Lecture notes

June 17, 2023

Contents

0.1 (Review) Lecture 1: Some category theory

Several people said they wanted to learn basic category theory. We already needed to review some category theory for this seminar, so we'll take the opportunity to review enough theory to hopefully help everyone.

The short-term plan is to review some category theory, Grothendieck sites, and sheaves. Then we will start actually talking about condensed things. I only reluctantly call this "review," since we are *not* assuming everyone is familiar with all the material moving forward. The goal is only to catch folks up to a working knowledge.

Probably everything we say here is said better somewhere in Emily Riehl's *Category Theory in Context*. That book also has many examples. You should read it.

Definition 0.1.1. A *category* C consists of the following data.

(1) A collection $Ob(\mathsf{C})$ we call *objects*.

(2) A collection $Mor(\mathsf{C})$ we call *morphisms*.

(3) For each morphism f, a *source* and *target* object. (We write $f : X \to Y$ to express that f is a morphism with source X and target Y.)

(4) For each object X, a distinguished morphism $\mathrm{id}_X : X \to X$ we call the *identity morphism*.

(5) For each pair of morphisms $f : X \to Y, g : Y \to Z$ such that $target(f) = source(g)$, a distinguished morphism $gf : X \to Z$ we call the *composite morphism*.

And this data must satisfy the following properties.

- Given a morphism $f : X \to Y$, we have $f = \mathrm{id}_Y f = f\mathrm{id}_G$.
- Given morphisms f, g, h, we have $(fg)h = f(gh)$ (when the source/targets match appropriately).

In practice, we think of categories as "like a collection of objects and maps between them, with all the structure that should accompany the word *maps*—identity self-maps, composites, associativity."

0.1.1 Examples of categories

(See Riehl for more)

Example 0.1.2. The category of sets Set has sets as objects and functions as morphisms. The rest of the structure is "obvious": source/target is domain/range, the composition of morphisms is defined as the composition of functions, and identity morphisms are the identity functions.

Example 0.1.3. The category of spaces Top has topological spaces as objects and continuous maps as morphisms. Again, the rest of the structure is "obvious."

Example 0.1.4. Define the category Top_* to have *based spaces*[1] as objects and *based maps*[2] as morphisms.

Example 0.1.5. Define the category Grp to have groups as objects and homomorphisms as morphisms.

Example 0.1.6. Define the category Ab to have abelian groups as objects and homomorphisms as morphisms.

Example 0.1.7. Denote by k a field (e.g., $k = \mathbb{R}$). Define the category $Vect_k$ to have k-vector fields as objects and k-linear maps as morphisms.

Example 0.1.8 (Morphisms do not have to be functions!)**.** Define a category $\mathsf{Naturals}$ to have

[1] A based space is pair (X, x) where X is a space and $x \in X$.

[2] A based map $f : (X, x) \to (Y, y)$ is a continuous map $f : X \to Y$ such that $f(x) = y$.

- As objects, the natural numbers $Ob(\mathsf{Naturals}) := \{0, 1, 2, \dots\}$; and
- A morphism $a \to b$ for each pair of numbers (a, b) such that $a \le b$.

Thus, given objects $a, b \in \{0, 1, 2, \dots\}$, there is at most one arrow $a \to b$, and it exists iff $a \le b$.

Example 0.1.9 (Morphisms do not have to be functions!)**.** Define a category $\mathsf{Skel}(\mathsf{FinSet}_{\mathsf{iso}})$ to have

- Natural numbers n as objects; and
- A morphism $n \to n$ for each element of the symmetric group Σ_n, and no morphisms $m \to n$ when $m \neq n$.

Example 0.1.10 (Example from Peter's talk)**.** A *poset* is a set S with a relation $\le$ that is reflexive, transitive, and antisymmetric. A *morphism of posets* $f : (S, \le) \to (S', \le')$ is a function $f : S \to S'$ that respects the partial orderings, i.e. $x \le y \iff f(x) \le' f(y)$. We denote by $Poset$ the category of posets and morphisms of posets.

0.1.2 Isomorphisms

All sorts of objects—groups, rings, sets, spaces—have a notion of "sameness." In an arbitrary category, we formalize this notion as *isomorphisms*.

Definition 0.1.11. Let C be a category. Suppose that $f : c \to c'$ is a morphism and that there exists a morphism $g : c' \to c$ such that $fg = \mathrm{id}_{c'}$ and $gf = \mathrm{id}_c$. Then we say f and g are *isomorphisms* and we say that c, c' are *isomorphic*.

Example 0.1.12. Isomorphisms in Set are bijections. Isomorphisms in Top are homeomorphisms. Isomorphisms in Top_* are based homeomorphisms. Isomorphisms in Grp and Ab are group isomorphisms.

Exercise: Figure out what the isomorphisms in $Poset$ are.

0.1.3 Functors

Definition 0.1.13. Let C, D be categories. A *functor from* C *to* D (which we write $F : \mathsf{C} \to \mathsf{D}$) is "a map of objects and morphisms that preserves categorical structure, i.e. sources, targets, composites, and identities." Formally, it consists of the following data.

(1) An object $FX \in \mathsf{D}$ for each object $X \in \mathsf{C}$.

(2) For each morphism $f : X \to Y$ in C, a morphism $Ff : FX \to FY$ in D.

And this data must satisfy the following properties.

- For any pair of composable morphisms f, g in C, we have $F(gf) = F(g)F(f)$.
- For any identity morphism id_X in C, we have $F(\mathrm{id}_X) = \mathrm{id}_{FX}$.

Very often, we say "(one type of object) are the same thing as (another type of objects)." Categories give us a great, concrete way to talk about "types of objects." Functors give us a way to "modify and compare" objects of different types. Can functors tell us when (one type of object) are "the same thing as" (another type)? Yes, and this is a very useful notion.

For the following definition, we will denote by $Mor(X, Y)$ the set of morphisms $X \to Y$ between objects $X, Y \in \mathsf{C}$.

Definition 0.1.14. Let C, D be categories. An *equivalence of categories* is a functor $F : \mathsf{C} \to \mathsf{D}$ that is

1. *Full*: for every pair of objects $X, Y \in \mathsf{C}$, the mapping $f \mapsto F(f)$ defines a surjection $Mor(X, Y) \to Mor(FX, FY)$;

2. *Faithful*: for every pair of objects $X, Y \in \mathsf{C}$, the mapping $f \mapsto F(f)$ defines an injection $Mor(X,Y) \to Mor(FX, FY)$; and
3. *Essentially surjective*: for every object $d \in \mathsf{D}$, there exists some $c \in \mathsf{C}$ such that $Fc \cong d$.

Remark 0.1.15. The analogy is, "full and faithful is like injectivity" and "essentially surjective is like surjectivity." If you have both, you have an isomorphism. Note that a full and faithful functor need not actually be surjective *on objects*.

Remark 0.1.16. There are other common, equivalent definitions of an equivalence of categories.

0.1.4 Examples of functors

(see also Riehl, p. 13)

Example 0.1.17. Let C denote one of the categories $\mathsf{Top}, Grp, \mathsf{Ab}$, or $Vect_k$. We can define a functor $U : \mathsf{C} \to \mathsf{Set}$ by mapping objects to their underlying sets and morphisms to their underlying set-maps. We call U the *forgetful functor*. (We generally refer to any functor that "tosses out" structure, e.g. a topology on a set, as a *forgetful functor*.)

Example 0.1.18. If C is any category, then we can form its *opposite category* C^{op} to have the same objects but with "flipped arrows," i.e. swapped source/targets of C's morphisms. There is a functor $\mathsf{C} \to \mathsf{C}^{\mathrm{op}}$ that takes objects to themselves and morphisms to their "flip."

Exercise: prove that $(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$ is equivalent to C.

Example 0.1.19. For $V \in Vect_k$, recall that its *dual* is defined as the vector space $V^* := \{\text{linear maps } V \to k\}$. Given a linear map $f : V \to W$, there is induced a map $f^* : W^* \to V^*$ that sends $v : W \to k$ to $v \circ f : V \to k$. The mapping $V \mapsto V^*, f \mapsto f^*$ defines the *dualization functor* $(-)^* : Vect_k \to Vect_k^{\mathrm{op}}$.

Remark 0.1.20. You have heard probably heard that we have an isomorphism $V \cong V^{**}$ that is "canonical" or "natural" or "very nice," but that we do not have such an isomorphism $V \cong V^*$. (Although the two are isomorphic.) This can be expressed very concretely as a statement about the functor $(-)^*$ and its self-composite $(-)^{**}$. We do not yet have the language for this (natural transformations); the non-categorical reason is that an isomorphism $V \cong V^*$ requires a choice of basis, but there is an isomorphism $V \cong V^{**}$ that does not need any choice.

Many—and historically, the motivating—examples of functors come from algebraic topology.

Example 0.1.21. Let (X, x) be a based space (i.e., X is a space and $x \in X$). We define the *fundamental group* $\pi_1(X, x)$ as the set of based continuous maps $\ell : [0,1] \to X$ such that $\ell(0) = \ell(1) = x$, modulo homotopy equivalence. The group structure is loop concatenation: given $\ell, \ell' : [0,1] \to X$, define $\ell'\ell : [0,1] \to X$ to do one loop over $[0.5]$ then the other over $[0.5, 1]$. Given a based map $f : (X, x) \to (Y, y)$, there is induced a map $f_* : \pi_1 X \to \pi_1 Y$ given by $\ell \mapsto f \circ \ell$. This defines a functor

$$\pi_1(-) : \mathsf{Top}_* \to \mathsf{Grp}.$$

Example 0.1.22. For each n, singular homology defines a functor $H_n(-) : \mathsf{Top} \to \mathsf{Ab}$. Similarly, singular cohomology defines a functor $H^n(-) : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Ab}$.

0.1.5 Natural transformations

We often want to compare *functors*. This will help us explain why e.g. "an arbitrary vector space is not *naturally* isomorphic to its dual" but "an arbitrary vector space *is* naturally isomorphic to its double-dual."

There are more serious examples where we *really* care about comparisons between functors. For example, the *Hurewicz homomorphism* from algebraic topology is a comparison $h_X : \pi_n(X) \to H_n(X)$ for every space X. But in fact, more can be said—for any continuous based map $f : X \to Y$, the Hurewicz homomorphism satisfies $h_Y \circ \pi_n(f) = H_n(f) \circ h_X$. (Here, $\pi_n(f)$ and $H_n(f)$ are the maps induced by f on pi_n and H_n.) This is a seriously useful fact that is not "formally guaranteed" to be true. One might phrase this as, "the Hurewicz homomorphism compares objects $\pi_n(X) \to H_n(X)$ in a way that respects how maps induce homomorphisms via the functors $\pi_n(-), H_n(-)$."

Natural transformations give a simple way to express this.

Definition 0.1.23. Let $F, G : \mathsf{C} \to \mathsf{D}$ be two functors. A *natural transformation from F to G*, which we denote as $\alpha : F \implies G$, is the data of

- For each object $c \in \mathsf{C}$, a morphism $\alpha_c : F(c) \to G(c)$ in D

such that for every morphism $f : c \to c'$ in C, one has $G(f) \circ \alpha_c = \alpha_{c'} \circ F(f)$. In other words, the following diagram commutes.

$$\begin{tikzcd} F(c) \arrow[r, "F(f)"] \arrow[d, "\alpha_c"] & F(c') \arrow[d, "\alpha'_c"] \\ G(c) \arrow[r, "G(f)"] & G(c') \end{tikzcd}$$

It turns out, natural transformations can be composed!

Definition 0.1.24. Let $F, G, H : \mathsf{C} \to \mathsf{D}$ be three functors, and suppose we are given two natural transformations $\eta : F \Rightarrow G$ and $\varepsilon : G \Rightarrow H$. Define the *composition* $\varepsilon \circ \eta$ whose component at some $c \in \mathsf{C}$ is given by $(\varepsilon \circ \eta)_c := \varepsilon_c \circ \eta_c$.

Exercise 0.1.25. Show that given natural transformations $\eta : F \Rightarrow G$ and $\varepsilon : G \to H$, that the composition $\varepsilon \circ \eta$ as defined above is actually a natural transformation. In other words, verify that the naturality condition is satisfied.

Exercise 0.1.26. Show that given two categories C and D, you can define the **functor category** $\mathrm{Fun}(\mathsf{C}, \mathsf{D})$ (also sometimes denoted as $[\mathsf{C}, \mathsf{D}]$ or D^{C}) whose objects are functors $\mathsf{C} \to \mathsf{D}$ and whose morphisms are natural transformations $\eta : F \Rightarrow G$.

You will need to check that:

- Given any functor F in $\mathrm{Fun}(\mathsf{C}, \mathsf{D})$, there exists an **identity natural transformation** $\mathrm{Id}_F : F \Rightarrow F$.
- Composition of natural transformations is both associative and unital with respect to your identity transformations.

0.2 (Review) Lecture 2: More category theory

The first part of the talk on limits and colimits (sections 0.2.1 – 0.2.3 below) will be given by Isaiah (who is also writing the notes for these sections). If you have any questions about anything below, please do not hesitate to reach out to me on discord, my username there is isaiahtx.

0.2.1 Reminder about natural transformations

First, I will review the definition of a natural transformation. If I have time, at some point during the presentation I will introduce the idea of a *category associated to a preorder*:

Definition 0.2.1. A **preorder** is pair $(P, \leq)$ where P is a set and $\leq$ is a reflexive and transitive relation on P, i.e., given $x \in P$, $x \leq x$, and if $x, y, z \in P$ satisfy $x \leq y$ and $y \leq z$, then $x \leq z$.

Definition 0.2.2. A **preorder** is a category such that there is at most one morphism between any two objects.

Exercise 0.2.3. Understand that the above two definitions are equivalent. In other words, show that given a preorder defined as in Definition 0.2.1 gives rise to a preorder as defined in Definition 0.2.2, and vice-versa.

0.2.2 Limits and colimits

Definition 0.2.4. Let c be an object in a category C.

We say c is *initial* if, given any object c' in C, there is a unique morphism $c \to c'$.

Conversely, we say c is *terminal* if, given an object c' in C, there is a unique morphism $c' \to c$.

Definition 0.2.5. Recall: an arrow $f : x \to y$ in a category C is called an **isomorphism** if there exists an arrow $g : y \to x$ such that $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$. We say two objects are **isomorphic** if there exists an isomorphism between them.

Exercise 0.2.6. Show that in a category C, given two initial objects c and c', there is a unique isomorphism $c \to c'$. Similarly, terminal objects in C are unique up to unique isomorphism. Thus, it makes sense to talk about *the* initial/terminal object in a category.

Hint: If c and c' are initial objects, there is a unique arrow $c \to c'$ and a unique arrow $c' \to c$ (why?). What can you say the compositions of these arrows?

Example 0.2.7. In the category Set, the initial object is the empty set and the terminal object is the singleton set.

Example 0.2.8. In the categories Grp and Ab, the trivial group is both initial and terminal.

Example 0.2.9. Given a preorder P, the terminal object, if it exists, is called the *top object*. The initial object is called the *bottom object*.

The top object is greater than or equal to every other object in the preorder. The bottom object is less than or equal to every other object in the preorder.

Given categories J and C, we often call a functor $F : \mathsf{J} \to \mathsf{C}$ a *diagram of shape* J *in* C.

Definition 0.2.10. Given two categories J and C and an object c in C, let $\underline{c} : \mathsf{J} \to \mathsf{C}$ denote the *constant functor on* c which sends every object in J to c, and every morphism in J to the identity morphism id_c on c.

Definition 0.2.11. Let J be a small category, and $F : \mathsf{J} \to \mathsf{C}$ be a functor.

A *cone under* F is a pair (λ, c), where c is an object in C and λ is a natural transformation $\lambda : F \Rightarrow \underline{c}$. We call c the **nadir** of the cone.

A *cone over* F is a pair (c, λ), where c is an object in C and λ is a natural transformation $\lambda : \underline{c} \Rightarrow F$. We call c the **summit** or **apex** of the cone.

Explicitly, the data of a cone λ under $F : \mathsf{J} \to \mathsf{C}$ with nadir c is a collection of morphisms $\lambda : F(j) \to c$, indexed by the objects j in J, such that for any morphism $f : j \to k$ in J, the following triangle commutes in C

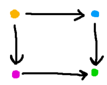
Oftentimes, you will see the word "cocone" instead of "cone under F", and in this context usually the word "cone" will refer explicitly to cones over F.

$$\begin{tikzcd} F(j) \arrow[rr, "F(f)"] \arrow[dr, "\lambda_j"'] & & F(k) \arrow[dl, "\lambda_k"] \\ & c & \end{tikzcd}$$

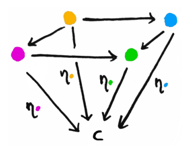
Dually, the data of a cone λ over $F : \mathsf{J} \to \mathsf{C}$ with apex c is a collection of morphisms $\lambda_j : c \to F(j)$, indexed by objects j in J, such that for any morphism $f : j \to k$ in J, the following triangle commutes in C

$$\begin{tikzcd} & c \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ F(j) \arrow[rr, "F(f)"] & & F(k) \end{tikzcd}$$

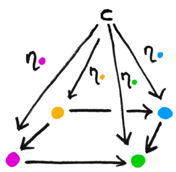
Typically, we think of limits and colimits of functors $F : \mathsf{J} \to \mathsf{C}$ when J is a relatively "small" or "simple" category. Maybe J looks something like this



Then if (c, η) is a cone under F, we have the following image in C:

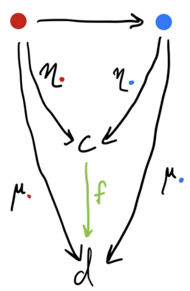


And if (η, c) is a cone over F, we have the following image in C:



Definition 0.2.12. Given a diagram F of shape J in a category C (so a functor $F : \mathsf{J} \to \mathsf{C}$) and two cones (η, c) and (μ, d) under F, a *morphism of cones under F* is a morphism $f \in \mathrm{Mor}(c, d)$ such that for all objects j in J, $\mu_j = f \circ \eta_j$.

Pictorally, a morphism of cones under F connects the nadirs of the cones.



Of course, we have a dual definition for cones over F, which connect the apexes of cones. Can you draw a picture?

Definition 0.2.13. Given a diagram F of shape J in a category C (so a functor $F : \mathsf{J} \to \mathsf{C}$) and two cones (c, η) and (d, μ) over F, a *morphism of cones over* F is a morphism $f \in \mathrm{Mor}(c, d)$ such that for all objects j in J, $\mu_j \circ f = \eta_j$.

Definition 0.2.14. Let F be a diagram of shape J in a category C.

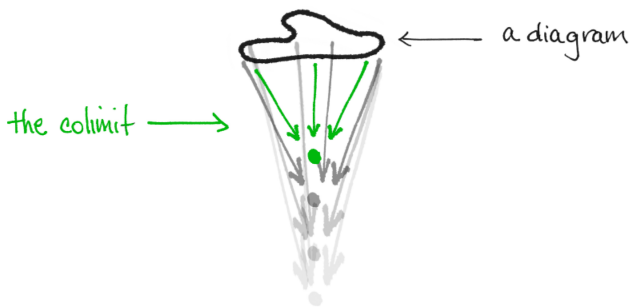
Define $\mathrm{Cone}_{\mathsf{C}}(F)$ to be the category whose objects are cones under F, and morphisms are morphisms of cones under F.

Conversely, define $\mathrm{Cone}^{\mathsf{C}}(F)$ to be the category whose objects are cones over F, and morphisms are morphisms of cones over F.

Definition 0.2.15. Given a diagram F of shape J in a category C, the *colimit cone* for F is the initial object in $\mathrm{Cone}^{\mathsf{C}}(F)$ (if it exists).

If (η, c) is a colimit cone for F, then we call the object c the *colimit* of F, and denote it by any of the following:

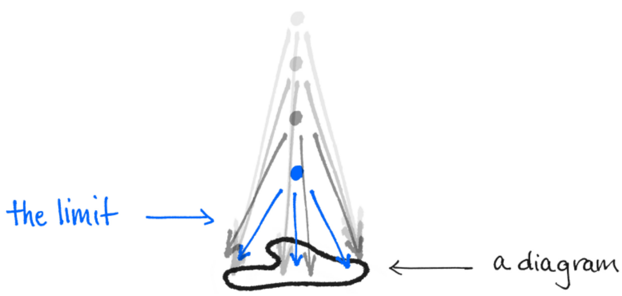
$$\operatorname*{colim}_{\mathsf{J}} F, \qquad \operatorname*{colim}_{\longrightarrow} F, \quad \text{or} \quad \varinjlim F$$



Definition 0.2.16. Given a diagram F of shape J in a category C, the *limit cone* for F is the terminal object in $\mathrm{Cone}^{\mathsf{C}}(F)$ (if it exists).

If (c, η) is a limit cone for F, then we call the object c the *limit* of F, and denote it by any of the following:

$$\lim_{\mathsf{J}} F \quad \text{or} \quad \varprojlim F$$



0.2.3 Examples of limits and colimits

I'll take a moment to say something about category theory: the above definitions were quite technical and abstract. Like most definitions in category theory, the definition of the (co)limit cannot be internalized or understood by reading it. You need to work through examples, preferably as many as possible. Thankfully, you already know lots of examples of limits and colimits!

I am going to explicitly give an example of computing a colimit and a limit in Set.

Example 0.2.17. Let J be a category with $n > 0$ objects and no non-identity morphisms. Then a functor $F : \mathsf{J} \to \mathsf{C}$ is the data of a choice of n objects $X_1, \ldots, X_n$ in C. Then the (co)limit of F is called the **(co)product** of the X_i's.

In Set, coproducts are disjoint unions and products are cartesian products of sets.

In Ab, the coproduct is given by direct sum, and the product is given by the product of groups.

In Top, the coproduct is given by the disjoint union of spaces with the disjoint union topology, while the product is given by the cartesian product with the product topology.

In Top_*, the coproduct is given by the wedge product (join spaces at their basepoint), while the product is the regular product in Top.

Example 0.2.18. An **equalizer** is a limit of a diagram indexed by the **parallel pair**, the category $\bullet \rightrightarrows \bullet$ with two objects and two parallel non-identity morphisms between them.

Example 0.2.19 (Limits in Set)**.** In general, limits in Set can be described completely as follows:

Given a diagram $S : \mathsf{J} \to \mathsf{Set}$, define the set

$$\varprojlim S := \{(s_i)_i \in \prod_{\mathsf{J}} Si : \forall \phi : i \to i', (S\phi)(s_i) = s_{i'}\}.$$

One can check that this is a limit of F.

Example 0.2.20 (Colimits in Set)**.** Given a diagram $S : \mathsf{J} \to \mathsf{Set}$, define the set

$$\operatorname{colim}_{\longrightarrow} S := \left(\coprod_{\mathsf{J}} Si\right) / (s_i \sim s_{i'} \text{ if } \exists \phi : (S\phi)(s_i) = s_{i'}).$$

One can check that this is a colimit of F.

Example 0.2.21 ((Co)limits in Top)**.** Like Set, the category Top is also complete and cocomplete. A limit in Top is formed by taking the limit of underlying sets and endowing it with the subspace topology. Likewise, a colimit in Top is formed by taking the colimit of underlying sets and endowing it with the quotient topology.

0.2.4 The Hom functor

Let X, Y be objects in C. We consider the set

$$\mathrm{Hom}_{\mathsf{C}}(X, Y)$$

of all morphisms $X \to Y$ in C. (When C is understood, we just write $\mathrm{Hom}(X, Y)$.) Given a morphism $f : Y \to Y'$, post-composition defines a function $f \circ - : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y')$. Given a morphism $g : X \to X'$, pre-composition defines a function $- \circ g : \mathrm{Hom}(X', Y) \to \mathrm{Hom}(X, Y)$. Notice that this map goes the "other way."

This describes two important functors.

Definition 0.2.22. Let C be a category. For each $c \in \mathsf{C}$, we define the *covariant* Hom *functor*

$$\mathrm{Hom}_{\mathsf{C}}(c, -) : \mathsf{C} \to \mathsf{Set}.$$

Definition 0.2.23. Let C be a category. For each $c \in \mathsf{C}$, we define the *contravariant* Hom *functor*

$$\mathrm{Hom}_{\mathsf{C}}(-, c) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}.$$

Exercise 0.2.24. Think about the contravariant Hom functor.

Exercise 0.2.25. In a category C, prove that a morphism $f : X \to Y$ is an isomorphism $\iff$ for every object $Z \in \mathsf{C}$, the morphism $\mathrm{Hom}(Z, f) : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$ is an isomorphism. (I.e., iff the function $f \circ -$ is a bijection.)

(Not totally essential part.) The two Hom functors fit together in such a way that we can turn the co/contravariant Hom functors into a single Hom functor. We mean the following.

Proposition 0.2.26. *If $f : X \to Y$, $g : X' \to Y'$ are morphisms in C, then the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(X, Y) \arrow[r, "\mathrm{Hom}(X,g)"] \arrow[d, "\mathrm{Hom}(f,Y)"'] & \mathrm{Hom}_{\mathsf{C}}(X, Y') \arrow[d, "\mathrm{Hom}(f,Y')"] \\
\mathrm{Hom}_{\mathsf{C}}(X', Y) \arrow[r, "\mathrm{Hom}(X',g)"] & \mathrm{Hom}_{\mathsf{C}}(X', Y')
\end{tikzcd}$$

Thus, we have a functor $\mathrm{Hom}_{\mathsf{C}}(-, -) : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$.

0.2.5 Adjunctions

There is a notion of adjunctions. The slogan is, "The slogan is, *adjunctions are everywhere.*" There are several equivalent definitions. None is "the best," they are very much so all useful. The idea is that two functors $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ may "undo each other on the level of hom sets;" we make this precise.

Definition 0.2.27. A *hom-set adjunction* is a pair of functors $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ together with a natural isomorphism

$$\Phi : \mathrm{Hom}_{\mathsf{D}}(F-, -) \cong \mathrm{Hom}_{\mathsf{C}}(-, G-).$$

Definition 0.2.28. A *unit-counit adjunction* is a pair of functors $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ together with natural transformations $\eta : \mathrm{id}_{\mathsf{C}} \implies GF$ and $\epsilon : FG \implies \mathrm{id}_{\mathsf{D}}$ such that the following diagrams commute (we call these the *triangle identities*).

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "\mathrm{id}_F"'] & FGF \arrow[d, "\epsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "\mathrm{id}_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

Proposition 0.2.29. *Let (F, G, Φ) be a hom-set adjunction. Define its canonical unit-counit structure as follows.*

- *For $c \in \mathsf{C}$, we define the morphism $\eta_c := \Phi_{c,Fc}(\mathrm{id}_{Fc})$; and*
- *For $d \in \mathsf{D}$, we define the morphism $\epsilon_d := \Phi^{-1}_{Gd,d}(\mathrm{id}_{Gd})$.*

The claim is that (F, G, η, ϵ) is a unit-counit adjunction, i.e. the η_c, ϵ_d's assemble to natural transformations satisfying the triangle identities.

Proposition 0.2.30. *Let (F, G, η, ϵ) be a unit-counit adjunction. Define its canonical hom-set adjunction structure as follows. For each $c \in \mathsf{C}$, $d \in \mathsf{D}$, and $f \in \mathrm{Hom}_{\mathsf{D}}(Fc, d)$, define $\Phi(f) := \eta_c \circ Gf \in \mathrm{Hom}_{\mathsf{C}}(c, Gd)$. The claim is that (F, G, Φ) is a hom-set adjunction.*

The last two propositions say that every hom-set adjunction gives rise to a unit-counit adjunction and vice-versa.

Proposition 0.2.31. *Hom-set adjunctions are "the same thing as" unit-counit adjunctions.*

Example 0.2.32 (Cool example from Peter's talk)**.** We work in Top^*, the category of based spaces.[3] Given based spaces X, Y, we may ask for some based space $G(X, Y)$ such that for every Z,

$$\mathrm{Hom}(G(X, Y), Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)). \tag{33}$$

In this sense, $G(X, Y)$ "undoes" the functor "take the space of based maps out of Y" on the level of hom-sets. In fact, such a based space $G(X, Y)$ exists for every X and Y, and the isomorphisms (32) are such that they describe an adjunction. (In the language above, a *hom-set adjunction*.) The space $G(X, Y)$ is the *smash product of based spaces*, defined as

$$G(X, Y) = X \wedge Y := (X \times Y)/(X \vee Y).$$

Precisely, the functors $- \wedge Y : \mathsf{Top}^* \to \mathsf{Top}^*$ and $\mathrm{Hom}(Y, -) : \mathsf{Top}^* \to \mathsf{Top}^*$ are adjoint. Smashing with Y is left adjoint to homming out of Y.

This matters because if we take $Y = S^1$, our adjoint functors specialize to important constructions: given a space X, one has

$$Y \wedge X = \Sigma X \quad \text{and} \quad \mathrm{Hom}(Y, X) = \Omega X.$$

The spaces $\Sigma X, \Omega X$ are called the *suspension* and *loop space* of X, respectively. They are essential to doing algebraic topology and homotopy theory. Then, being instances of functors which are adjoint, the adjunction tells us that for any space W we get

$$\mathrm{Hom}(\Sigma X, W) \cong \mathrm{Hom}(X, \Omega W)$$

and these isomorphisms are *natural*. I could go on about why this is great.

[3] Objects are spaces with a chosen point (X, x_0) and morphisms $(X, x_0) \to (Y, y_0)$ are continuous maps $f : X \to Y$ such that $f(x_0) = y_0$.

0.2.6 More Hom functor

Are hom-sets between (co)limits the (co)limits of hom-sets? The important answer is yes, and we will be thinking about questions like this more later.

Proposition 0.2.34. *For every object X, the functors $\mathrm{Hom}(X, -)$ and $\mathrm{Hom}(-, X)$ commute with limits. That is, if $\mathsf{I} \to \mathsf{C}$ is a diagram in C and $\varprojlim_{\mathsf{I}} c_i$ exists, then for any object X one has*

$$\mathrm{Hom}_{\mathsf{C}}(X, \varprojlim_{\mathsf{I}} c_i) \cong \varprojlim_{\mathsf{I}} \mathrm{Hom}_{\mathsf{C}}(X, c_i) \qquad \textit{and}$$
$$\mathrm{Hom}_{\mathsf{C}}(\varprojlim_{\mathsf{I}} c_i, X) \cong \operatorname{co}\varinjlim_{\mathsf{I}} \mathrm{Hom}_{\mathsf{C}}(c_i, X).$$

Proposition 0.2.35. *For every object X, the functors $\mathrm{Hom}(X, -)$ and $\mathrm{Hom}(-, X)$ commute with colimits. (The precise statement of this is dual to that in the previous proposition.)*