : \mathbf{\Delta}^{op} \to \mathsf{Set}$ be a simplicial set. We define the *opposite simplicial set* as $X^{op} := X \circ \mathrm{op}$.

One sees that $(\Delta^n_j)^{op} \cong \Delta^n_{n-j}$ and that $(NC)^{op} = N(C^{op})$. The former fact ensures that opposites of quasicategories are quasicategories. The latter ensures that this notion of opposites restricts to the usual 1-categorical notion.

0.3.3 (2/7) Examples of ∞-categories

Example 0.3.1. The nerve NC of a category C is a quasicategory. This is immediate by our characterization of nerves (Theorem 0.2.8).

Example 0.3.2. The singular complex $\mathrm{Sing}(X)$ of a space X is a quasicategory. In fact, we can say a little bit more. Denote by $(\Lambda^n_j)_{top}$ the *topological horn*, defined as you might expect:

$$(\Lambda^n_j)_{top} := \{t \in \Delta^n : t_i = 0 \text{ for some } i \in [n]/\{j\}\}.$$

It is clear that for any j, the simplex Δ^n_{top} retracts onto $(\Lambda^n_j)_{top}$. Thus by precomposing with the retract we get, *for every* j, an inverse to the restriction

$$\mathrm{Hom}(\Delta^n, \mathrm{Sing}X) \to \mathrm{Hom}(\Lambda^n_j, \mathrm{Sing}X).$$

In other words, we can fill every $\Lambda^n_j \to \mathrm{Sing}X$ via the retract. This shows that $\mathrm{Sing}(X)$ is a quasicategory. In fact, we've shown all horns fill, not just inner horns. We call such simplicial sets *Kan complexes*.

Example 0.3.3. Let A be an abelian group and $d \geq 0$ an integer. In spaces, the *Eilenberg-Maclane spaces* $K(A, d)$ represent $H^d(-; A)$. We will define an analogous simplicial set $K = K(A, d)$, the *Eilenberg-Maclane objects* in sSet, like so.

- An element of K_n is a collection $(a_\delta \in A)_\delta$, where the index δ occurs over all operators $\delta : [d] \to [n]$, so that
  - If δ is not injective, $a_\delta = 0$; and
  - For each operator $\gamma : [d+1] \to [n]$ we have $\sum_{j=0}^{d+1} (-1)^j a_{\delta d^j} = 0$.
- For each operator $f : [m] \to [n]$ and $a \in K_n$, we define $(af)_\delta := a_{f\delta}$.

These $K(A, d)$'s are ∞-categories. In fact, they are simplicial abelian groups, which are always Kan complexes. In sSet, they represent *normalized d-cocycles with values in A.* (See Charles' notes, p. 29.)

Example 0.3.4. There is a simplicial set of ordinary categories, denoted Cat_1. We define it like so.

- Each n-cell $(\mathsf{Cat}_1)_n$ is the data of $(C_i, F_{ij}, \zeta_{ijk})$ where
  - For each $i \in [n]$, C_i is a category,
  - For each $i \leq j$ in $[n]$, $F_{ij} : C_i \to C_j$ is a functor, and
  - For each $i \leq j \leq k$ in $[n]$, $\zeta_{ijk} : F_{ik} \to F_{jk} F_{ij}$ is a natural isomorphism,
  - And furthermore, these data are subject to certain basic properties (e.g. $F_{ii} = \mathrm{id}_{C_i}$).
- Each operator $f : [m] \to [n]$ acts on an n-cell $(C_i, F_{ij}, \zeta_{ijk})$ by composing with the indices.

The simplicial set Cat_1 is an ∞-category . Let's discuss fillers.

- A 2-horn $\Lambda^2_1 \to \mathsf{Cat}_1$ is the data of functors $C_0 \xrightarrow{F_{01}} C_1 \xrightarrow{F_{12}} C_2$. An extension is the data of a functor $F_{02} : C_0 \to C_2$ and a natural isomorphism $\zeta_{012} : F_{12}F_{01} \implies F_{02}$. An obvious *but not necessarily unique* candidate is $F_{02} := F_{12}F_{01}$.

- The data of a 3-horn $\Lambda^3_1 \to \mathsf{Cat}_1$ is a bit of a picture. A filler amounts to finding a natural isomorphism to fill the following diagram.

$$\begin{tikzcd} F_{03} \arrow[r, "\zeta_{013}"] \arrow[d, dashed] & F_{13}F_{01} \arrow[d, "\zeta_{123}F_{01}"] \\ F_{23}F_{02} \arrow[r, "F_{23}\zeta_{012}"'] & F_{23}F_{12}F_{01} \end{tikzcd}$$

We can always find this and it is unique, since we required the ζ's to be natural *isomorphisms*.

0.3.4 (2/8) The fundamental category of a simplicial set

The fundamental groupoid $\pi_{\leq 1}X$ of a space X can be recovered from its singular complex $\mathrm{Sing}(X)$. We will recast this construction $\pi_{\leq 1}X$ for any ∞-category. The result will no longer be a groupoid in general (it will only be so for Kan complexes, I think). Let's see how far we get.

First we will look at a certain construction for all simplicial sets. By its definition, it's essentially a left adjoint to the nerve functor.

Definition 0.3.9. Let X be a simplicial set. A *fundamental category of X* is a category hX and a map $\alpha : X \to N(hX)$ such that for every nerve NC, the restriction

$$\alpha^* : \mathrm{Hom}(N(hX), NC) \to \mathrm{Hom}(X, NC)$$

Is a bijection. This characterizes hX up to unique isomorphism, if it exists. (It always does.)

Proposition 0.3.10. *Every simplicial set has a fundamental category.*

Proof. Charles sketches this on p. 30. The objects of our category are X_0. The morphisms are (those generated by) the edges X_1, where we identify composites according to the 2-cells of X. So we turn X into a category in the most obvious way, flattening the higher-categorical structure in the process. The map $\alpha : X \to N(hX)$ is the one you'd expect. □

Proposition 0.3.11. *The fundamental category describes a functor $h : \mathsf{sSet} \to \mathsf{Cat}$, and this functor is left adjoint to the nerve functor N.*

0.3.5 (2/9) Homotopy for ∞-categories

Now we start down a long, dark path. Neither of these adjectives matter up-to-homotopy, however.

Definition 0.3.12. Let C denote an ∞-category and let $f, g : x \to y$ be two morphisms between objects x, y in C. A *homotopy from f to g* is a 2-cell $a \in C_2$ such that $a_{01} = f$, $a_{12} = \mathrm{id}_y$, and $a_{02} = g$.

Proposition 0.3.13. *The homotopy relation is is an equivalence relation on $\mathrm{Hom}_C(x, y)$, i.e. the set of edges i with $i_0 = x$ and $i_1 = y$. Thus we may unamibguously say maps are (or are not) homotopic and speak of homotopy classes.*

Remark 0.3.14. The existence of inner horn extensions is necessary for this relation to be symmetric and transitive. So quasicategories stand out amongst simplicial sets as those having a good notion of homotopy.

Proposition 0.3.15. *Maps f, g are homotopic in C if and only if they are homotopic in C^{op}.*

It's maybe a little weird that "f homotopic to g" is a slightly asymmetric definition, in that even if f is homotopic to g, the data of a homotopy does not suffice to get a homotopy from f to g in C^{op}. I don't think this matters much, in light of the previous proposition. Lurie also gives an alternate, symmetric notion of homotopy to address this point [Lur22, Tag 00V0].

Suppose as given $f \in \mathrm{Hom}_C(x, y)$ and $g \in \mathrm{Hom}_C(y, z)$. We say an edge $h \in \mathrm{Hom}_C(x, z)$ is a *composite* of (g, f) if there exists a 2-cell a such that (what you expect).

Proposition 0.3.16. *Composition respects the homotopy relation on morphisms. Thus, composites are unique up to homotopy.*

Definition 0.3.17. Let C be an ∞-category . Its *homotopy category hC* is the category having as objects C_0 and as morphisms the homotopy classes of morphisms of C.

Now we have defined the *fundamental category of a simplicial set* and the *homotopy category of an ∞-category* . The fundamental category is supposed to be the homotopy category to some extent, so we should compare these two notions where they are both defined (∞-categories).

Definition 0.3.18. Let C be an ∞-category . There is a natural map $\pi : C \to N(hC)$ that "passes to homotopy." It acts like so.

- An object is sent to itself (note $C_0 = (hC)_0 = N(hC)_0$).
- A morphism f is sent to its homotopy class.
- An n-cell $a \in C_n$ is sent to the unique $\pi(a) \in N(hC)_n$ that satisfies $\pi(a)_{i-1,i} = \pi(a_{i-1,i})$ for all i. (See also [Lur22, Construction 004G].)

This map $\pi : C \to N(hC)$ is compatible with simplicial operators, in the sense that given a cell $a \in C_n$, one has $[a_{01}] \circ \cdots \circ [a_{n-1,n}] = [a_{0n}]$.

Proposition 0.3.19. *If C is an ordinary category, then $f \simeq g$ iff $f = g$. Thus if an ∞-category C is isomorphic to a nerve of a 1-category, then $\pi : C \to N(hC)$ is an isomorphism, so it must be isomorphic to the nerve of its homotopy category.*

Proposition 0.3.20 (Universal property of homotopy category)**.** *Let C be an ∞-category and D a small category. If $f = C \to ND$ is a map of simplicial sets, then there exists a unique map $g : N(hC) \to ND$ such that $f = g \circ \pi$.*

Proof. We will construct g. We will do so by constructing a functor $g : hC \to D$. On objects $c \in ob(hC) = C_0$, we define $g(c) := f(c) \in (ND)_0 = ob(D)$. On morphisms, we define $g([h]) := f(h)$. This is well-defined, for if $h \simeq h'$ exhibited by some $a \in C_2$, then $\phi(a) \in (ND)_2$ exhibits the identity $f(h') = \mathrm{id} \circ h$. $\square$

Corollary 0.3.21. *The homotopy category construction is left adjoint to the nerve functor:*

$$h : \mathsf{qCat} \rightleftarrows \mathsf{Cat} : N$$

(Easy to-do: homotopy category of products.)

0.3.6 (2/12) About composition in ∞-categories

Let C be an ∞-category . Let $x, y, z \in C_0$ be objects and let $f \in \mathrm{Hom}_C(x, y)$ and $g \in \mathrm{Hom}_C(y, z)$ be morphisms (i.e. 1-cells starting/ending at their domains/targets.) Last time we defined a *composite of morphisms f, g in C* to be any $h \in \mathrm{Hom}_C(x, z)$ such that there exists a 2-cell $a \in C_2$ such that $a_{01} = f$, $a_{12} = g$, and $a_{02} = h$. Composites exist and are unique up-to-homotopy (thus we can compose homotopy classes), but are not uniquely determined in general. We may ask whether every representative of a homotopy class of a composite can be realized on-the-nose as the extension of its (compositees?) The answer is yes.

Proposition 0.3.22. *If $f : x \to y, g : y \to z$, and $h : x \to z$ are morphisms in an ∞-category C, then $h \in [g] \circ [f]$ if and only if h is a composite of f with g, i.e. there exists $u \in C_2$ satisfying*

$$u|_{\Delta^{0,1}} = f, \quad u|_{\Delta^{1,2}} = g, \quad u|_{\Delta^{0,2}} = h.$$

The proof is nice. I'd reproduce it here but I don't feel like making that diagram right now.

0.3.7 (2/13) Isomorphisms and inverses in ∞-categories

Denote by C an ∞-category and $f : x \to y$ a morphism in C. We say f is an *isomorphism* or an *equivalence* if $[f]$ is an isomorphism in hC. Unwinding a bit, this is equivalent to the existence of a $g : y \to x$ such that $[f] \circ [g] = [1_y]$ and $[g] \circ [f] = [1_x]$. The property of being an isomorphism is related to inverses; the following is elementary.

Proposition 0.3.23. *Let $f : x \to y$ be a morphism in an ∞-category C. A morphism $g : y \to x$ is called a preinverse to f if $[f] \circ [g] = [\mathrm{id}_y]$, and a postinverse if $[g] \circ [f] = [\mathrm{id}_x]$. If g is both, we call it an inverse. TFAE.*

1. *f is an isomorphism.*
2. *f has an inverse.*
3. *f has a preinverse and a postinverse.*
4. *f has a preinverse with a preinverse.*
5. *f has a postinverse with a postinverse.*

As for composites, inverses are not generally unique, but are so up-to-homotopy.

Proposition 0.3.24. *If $F : C \to D$ is a map of quasicategories, then F sends isomorphisms to isomorphisms.*

Proof. Suppose that $f : x \to y$ is an isomorphism in C. By Proposition 0.3.23, f admits an inverse $g : y \to x$. By definition, it satisfies $[f] \circ [g] = [\mathrm{id}_y]$, so by Proposition 0.3.22 there exists $u \in C_2$ witnessing $\mathrm{id}_y = f \circ g$. By this, I mean the obvious identities with simplicial operators hold. Morphisms of simplicial sets commute with operators, hence $F(u)$ witnesses $\mathrm{id}_{F(y)} = F(f) \circ F(g)$. This shows that $F(g)$ is a preinverse to $F(f)$. By an identical argument, one sees that $F(g)$ is a postinverse. So $F(g)$ is an inverse and we are done. □

0.3.8 (2/13) ∞-groupoids, cores, and Kan complexes

Here are some definitions.

- An ∞-category C is called a *quasigroupid* or *∞-groupoid* if hC is a groupoid, i.e. if every morphism is an isomorphism.
- For an ordinary category C, its *core* C^{core} is the subcategory with the same objects and only the isomorphisms of C.
- For an ∞-category C, its *core* C^{core} is the simplicial subset consisting of all cells of C whose edges are all isomorphisms.
- Recall that a simplicial set is called a *Kan complex* if all horns (not necessarily inner) have extensions.

Note that $N(C^{core}) = (NC)^{core}$, so our terminology is justified. Now here are some facts.

Proposition 0.3.25. *If C is an ∞-category , then $\pi_0 C^{core} = \{\text{objects of } C\}/\cong$.*

Proposition 0.3.26. *If C is an ∞-category , then C^{core} is a subcategory and an ∞-groupoid . Furthermore, every sub-∞-groupoid is contained in C^{core}. In other words, C^{core} is the maximal subcategory which is also an ∞-groupoid .*

Proposition 0.3.27. *Every Kan complex is an ∞-groupoid .*

Proof. Suppose that K is a Kan complex and $f : x \to y$ is a morphism in K. Consider the horn $u : \Lambda^2_0 \to K$ with $u_{01} = f$ and $u_{02} = id_x$. Extending this horn gives us $g := u_{12}$ with $[g] \circ [f] = [\mathrm{id}_x]$. Thus, every morphism admits a preinverse, and this turns out to be sufficient for an ∞-category to be an ∞-groupoid . □

Proposition 0.3.28 (Joyal's theorem; harder)**.** *Every ∞-groupoid is a Kan complex.*

Definition 0.3.29. Since $\mathrm{Sing}\, X$ is a Kan complex, the proposition allows us to define the *fundamental ∞-groupoid* of a space X as $\mathrm{Sing}\, X$.

0.3.9 (2/15) The functor quasicategory

The nerve functor $N : \mathsf{Cat} \to \mathsf{sSet}$ is fully faithful, so functors of categories correspond to morphisms of their nerves. Maybe this suggests that if we want a "mapping space," or really a "mapping ∞-category ," it's 0- and 1-categorical structure should consist of morphisms $X \to Y$ of simplicial sets and natural transformations between them. Morphisms are the same thing as maps $X \times \Delta^0 \to Y$. A natural transformation is a suitable map $X \times \Delta^1 \to Y$. This suggests the following.

Definition 0.3.30. Let X, Y be simplicial sets. Their *function complex* is the simplicial set $\mathrm{Fun}(X, Y)$ with

$$(\mathrm{Fun}(X,Y))_n := \mathrm{Hom}_{\mathsf{sSet}}(\Delta^n \times X, Y).$$

Proposition 0.3.31. *There is a natural bijection*

$$\mathrm{Hom}(X \times Y, Z) \xrightarrow{\sim} \mathrm{Hom}(X, \mathrm{Fun}(Y, Z)).$$

Proposition 0.3.32. *For ordinary C and D, one has $N(\mathrm{Fun}(C, D)) \cong \mathrm{Fun}(NC, ND)$.*

Eventually we will see that *function complexes to an ∞-category are ∞-categories* . This is what we want. We can't prove this yet.

0.3.10 (2/16) Lifting properties time — weakly saturated classes

My assessment to become a personal fitness trainer is approaching, so I'm shifting focus to that for a little while. I'll probably be leaving more to-do's than I should be.

Quasicategories are defined in terms of lifting properties. Now we will take some time to generally study lifting properties, which will be useful for studying quasicategories.

Definition 0.3.33. Let C be a category admitting small colimits. A class of morphisms A is called a *weakly saturated class* if it

1. Contains all isomorphisms,
2. Is closed under cobase change (also called pushouts), composition, transfinite composition, coproducts, and retracts. (See Charles p. 38 for the definitions.)

Given any class of morphisms S, its *weak saturation* $\bar{S}$ is the smallest weakly saturated class containing S.

Example 0.3.5. Take $C = \mathsf{Set}$. The weak saturation of $\{\{0, 1\} \to \{1\}\}$ is the class of surjective maps. The weak saturation of $\{\emptyset \to \{1\}\}$ is the class of injective maps.

Example 0.3.6. Take $C = \mathsf{Set}$. The surjections/injections also arise as weak saturations. Of what?

Proposition 0.3.34. *Let S be a category with small colimits and let $\mathcal{C}$ be a class of objects. Let $\mathcal{A}$ be the class of maps with the following lifting property: if $i : A \to B$ is in $\mathcal{A}$, then for every $f : A \to C$ to an object of $\mathcal{C}$, we can fill the following diagram:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \\ B \arrow[ur, dashed] & \end{tikzcd}$$

Then $\mathcal{A}$ is a weakly saturated class. (Example: $S = \mathsf{sSet}$, $\mathcal{C} = \{\infty$-categories $\}$.

Proof. To-do. (Worked out in meeting.) □

0.3.11 (2/16) Classes of horns, anodyne morphisms

As indicated, we are interested in ∞-categories , so we ought to study lifting properties of horns in particular. We make some definitions for this.

Definition 0.3.35. We define the following sets of horns.

$$\begin{aligned}
\mathrm{InnHorn} :=& \{\Lambda^n_k \hookrightarrow \Delta^n : 0 < k < n, n \geq 2\}, \\
\mathrm{Horn} :=& \{\Lambda^n_k \hookrightarrow \Delta^n : 0 \leq k \leq n, n \geq 1\}, \\
\mathrm{RHorn} :=& \{\Lambda^n_k \hookrightarrow \Delta^n : 0 < k \leq n, n \geq 1\}, \\
\mathrm{LHorn} :=& \{\Lambda^n_k \hookrightarrow \Delta^n : 0 \leq k < n, n \geq 1\}.
\end{aligned}$$

We call their weak saturations in sSet the *(inner, right, left) anodyne morphisms*.

Proposition 0.3.36. *Monomorphisms of simplicial sets form a weakly saturated class. Therefore, since* Horn *consists of monomorphisms, its weak saturation must too. So (inner, right, left) anodyne maps are always monomorphisms.*

Proposition 0.3.37. *Let C be an ∞-category . If $A \hookrightarrow B$ is an inner anodyne inclusion, then every $f : A \to C$ extends to B.*

Proof. Let $\mathcal{A}$ denote the set of maps of simplicial sets $X \to Y$ which extend along every map $X \to C$. (C is fixed here.) Since C is a quasicategory, we have $\mathrm{InnHorn} \subseteq \mathcal{A}$. By Proposition 0.3.34, the class $\mathcal{A}$ is weakly saturated, so $\overline{\mathrm{InnHorn}} \subseteq \mathcal{A}$. That is what we wanted to show. ☐

Proposition 0.3.38. *(To-do: Prop 16.10.)*

Example 0.3.7. Here are some examples of inner anodyne morphisms. (Important to-do: finish.)

- The inclusions of spines $I^n \hookrightarrow \Delta^n$ are inner anodyne for every n. In particular, if C is an ∞-category , every map $I^n \to C$ extends to a $\Delta^n \to C$.
- $\vdots$

0.3.12 (2/16) Lifting calculus

Here's a definition I expected a bit earlier in Charles' notes.

Definition 0.3.39. Say an object X *satisfies the extension property* for $f : A \to B$ if for every $u : A \to X$ we can find an extension $B \to X$.

Definition 0.3.40. Suppose as given maps $f : A \to B$ and $g : X \to Y$. A *lifting problem* for (f, g) is a pair of maps $u : A \to X$ and $v : B \to Y$ making a commutative square. A *lift* for the lifting problem is a fill s to the obvious diagram:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r, "v"'] \arrow[ur, dashed, "s"] & Y
\end{tikzcd}$$

Definition 0.3.41. Let f, g be morphisms in a category. We write *$f \boxslash g$* if every lifting problem for (f, g) admits a lift. We call this the *lifting relation* on morphisms. If $f \boxslash g$, we say:

- f has the left lifting property rel. to g, or
- g has the right lifting property rel. to f, or
- f lifts against g.

Definition 0.3.42. Let A be a class of morphisms. We define the *right complement* $A^{\boxslash} := \{g : a \boxslash g, \forall a \in A\}$. We define the *left complement* ${}^{\boxslash}A$ similarly.

Proposition 0.3.43. *Let $\mathcal{A}$ be any class of morphisms in a category with small colimits. The left complement ${}^{\boxslash}\mathcal{A}$ is weakly saturated, and the right complement $A^{\boxslash}$ is weakly cosaturated.*

Proof. (To-do: prove. Also, Charles' related exercises.) $\square$

Example 0.3.8. Let C be an abelian category, let $\mathcal{P} = \{0 \to P : P \text{ projective}\}$, and let $\mathcal{B}$ be the class of epimorphisms in C. By the definition of projective objects, we have $\mathcal{P} \boxslash \mathcal{B}$. Thus $\mathcal{B} \subseteq \mathcal{P}^{\boxslash}$.

(To-do: show converse?)

0.3.13 (2/17) Inner fibrations

A map p of simplicial sets is called an *inner fibration* if $\mathrm{InnHorn} \boxslash p$. Thus, $\mathrm{InnFib} = \mathrm{InnHorn}^{\boxslash}$.

Proposition 0.3.44. *A simplicial set C is an ∞-category iff $C \to *$ is an inner fibration.*

Proposition 0.3.45. InnFib *is defined as a right complement, thus* InnFib *is weakly cosaturated. This implies, for instance, that if C is an ∞-category and $D \to C$ is an inner fibration, then D is an ∞-category .*

Proposition 0.3.46 (. *kerodon] If X is a simplicial set, then a morphism $X \to ND$ is an inner fibration iff X is an ∞-category .*

Proof. (To-do.) $\square$

0.3.14 (2/20) Factorizations

Recall that we defined *inner fibrations* InnFib as the right complement of $\mathrm{InnHorn}$. This tells us something—maybe this is why we call it a "complement."

Proposition 0.3.47 (Small object argument)**.** *Let S be a set of morphisms in* sSet. *Then every map f of simplicial sets admits a factorization $f = p \circ j$ with $p \in S^{\boxslash}$ and $j \in \overline{S}$.*

Corollary 0.3.48. *If S is any set of morphisms in* sSet, *then $\overline{S} =^{\boxslash} (S^{\boxslash})$.*

Proof. Since ${}^{\boxslash}(S^{\boxslash})$ is a left complement, it is weakly saturated (Proposition 0.3.43), thus $\overline{S} \subseteq^{\boxslash} (S^{\boxslash})$.

Now suppose that $f \boxslash S^{\boxslash}$. By the previous proposition, we may write $f = pj$ for $p \in S^{\boxslash}$ and $j \in \overline{S}$, and by assumption f admits a lift in the following diagram.

$$\begin{tikzcd}
\bullet \arrow[r, "j"] \arrow[d, "f"'] & \bullet \arrow[d, "p"] \\
\bullet \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "s"] & \bullet
\end{tikzcd}$$

Thus, we get the following commutative diagram.

$$\begin{tikzcd}
\bullet \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] & \bullet \arrow[r, "\mathrm{id}"] \arrow[d, "j"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "s"'] & \bullet \arrow[r, "p"'] & \bullet
\end{tikzcd}$$

This exhibits f as a retract of j. Since $j \in \overline{S}$ and weak saturations are closed under retracts, we have $f \in \overline{S}$. $\square$

Corollary 0.3.49. *Every map f of simplicial sets can be factored $f = pj$ with p an inner fibration and j inner anodyne.*

0.3.15 (2/21) Factorization systems and unique lifts

Last time, we proved that for a class S of maps in sSet, we have $\overline{S} = {}^{\boxslash}(S^{\boxslash})$ and we can factor every map as a composite of one map from $\overline{S}$ and $S^{\boxslash}$. Starting with $S = \mathrm{InnHorn}$, we got a factorization of an arbitrary map as an inner anodyne followed by an inner fibration. We study such systems in general.

Definition 0.3.50. A *weak factorization system* in a category is a pair of classes of maps $(\mathcal{L}, \mathcal{R})$ with the following properties.

1. Every morphism factors as rl for $l \in \mathcal{L}$ and $r \in \mathcal{R}$; and
2. $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

Example 0.3.9. The pair $(\overline{\mathrm{InnHorn}}, \mathrm{InnHorn}^{\boxslash})$ is a weak factorization system.

We would like to understand lifting problems with unique solutions.

Definition 0.3.51. In a category with coproducts, let $f : A \to B$ be a morphism. We define the *fold* of f, denoted $f^{\wedge}$: $B \coprod_A B \to B$, as the unique map making the following diagram commute.

$$\begin{tikzcd}
B & & \\
& B \coprod_A B \arrow[ul, dashed, "f^\vee"] & B \arrow[l] \arrow[llu, "\mathrm{id}"'] \\
& B \arrow[u] \arrow[uul, "\mathrm{id}"] & A \arrow[l, "f"'] \arrow[u, "f"']
\end{tikzcd}$$

Proposition 0.3.52. *Let f, g be morphisms. The following are equivalent.*

1. *We have $f \boxslash g$ and $f^{\vee} \boxslash g$.*
2. *The solution to any lifting problem for (f, g) exists and is unique.*

Proof. $(2) \implies (1)$ is obvious. For $(1) \implies (2)$, existence is assumed, so we need only show uniqueness. Wait, what does uniqueness mean in an arbitrary category? □

Definition 0.3.53. A weak factorization system $(\mathcal{L}, \mathcal{R})$ is called an *orthogonal factorization system* if $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$ are realized by *unique* lifts.

Proposition 0.3.54. *The factorization $f = rl$ is unique up to unique isomorphism in an orthogonal factorization system.*

Proposition 0.3.55. $(\{$*surjections*$\}, \{$*injections*$\})$ *form an orthogonal factorization system in* Set. *(Proof is obvious.)*

Proposition 0.3.56. *For any class of simplicial maps S, the pair $(\overline{S \cup S^{\vee}}, (S \cup S^{\vee})^{\boxslash})$ is an orthogonal system.*

0.3.16 (2/23) Degenerate cells

We want to concretely understand monomorphisms of simplicial sets. For this, recall that we defined the *boundary of* Δ^n as the subcomplex of Δ^n whose k-cells are the non-surjective maps $[k] \to [n]$. Write Cell for the class of inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ and $\mathrm{InnFib} := \mathrm{Cell}^{\boxslash}$. Since Cell consists of monos, we know $\overline{\mathrm{Cell}}$ contains all monomorphisms. Our main theorem is the converse.

Proposition 0.3.57. *The class $\overline{\mathrm{Cell}}$ is exactly the class of monomorphisms of simplicial sets.*

We'll prove this (Proof 0.3.17) once we've set some stuff up.

Toward proving this, recall the notion of degenerate cells: a cell $\sigma : \Delta^n \to X$ is called *degenerate* if there exists a non-injective operator $f : [m] \to [n]$ such that $\sigma = \tau f$. Since every simplicial operator factors uniquely as $f = f^{inj} f^{surj}$, we see that if σ is degenerate if and only if there is some non-identity *surjective* f such that $a = bf$. A cell which is not degenerate is called *non-degenerate*. We write $X_n = X_n^{deg} \coprod X_n^{nd}$ for the decomposition of X_n into (non)-degenerate cells. Neither assemble to a subcomplex.

Proposition 0.3.58. *Here are some straightforward facts about degenerate cells.*

1. *If $f : X \to Y$ is a map of simplicial sets, then $f(X_n^{deg}) \subseteq Y_n^{deg}$.*
2. *If $f : X \to Y$ is a map of simplicial sets, then $f^{-1}(Y_n^{nd}) \subseteq X_n^{nd}$.*
3. *If $A \hookrightarrow X$ is a subcomplex, then*
$$A_n^{nd} = X_n^{nd} \cap A_n, \text{ and}$$
$$A_n^{deg} = X_n^{deg} \cap A_n.$$
4. *The elements of $(\Delta^n)_k^{nd}$ are in bijection with the subsets of $[n]$ of size k.*
5. *The simplicial n-sphere $\Delta^n/\partial\Delta^n$, defined as the pushout of $\Delta^n \hookleftarrow \partial\Delta^n \to \Delta^0$, has exactly two nondegenerate cells: its unique vertex and the generator $\langle 0, 1, \ldots, n\rangle$. In other words, the image of $\Delta^0 \to \Delta^n/\partial\Delta^n$ and of the generator in $\Delta^n \to \Delta^n/\partial\Delta^n$ in the pushout square:*

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, tail] \arrow[d] & \Delta^n \arrow[d] \\
\Delta^0 \arrow[r] & \Delta^n/\partial\Delta^n
\end{tikzcd}$$

The *Eilenberg-Zilber lemma* says that every cell a of X occurs *uniquely* as $a = b\sigma$ for a nondegenerate b and surjective operator σ. (This is not too complicated; the nontrivial part is uniqueness.) Let's state this in a slightly different, stronger form.

Proposition 0.3.59. *If X is a simplicial set, then for every n the map*

$$\coprod_{j\geq 0} X_j^{nd} \times \mathrm{Hom}_{\Delta^{surj}}([n],[j]) \to X_n$$

Given by $(j, a, \sigma) \mapsto a\sigma$ is a bijection. Furthermore, this map is natural with respect to surjective operators $[n'] \to [n]$ and with respect to monomorphisms of simplicial sets $X \to X'$.

There are various things to be said now. I think I will move on, and refer back to these things when they are needed.

0.3.17 (2/23) The skeletal filtration

If $\sigma : \Delta^n \to X$ is an n-cell, it uniquely factors as $\Delta^n \to \Delta^m \to X$ where the first map is surjective and the second is nondegenerate. So σ is "really" an m-cell, for some $m \leq n$. Now, we want a notion of the k-skeleton of X. Its n-cells should be the n-cells of X which are "really" j-cells for some $j \leq k$.

Definition 0.3.60. Let X be a simplicial set. The *k-skeleton* of X, written $Sk_k X$, is the smallest subcomplex containing all cells of dimension $\leq k$. Thus, we have

$$(Sk_k X)_n = \bigcup_{0\leq j\leq k} \{yf : y \in X_j \text{ and } f : [n] \to [j]\}.$$

A nondegenerate cell $\Delta^k \to X$ determines a cell $\Delta^k \to Sk_k X$. This map carries $\partial\Delta^{k-1}$ to $Sk_{k-1}X$.

Proposition 0.3.61. *The evident square*

$$\begin{tikzcd}
\coprod_{i \in X_k^{nd}} \partial\Delta^k \arrow[r] \arrow[d, tail] & Sk_{k-1}X \arrow[d, tail] \\
\coprod_{i \in X_k^{nd}} \Delta^k \arrow[r] & Sk_k X
\end{tikzcd}$$

Is a pushout square. More generally, if $A \subseteq X$ is a subcomplex, the following is a pushout square.

$$\begin{tikzcd}
\coprod_{i \in X_k^{nd}/A_k^{nd}} \partial\Delta^k \arrow[r] \arrow[d, tail] & A \cup Sk_{k-1}X \arrow[d, tail] \\
\coprod_{i \in X_k^{nd}/A_k^{nd}} \Delta^k \arrow[r] & A \cup Sk_k X
\end{tikzcd}$$

It is in this sense that simplicial sets are built out of standard simplices: a simplicial set X is filtered by $X_0 = Sk_0X \subseteq Sk_1X \subseteq Sk_2X \subseteq \cdots$, and each Sk_n is obtained from Sk_{n-1} by attaching copies of Δ^n as in Proposition 0.3.61.

Now we are ready to prove our characterization of monomorphisms (Proposition 0.3.57).

Proof. A monomorphism of simplicial sets is isomorphic to an inclusion $A \hookrightarrow X$. It is clear that $X \cong \operatorname{colim}_k A \cup Sk_kX$. But see that, by the above proposition, the maps $A \cup Sk_{k-1}X \to A \cup Sk_kX$ arise via cobase change from coproducts of maps in Cell. Then the inclusion is exhibited as a countable composition(?) of maps in $\overline{\mathsf{Cell}}$, thus is in $\overline{\mathsf{Cell}}$. $\square$

And so we have some handle on monomorphisms in sSet now.

Corollary 0.3.62 (Geometric realizations are CW)**.** *Recall that we constructed the geometric realization functor $|-| : \mathsf{sSet} \to \mathsf{Top}$ as a left adjoint. Left adjoints preserve colimits, hence we have a pushout diagram*

$$\begin{tikzcd}
\coprod_{a \in X_k^{nd}} \partial\Delta^k_{top} \arrow[r] \arrow[d, tail] & {|Sk_{k-1}X|} \arrow[d, tail] \\
\coprod_{a \in X_k^{nd}} \Delta^k_{top} \arrow[r] & {|Sk_k X|}
\end{tikzcd}$$

Additionally, we have $|X| = \lim_{\to} |Sk_kX|$. This describes a canonical CW structure on the geometric realization $|X|$ of a simplicial set. Evidently, cells of $|X|$ correspond to nondegenerate simplices of X.

0.3.18 (2/25) Pushout-products, pullback-homs

Let $f : A \to B$ and $g : K \to L$ be morphisms in sSet. We define the *pushout-product* of f and g, denoted fg, as the unique dotted map making the following pushout square diagram commute.

$$\begin{tikzcd}
A \times K \arrow[r, "\mathrm{id}\times g"] \arrow[d, "f\times\mathrm{id}"'] & A \times L \arrow[d] \arrow[ddr, "f\times\mathrm{id}"] & \\
B \times K \arrow[r] \arrow[drr, "\mathrm{id}\times g"'] & (B \times K)\coprod_{A\times K}(A \times L) \arrow[dr, dashed, "f\Box g"] & \\
 & & B \times L
\end{tikzcd}$$

Dually, we define the *pullback-hom* of f and g, denoted $f^{\square g}$, to be the unique dotted map making the following pullback square diagram commute.

$$\begin{tikzcd}
\mathrm{Fun}(L,A) \arrow[dr, dashed, "f^{\square g}"] \arrow[drr, "\mathrm{Fun}(g,\mathrm{id})"] \arrow[ddr, "\mathrm{Fun}(\mathrm{id},f)"'] & & \\
& \mathrm{Fun}(K,A) \times_{\mathrm{Fun}(K,B)} \mathrm{Fun}(L,B) \arrow[r] \arrow[d] & \mathrm{Fun}(K,A) \arrow[d, "\mathrm{Fun}(\mathrm{id},f)"] \\
& \mathrm{Fun}(L,B) \arrow[r, "\mathrm{Fun}(g,\mathrm{id})"'] & \mathrm{Fun}(K,B)
\end{tikzcd}$$

0.4 March 2023

0.4.1 (3/5) Pullback-hom as an enriched lifting problem

Suppose given $g : K \to L$ and $h : X \to Y$. We have the pullback-hom

$$h^{\square g} : \mathrm{Fun}(L, X) \to \mathrm{Fun}(K, X) \times_{\mathrm{Fun}(K,Y)} \mathrm{Fun}(L, Y).$$

The vertices of $\mathrm{Fun}(L, X)$ are morphisms $L \to X$ in sSet. The vertices of $\mathrm{Fun}(K, X) \times_{\mathrm{Fun}(K,Y)} \mathrm{Fun}(L, Y)$ are those pairs of morphisms $(s : K \to X, t : L \to Y)$ such that $hs = tg$, i.e. lifting problems for (g, h). The pullback-hom $h^{\square g}$ takes a morphism $w : L \to X$ and composes it to (wg, gh). This gives us a lifting problem for (g, h) that is solvable. Then the following is clear.

Proposition 0.4.1. *The pullback-hom $h^{\square g}$ is surjective on vertices iff $g \boxslash h$.*

In this sense, $h^{\square g}$ encodes an "enriched" lifting problem for (g, h). The target $\mathrm{Fun}(K, X) \times_{\mathrm{Fun}(K,Y)} \mathrm{Fun}(L, Y)$ parametrizes lifting problems for (g, h) while the source $\mathrm{Fun}(L, X)$ parametrizes families of lifting problems together with a chosen lift.

Also, let's talk about so-called *adjunctions of lifting problems*. The product and function complex constructions are adjoint. Ultimately, this leads to the following.

Proposition 0.4.2. *One has $(f\square g) \boxslash h$ if and only if $f \boxslash h^{\square g}$.*

Here's a special case. Take $K = \emptyset$ and $Y = *$. Then the proposition gives us that

$$(f \times \mathrm{id}_L) \boxslash (X \to *) \iff f \boxslash (\mathrm{Fun}(L, X) \to *).$$

0.5 April 2023

0.6 May 2023

0.6.1 (5/3) Operads for (Peter) May

For May, I will learn about operads and monads. Here are my motives.

(1) Peter **May** coined the term *operad*.

(2) They're interesting and fit into the ∞-categorical framework, eventually.

(3) I read some of Moerdijk-Weiss's *Dendroidal sets* for Charles' Kan Seminar and thought it was exicting.

(4) Peter May tipped me off that monads and operads would make a big appearance in his talks for the 2023 UChicago REU. (Probably related to his recent work with Ruoqi Zhang and Foling Zou.)

Here are some potential references.

(1) May, *The Geometry of Iterated Loop Spaces* (1971)

(2) Markl-Shnider-Stasheff, *Operads in Algebra, Topology, and Physics* (2000)

(3) Heuts, *Simplicial and Dendroidal Homotopy Theory* (2022)

(4) Markl, *Operads and PROPs* (2006)

(5) Lawson, *E_n-ring spectra and Dyer-Lashof operations*

Let me say what I *think* operads are supposed to do/be before I dive into it:
An operad abstracts away the structure of "an operation on a structure its identities/coherences." We'll see this worked out later, for the first time in Example 0.6.4. Here's a vague indication as to why this is useful: say an algebraic thing X has operations which "play nice" with its algebraic structure. If this occurs, it can have useful consequences. It's natural that we then (1) find and study objects for which this occurs, and (2) study them in concert. But this has problems: (A) it may require *lots* of data to verify or realize that X's operations "play nice" with its structure (*coherence data*), especially for complicated X, and/or especially if we're thinking up-to-homotopy, and (B) if we want to study such objects relative to each other, we'll have to compare several of these huge packages of data. Operads do the work for us: the idea is to say, "let $\mathcal{C}$ be the operad codifying the possession of coherent operations." Then, given an object X, a choice (if one exists) of such structure on X amounts to a morphism $\mathcal{C} \to \mathcal{E}\mathrm{nd}_X$, the latter being a canonical "endomorphism operad" associated to X. That morphism essentially says, "we can interpret the structure within $\mathcal{C}$ as some class of operations on X." In other words, for an algebraic structure X, an operad classifies "coherent" operations on X, the details (e.g., how coherent?) dependent on which operad you're considering.

Maybe another way of putting it is that operads represent the formal algebraic theory, while we're interested in "instantiations" of these theories, i.e. their representations—I think we call these "algebras over the operads."

OK, let me actually learn what these are now.

Definition 0.6.1. An *operad*, *symmetric operad*, or *classical operad* $\mathcal{C}$ is a collection of sets $(\mathcal{C}(i))_{i\geq 0}$ with a distinguished operation $1_{\mathcal{C}} \in \mathcal{C}(1)$ and functions $\gamma : \mathcal{C}(n) \times \mathcal{C}(k_1) \times \cdots \times \mathcal{C}(k_n) \to \mathcal{C}(\sum k_s)$, which we regard as *operations*, the *unit operation*, and as *composition*, respectively, and in this regard we require that these structures are suitably associative, unital, and equivariant up to reordering of inputs.

Concretely, $\mathcal{C}$ is the data of

- (Operations) For each $i \geq 0$, a set $\mathcal{C}(i)$ called *i-ary operations*; and
- (Composites) For each $n \geq 0$ and $k_1, \ldots, k_n \geq 0$, a *composition* map $\gamma : \mathcal{C}(n) \times \mathcal{C}(k_1) \times \cdots \times \mathcal{C}(k_n) \to \mathcal{C}(\sum k_s)$; and

- (Unit) A distinguished *identity* operation $1_{\mathcal{C}} \in \mathcal{C}(1)$; and
- (Symmetries) For each $i \geq 0$, an action $\Sigma_i \to \mathrm{Aut}\mathcal{C}(i)$

satisfying the following conditions.

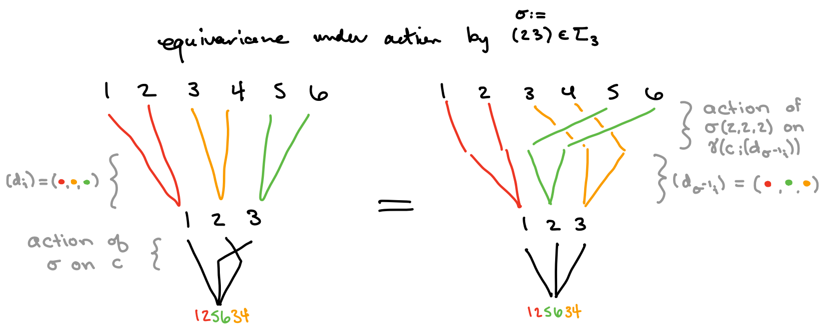
- (Unitality) For every operation $d \in \mathcal{C}(j)$, we have $\gamma(1; d) = d$ and $\gamma(d; 1^{\times j}) = d$.
- (Associativity) Blah blah blah
- (Σ_n-Equivariance) For each $\Sigma_1^k j_s = j, c \in \mathcal{C}(k)$, $d_s \in \mathcal{C}(j_s), \sigma \in \Sigma_k$, and $\tau_s \in \Sigma_{j_s}$, we have
$$\gamma(c\sigma; (d_i)) = \gamma(c; (d_{\sigma^{-1}i})) \cdot \sigma(j_1, \ldots, j_s), \quad \text{and}$$
$$\gamma(c; (d_i \tau_i)) = \gamma(c; (d_i))(\tau_1 \oplus \cdots \oplus \tau_k).$$
Here, $\sigma(j_1, \ldots, j_s) :=$ the permutation of j letters given by permuting the k blocks of letters determined by the partition $j = \Sigma j_s$ according to σ.

Definition 0.6.2. An operad with no Σ_i-actions or equivariance is called *plain* or *non-Σ* or *non-symmetric*.

Definition 0.6.3. Above, we defined an operad in Set. We can make an analogous definition in any bicomplete symmetric monoidal category $(\mathsf{C}, \otimes, \mathbf{1})$. In this case, the unit/identity is a distinguished morphism $\mathbf{1} \to \mathcal{C}(1)$. The symmetries become maps $\Sigma_i \to \mathrm{Iso}(\mathcal{C}(i), \mathcal{C}(i))$. Such a thing is called an *operad in* C. This is like "enriching" an operad over a category.

Definition 0.6.4. A *morphism of operads* $\mathcal{C} \to \mathcal{C}'$ is a collection of maps $f_i : \mathcal{C}(i) \to \mathcal{C}'(i)$ such that $f_1(1) = 1$ and (equivariance, compatibility with composition).

Remark 0.6.5. As indicated, we think of $\mathcal{C}(i)$ as a set of i-ary operations, and the functions $\gamma : \mathcal{C}(k) \times \mathcal{C}(n_1) \times \cdots \times \mathcal{C}(n_k) \to \mathcal{C}(\sum n_s)$ as taking a k-ary operation c and plugging in k other operations (d_i). The Σ_n-equivariance demands that if we tamper with the inputs for c then plug in the (d_i), that is the same as plugging in the (d_i) in a different order then tampering with the order of their inputs. See the little picture.



An operad's purpose in life is to help define *algebras over an operad*. Such a thing establishes an "algebraic structure representing the operad" upon an object. Here is the definition.

Definition 0.6.6. Let $\mathcal{C}$ be an operad in a symmetrical monoidal C. A *$\mathcal{C}$-algebra* A is an object A and maps $\mathcal{C}(i) \otimes A^{\otimes i} \to A$ that are suitably associative, unital, and equivariant. (We take $A^{\otimes 0} = \mathbf{1}_{\mathsf{C}}$.)

I'll go over many examples soon. Some of these will let us reinterpret some of the above structures. But for the rest of today, I'll just make a little remark.

Remark 0.6.7 (Classical operads generalize monoids)**.** Let $\mathcal{C}$ be a classical operad. (A non-Σ operad works too.) There is an associated category $j_!\mathcal{C}$ with one object and morphisms given by $\mathcal{C}(1)$, the unary operations. Given $f, g \in \mathcal{C}(1)$, their composite $g \circ f :=$ their image in $\mathcal{C}(1) \times \mathcal{C}(1) \to \mathcal{C}(1)$, and the unit is the identity operation $1_{\mathcal{C}} \in \mathcal{C}(1)$. This checks out thanks to the unitality and associativity axioms.

Conversely, given a one-object category M, i.e. a monoid, we may form an operad $j^*\mathsf{M}$ with solely unitary operations, given by $(j^*\mathsf{M})(1) := \mathrm{Hom}_{\mathsf{M}}(*, *)$. The unit and composition functions are obvious, and the Σ_i-actions are trivial.

Altogether we get an *adjunction*

$$\begin{tikzcd} \mathsf{Monoids} \arrow[r, shift left, "j_!"] \arrow[r, phantom, "\perp"] & \mathsf{Operads} \arrow[l, shift left, "j^*"] \end{tikzcd}$$

0.6.2 (5/5) Basic examples of operads

In what follows, let $(\mathsf{C}, \otimes, \mathbf{1})$ denote a symmetric monoidal category.

Example 0.6.1. Let A be an object of C. If C is closed,[1] we denote by $\mathcal{E}\mathrm{nd}_A$ the *endomorphism operad* of A, defined by $(\mathcal{E}\mathrm{nd}_A)(i) := \mathrm{Hom}(A^{\otimes i}, A)$. The unit is $\mathrm{id}_A \in \mathcal{E}\mathrm{nd}_A(1)$ and the compositions are given by composing tensor product maps. The right Σ_i-action is given by the left Σ_i-action on tensor powers.

Proposition 0.6.8. *Let $A \in \mathsf{C}$ and let $\mathcal{C}$ be an operad in C. Via the tensor-*Hom *adjunction, a $\mathcal{C}$-algebra structure on A is "the same thing as" a morphism $\mathcal{C} \to \mathcal{E}\mathrm{nd}_A$.*

Example 0.6.2. We denote by $\mathcal{C}\mathrm{omm}$ the *commutative operad* in Set. It is defined to have a single operation $\mathcal{C}\mathrm{omm}(i) := \{*\}$ for every i.

Example 0.6.3. We denote by $\mathcal{A}\mathrm{ssoc}$ the *associative operad* in Set. It is defined to have $\mathcal{A}\mathrm{ssoc}(i) := \Sigma_i$ for every i. The unit and Σ_i-action are obvious. The maps $\gamma : \Sigma_n \times \Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \to \Sigma_{k_1+\cdots+k_n}$ are defined as follows: given $\sigma \in \Sigma_n$ and $\tau_j \in \Sigma_{k_j}$ regarded as matrices, one inserts τ_j in place of the 1 in the j-th column of σ, for each $1 \leq j \leq n$.

Proposition 0.6.9. *In* Set*, $\mathcal{A}\mathrm{ssoc}$-algebras (resp. $\mathcal{C}\mathrm{omm}$-algebras) are precisely monoids (resp. commutative monoids).*

In fact, we can encode monoids in the arbitrary C with operad actions. If $(\mathsf{C}, \otimes, \mathbf{1})$ has finite coproducts, for a finite set S let $\mathbf{1}[S]$ denote the coproduct $\coprod_S \mathbf{1}$.

Definition 0.6.10. We denote by $\mathcal{C}\mathrm{omm}$ the *commutative operad* in C. It is defined to have $\mathcal{C}\mathrm{omm}(i) := \mathbf{1}$.

Definition 0.6.11. If C has finite coproducts, we denote by $\mathcal{A}\mathrm{ssoc}$ the *associative operad* in C. It is defined to have $\mathcal{A}\mathrm{ssoc}(i) := \mathbf{1}[\Sigma_i]$. The rest of its structure is mostly obvious.

Proposition 0.6.12. *In* C*, the $\mathcal{C}\mathrm{omm}$-algebras are precisely the monoids in* C*. If* C *has finite coproducts, then the $\mathcal{A}\mathrm{ssoc}$-algebras are precisely the commutative monoids in* C*.*

Remark 0.6.13 (Algebras over symmetric vs. plain operads)**.** Above, we are regarding $\mathcal{A}\mathrm{ssoc}$ and $\mathcal{C}\mathrm{omm}$ as *symmetric* operads, and this is manifest in the structure of algebras over them. We can instead skip any mention of Σ_n's and consider $\mathcal{A}\mathrm{ssoc}$, $\mathcal{C}\mathrm{omm}$ as *plain* operads. If we do, then $\mathcal{C}\mathrm{omm}$-algebras become precisely monoids in C. And $\mathcal{A}\mathrm{ssoc}$-algebras become...something? Maybe this indicates we should avoid plain operads if possible.

0.6.3 (5/6) Warm-up: monoids are $\mathcal{A}\mathrm{ssoc}$-algebras

Let me work out a concrete example of how monoids are "the same thing as" $\mathcal{A}\mathrm{ssoc}$-algebras.

Example 0.6.4 (How do $\mathcal{A}\mathrm{ssoc}$-algebras encode monoids?)**.** Let X be a set. Suppose it is a monoid, i.e. that we have chosen a unital, associative product $\mu : X \times X \to X$. As I said yesterday, the monoid (X, μ) is "the same thing as" an $\mathcal{A}\mathrm{ssoc}$-algebra which is "the same thing as" a choice of morphism $f : \mathcal{A}\mathrm{ssoc} \to \mathcal{E}\mathrm{nd}_X$.

I'll describe the morphism f. Let me write e_n for the identity in Σ_n.

- f sends $\mathcal{A}\mathrm{ssoc}(1) = \{*\}$ to the identity $\mathrm{id}_X \in \mathcal{E}\mathrm{nd}_X(1)$.
- f sends $e_2 \in \mathcal{A}\mathrm{ssoc}(2) = \Sigma_2$ to $\mu \in \mathcal{E}\mathrm{nd}_X(2)$.
- Where does f send $\sigma \in \mathcal{A}\mathrm{ssoc}(2) = \Sigma_2$? Equivariance demands that $f(\sigma) = f(e_2)\sigma$ ="swap inputs then do $f(e_2) = \mu$."
- Similarly, the value of f on $\mathcal{A}\mathrm{ssoc}(3)$ is already totally determined by its value $f(e_2) = \mu$. Here's why. In $\mathcal{A}\mathrm{ssoc}$, the composition $\gamma : \mathcal{A}\mathrm{ssoc}(2) \times \mathcal{A}\mathrm{ssoc}(2) \times \mathcal{A}\mathrm{ssoc}(1) \to \mathcal{A}\mathrm{ssoc}(3)$ satisfies $\gamma(e_2, e_2, e_1) = e_3$. Since f respects composition, it is determined at e_3:

$$f(e_3) = \gamma(f(e_2), f(e_2), f(e_1)) = \gamma(\mu, \mu, e) = \big((a, b, c) \mapsto \mu(\mu(a, b), c)\big).$$

[1] I.e., if it has an internal Hom

Moreover, notice that $\gamma(e_2, e_1, e_2) = e_3$. Then by the same argument, we find that

$$f(e_3) = \big((a,b,c) \mapsto \mu(a, \mu(b,c))\big).$$

So, $(ab)c = a(bc)$, where we're suppressing μ from notation. Associativity! Similarly, we could take into account the Σ_3-action and show the other associativity identities hold, e.g. $a(cb) = (ac)b$. So, the composition and equivariance conditions gave us associativity (with three inputs)!

- We could repeat the above to see that f's values on $\mathcal{A}\mathrm{ssoc}(n)$ are forced by $f(e_2) = \mu$. Again, the various equalities $\gamma(e_k, e_{n_1}, \ldots, e_{n_k}) = e_{n_1+\cdots+n_k}$, the equivariance equalities, and the fact that f must respect these force all the associativity laws.

Remark 0.6.14. The above example was nice in that after we specified $f(e_2 \in \mathcal{A}\mathrm{ssoc}(2)) = \mu$, we could determine the rest of f based on composition laws, equivariance, and the fact that f must respect those. This is because in some sense, $\mathcal{A}\mathrm{ssoc}$ is "generated by" the element $e_2 \in \mathcal{A}\mathrm{ssoc}(2)$.

So there's an example of how we can use an operad to describe an algebraic operation with coherence. In this case, coherence was strict associativity: we had equalities such as

$$(ab)c = a(bc),\ a(cb) = (ac)b,\ \text{ and } a((bc)d) = ((ab)c)d.$$

These equalities were present "formally" in $\mathcal{A}\mathrm{ssoc}$, and since $\mu : X \times X \to X$ was associative, we could find a corresponding $f : \mathcal{A}\mathrm{ssoc} \to \mathcal{E}\mathrm{nd}_X$ implementing μ.

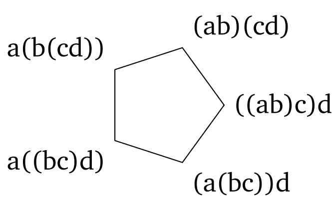
That example was overkill. We "just know" what the "coherence" is—it's associativity. The utility of operads arises when the coherences are more complicated. Next time I will look at an example wherein we'll replace the set X with a space, μ with a continuous map, and strict associativity with "associativity up to specified homotopy." In this case, we'll have e.g. $(ab)c \simeq a(bc)$ *and the data of a homotopy realizing this equivalence.*

0.6.4 (5/9) Associativity up to homotopy, Stasheff associahedra, and A_∞-operads

Let Y denote a based space. If $X = \Omega Y$, then X has a multiplication (loop concatenation). Let's parameterize it like so: for $x, y \in X$, define $xy : [0,1]/\sim \to Y$ as the loop "do y over the first half of the interval, then x over the other."

Generally, we have $(ab)c \neq a(bc)$, thus X is not an $\mathcal{A}\mathrm{ssoc}$-algebra. Loop concatenation is not "strictly associative." But clearly $(ab)c \simeq a(bc)$, realized by (say) a linear reparametrization of $[0,1]$. That would be a map $[0,1] \times I \to X$ starting at $a(bc)$ and landing at $(ab)c$. In $\mathrm{Fun}(X^3, X)$, in which $(ab)c$ and $a(bc)$ reside,[2] such a homotopy is a path from $(ab)c$ to $a(bc)$.

Now say we're concatenating four loops. There are five ways to do this (by reordering parentheses). Say we want a *choice* of homotopy between each reordering. This is the same as taking the pentagon below and sending it to $\mathrm{Fun}(X^4, X)$, each vertex going to the labeled 4-ary operation. **This pentagon is not filled in.**



Again, we think of e.g. $((ab)c)d$ and $(ab)(cd)$ as points in $\mathrm{Fun}(X^4, X)$ and the image in $\mathrm{Fun}(X^4, X)$ of the edge between them as a choice of homotopy equivalence $((ab)c)d \simeq (ab)(cd)$.

Actually, if we chose a path in $\mathrm{Fun}(X^3, X)$ from $a(bc)$ to $(ab)c$ (i.e., if we've made a choice of homotopy equivalence realizing $(ab)c \simeq a(bc)$), that already gives us the five homotopy equivalences above—that is, where to send the pentagon above in $\mathrm{Fun}(X^4, X)$.

[2] That is, the maps $X^3 \to X$ given by "concatenate (a,b,c) to $(ab)c$, or to $a(bc)$."

Notice that there are determined TWO homotopy equivalences between e.g. $a(b(cd))$ and $(ab)(cd)$. One follows the path in $\mathrm{Fun}(X^4, X)$ from $a(b(cd))$ to $(ab)(cd)$ given by traversing the pentagon clockwise, the other counterclockwise. **Since X is a loop space, it turns out that there is a higher homotopy equivalence between these two homotopies.** (In general, this need not be the case, since $\pi_1\mathrm{Fun}(X^4, X)$ is not generally trivial.) Therefore, **there is determined a continuous map from the *solid* pentagon to** $\mathrm{Fun}(X^4, X)$. This is like a "higher" level of associativity, a "higher" level of coherence.

In fact, since X is a loop space, its multiplication is associative up to *all* higher homotopy coherences. Let's define an operad whose algebras are spaces with a multiplication that is associative up to all higher homotopy coherences.

Definition 0.6.15. Denote by $\mathcal{K}$ the non-Σ *Stasheff operad*. It is defined to have $\mathcal{K}(n) :=$ the convex $(n-2)$-dimensional polygon with a vertex for each parenthetization of n ordered letters. (The composition maps can be defined if we use a more explicit description; I won't give that.)

It is somewhat clear (to me, maybe everyone) that the Stasheff operad $\mathcal{K}$ works largely because $\mathcal{K}(n)$ is contractible for each n.

Definition 0.6.16. Let $\mathcal{C}$ be a non-Σ operad in Top. Say it is an *A_∞-operad* if each $\mathcal{C}(n)$ is contractible. Say that a space is an *A_∞-space* if it an algebra over an A_∞-operad.

Here's the main thing.

Theorem 0.6.17. *Up to weak equivalence,*

1. *A_∞-spaces are precisely the $\mathcal{K}$-algebras, and*
2. *Loop spaces are precisely the grouplike $\mathcal{K}$-algebras.*

0.6.5 (5/11) A_n-operad stuff

Let X be a space. Given an operation $\mu : X^2 \to X$, we may ask if it is...

(1) Associative *up to first homotopy*, i.e. we can choose an equivalence $\mu(\mu(-,-),-) \simeq \mu(-,\mu(-,-))$. A choice is the same data as a path $\mathcal{K}(3) = I \to \mathrm{Fun}(X^3, X)$ from $(ab)c$ to $a(bc)$.

(2) (Harder) Associative *up to second homotopy*, i.e. not only can we choose an equivalence $\mu(\mu(-,-),-) \simeq \mu(-,\mu(-,-))$, but can do so in such a way that the resulting homotopy equivalences between e.g. $\mu(\mu(\mu(-,-),-),-)$ and $\mu(-,\mu(\mu(-,-),-))$ are themselves realted by a higher homotopy equivalence. (A "second order" homotopy equivalence.) Such a choice amounts to (A) the structure described in (2), plus (B) a "compatible" map from the solid pentagon $\mathcal{K}(4)$ to $\mathrm{Fun}(X^4, X)$ which sends the vertices to the various parenthetizations of $abcd$. See (5/9).

$\vdots$

(∞) (Even harder) Associative *up to all higher homotopies*, i.e. we can choose a morphism $\mathcal{K} \to \mathcal{E}\mathrm{nd}_X$ such that the path $\mathcal{K}(2) = I \to \mathcal{E}\mathrm{nd}_X$ connects $(ab)c$ and $a(bc)$.

(?) (Too hard) *Strictly* associative, i.e. $\mu(\mu(-,-),-) = \mu(-,\mu(-,-))$.

The structure of (∞) on X is intuitively captured by a choice of $\mathcal{K}$-algebra structure on X ($\iff$ a choice of A_∞-algebra structure). We saw that if $X \simeq \Omega Y$ and we take $\mu =$ loop concatenation, then X has a $\mathcal{K}$-algebra structure. Moreover, up to (some notion of equivalence between $\mathcal{K}$-algebras?), all grouplike $\mathcal{K}$-algebras arise from a loop space and loop concatenation. That's the $n = 1$ case of May's recognition principle.

We're often thinking about the case where μ is only associative up to n-th homotopy for some $n < \infty$. For example, *homotopy associative H-spaces* are precisely those with an operation that is associative up to first homotopy.[3] (And, in my notation, there is something which might be called "associative up to no homotopies," whose algebras are precisely not-necessarily-homotopy-associative H-spaces.)

[3] Actually, this is wrong as I've described "associativity up to n-th homotopy," but I'm just trying to give some intuition for A_n-algebra structures, so I'll gloss over this point as it'll all work when I actually get to A_n-operads.

We want to characterize these structures with operads too. This means taking the data of an A_∞-space and truncating its coherence data at the n-th level. So for example, an "A_3-space" should be a based space X, a homotopy monoidal structure $\mu : X^2 \to X$, and a homotopy equivalence $\mathcal{K}(3) = I \to \mathrm{Fun}(X^3, X)$ from $(ab)c$ to $a(bc)$. Then an "A_n-operad" should be an operad whose algebras are precisely A_n spaces, modulo weak homotopy equivalence.

Our flagship A_∞-operad is the Stasheff operad $\mathcal{K}$. Each associahedron $\mathcal{K}(i)$ is a CW complex (in fact a simplicial complex) in an obvious way. My first thought was, "maybe taking the $(n-2)$-skeleton of every $\mathcal{K}(i)$ will produce an A_n-operad." After all, taking e.g. $n = 3$, if we take the 1-skeleton of $\mathcal{K}$, call it $\mathcal{K}_3$, then an algebra over this operad will have a homotopy monoidal structure (specified by where we send $\mathcal{K}_3(2) = *$) that is homotopy associative (specified by where we send $\mathcal{K}_3(3) = I$), but it will NOT be "associative up to second homotopy" since $\mathcal{K}_3(4)$ is the *hollow* pentagon. (The image of its interior is what *would have* specified the "second-order homotopy coherence" for associativity.)

BUT, this thing $\mathcal{K}_3$ is NOT an operad. For a simple reason: if $\mathcal{K}_3(4)$ contains intervals, then composition necessitates that $\mathcal{K}_3(5)$ contains products of intervals. (We haven't precisely defined composition, but no matter.) But this proposal for $\mathcal{K}_3$ is such that $\mathcal{K}_3(5)$ has no products of intervals, so it doesn't work.

I don't think we're far off, though. There's a *free operad* construction on "collections of objects" (yet-undefined) that I suspect will recover an operad from an appropriate n-truncation of $\mathcal{K}$ which should be rightfully called an A_n-operad.

0.6.6 (5/13) Rings via operads?

Tangent today. So far, I've thought of operads as devices for studying the structure of "an operation with coherences" on a given object. We also care about *rings* and *ringlike* structures. These have TWO operations, plus coherence data for identities that may involve BOTH operations at once. (Distributivity.)

Question 0.6.18. Can we use operads to capture the structure of rings?

The answer depends on the base category! We cannot use operads to characterize rings "all at once," i.e. in Set. We can instead try changing our base category to "handle" some of one operation first, though—we'll find that we can recover rings as algebras over operads in Ab.

Let's start in Set. We want an operad $\mathcal{R}$ such that $\mathcal{R}$-algebras are precisely rings. Or commutative rings—it won't matter, since $\mathcal{R}$ does not exist in either case. I'll give two proofs why.

The first proof is moral, not a real proof. Suppose $\mathcal{R}$ is such that $\mathcal{R}$-algebras are precisely rings. Then there should be "addition" and "multiplication" elements $A, M \in \mathcal{R}(2)$ such that $\mathcal{R}$'s compositions realize the identity $a(b+c) = ab + ac$. The issue is that the 3-ary function $ab + ac$ calls a in more than one input spot. **This is NOT expressible using an operad's notion of composition.** The closest we can get is $\gamma(A, M, M) \in \mathcal{R}(4)$, but this is $ab + cd$, not really what we wanted. Thus, we cannot impose distributivity.

Here's a formal proof. First note: there's an operad $\mathcal{S}\mathrm{et}^\times$ whose i-th object is a set of size i and whose composition arises from the Cartesian product. For an operad $\mathcal{C}$, a $\mathcal{C}$-algebra is precisely a morphism $\mathcal{C} \to \mathcal{S}\mathrm{et}^\times$. The category Operad has a terminal object whose algebras are monoids,[4] so there's a functor S : Monoid $\to$ $\mathsf{Alg}_\mathcal{C}$. It takes each map $* \to \mathcal{S}\mathrm{et}^\times$ and postcomposes it with the unique map $\mathcal{C} \to *$. **I THINK** the functor S : Monoid $\to$ $\mathsf{Alg}_\mathcal{C}$ just takes a monoid, forgets its structure, and endows it with the structure of a $\mathcal{C}$-algebra; THUS, this functor fits into a commutative triangle with the forgetful functor. BUT, the following proposition says that there are NO functors Monoid $\to$ Ring commuting with the forgetful functor, so there must not exist an operad $\mathcal{C}$ in Set such that $\mathsf{Alg}_\mathcal{C} \cong$ Ring.

Proposition 0.6.19. *There are no functors* Monoid $\to$ Ring *commuting with the forgetful functors to* Set.

Zhen Lin proves this in his MSE answer here.

So there are no operads in Set whose algebras are rings. Here's a workaround: we can replace Set with a category of objects which have addition already built in, that would be Ab. Then we can finish the job with an operad in Ab and its algebras will recover Ring.

[4]If we're thinking of symmetric operads, the terminal operad is $\mathcal{C}\mathrm{omm}$. If we're thinking of plain operads, it is $\mathcal{A}\mathrm{ssoc}$.

Let $\mathcal{C}\mathrm{omm}$ denote the commutative monoid operad in Ab. Its objects are trivial groups and its actions are trivial also.

Suppose G is a $\mathcal{C}\mathrm{omm}$-algebra, i.e. an abelian group with a map $f : \mathcal{C}\mathrm{omm} \to \mathcal{E}\mathrm{nd}_G$. This data distinguishes two elements $1_G := f_0(*) \in G$ and $\times := f_2(*) \in \mathrm{Hom}(G^2, G)$. Since $\mathcal{C}(1) = \{*\}$, we have that

$$\gamma(\times, \mathrm{id}, 1_G) = \gamma(\times, 1_G, \mathrm{id}) \in \mathcal{C}(1).$$

Since f must preserve composition, we get that $g \times 1_G = 1_G \times g = g$ for every $g \in G$. (How to finish???)

OK, in the above I started with and tried to show its algebras in Ab are commutative rings, I should've started with $\mathcal{A}\mathrm{ssoc}$ and tried to show its algebras in Ab are rings, but I'm getting tired. (I couldn't figure out how to get the distributive property anyway...)

Anyway, I'll remark that we do something similar in homotopy theory. We want a good notion of "spectra that are ringlike up-to-homotopy," and we must play the same game: find a category of objects with addition built-in (that would be Sp) and then take algebras over an operad for commutativity "up to homotopy." (I think that's little disks?)

We are foreshadowing!

0.6.7 The rest of May

There are some notes I have not texed, and I also spent some time working on a condensed math seminar I'll be organizing at the University of Chicago in June. (I will also be there, probably studying more operad stuff with Peter May.) Maybe I will upload the rest of my May notes eventually?

0.7 June 2023

0.7.1 (6/13) June activities, monoidal categories

There are a few things going on.

1. Peter is giving lectures on operads and algebraic K-theory.
2. I'll probably be reading some of these lecture notes about algebraic K-theory.
3. I'm organizing a seminar on elementary condensed math. (And teaching quite a bit of basic category for that, as well as for other REU participants who just want to learn basic category theory.)

All this will be taking up most of my time. And it all somehow relates somehow to my goal of understanding higher category theory, especially (2). So I'll be sporadically writing here my inner monologue as I learn/do this stuff.

Today Peter spoke about A_∞-spaces. I already wrote about those. But Peter also mentioned monoidal categories, and this led me to a little question we were not sure about.

Let me get to explaining my thought.

Definition 0.7.1. A *monoidal category* is the data of a category C together with

- A functor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ called the *tensor product*;
- A distinguished object $\mathbf{1} \in \mathsf{C}$ called the *unit*;
- Isomorphisms $\lambda : \mathbf{1} \otimes - \to -$ and $\rho : - \otimes \mathbf{1} \to -$, we call the *left/right unitor* and whose components we denote λ_X, ρ_X; and
- An isomorphism
$$A : (- \otimes -) \otimes - \cong - \otimes (- \otimes -)$$
which we call the *associator* and whose components we write $A_{X,Y,Z}$.

And these data must have the following properties.

- The *triangle identity* holds:

$$\begin{tikzcd}
(X \otimes \mathbf{1}) \otimes Y \arrow[rr, "A_{X,\mathbf{1},Y}"] \arrow[dr, "\rho_X \otimes \mathbf{1}"'] & & X \otimes (\mathbf{1} \otimes Y) \arrow[dl, "\mathbf{1} \otimes \lambda_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

- The *pentagon identity* holds: Put it in?

The pentagon identity looks like

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