LIMITS AND COLIMITS

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Abstract. We develop limits and colimits.

(Co)limits can be developed categorically or algebraically. Category theory is all the rage, so we take the categorical route. In any case, the concept of (co)limits is fairly abstract.

With (co)limits we can describe very generally how to construct new objects from old, in such a way that similarities between important constructions in distinct areas of study appear. One marvelous advantage to understanding (co)limits is that we can take constructions we know in familiar settings and perform analogous constructions in unfamiliar settings. Even better, by virtue of category theory these constructions mostly free us from setting-specific details, which otherwise might slow us down. In other words, we are capturing only what is 'essential.'

1 LIMITS

It will help to have a precise notion of a diagram. Let J be any category; then a **diagram of shape** J in a category C is simply a functor $F : J \to C$. We think of J as *indexing* the diagram. The notion of a *limit* is developed relative to a diagram, i.e., a diagram is all the data that specify a limit.

Next, recall from an earlier note what a cone is.

> **Definition.** The **constant functor** $c : J \to C$ is the functor that sends every object in J to the object $c \in C$, and every morphism to id_c. A **cone over a diagram** $F : J \to C$ **with summit or apex** $c \in C$ is a natural transformation $\alpha : c \implies F$ whose domain is the constant functor c. The components of α are called its **legs**.

(Why call it a cone? See the linked note.)

We are ready to define limits.

> **Definition.** The **limit** of a functor $F : J \to C$ is a cone $\alpha : c \implies F$ over F with the property that if $\beta : c' \implies F$ is any other cone over F, then there exists a unique morphism $\pi : c' \to c$ through which each leg of β factors, i.e. $\beta = \alpha \circ \pi$. We denote by $\lim F$ the object c.

In other words, given a diagram F in C, its limit consists of the following data:

- An object $\lim F \in C$, and
- A natural transformation $\alpha : \lim F \implies F$,

Subject to the following condition:

- If $\beta : c' \implies F$ is a natural transformation between any other object $c' \in C$ and F, then there is a unique morphism from c' to $\lim F$ through which the components of β factor.

So, $\lim F$ is a cone over F, and it is 'universal' in the sense that every other cone over F has $\lim F$ in its metadata, just as every even integer has a factor of 2. This actually inspires a common definition of the limit:

Definition. The **limit** of $F : J \to C$ is the universal cone over F.

This can all be understood diagrammatically as follows. Let:

- $F : J \to C$ be our diagram of shape J;
- $d : X \to Y$ be any morphism in J;
- $(c, \alpha), (c', \beta)$ be two cones over F, so c, c' are objects and α, β natural transformations from c, c' to F, respectively.

Then to assert (c, α) is the limit of F means to say that there is a unique[1] morphism $\pi : c' \to c$ so that the diagram below commutes, where we denote c by $\lim F$.

$$\begin{tikzcd}
& c' \arrow[ddl, bend right, "\beta_X"'] \arrow[d, dotted, "\pi"] \arrow[ddr, bend left, "\beta_Y"] & \\
& \lim F \arrow[dl, "\alpha_X"] \arrow[dr, "\alpha_Y"'] & \\
F(X) \arrow[rr, "F(d)"'] & & F(Y)
\end{tikzcd}$$

Hopefully, this helps explain what 'limiting' process the word 'limit' is meant to describe: if one looks at all the cones over a diagram F in a category, its limit is the cone whose summit has no morphism from it to another cone over F.[2]

2 COLIMITS

Definition. The **colimit** of $F : J \to C$ is the universal cocone under F.

Like before, we refer to a colimit as the object (its summit) $\mathrm{colim} F \in C$ together with the natural transformation $\alpha : F \to \mathrm{colim} F$. It is essentially a limit, except we have a slightly different necessarily-commutative diagram:

$$\begin{tikzcd}
F(X) \arrow[rr, "F(d)"] \arrow[dr, "\alpha_X"] \arrow[ddr, bend right, "\beta_X"'] & & F(Y) \arrow[dl, "\alpha_Y"'] \arrow[ddl, bend left, "\beta_Y"] \\
& \lim F \arrow[d, dotted, "\pi"] & \\
& c' &
\end{tikzcd}$$

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[1] And not dependent on our choice of d, X, Y, c', or β.

[2] There are a few devils in the details. This construction is essentially well-defined, but so far, the why/how I have provided is patchy.

3 SPECIAL CASES

Limits and colimits of some diagrams are given special names.

Example 3.1 (Terminal object)**.** A cone over an empty diagram is simply an object (the summit), and morphisms between cones are just morphisms between objects. In this case, the limit (if it exists) is the object with a unique arrow *to* it from every other object. This defines the **terminal object** of a category.

Said differently, a terminal object c of C is an object such that for all $x \in C$, there is a unique morphism $x \to c$. It is easily shown to be unique.

Example 3.2 (Products)**.** A discrete category has only identity morphisms as morphisms. A **discrete diagram** is a diagram $F : J \to C$ such that J is discrete. (Such diagrams are usually not named.) The discrete diagram F is simply a collection of objects $F_j \in C$, indexed by $j \in J$.

We call the limit of a discrete diagram a **product**. A cone over a discrete diagram is a collection of morphisms $(\alpha_j : c \to F_j)_{j \in J}$. The limit is usually denoted $\Pi_{j \in J} F_j$, and its legs are maps

$$\pi_k : \prod_{j \in J} F_j \to F_k$$

where k is an object in J. The π_k are called **(product) projections**.

Example 3.3 (Pullback)**.** Let P denote the poset category with three objects and two non-identity morphisms shaped like so:

$$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

Denote by f, g the morphisms in the image of a P-shaped diagram $F : P \to C$. A cone over F with summit D consists of three maps a, b, c, one for each object of P, such that both triangles in the following diagram commute.

$$\begin{tikzcd} D \arrow[r, "c"] \arrow[d, "b"'] \arrow[dr, "a"] & C \arrow[d, "g"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

Commutativity of the triangles implies $f \circ b = g \circ c$. Hence, a cone over a poset-shaped diagram is completely described as 'a pair of morphisms $B \xleftarrow{b} D \xrightarrow{c} C$ that defines a commutative square.' This is more succinct than our previous description (as three morphisms a, b, c.)

So, we have slightly simplified the definition of a cone in the specific case that it is over $F : P \to C$; similarly, we can simplify what the universal cone (limit) is. It is the commutative square $fh = gk$ such that given any commutative square as above, there is a unique factorization of its legs through the summit of the limit cone, i.e., the diagram

$$\begin{tikzcd}
D \arrow[drr, bend left, "c"] \arrow[dr, dashed, "\exists!"] \arrow[ddr, bend right, "b"'] & & \\
& \lim F \arrow[r, "k"] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
& B \arrow[r, "f"'] & A
\end{tikzcd}$$

commutes. The limit of a P-shaped diagram (or, one might say the limit over f and g) is called a **pullback**. (The right angle symbol indicates that the square is a limit diagram.) With the notation we've been using, pullback is often denoted $B \times_A C$. It also called the **fiber product**.

Example 3.4. Dual to pullback, a **pushout** is the colimit of a diagram that is shaped like the below poset category.

$$\bullet \longleftarrow \bullet \longrightarrow \bullet$$

In this case, to indicate a commutative square is pushout, we again use the right angle symbol like so:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\ulcorner", very near end] & B \arrow[d, "h"] \\
C \arrow[r, "k"'] & \lim F
\end{tikzcd}$$

4 EXAMPLES