Sheaves and sheaf cohomology

Reading group notes

June 26, 2022

0.1 (Week 2) June 22 - Of Presheaves and Sheaves

First meeting. Opened with some history. (History motivates; see paper.)

Q: What is a sheaf? Furthermore, what (and I cannot stress this enough) *is* a sheaf?

Let X be a topological space. Define $\mathrm{Op}(X) :=$ category of its open sets, morphisms given by inclusions.

WANT: A systematic way to 'track' the 'constructions' possible on open sets. E.g., given an open set U, what is the set of cts functions $U \to \mathbb{R}$? How do these sets vary as U varies? How are the resulting sets related, w.r.t. the relation of their underlying open sets? (Try replacing 'cts function' with: k-forms, exact k-forms, solutions to a the PDE arising from a fixed vector field, ...)

Cts functions are the prototypical notion of a 'construction.' Among other things, they have a nice property: if $U \hookrightarrow V$ then a cts $f : V \to \mathbb{R}$ restricts to a cts $f|_U : U \to \mathbb{R}$. (This answers 'how are the resulting sets related?'–they are related by a *restriction map*.)

Any notion of 'construction' should have this restriction property. And this restriction property was just (contravariant) functoriality w.r.t. inclusions.

Definition. A *presheaf* is a contravariant functor $\mathcal{F} : \mathrm{Op}(X) \to \mathrm{D}$, where D is any (concrete) category. Usually it will be sets, rings, abelian groups, ... We call the image of an inclusion under $\mathcal{F}$ a *restriction*.

So a presheaf describes 'constructions' that come with restrictions. Continuous functions have more than just restrictions, though. Two properties in particular stand out (or rather, are so obvious that the need to identify them stands out):

1. if $f_1 : U_1 \to R$ and $f_2 : U_2 \to R$ agree on intersection, then there exists a global $f : U_1 \cup U_2 \to R$ that restricts to f_1 or f_2.
2. If $f, g : U_1 \cup U_2 \to R$ agree when restricted to U_1 or U_2, then $f = g$.

Generalizing (1) and (2) to arbitrary covers leads us to define a sheaf.

Definition. A *sheaf* is a presheaf satisfying two axioms.

1. (Identity) Let $\{U_i\}$ be any open covering of any open set U. If $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ for each U_i, then $s = t$.

2. (Gluing) Let $\{U_i\}$ be any open covering of any open set U. If one can choose a section s_i from each U_i such that s_i, s_j agree when restricted to $U_i \cap U_j$ for each i, j, then there is a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for each i.

EXAMPLES–important, see paper, end of §1.

Remark. (Pre)sheaves form categories $\mathsf{PSh}(X)$, $\mathsf{Sh}(X)$. Objects are (pre)sheaves, morphisms are natural transformations. The inclusion functor $\mathsf{Sh}(X) \hookrightarrow \mathsf{PSh}(X)$ has a left adjoint called *sheafification.* Thus, sheafifying a presheaf gives the 'best' sheaf approximate. The exact construction of sheafification is not widely useful; people (algebraic geometers) usually just need to know the universal property relating a presheaf to its sheafification. (The one described by adjunctions.)

0.2 (Week 2) June 26 - Abelian Categories I

Opened with historical refresher: homological algebra had just systematized our understanding of modules, at least in the context of their use in algebraic topology. This clearly worked for objects besides modules, but a rigorous abstraction was not made. In particular, *sheaves* had 'homological algebra-like' properties, e.g. kernels, quotients. And sheaves had JUST effected advances in complex/algebraic geometry (Serre, Cartan, ...) The big question: can we figure out how to do homological algebra generally, fully fit sheaves into their framework, then recruit the techniques of algebraic topology (cohomology!) to study sheaves? Short answer: yes.

Today, we want to identify what makes homological algebra work. Or rather, where it works.

The base case is Mod_R, the category of R-modules–this is where we first understood homological algebra. In fact, we can specialize to Ab, the category of abelian groups. What are its properties?

1. Given two groups G, H, the set $\mathrm{Hom}(G, H)$ is an abelian group.

2. With respect to the group structure above, morphisms compose bilinearly; $h \circ (f + g) = h \circ f + h \circ g$, similarly with $(f + g) \circ h$.

3. It turns out that (1) and (2) imply that the (finite) product and coproduct of two groups are isomorphic, *if either exists.* Assuming they exist, we have a natural categorical description of the addition law on hom-sets (that's good!), see the reading p. 5-6, starting at "Getting categorical, ..." And Ab has *all* finite products and coproducts. (That's the property.)

4. It has a *zero object*, to mean an object with a unique morphism to AND from all other objects. (It's the trivial group.) This gives rise to e.g. *zero morphisms.*

Then the Reg closed and some of us got dinner. We'll finish listing properties next time.