Title tbd

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0.1 December 2022

0.1.1 (11/28) Ok, let's give this a try

I want to read *Higher Topos Theory* (HTT). That book is > 700 pages, and fairly dense ones, so it's a bit of a project. On top of that, HTT isn't really a self-contained read (and that's sort of the nature of the subject). So I'll be drawing on lots of additional material. The subject seems well worth learning, even unavoidable at times (e.g., in the areas I am interested in), so this should be a productive little activity.

...

I need to cover some ground before actually opening HTT. First I need to think about *simplicial sets*. Their role in higher category theory is ubiquitous. They give us a combinatorial model for the homotopy theory of spaces, and also a model for ∞-categories. (On the list of things to do is make precise sense of those statements.) Some references are [Rie], [Mat], [Fri08], and `kerodon.net`.

Definition 0.1.1. Denote by $\mathbf{\Delta}$ the *simplex category*, defined to have...

- As objects, the ordered set $[n] := \{0 < 1 < \cdots < n\}$ for each $n \geq 0$; and
- As maps, the weakly order-preserving set maps.

Definition 0.1.2. A *simplicial set* is a contravariant functor $\mathbf{\Delta} \to \mathsf{Set}$. The *category of simplicial sets*, denoted sSet, is the functor category $\mathsf{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set})$.

Notation 0.1.1. Let $X : \mathbf{\Delta}^{op} \to \mathsf{Set}$ be a simplicial set. We may write it $X_\bullet$, and denote by X_n the set $X([n])$. We call the elements of X_n the *n-simplices* of X.

Notation 0.1.2. We may write $\langle f_0, \ldots, f_n \rangle$ to denote the function $[n] \to [m]$ given by $n \mapsto f_n$.

Simplicial sets are not just simplices. They carry additional structure, that arising from morphisms in $\mathbf{\Delta}$. We can give a simple description of $\mathbf{\Delta}$. This in turn gives some intuition for what a simplicial set "is."

Proposition 0.1.3 (The structure of $\mathbf{\Delta}$). *For each $n \geq 0$ and $0 \leq i \leq n$, define the*

$$i\text{-th face map } d^i : [n-1] \to [n] \text{ as } \langle 0, \ldots, \hat{i}, \ldots n \rangle, \text{ and the}$$
$$i\text{-th degeneracy map } s^i : [n+1] \to [n] \text{ as } \langle 0, \ldots, i, i, \ldots, n \rangle.$$

Every morphism in $\mathbf{\Delta}$ may be written as a composition of face and degeneracy maps. (Also, the face/degeneracy maps satisfy various relations, the simplicial identities; in fact $\mathbf{\Delta}$ is the category generated by those maps, subject to these identities.)

Thus a simplicial set $X_\bullet$ can be described as a collection of sets X_n (n-cells) together with face and degeneracy maps which satisfy the "simplicial identities." I should write more about how this notion arises from topology, in particular the singular complex. That in turn would be a good time to relate all this back to topology (nerves, geometric realization, ...) which is important.

0.1.2 (12/1) Why simplicial sets, simplicial complexes

I had stuff written here. But it was incomplete, and the "story" here is an aside I want to write about a bit more carefully at some point. I'm leaving this day blank for the time being.

0.1.3 (12/4) Basic structure in sSet

We need to make some terminology regarding / record examples of simplicial sets.

Definition 0.1.4. The *standard n-simplex* Δ^n is the simplicial set represented by $[n]$, i.e. $\Delta^n := \mathrm{Hom}_{\mathbf{\Delta}}(-, [n])$.

Definition 0.1.5. Let $X_\bullet, Y_\bullet$ be simplicial sets. We say $Y_\bullet$ is a *simplicial subset* of $X_\bullet$ if $Y_n \subseteq X_n$ and $Xf|_{Y_n} = Yf$ for every $n \geq 0$ and simplicial operator f. In other words, the action of operators on Y is the restriction of their action on X. In other words, $Y_\bullet$ is a subfunctor of $X_\bullet$.

Proposition 0.1.6. *Let $X_\bullet$ be a simplicial set. The Yoneda lemma asserts a bijection* $\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, X_\bullet) \cong X_n$. *Under this bijection, each n-cell $a \in X_n$ corresponds to a map $f_a : \Delta^n \to X_\bullet$ satisfying $f_a(\mathrm{id}_{[n]}) = a$.*

Definition 0.1.7. Let $X_\bullet$ be a simplicial set. By the above, we may identify its n-cells with maps $\Delta^n \to X_\bullet$. Call a cell $a \in X_n$ *degenerate* if it factors as $\Delta^n \to \Delta^m \to X_\bullet$ for some $m < n$. (See [Lur22, Tag 0011] for equivalent conditions.)

Proposition 0.1.8. *The standard simplex Δ^n has a unique non-degenerate n-simplex, that arising from $\mathrm{id}_{[n]}$. We may call this the generator of Δ^n.*

Definition 0.1.9 (Boundary of Δ^n)**.** Define a simplicial subset $\partial\Delta^n$, the *boundary of Δ^n*, by

$$(\partial\Delta^n)_k := \{\text{non-surjective maps } [k] \to [n]\} \subseteq \mathrm{Hom}_{\mathbf{\Delta}^{op}}([k],[n]).$$

Proposition 0.1.10. *The boundary of Δ^n is the maximal proper simplicial subset of Δ^n.*

Definition 0.1.11 (Horns in Δ^n)**.** For $0 \le i \le n$, define a simplicial subset Λ^n_i, the *i-th horn in Δ^n*, by

$$(\Lambda^n_i)_k := \{f \in \mathrm{Hom}_{\mathbf{\Delta}^{op}}([k],[n])) : f([k]) \neq [n] \cup \{i\}\}.$$

A horn Λ^n_i is called *outer* if $i \neq 0, n$ and *inner* otherwise.

Any simplicial operator $f : [m] \to [n]$ factors through its image, i.e. we can uniquely write $f = f^{inj} f^{surj}$, a surjection followed by an injection. Furthermore, this is unique. We get the following.

Proposition 0.1.12. *Let $\sigma : \Delta^n \to X_\bullet$ be an n-cell of $X_\bullet$. Then σ factors uniquely as*

$$\Delta^n \xrightarrow{\alpha} \Delta^m \xrightarrow{\tau} X_\bullet,$$

Where α represents a surjection $[n] \to [m]$ and τ is not degenerate. Call m the dimension of the cell σ. (My notation, maybe poor, not that important.)

So, degenerate n-simplices are just non-degenerate simplices in a lower dimension (their "dimension"), trying to bite off more than they can chew.

Definition 0.1.13 (Skeleta)**.** Let $X_\bullet$ be a simplicial set. For $k \ge -1$, define a simplicial subset $\mathrm{sk}_k(X_\bullet)$, the *k-skeleton* of $X_\bullet$, by

$$(\mathrm{sk}_k(X_\bullet))_n := \{\, n\text{-simplices of } X_\bullet \text{ with dimension at most } k\,\}.$$

Remark 0.1.14. The face maps $d^i : [n-1] \to [n]$ induce maps $d^i : \Delta^{n-1} \to \Delta^n$ via post-composition. Now, consider an n-cell $a \in X_n$ and its representation $a : \Delta^n \to X_\bullet$. We have that $d_i(a) \in X_{n-1}$ is represented by ad^i.

0.1.4 (12/6) Colimits in/functors out of sSet

Today I want to understand part of Akhil's notes, about functors out of sSet. This is closely related to understanding colimits in sSet, by general theory for presheaf categories. So we also want to understand colimits in sSet. (And we should want to understand these regardless.) Let's go over this.

Here's a standard structure result for presheaf categories.

Proposition 0.1.15. *If a category C is small, then every presheaf on C is canonically the colimit of representable presheaves. In particular, every simplicial set is canonically the colimit of standard simplices.*

Proof. This is written out in Akhil's notes. I'll give the idea. Also see [Lur22, Remark 00X5].
Consider a presheaf $F : \mathsf{C}^{op} \to \mathsf{Set}$. We associate to F the category D_F with

- Objects: morphisms from represented presheaves to F, i.e. arrows $[-, X] \to F$; and
- Morphisms: morphisms between represented presheaves such that the obvious triangle commutes.

There is a functor $\phi_F : \mathsf{D}_F \to \mathsf{PShv}(\mathsf{C})$ which sends objects $[-, X] \to F$ to $[-, X]$. By construction, for each object $c \in \mathsf{D}_F$, there is a morphism $\phi_F(c) \to F$, and the diagram described by ϕ_F together with these morphisms commutes. We therefore have a distinguished morphism

$$\lim_{\mathsf{D}_F} \phi_F \to F.$$

This map turns out to be an isomorphism. □

Hereafter, denote by $\overline{\mathsf{C}}$ the category of presheaves on C.

Suppose D is cocomplete. We want to understand functors $\overline{F} : \overline{\mathsf{C}} \to \mathsf{D}$. The previous proposition says that objects in $\overline{\mathsf{C}}$ are colimits of representables. So, if $\overline{F}$ preserves colimits, then $\overline{F}$ is determined by $\overline{F}|_{\mathsf{C}}$, i.e. what it does to C (embedded via Yoneda). We've described an injection of sets

$$\mathrm{Fun}'(\overline{\mathsf{C}}, \mathsf{D}) \hookrightarrow \mathrm{Fun}(\mathsf{C}, \mathsf{D}). \tag{16}$$

Here, Fun′ denotes the set of colimit-preserving functors.

Conversely, suppose given a functor $F : \mathsf{C} \to \mathsf{D}$. Does it extend along the Yoneda embedding to a functor $\overline{F} : \overline{\mathsf{F}} \to \mathsf{D}$? We can do something here, let me write it out:

(1) As above, for each presheaf $G : \mathsf{C}^{op} \to \mathsf{Set}$, consider it as a colimit of $\phi_G : \mathsf{D}_G \to \mathsf{C}$. (We can do this because it lands in represented functors.)

(2) This is 'functorial' in the following sense: a morphism $G \to H$ induces a functor $\mathsf{D}_G \to \mathsf{D}_H$ such that the obvious triangle commutes.

(3) Define a functor $\overline{F} : \overline{\mathsf{C}} \to \mathsf{D}$ by
$$\overline{F}(G) := \operatorname*{colim}_{\mathsf{D}_G} F \circ \phi_G.$$
This is a functor because of (2).

This functor $\overline{F}$ really extends F, i.e. the obvious diagram commutes. For suppose $G = [-, c]$; then D_G has a final object $[-, c] \to [-, c]$, therefore $\overline{F}(G) = \mathrm{colim}_{\mathsf{D}_G} F \circ \phi_G = F(G)$.

Proposition 0.1.17. *Suppose given a functor $F : \mathsf{C} \to \mathsf{D}$ to a cocomplete category. Then the associated functor $\overline{F} : \overline{\mathsf{C}} \to \mathsf{D}$ constructed above preserves all colimits. In fact, $\overline{F}$ is a left adjoint. The right adjoint to $\overline{F}$ is the functor defined by*
$$\mathsf{D} \ni d \mapsto (c \mapsto \mathrm{Hom}_{\mathsf{D}}(Fc, d)) \in \overline{\mathsf{C}}.$$

Proposition 0.1.18. *Suppose given a functor $F : \mathsf{C} \to \mathsf{D}$ to a cocomplete category. Then the mapping $F \mapsto \overline{F}$ describes a bijection of sets*
$$\mathrm{Fun}(\mathsf{C}, \mathsf{D}) \xrightarrow{\sim} \mathrm{LeftAdjoints}(\overline{\mathsf{C}}, \mathsf{D}).$$

The proofs are short and formal.

Corollary 0.1.19. *If a functor $\overline{F} : \overline{\mathsf{C}}^{op} \to \mathsf{Set}$ takes colimits to limits, then it is representable.*

Proof. Suppose as given $\overline{F}$. By the above, it is left adjoint to some $\overline{G} : \mathsf{Set}^{op} \to \mathsf{C}$. Define $f := \overline{G}(\{pt\})$, the image of the terminal object in Set^{op}. I claim that f represents F. (Insert short, formal proof; it's in Akhil's notes.) □

0.1.5 (12/23) The singular complex and geometric realization

Finals are over and I've had some time to wind down at home. Last time I worked through part of Akhil's notes about functors out of sSet. (Emily Riehl talks about something similar in her notes, but I have not gotten through those, so let me say nothing about that right now.)

Next I want to relate Top, Cat, and sSet. This is the backdrop for the idea that higher categories "bridge" topology/homotopy theory and ordinary categories. Today I'll go over the relation of sSet to Top, by which I mean the adjunction

$$\text{the geometric realization functor} \dashv \text{the total singular complex functor.}$$

There are a few ways to introduce this adjunction. Lurie, Charles, and Akhil each do it differently. As a matter of taste, I prefer Akhil's approach. (Possibly related: [Lur22, Tag 002D].) Lurie's approach has some important ideas behind it too, I think, but that is overruled because I am feeling sleepy today.

Definition 0.1.20. Define a functor $|-| : \mathbf{\Delta} \to \mathsf{Top}$ as follows.

- Each object $[n]$ is sent to the *topological n-simplex* $\Delta^n_{top} \subseteq \mathbb{R}^{n+1}$, defined as those $(t_0, \dots, t_n)$ satisfying $t_i \geq 0$ and $\sum t_i = 1$ and given the subspace topology.
- Each morphism $f : [m] \to [n]$ is sent to the map

$$(t_0, \dots, t_n) \mapsto (u_j), \quad u_j = \sum_{i : f(i) = j} t_i.$$

Definition 0.1.21 (Geometric realization)**.** Since Top is cocomplete, according to Proposition 0.1.17 and 0.1.18 the functor of Definition 0.1.20 extends uniquely to a left adjoint $|-| : \mathsf{sSet} \to \mathsf{Top}$. We call this *geometric realization*.

In fact, as in Proposition 0.1.18, we know the right adjoint to geometric realization. It sends a space X to the simplicial set $[n] \mapsto \mathrm{Hom}_{\mathsf{Top}}(|[n]| = \Delta^n_{top}, X)$. This is an important construction I maybe should have defined earlier.

Definition 0.1.22 (Singular complex)**.** Let X be a space. Denote by $\mathrm{Sing}(X)_\bullet$ the simplicial set given as follows.

- The n-cells are the continuous maps $\Delta^n_{top} \to X$, and
- Each simplicial operator $f : [m] \to [n]$ acts by precomposing with the continuous map

$$\Delta^m_{top} \to \Delta^n_{top}, \quad (t_j) \mapsto (u_j = \sum_{f(i)=j} t_i).$$

We call $\mathrm{Sing}(X)_\bullet$ the *singular complex* of X. We define a functor $\mathrm{Sing}(-)_\bullet : \mathsf{Top} \to \mathsf{sSet}$ in the obvious way.

Proposition 0.1.23. *Prior discussion tells us that geometric realization $|-| : \mathsf{sSet} \to \mathsf{Top}$ is left adjoint to the singular complex functor.*

Proposition 0.1.24. *Since geometric realization is a left adjoint, it commutes with colimits. Furthermore, geometric realization commutes with finite limits of compactly generated spaces.*

We will see later that this adjunction is homtopically well-behaved.

I next want to describe geometric realization. We already have the general construction laid out for us by Proposition 0.1.17 and the preceding discussion. Given $X_\bullet$, we will form the *category of simplices*, also called its *category of elements*, whose elements are the morphisms $\Delta^n \to X_\bullet$ (i.e., the cells of $X_\bullet$), and we take the colimit of $|-|$ restricted to this subcategory. (This is not circular since we are really applying the "baby" geometric realization to the simplex category, Yoneda embedded.)

Definition 0.1.25. Given $X_\bullet \in \mathsf{sSet}$, its *category of simplices* or *category of elements* has as objects all morphisms $\Delta^n \to X_\bullet$ for every n, and morphisms the maps $\Delta^m \to \Delta^n$ making the obvious diagram commute. We write this category $\mathsf{el}(X)$.

This category of elements/simplices $\mathsf{el}(X)$ is precisely the category D_X described on (12/6) with $\mathsf{C} = \mathbf{\Delta}$. (Lurie writes this $\mathbf{\Delta}_X$.) Also as noted there, there is a natural functor $\phi_X : \mathsf{el}(X) \to \mathsf{sSet}$. Geometric realization is by definition the colimit

$$|X| \cong \operatorname*{colim}_{\mathsf{el}(X)} |-| \circ \phi_X.$$

Here, we are thinking of the "baby" geometric realization defined only on $\mathbf{\Delta}$.

Remark 0.1.26. General machinery gave us geometric realization. I think there are a few things worth saying about this, but I don't totally know what. I'll leave this remark here as a "to-do." Some possibly related keywords and references: "Grothendieck construction," "Kan extension," nLab, [Rie, §4], and Subsection 01Q7.

0.2 January 2023

0.2.1 (1/23) Plans have changed, nerves of categories

I've gone radio silent for a month. One big reason why is that I am busy this semester. Another is that some mutuals want to organize a reading group/seminar similar, but not identical to, what I've been trying to do here, and I may join them. Maybe the biggest difference is that they want to focus on Charles' quasicategory notes (under Charles' supervision).

In any case, I want to talk about the nerve of a category. This is part of the basic "Spaces, categories, and simplicial sets" picture. In particular, the nerve of a category is a simplicial set encoding that category.

Definition 0.2.1. Let C be a category. Define a simplicial set $N\mathsf{C}$, the *nerve* of C to have as cells $(N\mathsf{C})_n := \mathrm{Hom}_{\mathsf{Cat}}([n], C)$ and so that operators $f : [m] \to [n]$ act by precomposition. This defines the *nerve functor* $N : \mathsf{Cat} \to \mathsf{sSet}$.

Here's a feel for the structure of a nerve. The n-cells of $N\mathsf{C}$ may be canonically identified with the set of length n tuples of composable arrows in C. The 0-cells in particular may be identified with objects of C. An operator $f : [m] \to [n]$, or in Charles' notation $\langle f_1, \ldots, f_m \rangle$, acts by taking n-strings of composable arrows and "collapsing edges" by composing the arrows, those collapsed edges being determined by the f_i. (At least that's how I try to think about it. I think that's correct. UPDATE: Yes this is correct, see Charles' notes, Proposition 4.4.) For instance, $\langle 0, 2 \rangle^*$ takes a pair of composable arrows $(f, g) \in (N\mathsf{C})_2$ and sends them to their composite $gf \in (N\mathsf{C})_1$. See also Charles notes, p. 13.

Now we ask a nascent question: can we characterize nerves of categories?

Proposition 0.2.2. *Let X be a simplicial set. For $n \geq 2$, consider the function*

$$\phi_n : X_n \to \{(g_i) \in (X_1)^n : g_j\langle 1 \rangle = g_{j+1}\langle 0 \rangle \text{ for all } i\}$$

(The latter set being the collection of "n-paths" of 1-cells in X) Which acts by $a \mapsto (a\langle 0, 1\rangle, \ldots, a\langle n-1, n\rangle)$. These ϕ_n are bijections for all $n \geq 2$ if and only if X is the nerve of a category.

Maybe a lazy way to digest this is that "a simplicial set is the nerve of a category iff, thinking of n-cells as length n strings of arrows, their 1*-dimensional structure* exactly reflects the structure that should arise from the existence and uniqueness of composites."

Proposition 0.2.3. *The nerve functor $N : \mathsf{Cat} \to \mathsf{sSet}$ is fully-faithful. That is, morphisms of nerves $NC \to ND$ correspond exactly to functors $C \to D$.*

0.2.2 (1/26) Spines

We characterized nerves as those simplicial sets whose n-cells were exactly determined by the collection of length n strings of "composable" 1-cells, in the obvious way. This captures the existence and uniqueness of composites for morphisms in a category. We can go about this characterization a bit more systematically.

Definition 0.2.4. Let $n \geq 0$. The *spine* of the standard n-simplex Δ^n is the simplicial subset defined by

$$(\mathrm{Spine}^n)_k := \{\langle f_0, \ldots, f_k \rangle : [k] \to [n] : f_k \leq f_0 + 1\} \subseteq \Delta^n_k.$$

Informally, the spine is the set of vertices of Δ^n together with the arrows between adjacent vertices (considered with their total ordering).

Proposition 0.2.5. *Let X be a simplicial set. For every $n \geq 0$, the map*

$$\mathrm{Hom}(\mathrm{Spine}^n, X) \to \{(a_i) \in (X_1)^n : a_i\langle 1 \rangle = a_{i+1}\langle 0 \rangle\} \tag{6}$$

Given by $f \mapsto (f\langle 0, 1\rangle, \ldots, f\langle n-1, n\rangle)$ is a bijection.

Pictorially, I think this is obvious. Here's a clean proof.

Proof. One point we need: we previously talked about colimits in sSet. Or at least I intended to. Here's the main fact: *a colimit of simplicial sets X_α exists and has as its n-cells the colimit of the n-cells of the X_α.* This is true for presheaves in general; we say their (co)limits are "computed objectwise."

Another point we need: here's a definition. Suppose S is a totally ordered set. We denote by Δ^S the simplicial set having $(\Delta^S)_n :=$ {order-preserving maps $[n] \to S$}. If S is finite and nonempty, there is a unique isomorphism $\Delta^{|S|-1} \cong \Delta^S$. In the case that $S \subseteq [n]$, this is a good way to notate subcomplexes of Δ^n.

Here's a fact I won't prove: *given a subcomplex $K \subseteq \Delta^n$, writing A for the poset of $S \subseteq [n]$ such that $\Delta^S \subseteq K$, the canonical map* $\operatorname{colim}_{S\in A} \Delta^S \to K$ *is an isomorphism.*

Finally, our proposition: in the case that $K = \mathrm{Spine}^n$, the poset A consists of sets of the form $\{j\}$ and $\{j+1\}$, and we have that $\operatorname{colim}_{S\in A} \Delta^S \cong \mathrm{Spine}^n$. Now:

$$\mathrm{Hom}(\mathrm{Spine}^n, X) \cong \mathrm{Hom}(\operatorname*{colim}_{S\in A} \Delta^S, X) \cong \lim_A \mathrm{Hom}(\Delta^S, X).$$

I don't think it's hard to see that the latter set is precisely the RHS of (6). □

Maybe the key observation is that Δ^n is "generated" precisely by the arrows of Spine^n. (Make this formal? Say this better? Well, this is how I think about it.)

Proposition 0.2.7. *A simplicial set X is the nerve of some category if and only if for each $n \geq 2$, every morphism $f : \mathrm{Spine}^n \to X$ uniquely extends along the inclusion $\mathrm{Spine}^n \hookrightarrow \Delta^n$.*

Bibliography

[Fri08] Greg Friedman. An elementary illustrated introduction to simplicial sets. 2008.

[Lur22] Jacob Lurie. Kerodon. https://kerodon.net, 2022.

[Mat] Akhil Mathew. The dold-kan correspondence. https://math.uchicago.edu/~amathew/doldkan.pdf.

[Rie] Emily Riehl. A lesuirely introduction to simplicial sets. https://math.jhu.edu/~eriehl/ssets.pdf.