Sheaves and Sheaf Cohomology

Reading group notes

June 29, 2022

Contents

0.1 (Week 2) June 22 - Of Presheaves and Sheaves

Reading: my paper [1].

First meeting. Opened with some history. (History motivates; see paper.)

Q: What is a sheaf? Furthermore, what (and I cannot stress this enough) *is* a sheaf?

Let X be a topological space. Define $\mathrm{Op}(X) :=$ category of its open sets, morphisms given by inclusions.

WANT: A systematic way to 'track' the 'constructions' possible on open sets. E.g., given an open set U, what is the set of cts functions $U \to \mathbb{R}$? How do these sets vary as U varies? How are the resulting sets related, w.r.t. the relation of their underlying open sets? (Try replacing 'cts function' with: C^∞ functions, k-forms, exact k-forms, sections of a vector bundle, solutions to a the PDE arising from a fixed vector field, ...)

Cts functions are the prototypical notion of a 'construction.' Among other things, they have a nice property: if $U \hookrightarrow V$ then a cts $f : V \to \mathbb{R}$ restricts to a cts $f|_U : U \to \mathbb{R}$. (This answers 'how are the resulting sets related?'–they are related by a *restriction map*.)

Any notion of 'construction' should have this restriction property. And this restriction property was just (contravariant) functoriality w.r.t. inclusions.

Definition. A *presheaf* is a contravariant functor $\mathcal{F} : \mathrm{Op}(X) \to \mathsf{D}$, where D is any (concrete) category. Usually it will be sets, rings, abelian groups, ... We call the image of an inclusion under $\mathcal{F}$ a *restriction*.

So a presheaf describes 'constructions' that come with restrictions. Continuous functions have more than just restrictions, though. Two properties in particular stand out (or rather, are so obvious that the need to identify them stands out):

1. if $f_1 : U_1 \to R$ and $f_2 : U_2 \to R$ agree on intersection, then there exists a global $f : U_1 \cup U_2 \to R$ that restricts to f_1 or f_2.
2. If $f, g : U_1 \cup U_2 \to R$ agree when restricted to U_1 or U_2, then $f = g$.

Generalizing (1) and (2) to arbitrary covers leads us to define a sheaf.

Definition. A *sheaf* is a presheaf satisfying two axioms.

1. (Identity) Let $\{U_i\}$ be any open covering of any open set U. If $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ for each U_i, then $s = t$.
2. (Gluing) Let $\{U_i\}$ be any open covering of any open set U. If one can choose a section s_i from each U_i such that s_i, s_j agree when restricted to $U_i \cap U_j$ for each i, j, then there is a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for each i.

EXAMPLES–important, see paper, end of §1.

Remark. (Pre)sheaves form categories $\mathsf{PSh}(X)$, $\mathsf{Sh}(X)$. Objects are (pre)sheaves, morphisms are natural transformations. The inclusion functor $\mathsf{Sh}(X) \hookrightarrow \mathsf{PSh}(X)$ has a left adjoint called *sheafification.* Thus, sheafifying a presheaf gives the 'best' sheaf approximate. The exact construction of sheafification is not widely useful; people (algebraic geometers) usually just need to know the universal property relating a presheaf to its sheafification. (The one described by adjunctions.)

0.2 (Week 2) June 26 - Abelian Categories I

Still reading [1].

Opened with historical refresher: homological algebra had just systematized our understanding of modules, at least in the context of their use in algebraic topology. This clearly worked for objects besides modules, but a rigorous abstraction was not made. In particular, *sheaves* had 'homological algebra-like' properties, e.g. kernels, quotients. And sheaves had JUST effected advances in complex/algebraic geometry (Serre, Cartan, ...) The big question: can we figure out how to do homological algebra generally, fully fit sheaves into their framework, then recruit the techniques of algebraic topology (cohomology!) to study sheaves? Short answer: yes.

Today, we want to identify what makes homological algebra work. Or rather, where it works.

The base case is Mod_R, the category of R-modules–this is where we first understood homological algebra. Let's specialize to Ab, the category of abelian groups. What are its properties?

1. Given two groups G and H, the set $\mathrm{Hom}(G, H)$ is an abelian group.
2. With respect to the group structure above, morphisms compose bilinearly; $h \circ (f + g) = h \circ f + h \circ g$, and similarly with $(f + g) \circ h$.
3. It turns out that (1) and (2) imply that the (finite) product and coproduct of two groups are isomorphic, *if either exists.* (Here, I mean the categorical (co)product; in strictly group-theoretic terms, these are the direct sum/product.) Assuming they exist, we have a natural categorical description of the addition law on hom-sets (that's good!), see the reading p. 5-6, starting at "Getting categorical,

. . . ." And it turns out that Ab *does* have all finite products and coproducts. (That's the property.)

4. It has a *zero object*, to mean an object with a unique morphism to AND from all other objects. (It's the trivial group.) This gives rise to e.g. *zero morphisms.*

Then the Reg closed and some of us got dinner. We'll finish listing properties next time.

0.3 (Week 3) June 29 - Abelian Categories II

Picking up where we left off—what are the good properties of Ab, insofar as homological algebra is concerned?

5. Kernels and cokernels of morphisms are abelian (sub)groups. Furthermore, they satisfy a 'canonical relationship' to the original groups (If this doesn't make sense, ignore it. It will be clear what I'm trying to get at when we abstract to categorical (co)kernels.)

6. The first isomorphism theorem: if $f : G \to H$ is a hom, then $G/\ker f \cong \operatorname{im} f$. Another name for $G/\ker f$ is the *coimage* of f. This notation is good to know.

Facts (1)-(6) let us do 'homological algebra' with abelian groups–we can talk about chain complexes, quotients, kernels, etc. Now we reformulate these properties for arbitrary categories. This requires we 'categorify' certain terms above. Here's what we get:

Definition. Consider the following properties a category may have.

(PA1) Every hom-set has an abelian group structure.

(PA2) (PA1), and composition is bilinear with respect to hom-set addition law.

(A) (AB2), and it has all finitary products and coproducts. **Consequences**: we mentioned that this implies the (co)product of two objects coincide. We also mentioned that this inicidence gives us a categorical construction of the addition law on hom-sets.

(AB0) It has a *zero object.*

(AB1) (AB0), and it has all *kernels* and *cokernels.* **Definitions**: the *kernel* of a morphism $f : X \to Y$ is the equalizer of f and the zero map $X \to Y$. Equivalently, it is a map $k : K \to X$ such that (i) $f \circ k$ is the zero map, and (ii) for any map $k' : K' \to X$ such that $f \circ k'$ is zero, there is a unique map $K' \to K$ making that whole diagram commute. Dually, the *cokernel* of f is the coequalizer of f and the zero map $X \to Y$.

(AB2) *Images* and *coimages* are isomorphic. **Definitions**: Given f, the *image* of f is defined is the kernel of its cokernel. Dually, the *coimage* of f is the cokernel of its kernel. Actually, I think the isomorphism is 'canonical': by the definitions of (co)images, there appears a map $\operatorname{coker}\ker f \to \ker\operatorname{coker} f$, which gives the isomorphism(?)

Definitions: A category satisfying (PA2) is called *preadditive.* A category satisfying (A) is called *additive.* A category satisfying (A) and (AB2) is called *abelian.*

Abelian categories have homological algebra!

0.3.1 Digression: homological algebra?

At this point, I should stop saying 'homological algebra' without any indication of what it is. Especially since some of us might not know examples of (co)homology. I'll introduce *singular (co)homology*. It is a good 'toy example' witnessing the emergence of algebra in the study of spaces.

Let X be a topological space. Define $C_n(X) := FAb\{\text{continuous } \sigma : \Delta^n \to X\}$.[1] The n-simplex Δ^n has faces, so $\sigma \in C_n(X)$ can be restricted to a face, which defines a map $\Delta^{n-1} \to X$. Define a homomorphism $\partial : C_n(X) \to C_{n-1}(X)$ by

$$\sigma \mapsto \sum (-1)^i \sigma_{\text{its } i\text{-th face}}.$$

That $(-1)^i$ accounts for orientation, which makes everything work. We can forget about it.

Now we look at the following 'chain':

$$\cdots \to C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} C_{n-2} \to \cdots \to C_1 \xrightarrow{\partial} C_0.$$

Note that $\partial^2 = 0$; in this case, we call the chain a *chain complex of abelian groups*. Since $\partial^2 = 0$, we know $\mathrm{im}\partial \subseteq \ker\partial$. So we may define $H_n(X) := \ker\partial/\mathrm{im}\,\partial$. These are the *homology groups* of X.

We wish to know things about $H_n(X)$. It's definition is in terms of that chain complex—*it turns out we can just study abstract chain complexes and learn a lot about $H_n(X)$ as a result.* That's homological algebra!

***Example**: one learns that given a pair of spaces $A \subset X$, there is a corresponding 'long exact sequence' (LES) of singular homology groups of A, X, and (A, X). The proof is pure (homological) algebra, making no reference to how C_n is defined. It's an example of a more general fact: given a SES of chain complexes, there is an induced LES in homology. In this case, $A \to X \to X/A$ is a SES of spaces, which we hit with $C_*(-)$ to get a SES of chain complexes, to get the LES in homology. The proof depends on the *snake lemma* and *zig-zag lemma* from hml. algebra.

0.3.2 The furniture of abelian categories

Finally, we have defined abelian categories. They come with:

- A first isomorphism theorem (by assumption).
- Quotients (and 'subobjects').
- (Co)chain omplexes. And since we have quotients, we can define the *(co)homology* of a chain (co)complex.
- Since we have (co)kernels, we can talk about *exactness*.
- A snake lemma and zig-zag lemma.

This exactly extends the language we used to define and study singular homology to work in other categories (e.g., modules). This corroborates our history lesson: they were coming up with algebra to help them do 'combinatorial topology,' then wanted to seperate and systematize all that algebra.

[1] FAb means 'free abelian group on this set.' And Δ^n is the n-simplex; Google its definition.

0.3.3 Upshot

Here's vaguely how we use singular homology:

1. Take a space, (e.g. X)
2. Construct sufficiently algebraic things from that space, (e.g. $C_n(X)$, ∂)
3. Use homological algebra to study those 'things,' (e.g. $H_n(X)$, derivation of LES induced by SES)
4. Leverage the relationship between X's topology and the $H_n(X)$ to learn about X.

Step (3) was very important. On one level, this is because the *tools* it provides (e.g. the LES) are indispensable. But more to the point: a lot of what we do (especially if we defined singular cohomology) would be incoherent if we weren't certain *kernels*, *images*, *quotients*, etc. 'worked' (e.g., existed).

Step (4) was also important. A priori, an 'algebraic thing' constructed from a space need not say anything about its topology–but for singular homology, the definition is intrinsically related to the topology.

With abelian categories defined, we can be a bit more creative. We may try to:

- Take a space X,
- Construct an algebraic gadget that tracks when we can 'do something we care about' on parts of our space,
- See if those gadgets form an abelian category, and if so, study them from the POV of homl. algebra, perhaps deriving algebraic objects K related to X,
- Leverage the relationship between X and the K to learn about X, AND learn about K.

References

[1] Matthew A. Niemiro. *The derived functor approach to sheaf cohomology*. University of Chicago Mathematics REU. 2020.