Title tbd

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0.1 December 2022

0.1.1 (11/28) Ok, let's give this a try

I want to read *Higher Topos Theory* (HTT). That book is > 700 pages, and fairly dense ones, so it's a bit of a project. On top of that, HTT isn't really a self-contained read (and that's sort of the nature of the subject). So I'll be drawing on lots of additional material. The subject seems well worth learning, even unavoidable at times (e.g., in the areas I am interested in), so this should be a productive little activity.

...

I need to cover some ground before actually opening HTT. First I need to think about *simplicial sets*. Their role in higher category theory is ubiquitous. They give us a combinatorial model for the homotopy theory of spaces, and also a model for ∞-categories. (On the list of things to do is make precise sense of those statements.) Some references are [Rie], [Mat], [Fri08], and `kerodon.net`.

Definition 0.1.1. Denote by $\mathbf{\Delta}$ the *simplex category*, defined to have...
- As objects, the ordered set $[n] := \{0 < 1 < \cdots < n\}$ for each $n \geq 0$; and
- As maps, the weakly order-preserving set maps.

Definition 0.1.2. A *simplicial set* is a contravariant functor $\mathbf{\Delta} \to \mathsf{Set}$. The *category of simplicial sets*, denoted sSet, is the functor category $\mathsf{Fun}(\mathbf{\Delta}^{op}, \mathsf{Set})$.

Notation 0.1.1. Let $X : \mathbf{\Delta}^{op} \to \mathsf{Set}$ be a simplicial set. We may write it $X_\bullet$, and denote by X_n the set $X([n])$. We call the elements of X_n the *n-simplices* of X.

Notation 0.1.2. We may write $\langle f_0, \dots, f_n \rangle$ to denote the function $[n] \to [m]$ given by $n \mapsto f_n$.

Simplical sets are not just simplices. They carry additional structure, that arising from morphisms in $\mathbf{\Delta}$. We can give a simple description of $\mathbf{\Delta}$. This in turn gives some intuition for what a simplicial set "is."

Proposition 0.1.3 (The structure of $\mathbf{\Delta}$). *For each $n \geq 0$ and $0 \leq i \leq n$, define the*

$$i\text{-th } \textit{face map } d^i : [n-1] \to [n] \text{ as } \langle 0, \dots, \hat{i}, \dots n \rangle, \text{ and the}$$
$$i\text{-th } \textit{degeneracy map } s^i : [n+1] \to [n] \text{ as } \langle 0, \dots, i, i, \dots, n \rangle.$$

Every morphism in $\mathbf{\Delta}$ may be written as a composition of face and degeneracy maps. (Also, the face/degeneracy maps satisfy various relations, the simplicial identities; in fact $\mathbf{\Delta}$ is the category generated by those maps, subject to these identities.)

Thus a simplicial set $X_\bullet$ can be described as a collection of sets X_n (n-cells) together with face and degeneracy maps which satisfy the "simplicial identities." I should write more about how this notion arises from topology, in particular the singular complex. That in turn would be a good time to relate all this back to topology (nerves, geometric realization, ...) which is important.

0.1.2 (12/1) Why simplicial sets, simplicial complexes

Today I want to ask and answer, "why simplicial sets?" This should naturally lead into important basic properties of simplicial sets, namely their relationship with algebraic topology.

Poincaré was very smart. In 1895 he gave several definitions of $\pi_1(X)$ using...
1. Deck transformations of covering spaces;
2. Analysis;
3. Homotopy classes of loops; and
4. For nice, "polyhedral" spaces, an explicit generators-and-relations construction.

It will do us well to focus on (4). There, the "polyhedral spaces" we may study are *simplicial complexes*. These spaces are elementary, combinatorial, and quite natural to study. Informally, they are "wire frames." Pictures:

(TO-DO: make pictures)

You know a simplicial complex when you see one. How to define them? Notice that a simplicial complex is determined by (1) its vertices, together with (2) the data of which sets of vertices span a face. Using this fact, we can define a simplicial complex as a certain *combinatorial structure*.

Definition 0.1.4. An *abstract simplicial complex* K consists of

- A set K^0, and
- A set K^n of $(n+1)$-sized subsets of K^0 for every $n > 0$, the "n-simplices,"

With the property that every subset of K^n is an element of some K^i, for every n.

Remark 0.1.5. An abstract simplicial complex is the same thing as *a set of finite sets that is closed under taking subsets*. Given such a set W, we may form an a.s.c. by taking K^i to be the i-sized elements of W. Clearly every a.s.c. arises this way.

0.1.3 (12/4) Basic structure in sSet

We need to make some terminology regarding simplicial sets.

Definition 0.1.6. Let $X_\bullet, Y_\bullet$ be simplicial sets. We say $Y_\bullet$ is a *simplicial subset* of $X_\bullet$ if $Y_n \subseteq X_n$ and $Xf|_{Y_n} = Yf$ for every $n \geq 0$ and simplicial operator f. In other words, the action of operators on Y is the restriction of their action on X.

Definition 0.1.7. The *standard n-simplex* Δ^n is the simplicial set represented by $[n]$, i.e. $\Delta^n := \mathrm{Hom}_{\Delta}(-, [n])$.

Proposition 0.1.8. *Let $X_\bullet$ be a simplicial set. Yoneda's lemma provides a bijection $\mathrm{Hom}_{\mathsf{sSet}}(\Delta^n, X_\bullet) \cong X_n$. Under this bijection, each n-cell $a \in X_n$ corresponds to a map $f_a : \Delta^n \to X_\bullet$ satisfying $f_a(\mathrm{id}_{[n]}) = a$.*

Definition 0.1.9. Let $X_\bullet$ be a simplicial set. By the above, we may identify its n-cells with maps $\Delta^n \to X_\bullet$. Call a cell $a \in X_n$ *degenerate* if it factors as $\Delta^n \to \Delta^m \to X_\bullet$ for some $m < n$. (See [Lur22, Tag 0011] for equivalent conditions.)

Proposition 0.1.10. *The standard simplex Δ^n has a unique non-degenerate n-simplex, that arising from $\mathrm{id}_{[n]}$. We may call this the generator of Δ^n.*

Definition 0.1.11 (Boundary of Δ^n)**.** Define a simplicial subset $\partial\Delta^n$, the *boundary of Δ^n*, by

$$(\partial\Delta^n)_k := \{\text{non-surjective maps } [k] \to [n]\} \subseteq \mathrm{Hom}_{\Delta^{op}}([k], [n]).$$

Definition 0.1.12 (Horns in Δ^n)**.** For $0 \leq i \leq n$, define a simplicial subset Λ^n_i, the *i-th horn in Δ^n*, by

$$(\Lambda^n_i)_k := \{f \in \mathrm{Hom}_{\Delta^{op}}([k], [n])) : f([k]) \neq [n] \cup \{i\}\}.$$

A horn Λ^n_i is called *outer* if $i \neq 0, n$ and *inner* otherwise.

Remark 0.1.13. The face maps $d^i : [n-1] \to [n]$ induce maps $d^i : \Delta^{n-1} \to \Delta^n$ via post-composition. Now, consider an n-cell $a \in X_n$ and its representation $a : \Delta^n \to X_\bullet$. We have that $d_i(a) \in X_{n-1}$ is represented by ad^i.

0.1.4 (12/6) Colimits in/functors out of sSet

Today I want to understand part of Akhil's notes, about functors out of sSet. This is closely related to understanding colimits in sSet, by general theory for presheaf categories. So we also want to understand colimits in sSet. (And we should want to understand these regardless.) Let's go over this.

Here's a standard structure result for presheaf categories.

Proposition 0.1.14. *If a category C is small, then every presheaf on C is canonically the colimit of representable presheaves. In particular, every simplicial set is canonically the colimit of standard simplices.*

Proof. This is written out in Akhil's notes. I'll give the idea.
Consider a presheaf $F : \mathsf{C}^{op} \to \mathsf{Set}$. We associate to F the category D_F with

- Objects: morphisms from represented presheaves to F, i.e. arrows $[-, X] \to F$; and
- Morphisms: morphisms between represented presheaves such that the obvious triangle commutes.

There is a functor $\phi_F : \mathsf{D}_F \to \mathsf{PShv}(\mathsf{C})$ which sends objects $[-, X] \to F$ to $[-, X]$. By construction, for each object $c \in \mathsf{D}_F$, there is a morphism $\phi_F(c) \to F$, and the diagram described by ϕ_F together with these morphisms commutes. We therefore have a distinguished morphism

$$\lim_{\mathsf{D}_F} \phi_F \to F.$$

This map turns out to be an isomorphism. □

Hereafter, denote by $\bar{\mathsf{C}}$ the category of presheaves on C.

Suppose D is cocomplete. We want to understand functors $\bar{F} : \bar{\mathsf{C}} \to \mathsf{D}$. The previous proposition says that objects in $\bar{\mathsf{C}}$ are colimits of representables. So, if $\bar{F}$ preserves colimits, then $\bar{F}$ is determined by $\bar{F}|_{\mathsf{C}}$, i.e. what it does to C (embedded via Yoneda). We've described an injection of sets

$$\mathrm{Fun}'(\bar{\mathsf{C}}, \mathsf{D}) \hookrightarrow \mathrm{Fun}(\mathsf{C}, \mathsf{D}). \tag{15}$$

Here, Fun' denotes the set of colimit-preserving functors.

Conversely, suppose given a functor $F : \mathsf{C} \to \mathsf{D}$. Does it extend along the Yoneda embedding to a functor $\bar{F} : \bar{\mathsf{F}} \to \mathsf{D}$? We can do something here, let me write it out:

(1) As above, for each presheaf $G : \mathsf{C}^{op} \to \mathsf{Set}$, consider it as a colimit of $\phi_G : \mathsf{D}_G \to \mathsf{C}$. (We can do this because it lands in represented functors.)

(2) This is 'functorial' in the following sense: a morphism $G \to H$ induces a functor $\mathsf{D}_G \to \mathsf{D}_H$ such that the obvious triangle commutes.

(3) Define a functor $\bar{F} : \bar{\mathsf{C}} \to \mathsf{D}$ by
$$\bar{F}(G) := \operatorname*{colim}_{\mathsf{D}_G} F \circ \phi_G.$$
This is a functor because of (2).

This functor $\bar{F}$ really extends F, i.e. the obvious diagram commutes. For suppose $G = [-, c]$; then D_G has a final object $[-, c] \to [-, c]$, therefore $\bar{F}(G) = \mathrm{colim}_{\mathsf{D}_G} F \circ \phi_G = F(G)$.

Proposition 0.1.16. *Suppose given a functor $F : \mathsf{C} \to \mathsf{D}$ to a cocomplete category. Then the associated functor $\bar{F} : \bar{\mathsf{C}} \to \mathsf{D}$ constructed above preserves all colimits. In fact, $\bar{F}$ is left adjoint to "restriction to C."*

Proposition 0.1.17. *Suppose given a functor $F : \mathsf{C} \to \mathsf{D}$ to a cocomplete category. Then the mapping $F \mapsto \bar{F}$ describes a bijection of sets*
$$\mathrm{Fun}(\mathsf{C}, \mathsf{D}) \xrightarrow{\sim} \mathrm{LeftAdjoints}(\bar{\mathsf{C}}, \mathsf{D}).$$

The proofs are short and formal.

Corollary 0.1.18. *If a functor $\bar{F} : \mathsf{C}^{op} \to \mathsf{Set}$ takes colimits to limits, then it is representable.*

Proof. Suppose as given $\bar{F}$. By the above, it is left adjoint to some $\bar{G} : \mathsf{Set}^{op} \to \mathsf{C}$. Define $f := \bar{G}(\{pt\})$, the image of the terminal object in Set^{op}. I claim that f represents F. (Insert short, formal proof; it's in Akhil's notes.) □

0.1.5 (12/23) Geometric realization

Finals are over and I've had some time to wind down at home. Last time I worked through part of Akhil's notes about functors out of sSet. (Emily Riehl talks about something similar in her notes, but I have not gotten through those, so let me say nothing in that direction right now.)

Next I want to relate Top, Cat, and sSet. This is the backdrop for the idea that higher categories "bridge" topology/homotopy theory and ordinary categories. Today I'll go over the relation of sSet to Top, by which I mean the adjunction

the geometric realization functor $\dashv$ the total singular complex functor.

There are a few ways to introduce this adjunction. Lurie, Charles, and Akhil each do it differently. (That is not to say their ultimate definitions are very different.) As a matter of taste, I prefer Akhil's approach. (I think Lurie planned to say something along the lines of Akhil's approach, based on [Lur22, Tag 002D], but I don't know if he has.) Lurie's approach has some important ideas behind it too, I think, but that is overruled because I am feeling sleepy today.

Definition 0.1.19.

Bibliography

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