GROUPOIDS

MATTHEW A. NIEMIRO

Abstract. We begin development of the categorical notion of groupoids.

Definition. A **groupoid** is a category in which every morphism is an isomorphism.

Why groupoids? One boilerplate reason is that they enable more general or elegant versions of known results. Groupoids are a categorical generalization of groups, so a groupoid-theoretic perspective can fit into a richer framework what was originally only understood group-theoretically. But I do not mean to trivialize groupoids! They are of independent interest in pure category theory, and are useful well beyond simply extending group-theoretic results. There is a more specific and exciting exposition to be given, but I don't yet know the topic well enough to give it. The purpose of these notes is to fix that.

Example 0.1. A **group** is a groupoid with one object.

This is not just an example, but a definition.[1] Before learning category theory, groups are introduced as the glob of symmetries of an object (e.g., the rotational group of a snowflake.) This is taught with objects as concrete examples (the snowflake) for which group elements are described as actions on the object, but the group itself really possesses no data referencing any such object—what matters are the relations amongst the actions. The categorical definition of a group as a one-object groupoid suggests this element-independent perspective right away, since in general we avoid thinking about the elements of objects in a category.

An advantage of groupoids is that we may encode the interactions between multiple groups into a single groupoid. See the next example.

Example 0.2. A disjoint union of groups is a groupoid. That is, a category whose objects are groups and whose morphisms are isomorphisms form a groupoid.

This is not a complicated example, but it is an important one. To see why, step back and consider the fundamental mathematical problem of *classification*. In many situations, equality of objects (rings, groups, spaces, etc.) is too fine a relation, so we consider our objects *up to a classification weaker than equality*. We sacrifice information for traction, much to our benefit. Nevertheless, at times we must be more tactile, and the usual notion of 'up to a relation' must be replaced by something that does not forget as much information.

We advertise groupoids as the solution. Do groupoids fix our problem without keeping tabs on the many (equivalent) equivalencies between objects, whose vastness was the problem in the first place? Perhaps counter-intuitively, the answer is

[1] To understand the analogy between this definition of a group and the usual one, it may help to think of group elements as morphisms.

no. Groupoids instead organize this vastness to fit into the master plan of category theory. In the previous example, for instance, although there were (possibly) isomorphisms between our objects, we did not go out of our way to forget their multitude.

This has ups and downs. By far the biggest roadblock (common throughout category theory) is the fundamental task of actually figuring out what's going on and knowing how much abstraction is appropriate for specific purposes—and we're not even *trying* to develop ∞-groupoids higher-categorically (yet). But all this comes in time.

Returning from our peripherals, let's see an important groupoid.

Definition. The **fundamental groupoid** $\Pi_1(X)$ of a topological space X is the groupoid whose objects are the points in X and whose morphisms $x \to y$ are endpoint-preserving homotopy classes $[\gamma]$ of maps $\gamma : [0,1] \to X$ from x to y.

Taken at face, you might be disappointed. This object does not seem useful, and in fact this was the general consensus on the object for some time. Then in 1967, R. Brown showed how the Van Kampen theorem could be formulated for groupoids. We will improve the notion of $\Pi_1(X)$ to a new, relative object.

Definition. A **full** subcategory of a category is one that has some of the original objects, but all the original morphisms between those objects.

Definition. A **subgroupoid** of a groupoid is a subcategory of the groupoid that is also a groupoid.

Definition. The **fundamental groupoid on a set** A **of** X, denoted $\Pi_1(X, A)$, is the full subgroupoid of $\Pi_1(X)$ on the set A. In other words, it is the full subcategory of $\Pi_1(X)$ whose objects are exactly the $x \in A \cap X$.

Thus, $\Pi_1(X, A)$ consists of homotopy classes joining points in $A \cap X$.

Now, from what I can tell, a proper development of the groupoid-theoretic Van Kampen theorem would be somewhat technical and dense. So, we will not give a comprehensive treatment. (But if you'd like, knock yourself out.)