WHAT ARE (PRE)SPECTRA?

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Abstract. In algebraic topology, a "spectrum" may refer to the objects in one of several related categories, all of which give the same category after passage to homotopy. That would be the *stable homotopy category*. Here, I define the forerunner among these "spectra," which are sometimes called *prespectra*. Disadvantages: many, e.g. a non-commutative, non-associative smash product, and a tedious construction. Advantages: accessible definition, instructive.

These are just notes, subject to all nonliability clauses which usually preface notes. I used K.G. Wickelgren's notes and Adams' *Stable Homotopy and Generalized Homology*.

Definition. A *spectrum* E is a sequence of pointed spaces E_n together with maps

$$\epsilon_n : \Sigma E_n \ (\cong_{\text{homeo}} S^1 \vee E_n) \ \to E_{n+1}.$$

Remark. Let X be a pointed space, $\Omega X = [S^1, X]_*$ the space of based maps $S^1 \to X$. There is a natural equivalence $[\Sigma X, Y]_* \cong [X, \Omega Y]$; hence, a spectrum may equivalently consist of maps $\epsilon'_n : E_n \to \Omega E_{n+1}$.

Definition. A *degree r function of spectra* $f : E \to F$ is a sequence of maps $f_n : E_n \to F_{n-r}$ such that the corresponding diagram commutes (strictly, not just up to homotopy.)

With functions as morphisms, we have constructed a category of spectra. It is painfully rigid.[1] There are different constructions of a category of spectra which are better behaved (I don't know them.) Our version of spectra here are often called *prespectra*.

In any case, we want to do homotopy theory with spectra. We will construct, from our category of prespectra, a homotopy category of prespectra and call it the *stable homotopy category*. (This is actually the same category arising from other versions of "category of spectra" after passage to homotopy.)

Definition. A spectrum E is called a *CW-spectrum* if

(1) Each E_n is a CW-complex, and
(2) The structure maps $\Sigma E_n \to E_{n+1}$ are inclusions of subcomplexes.[2]

Definition. Let E be a CW-spectrum. A CW-spectrum F is called a *subspectrum* if F_n is a subcomplex of E_n, for all n. Furthermore, F is called *cofinal* if for each cell $e_n \in E_n$, there is an m so that $\Sigma^m e_n$ maps into F_{n+m} under the canonical map.

Definition. Let E be a CW-spectrum, E' and E'' cofinal subspectra. Functions $f' : E' \to F$ and $f'' : E'' \to F$ are called *equivalent* if they agree on $E' \cap E''$ (which is a cofinal subspectrum). This is an equivalence relation.

Definition. A *map of CW-spectra* $f : E \to F$ is an equivalence class of functions from cofinal subspectra of E to F, the relation being that above.

Fact. Taking maps of CW-spectra is well-defined with respect to the composition of functions of CW-spectra.

Definition. Let $f, g : E \to F$ be maps of CW-spectra. We will define a "homotopy" using mapping cylinders, like in baby algebraic topology. I proceed in steps.

[1] For instance, we cannot extend the Hopf fibration $\eta : S^3 \to S^2$ to a (degree one) function $\mathbb{S} \to \mathbb{S}$.
[2] Note that the suspension of an n-cell is an $(n + 1)$-cell.

(1) Denote by I^+ the disjoint union of $[0,1]$ and a basepoint. Define a functor $\mathsf{Cyl}(-)$ from spectra to spectra, summarized by the diagram below.

$$\begin{tikzcd}
\cdots \arrow[r] & E_{n-1} \arrow[r] & E_n \arrow[r] \arrow[d, rightsquigarrow, "\mathsf{Cyl}(-)"'] & E_{n+1} \arrow[r] & \cdots \\
\cdots \arrow[r] & I^+ \wedge E_{n-1} \arrow[r] & I^+ \wedge E_n \arrow[r] & I^+ \wedge E_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

The maps are $1 \wedge \epsilon_n$.

(2) Denote by i_0, i_1 the functions $E \to \mathsf{Cyl}(E)$ which include E_n into $\mathsf{Cyl}(E_n)$.

(3) Call f and g *homotopic* if there is a map $h : \mathsf{Cyl}(E) \to F$ so that $f = h \circ i_0$ and $g = h \circ i_1$.

Now we form the category whose objects are CW-spectra and whose morphisms are homotopy classes of maps. I think this is the **stable homotopy category**. A final remark: any spectrum is weakly equivalent to a CW-spectrum, so restricting to CW-spectra did not bother the homotopy theorists.

There is more to be said, especially if one wants to do anything useful with spectra. But we take baby steps.