SPECTRA: FIRST EXAMPLES

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Abstract. Follow-up to my prespectra notes. Some first examples of spectra.

These are just notes, subject to all nonliability clauses which usually preface notes. I used K.G. Wickelgren's notes and some of Neil Strickland's.

Example (Suspension spectrum)**.** For a topological space X, its *suspension spectrum* $\Sigma^\infty X$ has $\Sigma^\infty_n := \Sigma^n X$ and structure maps $\epsilon_n : \Sigma(\Sigma^n X) \to \Sigma^{n+1} X$ the identity. This defines a functor $\Sigma^\infty : \mathsf{Top}_* \to \mathsf{PreSpec}$. A map $f : X \to Y$ induces a family of maps $\Sigma^n f : \Sigma^n X \to \Sigma^n Y$ denoted $\Sigma^\infty f$. This is a function of spectra, i.e. a morphism.

Example (Loop spectrum)**.** There is an analogous functor $\Omega^\infty : \mathsf{Top}_* \to \mathsf{PreSpec}$ to Σ^∞.

Example (Eilenberg-Maclane spectrum)**.** Fix G abelian. Recall $K(G,n)$ denotes the space whose only nontrivial homotopy group is π_n, which is $\cong G$. We may represent it by a CW-complex, unique up to weak equivalence. Since

$$\pi_n(\Omega K(G,n+1)) = \pi_{n+1} K(G,n+1),$$

The space $\Omega K(G,n+1)$ is a $K(G,n)$. We may take it to be a CW-complex, so it is weakly equivalent to our CW representative of $K(G,n)$ by uniqueness. Thus, Whitehead's theorem begets a homotopy equivalence[1]

$$K(G,n) \simeq \Omega K(G,n+1). \tag{0.1}$$

This specifies a spectrum, the *Eilenberg-Maclane spectrum* HG of G, having spaces $HG_n = K(G,n)$ and structure maps the equivalences 0.1.

Other Important Examples. There are many. They deserve their own notes (to-do.) But to name some:

- Thom spectra MG. The complex cobordism spectrum MU is an important example of a Thom spectrum. Then, tmf.
- Brown Representability (to-do) associates a spectrum to any generalized cohomology theory.
- Let G be an abelian group. A Moore spectrum X is one having $\pi_n(X) = 0$ for $n < 0$ and $H_n(X) = 0$ for $n > 0$. Up to a non-canonical isomorphism, there is a unique Moore spectrum having $H_0 = G$.

[1] Relevant MO discussion here.