

NATURAL TRANSFORMATIONS

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ABSTRACT. We develop the categorical notion of a natural transformation.

Categories and functors were initially developed as part of an auxiliary framework in which the prevalent but then-informal notion of naturality could be precisely defined once and for all. Here is the definition.

Definition. Let $F, G : C \rightarrow D$ be functors. A **natural transformation** α from F to G , denoted $\alpha : F \Rightarrow G$, consists of:

- For each object $c \in C$, an arrow $\alpha_c : Fc \rightarrow Gc$ in D . These define the **components** of the natural transformation.

Such that for each morphism $f : x \rightarrow y$ in C ,

$$\alpha_x \circ F(f) = G(f) \circ \alpha_y.$$

In other words, the following square is always commutative.

$$\begin{array}{ccc} Fx & \xrightarrow{\alpha_x} & Gx \\ Ff \downarrow & & \downarrow Gf \\ Fy & \xrightarrow{\alpha_y} & Gy \end{array}$$

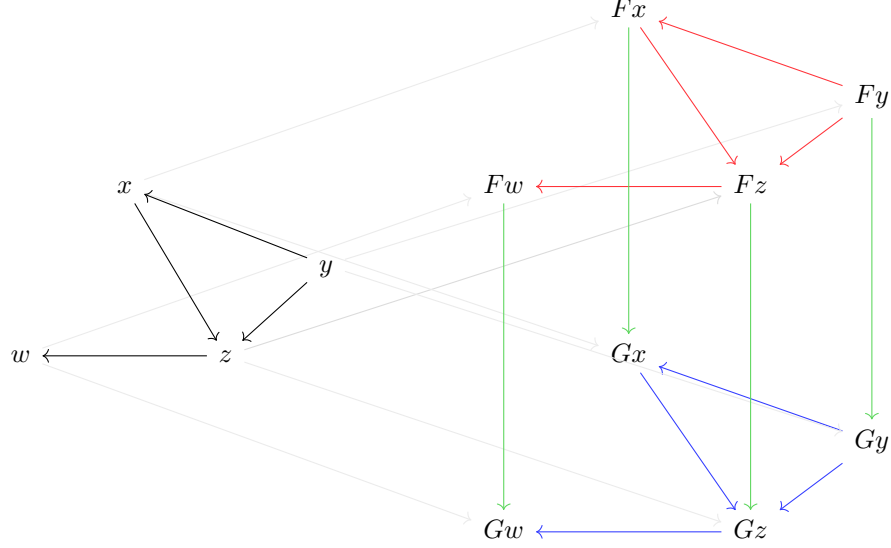
We could have equally required that the following diagram be commutative.

$$\begin{array}{ccccc} & x & & & \\ & \searrow & & \nearrow & \\ & Fx & \xrightarrow{\alpha_x} & Gx & \\ f \downarrow & & & & \\ & y & & & \\ & \searrow & & \nearrow & \\ & Fy & \xrightarrow{\alpha_y} & Gy & \end{array}$$

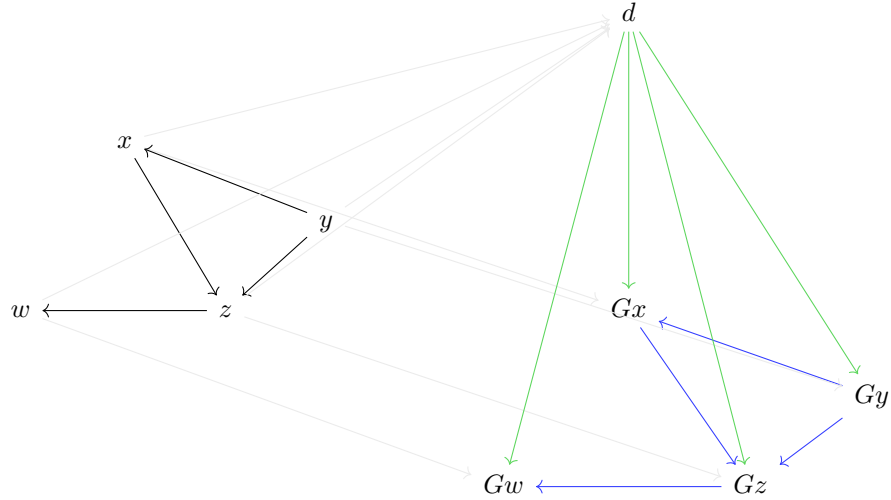
To me, this depiction is more illuminating. It gets at how I think of natural transformations, which is along these lines: *starting from a diagram of objects and morphisms between them in C , the functors F and G map the diagram to two diagrams in D ; a natural transformation consists of maps between the two image diagrams such the total diagram is commutative.*

To pound in this intuition, we will see similar diagrams in the following examples.

Example 0.1 (General example). Arbitrarily, let x, y, z, w be objects in a category C such that there are morphisms $y \rightarrow x$, $x \rightarrow z$, $y \rightarrow z$, and $z \rightarrow w$. Left below, the diagram in C consisting of these objects and maps is shown. Let F, G be two functors $C \rightarrow D$. Then there are two diagrams in D ; one consists of the image of the previous diagram under F , and likewise for G . These are the top and bottom diagrams below at right. A natural transformation from F to G is the collection of green arrows making total diagram is commutative.



Example 0.2 (F is constant). In the last example, we implicitly supposed $F, G : C \rightarrow D$ were injective (at least at x, y, z and w). For each $d \in D$, we have the **constant functor** denoted d which takes every object to d and every morphism to id_d . Suppose F is the functor d .



For obvious reasons, $\alpha : d \Rightarrow G$ is called the **cone over G with summit or apex d** . If G were constant instead of F , we would call α the cone *under* F . In either case, we call the components of α its **legs**.

Definition. A **natural isomorphism** is a natural transformation $\alpha : F \Rightarrow G$ whose components are all isomorphisms. In this case, we say F and G are **naturally isomorphic**, notated $F \cong G$.

On the other hand, if every component of α is an isomorphism, then it is natural if and only if $F(f) = \alpha_y^{-1} \circ G(f) \circ \alpha_x$. (Look familiar?)

Let's see some concrete examples.

Example 0.3 (Double-dual). Recall that the dual V^* of a finite-dimensional k -vector space is the space of linear maps $V \rightarrow k$, and the double-dual V^{**} is the space of linear maps $V^* \rightarrow k$. Both V^* and V^{**} are vector spaces over k .

The space V is isomorphic to V^* . However, the two are not naturally isomorphic, in the sense that there is no natural isomorphism from the identity functor to the dual functor $(-)^*$. A technical reason for this is because $\text{id}_{\text{Vect}_k}$ is covariant and $(-)^*$ is not. But even ignoring that, there are issues. To specify an isomorphism requires choosing a basis; if α were a natural transformation defined after such a choice and $T : V \rightarrow W$ is an arbitrary linear map, we must have $\alpha_W = T^* \circ \alpha_V^{-1} \circ T$. But this almost never holds, in fact there need not even exist inverses; indeed, unless T fixes the basis (in other words, is the identity map), α will fail to be a natural isomorphism. And there are certainly such T .

On the other hand, $V \cong V^{**}$ in a natural way. Define the **evaluation map** $\text{ev} : V \rightarrow V^{**}$ as the function which sends v to $\text{ev}_v : V^* \rightarrow k$, which itself evaluates a map at v . Symbolically,

$$\text{ev}(v) = (v^* \mapsto v^*(v)).$$

The evaluation map is an isomorphism from a vector space to its double-dual. To see that it is a natural transformation, we must check that the square

$$\begin{array}{ccc} V & \xrightarrow{\text{ev}} & V^{**} \\ T \downarrow & & \downarrow T^{**} \\ W & \xrightarrow{\text{ev}} & W^{**} \end{array}$$

commutes. Denoting an arbitrary vector of V by v , we have $\text{ev}_{Tv} = (Tv)^* : W^* \rightarrow k$, which maps the functional $w^* \in W^*$ to $w^*(Tv)$. Down the other path, T^{**} takes $\text{ev}_v : V^* \rightarrow k$ to the functional $(Tv)^* \mapsto (Tv)^*Tv$. These are the same maps.

Because the evaluation map is an isomorphism $V \cong V^{**}$ and (as we checked) the above square commutes for all T , we have shown there exists a natural isomorphism $\text{id}_{\text{Vect}_k} \Rightarrow (-)^{**}$.

Example 0.4 (Abelianization). Given a group G , we define its **abelianization** $G^{\text{ab}} \stackrel{\text{def}}{=} G/[G, G]$, where $[G, G]$ is its commutator subgroup. The (canonical) projection map $\pi_G : G \rightarrow G^{\text{ab}}$ is a homomorphism; in fact, π_G is *natural in G* in the sense that it defines the components of a natural transformation.

In detail, let $f : G \rightarrow H$ be a group homomorphism. Since a homomorphism into an abelian group kills the commutator subgroup, the kernel of $\pi_H \circ f$ contains $[G, G]$. By a [universal property](#), there is a unique $\phi : G^{\text{ab}} \rightarrow H^{\text{ab}}$ such that $\pi_H \circ f = \phi \circ \pi_G$. We define **the abelianization of a homomorphism** $f^{\text{ab}} \stackrel{\text{def}}{=} \phi$. As suggested by our notation, we are concerned with $(-)^{\text{ab}}$ as a functor. Then, the abelianization

(specifically, the projection) is natural in the sense that $\pi : \text{id}_{\text{groups}} \implies (-)^{\text{ab}}$ is a natural transformation. It is not a natural isomorphism.

Example 0.5 (Groupoids). A group G defines a category (in fact, a [groupoid](#)) BG consisting of one object, whose endomorphisms correspond to group elements. These morphisms behave under composition as G 's elements behave under multiplication.

Now, given a group G , consider what a functor $X : BG \rightarrow D$ does. Because BG has one object, X has a single object in its image; we suppose that object is X . (An abuse of notation.) Each endomorphism f in BG is sent to $f_* : X \rightarrow X$, which as a sequence of functoriality satisfy

- (1) $g_* f_* = (gf)_*$, and
- (2) $(1_G)_* = \text{id}_X$.

In this way, the functor $X : BG \rightarrow D$ defines an **action** of the group G on the object $X \in D$. If $D = \text{Set}$, the object X endowed with its action is a **G -set**. If $D = \text{Vect}_k$, it is called a **G -representation**. If $D = \text{Top}$, it is called a **G -space**.

Back to naturality. As discussed, a functor $X : BG \rightarrow D$ corresponds to an object X equipped with a (left) action by G . Given a second functor $Y : BG \rightarrow D$, what is a natural transformation $\alpha : X \implies Y$? Since the functors X and Y both have image an object, namely the objects X and Y , α has a single component, namely $\alpha : X \rightarrow Y$. This morphism is **G -equivariant**, meaning $g_* \alpha = \alpha g_*$ for each $g \in G$. See the square below.

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ g_* \downarrow & & \downarrow g_* \\ X & \xrightarrow{\alpha} & Y \end{array}$$