NATURAL TRANSFORMATIONS

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Abstract. We develop the categorical notion of a natural transformation.

Categories and functors were initially developed as part of an auxiliary framework in which the prevalent but then-informal notion of naturality could be precisely defined once and for all. Here is the definition.

Definition. Let $F, G : C \to D$ be functors. A **natural transformation** α from F to G, denoted $\alpha : F \implies G$, consists of:

- For each object $c \in C$, an arrow $\alpha_c : Fc \to Gc$ in D. These define the **components** of the natural transformation.

Such that for each morphism $f : x \to y$ in C,

$$\alpha_x \circ F(f) = G(f) \circ \alpha_y.$$

In other words, the following square is always commutative.

$$\begin{tikzcd}
Fx \arrow[r, "\alpha_x"] \arrow[d, "Ff"'] & Gx \arrow[d, "Gf"] \\
Fy \arrow[r, "\alpha_y"'] & Gy
\end{tikzcd}$$

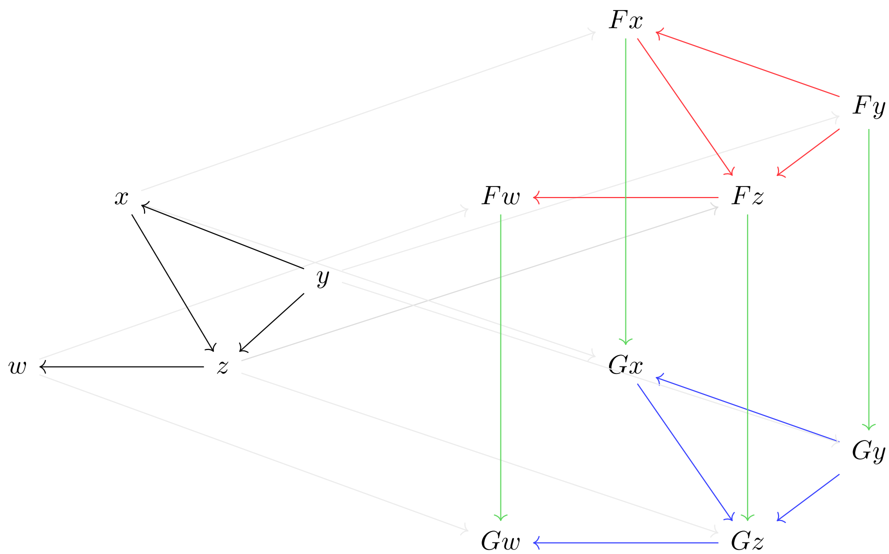
We could have equally required that the following diagram be commutative.

$$\begin{tikzcd}
x \arrow[dr] \arrow[drr] \arrow[dd, "f"'] & & \\
& Fx \arrow[r, "\alpha_x"] \arrow[dd, "Ff"'] & Gx \arrow[dd, "Gf"] \\
y \arrow[dr] \arrow[drr] & & \\
& Fy \arrow[r, "\alpha_y"'] & Gy
\end{tikzcd}$$

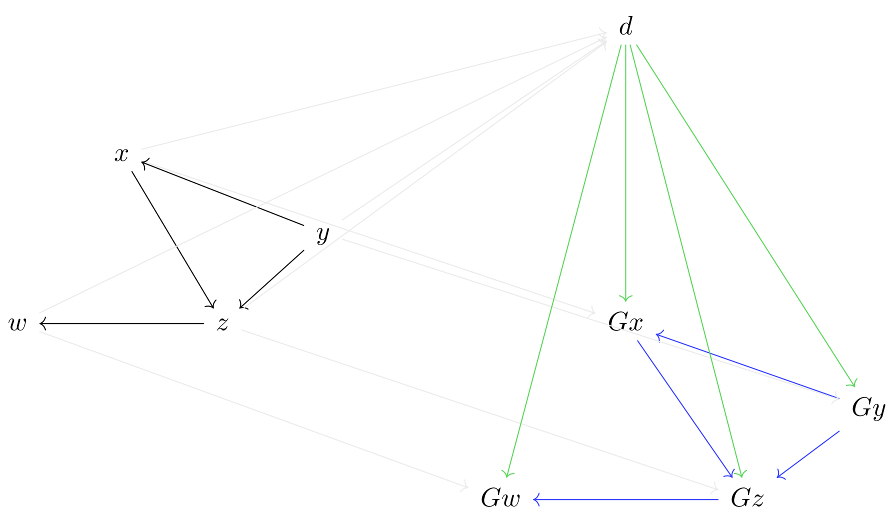
To me, this depiction is more illuminating. It gets at how I think of natural transformations, which is along these lines: *starting from a diagram of objects and morphisms between them in C, the functors F and G map the diagram to two diagrams in D; a natural transformation consists of maps between the two image diagrams such the total diagram is commutative.*

To pound in this intuition, we will see similar diagrams in the following examples.

Example 0.1 (General example)**.** Arbitrarily, let x, y, z, w be objects in a category C such that there are morphisms $y \to x$, $x \to z$, $y \to z$, and $z \to w$. Left below, the diagram in C consisting of these objects and maps is shown. Let F, G be two functors $C \to D$. Then there are two diagrams in D; one consists of the image of the previous diagram under F, and likewise for G. These are the top and bottom diagrams below at right. A natural transformation from F to G is the collection of green arrows making total diagram is commutative.



Example 0.2 (F is constant)**.** In the last example, we implicitly supposed $F, G : C \to D$ were injective (at least at x, y, z and w). For each $d \in D$, we have the **constant functor** denoted d which takes every object to d and every morphism to id_d. Suppose F is the functor d.



For obvious reasons, $\alpha : d \implies G$ is called the **cone over** G **with summit or apex** d. If G were constant instead of F, we would call α the cone *under* F. In either case, we call the components of α its **legs**.

Definition. A **natural isomorphism** is a natural transformation $\alpha : F \implies G$ whose components are all isomorphisms. In this case, we say F and G are **naturally isomorphic**, notated $F \cong G$.

On the other hand, if every component of α is an isomorphism, then it is natural if and only if $F(f) = \alpha_y^{-1} \circ G(f) \circ \alpha_x$. (Look familiar?)

Let's see some concrete examples.

Example 0.3 (Double-dual)**.** Recall that the dual V^* of a finite-dimensional k-vector space is the space of linear maps $V \to k$, and the double-dual V^{**} is the space of linear maps $V^* \to k$. Both V^* and V^{**} are vector spaces over k.

The space V is isomorphic to V^*. However, the two are not naturally isomorphic, in the sense that there is no natural isomorphism from the identity functor to the dual functor $(-)^*$. A technical reason for this is because $\mathrm{id}_{\mathrm{Vect}_k}$ is covariant and $(-)^*$ is not. But even ignoring that, there are issues. To specify an isomorphism requires choosing a basis; if α were a natural transformation defined after such a choice and $T : V \to W$ is an arbitrary linear map, we must have $\alpha_V = T^* \circ \alpha_W^{-1} \circ T$. But this almost never holds, in fact there need not even exist inverses; if T does not fix the basis (in other words, is not the identity map), α will fail to be a natural isomorphism. And there are certainly such T.

On the other hand, $V \cong V^{**}$ in a natural way. Define the **evaluation map** $\mathrm{ev} : V \to V^{**}$ as the function which sends v to $\mathrm{ev}_v : V^* \to k$, which itself evaluates a map at v. Symbolically,

$$\mathrm{ev}(v) = (v^* \mapsto v^*(v)).$$

The evaluation map is an isomorphism from a vector space to its double-dual. To see that it is a natural transformation, we must check that the square

$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}"] \arrow[d, "T"'] & V^{**} \arrow[d, "T^{**}"] \\ W \arrow[r, "\mathrm{ev}"'] & W^{**} \end{tikzcd}$$

commutes. Denoting an arbitrary vector of V by v, we have $\mathrm{ev}_{Tv} = (Tv)^* : W^* \to k$, which maps the functional $w^* \in W^*$ to $w^*(Tv)$. Down the other path, T^{**} takes $\mathrm{ev}_v : V^* \to k$ to the functional $(Tv)^* \mapsto (Tv)^* Tv$. These are the same maps.

Because the evaluation map is an isomorphism $V \cong V^{**}$ and (as we checked) the above square commutes for all T, we have shown there exists a natural isomorphism $\mathrm{id}_{\mathrm{Vect}_k} \implies (-)^{**}$.

Example 0.4 (Abelianization)**.** Given a group G, we define its **abelianization** $G^{\mathrm{ab}} \overset{\mathrm{def}}{=} G/[G,G]$, where $[G,G]$ is its commutator subgroup. The (canonical) projection map $\pi_G : G \to G^{\mathrm{ab}}$ is a homomorphism; in fact, π_G is *natural in G* in the sense that it defines the components of a natural transformation.

In detail, let $f : G \to H$ be a group homomorphism. Since a homomorphism into an abelian group kills the commutator subgroup, the kernel of $\pi_H \circ f$ contains $[G,G]$. By a universal property, there is a unique $\phi : G^{\mathrm{ab}} \to H^{\mathrm{ab}}$ such that $\pi_H \circ f = \phi \circ \pi_G$. We define **the abelianization of a homomorphism** $f^{\mathrm{ab}} \overset{\mathrm{def}}{=} \phi$. As suggested by our notation, we are concerned with $(-)^{\mathrm{ab}}$ as a functor. Then, the abelianization (specifically, the projection) is natural in the sense that $\pi : \mathrm{id}_{\mathrm{groups}} \implies (-)^{\mathrm{ab}}$ is a natural transformation. It is not a natural isomorphism.

Example 0.5 (Groupoids)**.** A group G defines a category (in fact, a groupoid) BG consisting of one object, whose endomorphisms correspond to group elements. These morphisms behave under composition as G's elements behave under multiplication.

Now, given a group G, consider what a functor $X : BG \to D$ does. Because BG has one object, X has a single object in its image; we suppose that object is X. (An abuse of notation.) Each endomorphism f in BG is sent to $f_* : X \to X$, which as a sequence of functoriality satisfy

(1) $g_* f_* = (gf)_*$, and
(2) $(1_G)_* = \mathrm{id}_X$.

In this way, the functor $X : BG \to D$ defines an **action** of the group G on the object $X \in D$. If $D = \mathrm{Set}$, the object X endowed with its action is a G**-set**. If $D = \mathrm{Vect}_{\mathrm{k}}$, it is called a G**-representation**. If $D = \mathrm{Top}$, it is called a G**-space**.

Back to naturality. As discussed, a functor $X : BG \to D$ corresponds to an object X equipped with a (left) action by G. Given a second functor $Y : BG \to D$, what is a natural transformation $\alpha : X \implies Y$? Since the functors X and Y both have image an object, namely the objects X and Y, α has a single component, namely $\alpha : X \to Y$. This morphism is G**-equivariant**, meaning $g_*\alpha = \alpha g_*$ for each $g \in G$. See the square below.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "g_*"'] & Y \arrow[d, "g_*"] \\
X \arrow[r, "\alpha"'] & Y
\end{tikzcd}$$