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2024 Noteb∞k

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I January

> Prior to the advent of the brain, there was no color and no sound in the universe, nor was there any flavor or aroma...before brains the universe was also free of pain and anxiety.
>
> *Roger Sperry*

New year. And a new notebook. Last one was getting tedious to compile, and it was about time I tweak the TEXanyway. I should start hearing back from graduate schools soon. I am pretty excited to start grad school, and now that the anxiety of applications is gone, I can feel and and appreciate how much time six years (the duration I plan to be in graduate school) is to do things. I have some math-related new year's resolutions: learn some combinatorics, GAP 4, probability theory and ergodic theory, and number theory. Maybe special mention to Cohen-Lenstra statistics. This list reflects personal interests I feel I have neglected. Bonus points if I can make things weird (i.e., categorical). This notebook remains a documentation of what I'm studying in higher category theory.

I.1 (1/1) Cardinality

A "set of all sets" is set-theoretically impossible, although there is a proper class T of such things. A group is a set, hence *the class of groups of order* 2 is at least the size of T since each set $S \in T$ gives rise to a group of order 2, namely $\{S, S\}$ with whatever[1] group structure you'd like. In particular, there is a *proper* class of groups of order 2. The lesson: for many purposes, we should count up to isomorphism.

Let X be a finite set and $\sim$ an equivalence relation. The quotient $X/\sim$ is finite, hence its *cardinality* is well-defined and equals the number of equivalence classes, which we can count "over X" via the easy formula

$$|X/\sim| = \sum_{x \in X} \frac{1}{|[x]|}.$$

Notice this formula does not use the relation $\sim$ in any meaningful way. In certain situations, we may not like this.

Observation I.1. Let X be a finite set and $\sim$ an equivalence relation. *A priori*, the natural distribution on a finite set X is the uniform one. You can also put the uniform distribution on $X/\sim$, but it is more natural that the likelihood of obtaining $[x]$ is induced by the likelihood of obtaining any $y \in [x]$. These quantities should be inversely proportional(?), hence $\mathbb{E}([x]) = \sum(|[x]|)^{-1}$.) Then another notion of "size" for $X/\sim$ is perhaps the sum $\sum_{[x]} |[x]|^{-1}$.

Example I.1. Finite group $G = \mathbb{Z}/2\mathbb{Z}$ acting on the set

$$X := \{1, 2, 3, 4, 5\}$$

as the cycle (15)(24). Then X/G has three points, those being the orbits $\{1, 5\}, \{2, 4\}$, and $\{3\}$. Both X and X/G are finite sets. If we concede that the natural dsitribution on a finite set is the uniform one, however, then the induced distribution on X/G is *not* the natural one. See, the induced distribution weighs the listed orbits $2/5, 2/5$, and $1/5$, respectively, but the uniform weighs them each $1/3$.

Our use of probability here is not essential. It just gives a natural language to talk about the "size" of objects—bigger objects should take up more of the distribution.

Remark I.1. Let G be any finite group acting on a finite set X. If the action is free, then there is no issue like the above, for freeness is equivalent to trivial stabilizers, which implies that the orbits of G have uniform size, namely the orbits are of size $|G|$.

[1] Which ever. Of the two.

Remark I.2. The category of finite sets is the "categorification" of the natural numbers. We can divide two natural numbers—how to divide two finite sets S and T? If T is a finite group acting on S, we may consider the quotient S/T. By the previous remark, if this action is free, then $|S/T| = |S|/|T|$. Hence, quotienting sets by finite free actions is a "categorification" of division. However, this process only produces finite sets (categorified naturals, even though we should get rationals), and furthermore does *not* work for non-free actions (e.g., in the example, $|S/T| = 3$ but $|S| = 5$ and $|T| = 3$). These issues are quite related—to address the latter, we will need a new type of object, for which we will need a new notion of cardinality, from which we get the rationals.

The problem is that $3 \in X$ has more automorphisms (a smaller orbit), so it "appears larger" in X/G than it "really is" because the set X/G does not see automorphisms. This suggests that the "size" of an element should be inversely proportional to its number of automorphisms. We therefore should replace sets with objects that carry automorphism data (groupoids) and extend the notion of cardinality to account for that data.

Definition I.1. Let X be an ordinary groupoid. Define its *homotopy cardinality* as the sum

$$|X| = \sum_{[x]\in\pi_0 X} \frac{1}{|\mathrm{Aut}(x)|}.$$

Example I.2. Let $G = \mathbb{Z}/2\mathbb{Z}$ act on a five-element X as the cycle $\sigma = (15)(24)$ again. Consider the associated *action groupoid* $X//G$ [nLab]. It has $\mathrm{Ob}(X//G) = X$ and a morphism $g : x \to y$ if and only if $\exists g : gx = y$. In this case, we can compute its homotopy cardinality as

$$|X| = \frac{1}{|\mathrm{Aut}(1)|} + \frac{1}{|\mathrm{Aut}(2)|} + \frac{1}{|\mathrm{Aut}(3)|} = \frac{1}{|\{\sigma^2 = \mathrm{id}\}|} + \frac{1}{|\{\sigma^2 = \mathrm{id}\}|} + \frac{1}{|\{\mathrm{id}, \sigma\}|} = 5/2.$$

This works. In the above example, it weights elements in the quotient correctly, namely prescribing 3 a weight of $1/2$. More generally, given $G \curvearrowright X$, we have $|X//G| = |X|/|G|$ as desired. One can crank out more exotic examples. You may ponder the cardinality of the core of your favorite category. For example, $|\mathsf{FinSet}^{\cong}| = e$.

Homotopy cardinality is homotopy invariant. Furthermore, it has the essential properties of ordinary cardinality: it is additive and multiplicative over products and coproducts, respectively. How to extend this notion to ∞-groupoids? Let me just give you the definition: for an ∞-groupoid X, we define its *homotopy cardinality* as the sum

$$|X| := \sum_{[x]\in\pi_0 X} \prod_{n=1}^{\infty} |\pi_n(X,x)|^{(-1)^k} = \sum_{[x]\in\pi_0 X} \frac{|\pi_2(X,x)| \cdot |\pi_4(X,x)| \cdots}{|\pi_1(X,x)| \cdot |\pi_3(X,x)| \cdots}.$$

This satisfies a more general multiplicativity property: given a fibration $F \to E \to B$ over a connected base B, the homotopy long exact sequence yields $|E| = |F||B|$. (This is more general because fiber bundles are "twisted" cartesian products.) This tells us, for instance, that

$$|BG| = \frac{1}{|G|}.$$

What's with the alternation in the product? Crudely, it is a manifestation of an iterated inclusion-exclusion argument. I think of this informally: the cardinality of the underlying set of X is the number of its points. But homotopy theory says to replace sets with 0-groupoids, in which case the cardinality is the number o f connected components. For simplicity, suppose X is connected; then its underlying 1-groupoid has one point up to isomorphism, but that point may have automorphisms, accounted for by $\pi_1 X$, and we know that cardinality should be inversely proportional to this. But wait—those automor mtomorphisms "smaller." But wait—the automorphisms of automorphisms have automorphisms, accounted for by $\pi_3 X$, which by parallel reasoning make $|X|$ *smaller*. So on and so forth *ad infinitum*.

Remark I.3. We are implicitly assuming the defining series for $|X|$ converges. Call spaces for which this is the case *tame*.

Recall that a space X is called *n-finite* if $\pi_{k>n}X = 0$ and $\pi_{k\leq n}X$ is finite, and that X is called *π-finite* if it is n-finite for some n. It is clear that π-finite spaces are tame. Thus, π-finite spaces seem like a good category of spaces in which to think about homotopy cardinality.

Definition I.2. Write $\mathsf{S}_{\mathrm{fin}}$ for the ∞-category generated by a point under finite colimits. We also consider $\mathsf{S}_{n-\mathrm{fin}}$ and $\mathsf{S}_{\pi-\mathrm{fin}}$, the full subcategories of n- and π-finite spaces.

Homotopy cardinality defines a functor[2] $\mathsf{S}_{\pi-\mathrm{fin}} \to \mathbb{Q}_{\geq 0}$. It is the unique extension of the cardinality of finite sets that is homotopy invariant, additive w.r.t. disjoint unions, and multiplicative w.r.t. fibrations. See the answer to my MO question.

I.2 (1/4) Colimit completions and filtering classes

Here's a story I really like. Consider an ordinary category C. Its presheaf category $\mathrm{PShv}(\mathsf{C}) := \mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ is the "free completion at colimits" of C in the following sense.

Theorem I.1. The Yoneda embedding $\mathrm{H} : \mathsf{C} \to \mathrm{PShv}(\mathsf{C})$ is a free cocompletion of C. That is, $\mathrm{PShv}(\mathsf{C})$ has small colimits, and if D admits small colimits, then restriction along H defines a natural equivalence

$$\mathrm{H}^* : \mathrm{Fun}^{cocts}(\mathrm{PShv}(\mathsf{C}), \mathsf{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathsf{C}, \mathsf{D}).$$

This is standard. I think it also characterizes H (and hence $\mathrm{PShv}(\mathsf{C})$) by the usual "thing satisfying universal property is unique" argument. Here is another characterization.

Definition I.3. Consider a presheaf $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. Its *category of elements* $\mathrm{el}\, F$ has (objects) transformations $\mathrm{H}c \to F$ and (arrows) transformations $\mathrm{H}c \to \mathrm{H}d$ such that the evident triangle commutes.

Proposition I.1 (Density)**.** There is a canonical map $\mathrm{el}\, F \to \mathsf{C}$. Its colimit canonically presents F, i.e.,

$$\mathrm{colim}(\mathrm{el}\, F \to \mathsf{C} \to \mathrm{PShv}(\mathsf{C})) \cong F.$$

The density theorem canonically associates to each presheaf $F : * \to \mathrm{PShv}(\mathsf{C})$ a diagram of representables $\mathrm{el}\, F \to \mathrm{PShv}(\mathsf{C})$, which we equivalently regard as its underlying diagram $\mathrm{el}\, F \to \mathsf{C}$. I think you can upgrade this to say "the category $\mathrm{PShv}(\mathsf{C})$ is *equivalent* to the category of diagrams in C."

We have roughly provided three equivalent definitions of $\mathrm{PShv}(\mathsf{C})$: it is (I) the functor category $\mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$, (II) the category of diagrams in C,[3] and (III) the free colimit completion of C. The equivalence (I) → (III) associates to F its diagram $\mathrm{el}\, F \to \mathsf{C}$. The equivalence (III) → (I) takes the colimit of the diagram. Either can be shown to fit the description (II).

Remark I.4. For a presheaf F, its category of elements can also be defined as the pullback below. The map $\mathrm{el}\, F \to \mathsf{C}$ is again evident. Perhaps because of this definition, we sometimes write $\mathsf{C}/F := \mathrm{el}\, F$.

$$\begin{tikzcd}
\mathrm{el}\, F \arrow[r, dashed] \arrow[d, dashed, "\pi_F"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{PShv}(\mathsf{C})_{/F} \arrow[d] \\
\mathsf{C} \arrow[r, "\mathrm{H}"] & \mathrm{PShv}(\mathsf{C})
\end{tikzcd}$$

Remark I.5. The category of elements C/F can also be defined as the *comma category* $* \downarrow F$. I honestly do not know about comma categories and am not sure I need to know about them right now.

Often, we want to think about the completion C at a *certain class* of colimits—say, filtered or sifted colimits. We described the colimit completion $\mathrm{PShv}(\mathsf{C})$ in three ways (I), (II), and (III) above; these suggest three ways to complete at a *chosen class* of colimits. Namely, the completion of C at a "nice" colimits should be...

[2] I don't think this is actually a functor. It is a function from the set of equivalence classes of π-finite Kan complexes to $\mathbb{Q}_{\geq 0}$.

[3] The description (III) needs more attention than we have provided. First of all, you want *small* diagrams. Then you must define morphisms. This requires some care. Will I get around to typing this out?

(I) A subcategory of “nice" presheaves in $\mathrm{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$;

(II) A subcategory of “nice" diagrams in C; and

(III) The “free nice colimit completion" of C (in the sense of a universal property).

All three models are useful. For example, *filtered colimits* are easiest to define as colimits over *filtered diagrams*, and this diagrammatic definition suggests we define the *ind-completion* $\mathrm{Ind}(\mathsf{C})$[4] as the category of filtered diagrams in C, in the likeness of (II).[5] (They are equivalently the diagrams whose colimits in Set commute with finite limits.) Meanwhile, *sifted colimits* are not so easily described by the shape[6] of their diagram; we say a diagram is *sifted* if its colimits commute with finite products (in Set). This suggests a definition of the sifted completion in the likeness of (I): when C has finite (co?)products, I think you can define its sifted completion as the full subcategory of presheaves commuting with finite products. (You can also define it as the subcategory of sifted diagrams in C, but I'm trying to make the point that (I) and (II) are both natural.)

Remark I.6 (Further reading)**.** I just said lots of stuff, but maybe left out lots of stuff. A little while ago, I gave a detailed account of the above picture in Lecture 6 of our condensed seminar, notes here.

I have not really thought about colimit completions for ∞-categories before. But, it seems the ordinary picture persists verbatim. Let me spell out an extended example, the motivating example in Charles' paper [Rez22], which is what made me start thinking about colimit completions again.

Example I.3 (Completion at κ-filtered colimits)**.** For a regular cardinal κ, an ∞-category C is called *κ-filtered* if every κ-small simplical set diagram $K \to \mathsf{C}$ admits an extension $K^{\triangleright} \to \mathsf{C}$ [Lur08, 5.3.1.7]. Parallel to the ordinary case, κ-filteredness is equivalent to commutation with all small limits in Space [Lur08, 5.3.3.3]. It turns out that κ-smallness is a generally well-behaved property (e.g., it is preserved by categorical equivalences), enough so that we may *complete* C *at κ-small colimits* to obtain $\mathrm{Ind}_\kappa(\mathsf{C})$. Also parallel to the ordinary picture, $\mathrm{Ind}_\kappa(\mathsf{C})$ admits models in the likeness of (I), (II), and (III) above:

(I) In terms of presheaves, $\mathrm{Ind}_\kappa(\mathsf{C})$ is the full subcategory of $\mathrm{PShv}(\mathsf{C})$ spanned by filtered colimits of representables [Lur08, 5.3.5.4]. Furthermore, if C admits small colimits, then Ind_κ can be more concretely described as the full subcategory spanned by preseheaves preserving finite limits.

(II) In terms of point categories, $\mathrm{Ind}_\kappa(\mathsf{C})$ consists of diagrams $\mathsf{J} \to \mathsf{C}$ such that J is a κ-filtered simplicial set(?)

(III) In terms of its universal property, $\mathrm{Ind}_\kappa(\mathsf{C})$ admits κ-filtered colimits, Yoneda factors as $\mathrm{H} : \mathsf{C} \to \mathrm{Ind}_\kappa(\mathsf{C})$, and if D admits κ-filtered colimits, then restriction along H defines an equivalence

$$\mathrm{Fun}_\kappa(\mathrm{PShv}(\mathsf{C}), \mathsf{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathsf{C}, \mathsf{D}).$$

In other words, if D admits κ-filtered colimits, then each arrow $\mathsf{C} \to \mathsf{D}$ admits an essentially unique extension to $\mathrm{Ind}_\kappa(\mathsf{C})$ [Lur08, 5.3.5.10].

Remark I.7. Lurie's verbatim definition [Lur08, 5.3.5.1] is “$\mathrm{Ind}_\kappa(\mathsf{C})$ is the full subcategory of $\mathrm{PShv}(\mathsf{C})$ spanned by presheaves F which classify right fibrations $\mathsf{C}/F \to \mathsf{C}$ such that C/F is κ-filtered." I *feel* like this is just (I) and (II) above at the same time. Lurie basicallly says in [Lur08, 5.3] that he exhibits model (II), but he does not seem to exhibit that explicitly. But I actually do not know anything about (un)straightening, so I cannot really navigate.

[4] Filtered colimits used to be called inductive limits, so we call the completion of C at filtered colimits its “ind-completion" or “indization."

[5] Then you can recharacterize it in the likeness of (I) and (III). For (I), it turns out $\mathrm{Ind}(\mathsf{C})$ is the full subcategory of $\mathrm{PShv}(\mathsf{C})$ spanned by filtered colimits of representable presheaves. If C is finitely cocomplete, so that C^{op} has finite limits, then we can be more explicit: $\mathrm{Ind}(\mathsf{C})$ is precisely those presheaves commuting with finite colimits. For (III), take the universal property of $\mathrm{PShv}(\mathsf{C})$ but replace “cocontinuous" with “preserves filtered colimits."

[6] I think there is such a description, but it is not nice.

Remark I.8. In the ordinary case, it is standard to characterize (say) ind-objects by their point category, i.e., to say that an ind-object is a filtered diagram in C, equivalently a presheaf $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ whose category of elements is filtered (and cofinal) [KS06, 3.3.13, 6.1.5].

I regard Lurie's definition of $\mathrm{Ind}_\kappa(\mathsf{C})$ as an amalgam of models (I) and (II). These are "concrete" models which one shows have the relevant universal property. Charles' paper [Rez22] considers the general question: for which classes of ∞-categories $\mathcal{F}$ do these "concrete" models still work (i.e., actually have the relevant universal property)? More precisely, which classes of small ∞-categories $\mathcal{F}$ are such that for every ∞-category C, the category $\{X \in \mathrm{PShv}(\mathsf{C}) : \mathsf{C}/X \in \mathcal{F}\}$ has the "free $\mathcal{F}$-colimit completion" property?

For this purpose, we make two definitions. First, we define $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) := \{X \in \mathrm{PShv}(\mathsf{C}) : \mathsf{C}/X \in \mathcal{F}\}$. Second, we define $\mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ as the *minimal full subcategory of* $\mathrm{PShv}(\mathsf{C})$ *generated by representables under* $\mathcal{F}$*-colimits,* to mean the smallest full subcategory containing $\mathrm{H}(\mathsf{C}) \subseteq \mathrm{PShv}(\mathsf{C})$ and closed under $\mathcal{F}$-colimits. Yoneda factors as $\mathrm{H} : \mathsf{C} \to \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$, and this functor is a *free* $\mathcal{F}$*-colimit completion* [Lur08, 5.3.6.2]. Then we seek to compare $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ and $\mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$.

This is not complicated. First, note that $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \subseteq \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ since any presheaf $X \in \mathrm{PShv}(\mathsf{C})$ is a colimit of $\mathsf{C}/X \to \mathsf{C} \to \mathrm{PShv}(\mathsf{C})$, thus X is an $\mathcal{F}$-colimit of representables as soon as $\mathsf{C}/X \in \mathcal{F}$. Now, if we can show that $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \subseteq \mathrm{PShv}(\mathsf{C})$ contains HC and is stable under $\mathcal{F}$-colimits, then we get the reverse inclusion are conclude

$$\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) = \mathrm{PShv}_{\mathcal{F}}(\mathsf{C}).$$

We consider sufficient conditions for $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ to obtain these properties.

- For which $\mathcal{F}$ does $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ always contain the representables? For an object c, we have $\mathsf{C}/\mathrm{H}c \cong \mathsf{C}_{/c}$. Hence, you could ask that $\mathcal{F}$ contains all ∞-categories with a terminal object. Alternatively(?): there c is terminal, so $\mathrm{H}c$ is a terminal presheaf. By definition, we in fact have $\mathrm{H}c \in \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$. Hence, if $\mathcal{F}$ contains D whenever $\mathrm{PShv}_{\mathcal{F}}(\mathsf{D})$ contains a terminal presheaf, then we would have $\mathrm{H}c \in \mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$.

- I have nothing to say about when $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ is stable under $\mathcal{F}$-colimits. See the proof of Prop. 4.2 in Charles paper [Rez22].

We say that $\mathcal{F}$ is *filtering* if it contains D whenever $\mathrm{PShv}_{\mathcal{F}}(\mathsf{D})$ contains a terminal presheaf. We seek this property (rather than just ask that $\mathcal{F}$ contain D whenever D has a terminal object) because filtration turns out to be closely related to our problem:

Proposition I.2. Suppose that a family of ∞-categories $\mathcal{F}$ is such that for any C, the full subcategory inclusion $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \subseteq \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ is an isomorphism. Then $\mathcal{F}$ is filtering.

Proof. If $1 \in \mathrm{PShv}(\mathsf{D})$ denotes a terminal presheaf, then $\mathsf{D}/1 \cong \mathsf{D}$. Therefore[7] $\mathsf{D}/1 \cong \mathsf{D}$ lies in $\mathrm{PShv}_{\mathcal{F}}(\mathsf{D})$ and thus $\mathrm{Ind}_{\mathcal{F}}(\mathsf{D})$ by assumption. This exactly says that $\mathcal{F}$ is filtering. □

To summarize, we explained that if $\mathcal{F}$ is filtering, then $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \subseteq \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ is an equivalence, and we proved the converse. Hence, given a class of ∞-categories $\mathcal{F}$, the "diagrammatic" model for the $\mathcal{F}$-colimit completion $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C})$ (presheaves whose point category belong to $\mathcal{F}$) is an actual $\mathcal{F}$-colimit completion (in the sense of possessing the relevant universal property, equivalently $\mathrm{Ind}_{\mathcal{F}}(\mathsf{C}) \hookrightarrow \mathrm{PShv}_{\mathcal{F}}(\mathsf{C})$ being an equality) if and only if $\mathcal{F}$ is filtering. I wonder what the diagrammatic model is good for in general. Charles also studies the notion of filtering classes, which I have not read yet. May also be interesting to look at [Du23].

Remark I.9. In light of this writing, my mind has changed to consider there as being only *two* truly distinct models: a *diagrammatic* model and a *universal property* model. Maybe it's not right to even call the latter a model, since it's really what we're trying *to model*, but that seems to be the language I've found most efficient.

Remark I.10. Someone should rename *filtering classes*. But I am not sure what a good name would be. I do not like *filtering* because we already have *filtered/filtering/filtrant* colimits, which are a very specific example of filtering classes. Something to reflect that $\mathcal{F}$-diagrams are special. Diagrammatic? Graphical? Figurative? "Graphical classes" has a nice ring to it. Or just call them "graphics." Or "complete graphics."

[7] Hmm. Why does $\mathsf{D}/1$ lie in $\mathrm{PShv}_{\mathcal{F}}(\mathsf{D})$? Does

I.3 (1/15) Reminding myself what presentability, dualizability, and stability are

Classes just started at UIUC, but I am in Chicago while Ishan and Efimov are giving lectures. Today I want to define a category $\mathsf{Pr}^{\mathrm{dual}}_{\mathrm{st}}$ which contains the category $\mathsf{Cat}^{\mathrm{perf}}_{\infty} = \{$small, stable, idempotent-complete ∞-categories$\}$ and out of which we can define *continuous ("Efimov") K-theory* $\mathsf{Pr}^{\mathrm{dual}}_{\mathrm{st}} \to \mathsf{Sp}$, extending the algebraic K-theory. I have not actually thought about presentability or dualizability before, nor how they interact with stability. I'm going to review these quickly. Here are some things I am looking at:

- Peter Haine, *Descent for sheaves on compact Hausdorff spaces* here.
- Hoyois' continuous K-theory notes here.
- Parts of Mortiz Groth's notes here, particularly the bits about presentability.
- Dustin Clausen's lectures about Efimov K-theory (on Youtube).
- He Li's Efimov K-theory notes here.
- Chapter 5 of HTT [Lur08].
- Alberto Garcìa-Raboso's notes on stable ∞-categories.
- Yonatan's notes on stable ∞-categories.

First I want to remember what presentability is. In nice cases, colimits are like unions. For instance, a group is the colimit of its directed poset of subgroups. In fact, you can just take that of its *finitely generated* subgroups. We can actually make a categorical equivalence

$$\mathrm{Ind}(\mathsf{Ab}_{\mathrm{fg}}) \cong \mathsf{Ab},$$

the point being that although Ab is large, it admits a small subcategory of ind-generators. This should be considered an essential detail of the structure of Ab, and it is both practically and philosophically important.

Toward pinning down this "big thing is secretly small and this helps us to work with it" idea, consider the *adjoint functor theorem*: a functor $F : \mathsf{A} \to \mathsf{B}$ between cocomplete categories admits a right adjoint if and only if F preserves colimits and satisfies the *solution set condition*. Without going into details, that condition is a certain smallness condition—then is reasonable to think that it is automatically fulfilled whenever A, B are themselves "small enough."

Definition I.4. Say that an ∞-category C is *presentable* if it is cocomplete and *accessible*: there exists a regular cardinal κ such that C has κ-filtered colimits and admits a small subcategory C_0 such that $\mathrm{Ind}_{\kappa}(\mathsf{C}_0) \cong \mathsf{C}$.

Theorem I.2 (Generalized adjoint functor theorem)**.** A functor between presentable, cocomplete ∞-categories admits a right adjoint if and only if it preserves colimits.

Remark I.11. Since $\mathrm{H} : \mathsf{D} \to \mathrm{Ind}(\mathsf{D})$ is fully faithful, accessibility amounts to the admittance of a small subcategory of ind-generators. Also, presentable categories admit *bilimits*.

Remark I.12. See [Lur08, §5.4] for a detailed review of accessbility.

Presentability is not a rare property, and it is generally well-behaved. It can be characterized in terms of presheaves:

Theorem I.3. An ∞-category C is presentable if and only if C is an *accessible localization* of $\mathrm{PShv}(\mathsf{D})$ for some small D: there exists a functor $F : \mathrm{PShv}(\mathsf{D}) \to \mathsf{C}$ such that (loc.) F admits a fully faithful right adjoint and (acs.) F commutes with all κ-filtered colimits for some regular κ.

I feel as if I should have something to say about this, but I do not. An idea: let D be small and regard the free colimit completion $\mathrm{PShv}(\mathsf{D})$ as a a "free, small collection of generators." Next, think of an accessible localization $\mathrm{PShv}(\mathsf{D}) \to \mathsf{C}$ as "imposing a small number of relations." Then our characterization seems natural: presentable categories are those with a "small" presentation by generators and relations. It is unclear to me how far one can expound this idea, what the role of accessibility is for the localization insofar as it makes this idea work, ... I do not want to waste time on these details right now.

Return to this.

Definition I.5. We organize presentable ∞-categories into an ∞-category. Its objects are (not necessarily small) presentable ∞-categories and its morphisms are the cocontinuous functors [Lur08, 5.5.3.1]. By the adjoint functor theorem, we can also say this category has left adjoints as morphisms. We write Pr^L for this category.

Misc properties.

Proposition I.3 ([Lur08, 5.5.3.8])**.** If C, D are presentable, then the full subcategory $\mathrm{Fun}^L(\mathsf{C}, \mathsf{D}) \subset \mathrm{Fun}(\mathsf{C}, \mathsf{D})$ spanned by left adjoints is presentable. (In fact, C can be any simplicial set.)

Proposition I.4. If C, D are presentable, then there exists a presentable category $\mathsf{C} \otimes \mathsf{D}$ which is the universal recipient of a functor from $\mathsf{C} \times \mathsf{D}$ that is colimit-preserving in both variables seperately. One presentation is $\mathsf{C} \otimes \mathsf{D} \cong \mathrm{Fun}^{cts}(\mathsf{C}^{\mathrm{op}}, \mathsf{D})$. It inherits this (symmetric monoidal) tensor product as a full subcategory (on the presentables) of $\hat{\mathsf{Cat}}_\infty(K)$, the ∞-category of ∞-categories with small colimits and colimit-preserving functors. (Which has the Lurie tensor product?) Maybe c.f. [Lur17, 4.8.1.5]. See also nLab.

Ok, now dualizability. In a monoidal category C, an object $x \in \mathsf{C}$ is called *dualizable* if (existence of "dual" with co/evaluation maps). We consider this in the case of vector spaces.

Example I.4. Consider a vector space $V \in \mathsf{Vect}_k$. A candidate for its dual is $V^* = \mathrm{Hom}(V, k)$ and there is an obvious map $\mathrm{ev} : V \otimes V^* \to k$. We would like a map $\mathrm{coev} : k \to V \otimes V^*$, which amounts to the choice of an element $v \in V \otimes V^*$. If $(e_i)_I$ is a basis for V, we have coordinates $v = \sum c_{i,j} e_i \otimes e_j^*$ such that only finitely many $c_{i,j}$ are nonzero. The axioms for co/evaluation imply that for each $i \in I$, some $c_{i,j}$ is nonzero, hence I must be finite, i.e., V must be finite-dimensional. Conversely, if V is finite-dimensional, then it is dualizable with dual V^*, in fact we may identify $V \otimes V^* \cong \mathrm{End}(V)$ (which requires a choice of basis) and define coev by $1_k \mapsto \mathrm{id}_V$.

As this suggests, dualizability is a sort of finiteness condition. But, apparently different from that of presentability. Dualizable objects inherit some of the theory of finite-dimensional vector spaces, for instance a notion of *traces of endomorphisms* and *dimension* (the trace of the identity).

Remark I.13. I think that if C is monoidal with internal homs, then an object $X \in \mathsf{C}$ is dualizable if and only if the canonical pairing $X \otimes \mathrm{Hom}(X, 1) \to \mathrm{End}(X)$ is an isomorphism. The category Vect_k has internal homs, and we saw that the map is an isomorphism therein, so this checks out. Also in Vect_k, we know that for non-dualizable (equivalently, infinite dimensional) V, that $V \otimes V^*$ is nicer than $\mathrm{End}(V)$ in general. In other monoidal categories *without* internal homs, I wonder if you can ever treat $X \otimes X^*$ like a well-behaved substitute for $\mathrm{End}(X)$. I had this thought during coffee with Anthony and Sam and they asked, "have you heard of a star-autonomous category?"

We want to think about dualizability in the full subcategory $\mathsf{Pr}_{\mathrm{st}} \subset \mathsf{Pr}^L$ spanned by the presentable, stable ∞-categories. A good question is, why are we making the stability hypothesis?

Recall that an ∞-category is *stable* if it has a zero object, has fibers and cofibers, and fiber sequences coincide with cofiber sequences. I can weakly explain where this comes from: even in classical homotopy theory, we care deeply about pushouts, pullbacks, and (co)fibrations. These are tricky notions when you work with spaces (not intractable), partly due to the rebellious nature of homotopy (co)limits in the classical stable homotopy category. This gives some impetus for *stable* homotopy theory: stable phenomena somehow simplify the story. I wish I could give a concrete, classical example of this, but all I can think of is "fibrations and cofibrations of spectra coincide." Maybe some helpful discussion here and here.[8]

Remark I.14. A functor between stable categories is called *reduced* if it preserves zero, and *exact* if in addition it preserves (co)fiber sequences. Since the zero object and (co)fibration sequences are (co)limits, it is preserved by left adjoints, hence these properties are superfluous in Pr^L and Pr^R.

Given a pointed category C, it has a *suspension functor* Σ_{C} once it has cofibers, and it has a *loop functor* Ω_{C} once it has fibers. Stability is characterized by either suspension or looping being an equivalence. This

Yonatan's notes have a bit about suspension and the triangulated structure for stable

[8]Think about these. Maybe think about connectivity results.

presents an idea: does formally inverting Σ_{C} or Ω_{C} present a "stabilization"? Note that Cat^{ex} occurs as a full subcategory of $\mathsf{Cat}_*^{\mathrm{fincolim}}$ (resp. $\mathsf{Cat}_*^{\mathrm{finlim}}$), spanned by those categories whose suspension (resp. loop) functor is an equivalence. "Stabilization" should mean an adjoint to these inclusions.

These inclusions in fact have left and right adjoints, and we get the left adjoints in the manner described.

Proposition I.5. Let C be pointed. If C has finite colimits (cofibers and suspension in particular), then the colimit $\mathsf{Sp}^{\Sigma}(\mathsf{C}) := \mathrm{colim}(\mathsf{C} \xrightarrow{\Sigma} \mathsf{C} \to \cdots) \in \mathsf{Cat}_*^{\mathrm{fincolim}}$ is stable, and this extends to a left adjoint to the inclusion $\mathsf{Cat}^{\mathrm{ex}} \hookrightarrow \mathsf{Cat}_*^{\mathrm{fincolim}}$. Dually, if C has finite limits, we get a left adjoint to $\mathsf{Cat}^{\mathrm{ex}} \hookrightarrow \mathsf{Cat}_*^{\mathrm{finlim}}$ given by $\mathsf{C} \mapsto \mathsf{Sp}^{\Omega}(\mathsf{C})$.

Rather strangely(?), even if C is (finitely) complete and cocomplete, their suspension-spectra and loop-spectra ponder
are not generally equivalent. That is, we do not always have $\mathsf{Sp}^{\Sigma}(\mathsf{C}) \simeq \mathsf{Sp}^{\Omega}(\mathsf{C})$. For example, finite spectra and Ω-spectra do not coincide. But...

Observation I.2. Presentability puts us somewhere nice: we get *all* limits and colimits, for instance. Also, the natural notion of morphisms of presentable categories (limit or colimit preserving, you choose) already fits that for categories we usually consider stabilizing (finite limit or colimit preserving, you choose). And if your presentable categories were already stable, either notion is automatically an exact functor.

Hence, at least formally, it seems easy and convenient to consider *presentable* stable ∞-categories. Since presentable demands (co)limits, the characterization of stability simplifies: *if* C *is a pointed, presentable* ∞*-category, then* $\Sigma_{\mathsf{C}}, \Omega_{\mathsf{C}}$ *are defined, and* C *is stable iff either of these is an equivalence.*

(Stuff; universal "stabilization" property; the realization of Ω-spectra as the finite shifts of finite suspension spectra; the definition of the ∞-category of spectra; the definition of its symmetric monoidal *smash product*!) Probably give this it's own day.

I.4 (1/24) Stabilization, the ∞-category of spectra, and the smash product

My notes previously got me thinking about stability, and then Charles told me some things about presentability and the smash product on the ∞-category of spectra, so now I am going to think a bit about all that. I have already defined *stable* ∞*-categories* at least twice, which behave "like spectra," or maybe "like chain complexes of abelian groups." Last time, I also thought a bit about how stability simplifies in the presence of presentability (although it seems I didn't actually write about that). This makes sense, since presentability forces existence of (co)limits, and stability cares about cetain (co)limits (and needs them for the essential Σ and Ω). But that does not do justice to the fun which presentability brings to the party. I'm going to be primarily reading Groth's notes.

For a pointed category C, recall the definition of *triangles*: they are the composable pairs of morphisms (g, f) together with 2-cells realizing $gf \simeq h$ and $h \simeq 0_{X,Y}$. These form a category, namely the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathsf{C})$ spanned by functors mapping the "bottom-left" vertex to 0. Consider that $\Delta^1 \times \Delta^1$ is both a left and right cone [Lur24, Tag 0165]:

$$(\Lambda_0^2)^{\triangleleft} \cong \Delta^1 \times \Delta^1 \cong (\Lambda_2^2)^{\triangleright}.$$

We say a triangle is *exact* if it is a limit as a left cone, and *coexact* if it is a colimit as a right cone. If C admits all finite (co)limits, we denote by

$$\mathsf{C}^{\Sigma} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathsf{C}) \quad \text{and} \quad \mathsf{C}^{\Omega} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathsf{C})$$

the full subcategories spanned by coexact and exact triangles *with the bottom-left and top-right corners zero*, respectively. To form a suspension ΣX, no data is needed beyond specifying X; diagramatically, a functor $F \in \mathsf{C}^{\Sigma}$ should be determined by its top-left corner. Dually for ΩX. Abstact nonsense says that indeed, $\mathrm{ev}_{(0,0)} : \mathsf{C}^{\Sigma} \to \mathsf{C}$ and $\mathrm{ev}_{(1,1)} : \mathsf{C}^{\Omega} \to \mathsf{C}$ are acyclic Kan fibrations.

Definition I.6. Suppose that C is pointed and admits finite (co)limits. Since $\mathrm{ev}_{0,0} : \mathsf{C}^{\Sigma} \to \mathsf{C}$ is acyclic, it admits a section $s_{\Sigma} : \mathsf{C} \to \mathsf{C}^{\Sigma}$, well-defined up to a contractible choice. We can thus define (up to a contractible choice) the *suspension functor* $\Sigma_{\mathsf{C}} = s_{\Sigma} \circ \mathrm{ev}_{(0,0)}$. We define the *loops functor* Ω_{C} identically.

The functors $\Sigma_{\mathsf{C}}, \Omega_{\mathsf{C}}$ are adjoint. Furthermore, if C is stable (i.e., exact triangles = coexact triangles), then they are inverse equivalences. The converse is also true.

We are interested in stabilizing categories. As stability is characterized by Ω_{C} being an equivalence, this means inverting Ω_{C}, i.e. by taking the colimit of $\cdots \to \mathsf{C} \to \mathsf{C}$. That is categorical. But there's an analogy here with algebraic topology, and we wonder about forming "spectrum objects" in C. In fact, we can do this, giving a more explicit model for the stabilization. We can apply this in the case of spaces—the categorical properties of the stabilization will give us the *symmetric monoidal smash product* of spectra, and the description as "spectrum objects" will make clear that we are talking *actual* spectra, the kind we care about from classical algebraic topology.

Let C be pointed and finitely (co)complete. A *prespectrum* in C is a functor $X : \mathbb{N} \times \mathbb{N} \to \mathsf{C}$ such that $X(i, j) = 0$ whenever $i \neq j$. A prespectrum X determines a sequence of triangles in C, and thus maps

$$\Sigma X_n \to X_{n+1} \quad \text{and} \quad X_n \to \Omega X_{n+1}.$$

We say that X is a *spectrum* if all the maps $X_m \to \Omega X_{m+1}$ are equivalences. We say that X is a *spectrum below n* if that is true for $m < n$. The following theorem says that (with mild hypotheses) spectrum objects model the stabilization.

Theorem I.4 (DAG I, 8.14)**.** For an arbitrary ∞-category C, we define its *stabilization* $\mathrm{Stab}(\mathsf{C}) := \mathsf{Sp}(\mathsf{C}_*)$. Note that if C is pointed, then $\mathsf{C}_* \to \mathsf{C}$ is a trivial Kan fibration, and so is the induced $\mathrm{Stab}(\mathsf{C}) \to \mathsf{Sp}(\mathsf{C})$. If C is pointed with finite limits, then

$$\mathsf{Sp}(\mathsf{C}) \cong \operatorname{colim}(\cdots \xrightarrow{\Omega} \mathsf{C} \xrightarrow{\Omega} \mathsf{C}).$$

Remark I.15. This presentation follows Moritz's notes and Lurie's DAG. However, Lurie's presentation in HA is different. There, he wants to faithfully reenact the story of *infinite loop spaces* (in particular, Brown representability and excision for cohomology) in the setting of ∞-categories. For C with finite limits, he defines $\mathsf{Sp}(\mathsf{C}) := \mathrm{Fun}(\S_*^{\mathrm{fin}}, \mathsf{C}) \subseteq \mathrm{Fun}(\dots)$ the full subcategory of reduced, excisive functors [Lur17, 1.4.2.8]. These approaches are equivalent. Given such a functor X, one may consider $X_n := X(S^n)$, and we get equivalences $X_n \xrightarrow{\sim} \Omega X_{n+1}$ by considering excision for the following pushout diagram.

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^{n+1} \end{tikzcd}$$

There are forgetful functors $\mathsf{Sp}(\mathsf{C}) \to \mathsf{Sp}_{\leq n}(\mathsf{C}) \to \mathsf{PSp}(\mathsf{C})$. We can ask about adjoints. Let's first ask about finding $\mathsf{PSp}(\mathsf{C}) \to \mathsf{Sp}_{\leq n}(\mathsf{C})$. Given a prespectrum X, one may imagine that "$L_n(X) :=$ the free Ω-spectrum below n on X" should be $X[k]$ in degrees $k \geq n$ and should be $\Omega^k X[n]$ in degrees $k < n$. Heuristically, $L_n(X)$ should be determined by the part of X in degrees $\geq n$, obtained by just chopping off lower degrees and refilling them by looping downward.

Lurie spells all this out in [Lur09, §8] (and these approximations $L_n X$ are needed to prove 8.14 above). The basic idea is as follows. For $-\infty \leq a \leq b \leq \infty$, we define

$$Q(a, b) := \{(i, j) : i \neq j \text{ or } a \leq i = j \leq b\} \subseteq \mathbb{N} \times \mathbb{N}.$$

We want to define $L_n(X) := \mathrm{Ran}_{Q(n,\infty) \hookrightarrow \mathbb{N}\times\mathbb{N}}(X|_{Q(n,\infty)})$. Since that inclusion $Q(n, \infty) \hookrightarrow \mathbb{N} \times \mathbb{N}$ is quite "finite," this is possible as soon as C has finite limits.

Proposition I.6 ([Lur09, 8.12])**.** C pointed with finite limits, $X_0 \in \mathsf{PSp}_a^\infty(\mathsf{C})$. Then

(1) There exists $X \in PSp_{a-1}^\infty(\mathsf{C})$ which is a right Kan extension of X_0.

(2) There exists $X \in PSp_{-\infty}^\infty(\mathsf{C})$ which is a right Kan extension of X_0.

(3) An object $X \in PSp^{\infty}_{-\infty}(\mathsf{C})$ is a right Kan extension of X_0 if and only if X is a spectrum below a.

Remark I.16. Lurie also states a characterization of $X \in PSp^{\infty}_{a-1}(\mathsf{C})$ right Kan-extending X_0, except there seems to be a typo that renders it unclear.

Hence, for C with finite limits, we have described a sequence of functors $\mathrm{id} \to L_0 \to L_1 \to \cdots$ such that

(1) $L_n X$ is a spectrum below n,

(2) For $m \geq n$, the map $X[m] \to L_n X[m]$ is an equivalence,

(3) If X is already a spectrum below n, then the map $X \to L_n X$ is an equivalence, and

(4) As a functor $\mathsf{PSp}(\mathsf{C}) \to \mathsf{PSp}_{\leq n}(\mathsf{C})$, each L_n is left-adjoint to the inclusion $\mathsf{PSp}_{\leq n}(\mathsf{C}) \hookrightarrow \mathsf{PSp}(\mathsf{C})$.

Properties (1) and (2) are immediate if you unwind everything. Property (3) follows from (1) and (2). Property (4) is not hard either. Next, toward an adjoint $\mathsf{PSp}(\mathsf{C}) \to \mathsf{Sp}(n)$, it is natural to ask about the colimit of this tower of approximations L_n. This works under some mild hypotheses.

Proposition I.7. If C is pointed and admits finite limits and countable colimits, and $\Omega_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}$ preserves sequential colimits, then $L := \operatorname{colim} L_n : \mathsf{PSp}(\mathsf{C}) \to \mathsf{PSp}(\mathsf{C})$ is a localization with essential image $\mathsf{Sp}(\mathsf{C})$. Under these conditions, we call L a *spectrafication functor*.

Finish this

I.5 (1/29) Ok, what's a localization?

I secretly never learned anything, ever. This includes most of algebraic topology—somehow, it was my third(?) undergrad course, so I spent that semester learning mathematical maturity, not actual algebraic topology. Rather than patiently fill the holes in my knowledge, I moved forward to homotopy theory, at which point I was mature enough to actually learn homotopy theory, although that was (and still is) complicated since my algebraic topology is lacking. The water settles and some holes get filled idly, but maybe not all of them. This is most apparent when I try to do chromatic things. All that is to say, today I want to review localizations, with an eye toward localizations at spectra (homology theories), and I am going to start from basics. Some references are:

- Tyler Lawson's expository article about Bousfield localization [Law20].
- nLab.
- Ishan's notes.
- Paul VanKoughnett's thesis here.

Fix a category C and a class $S \subseteq \mathrm{Mor}(\mathsf{C})$ of morphisms. We say that an object Z is *S-local* if for each $s \in S$, the pullback $s^* : \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is an equivalence. We say that a morphism $f : X \to Y$ is an *S-equivalence* if for each S-local Z, the pullback $f^* : \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is an equivalence. (Hence, each $s \in S$ is an S-equivalence.) We call a morphism $L : X \to Z$ an *S-localization* if it is an S-equivalence and Z is S-local.

Proposition I.8. If two S-localizations of an object X exist, then they are isomorphic under X.

Proof. Given two S-localizations $Z \xleftarrow{L} X \xrightarrow{K} Z'$. Since Z, Z' are S-local, their represented functors invert S-equivalences. In particular, we have bijections $K^* : \mathrm{Hom}(Z', Z) \cong \mathrm{Hom}(X, Z)$ and $L^* : \mathrm{Hom}(Z, Z') \cong \mathrm{Hom}(X, Z')$. So we may consider $(L^*)^{-1}K : Z \leftrightarrow Z' : (K^*)^{-1}L$. Pullback is just precomposition, whence the relevant triangles commute, so that these are *morphisms under* X. Furthermore, uniqueness implies that these two morphisms are inverse equivalences. □

Hence we speak of *the* S-localization of X, written $L^S X$ or just LX (the issue of existence notwithstanding). Assuming existence, the localization morphisms $X \to LX$ are functorial. Here's that formally:

Proposition I.9. Consider $\mathrm{Loc}^S(\mathsf{C})$ the category of S-localizing morphisms and commutative squares between them. The forgetful functor $\mathrm{Loc}^S(\mathsf{C}) \to \mathsf{C}$ given by $(X \to LX) \mapsto X$ is fully faithful.

In particular, if C has S-localizations, then $\mathrm{Loc}^S(\mathsf{C}) \to \mathsf{C}$ is an equivalence. In that case we may choose an inverse $\mathsf{C} \to \mathrm{Loc}^S(X)$ written as $X \mapsto (X \to LX)$, and forgetting gives us a *functorial localization functor*

$$\mathsf{C} \to L^S\mathsf{C}.$$

Proposition I.10. If C has S-localizations, then we have described a functorial choice of localization $X \mapsto LX$ which is left adjoint to the forgetful functor $LX \mapsto X$.

Some examples.

Example I.5. Consider Mon and the inclusion $f : \mathbb{N} \hookrightarrow \mathbb{Z}$. For a monoid M to be f-local, every map $\mathbb{N} \to M$ must extend (uniquely) to a map $\mathbb{Z} \to M$. In other words, each element of M must admit a (unique) inverse. (I parenthesize uniqueness of inverses here because that property is superfluous in monoids.) Hence,

$$L^{\mathbb{N} \hookrightarrow \mathbb{Z}}\mathsf{Mon} = \mathsf{Gp}.$$

Furthermore, group completion $M \mapsto M^{gc}$ presents the localization functor.

Example I.6. Consider Gp and the abelianization map $f : F_2 \to \mathbb{Z}^2$ given by $(1, 1) \mapsto (1, 1)$. A group G is f-local if every $F_2 \to G$ factors (uniquely) through $\mathbb{Z}^2$. This means that for each $x, y \in G$, the commutator $[x, y]$ vanishes, which occurs precisely when G is abelian. Therefore,

$$L^{F_2 \to \mathbb{Z}^2}\mathsf{Gp} = \mathsf{Ab},$$

and abelianization $G \mapsto G^{ab} = G/[G, G]$ presents the localization functor.

Example I.7. I think that $L^{\mathbb{N}^2 \hookrightarrow \mathbb{Z}^2}\mathsf{Mon} = \mathsf{CMon}$.

Example I.8. Consider $\mathsf{Ab}^{\mathrm{fg}} \subset \mathsf{Ab}$ and let $S = \{\mathbb{Z} \xrightarrow{p} \mathbb{Z} : p \text{ is prime}\}$. An abelian group G is S-local if every p is invertible in G, i.e., when G is a rational vector space. (Note that this description is valid in both Ab and $\mathsf{Ab}^{\mathrm{fg}}$ because $\mathsf{Ab}^{\mathrm{fg}}$ is a full subcategory.) Then within $\mathsf{Ab}^{\mathrm{fg}}$, the only S-local object is $G = 0$, whence S-localization is the zero map and all homomorphisms are S-equivalences. However, the S-localizations in Ab are given by rationalization.

Finish. At least get to Lawson's stuff on unstable and stable settings.

II February

II.1 (2/2) Phony multiplication

Some references (in order of discovery as I wrote) are

- Yigal Kamel's talk for the homotopy theory seminar at UIUC. Good talk!
- Thomason's article.
- Segal's paper "Categories and cohomology theories."
- Dmitri Pavlov's MO question about a higher $S^{-1}S$ construction.
- Gurski-Johnson-Osorno's paper "2-CATEGORICAL OPFIBRATIONS, QUILLEN'S THEOREM B, AND $S^{-1}S$."
- Dan Grayson's expository article "Quillen's work on algebraic K-theory."
- Zbigniew Fiedorowicz's proceedings article "The Quillen-Grothendieck Construction and Extensions of Pairings" c.f. [Fie78].
- Clayton Sherman's chapter "Group representations and algebraic K-theory," c.f. [She82], which has some random details worked out I found helpful.
- Daniel Harrer's thesis "Comparison of the Categories of Motives defined by Voevodsky and Nori" available here. This is mostly unrelated, but Harrer mentions that somewhere in motivic cohomology, one needs the swap map in a tensor category to be an equality, otherwise you get a problem analogous to what Thomason finds for algebraic K-theory. Nice
- Baas-Dundas-Rognes "Two-vector bundles and forms of elliptic cohomology" c.f. [BDR03].

Given a monoid M, we may "formally add inverses" to obtain its *group completion* M^{gp}. If M is commutative, then M^{gp} has a universal property: there exists a monoidal map $i : M \to M^{\mathrm{gp}}$ such that for every abelian group A, monoidal maps $M \to A$ extend surjectively and faithfully to homomorphisms $M^{\mathrm{gp}} \to A$. By virtue of universality, this map i is determined up to isomorphism(?) Alternatively, you could characterize $(-)^{\mathrm{gp}}$ as left adjoint to $U : \mathsf{Ab} \to \mathsf{CMon}$, and then a choice of maps $\{M \to M^{\mathrm{gp}}\}_M$ is a presentation of the left adjoint(?)

That's categorical. But we can do this with our hands. Two ways! Let's just take $M = \mathbb{N}$.

Construction II.1. We define $\mathbb{N}^{\mathrm{gp}}$ as the set of symbols $\{m-n : m, n \in \mathbb{N}\} = \mathbb{N}\times\mathbb{N}$ modulo the equivalence relation *generated by*[9] identifying $a - b \sim c - d$ when there exists $k \in \mathbb{N}$ such that $a + k = c$ and $b + k = d$. You can check that $\mathbb{N}^{\mathrm{gp}} \cong \mathbb{Z}$ and that both the maps $\mathbb{N} \to \mathbb{N}^{\mathrm{gp}}$ given by $m \mapsto m - 0$ and $m \mapsto 0 - m$ are group completions.

Construction II.2. We define $\mathbb{N}^{\mathrm{gp}}$ as $\mathrm{FAb}(\mathbb{N})/\langle (m + n) -_F (m +_F n)\rangle$, where by $\mathrm{FAb}(\mathbb{N})$ we mean the free abelian group on the set $\mathbb{N}$, and by $+_F, -_F$ we mean the group operations in $\mathrm{FAb}(\mathbb{N})$.

Both these constructions readily extend to an arbitrary commutative monoid M. Note that Construction II.2 works for noncommutative M. And there's more: if M is a semiring, to mean a monoid with a multiplication, then $(M, +, \times)^{\mathrm{gp}}$ has a canonical and well-defined multiplication $\times_{M^{\mathrm{gp}}}$ given as

$$((a - b), (c - d)) \mapsto (ac + bd, ad + bc).$$

We have described a functor $K_0 : \mathsf{CMon} \to \mathsf{Ab}$. (The second construction works for Mon.) With an eye toward (higher) K-theory, we would like to consider more general input data. A generation of mathematical work (in geometry, topology, number theory, ...) evidences that *many things have K-theory*: (homotopy commutative) rings, spaces, schemes, (various adjectives) categories, ... Moreover, we want highly structured output data, perhaps a *sequence* of abelian groups, or better yet a (nice) K-theory spectrum. Often, such things arise by constructing a (category, or higher category/space) C encoding the important

[9] "Generated by" here is necessary since $\sim$ as defined is not reflexive. There are very easy fixes to this, see e.g. the equivalent definitions of the relation on nLab, but this equivalent definition generalizes verbatim.

information, in such a way that the data unwinds into an infinite loop space structure on the geometric realization $BG = |NC|$.

Let M be a monoid. Following Grayson, we can easily prototype topological models for M^{gp}. For this, we will use simplicial sets, for with simplicial sets we can simultaneously encode the algebra in M and realize it spatially.

Construction II.3. Suppose that M is commutative. We may form the simplicial set $X_0(M)$ wherein vertices are formal differences $m - n$, edges connect $(m+k) - (n+k) \to m - n$ for every $k \in M$, faces fill every composable pair of the form shown below, and so on.

$$\begin{tikzcd} m-n \arrow[rr, "k+k'"] \arrow[dr, "k"'] & & m''-n'' \\ & m'-n' \arrow[ur, "k'"'] & \end{tikzcd}$$

Construction II.4. Do not assume that M is commutative. We may form the simplicial set $X_1(M)$ wherein there is one vertex, an edge for each $m \in M$, a face with edge m, m', and $m + m'$, and so on.

The constructions of $X_0(M)$ and $X_1(M)$ are analogous to Construction II.1 and Construction II.2, respectively. You can check that $\pi_0 X_0(M) \cong M^{\text{gp}}$ and $\pi_1 X_1(M) \cong M^{\text{gp}}$. If M is already a group (resp. abelian group), then $X_1(M)$ (resp. $X_0(M)$) has trivial higher homotopy and models the classifying space BM (resp. ΩBM?). So, we have described two constructions X_0, X_1 which (topologically) model group completion in the sense that $\pi_i X_i(M) \cong M^{\text{gp}}$. The X_1 construction works for noncommutative M, and if M is already group, then $X_0(M) = \Omega BM$ and $X_1(M) = BM$ (in some sense...).

Remark II.1. Group completion also changes homology, see e.g. MO. This persists for categories!

We would like to replace M by a symmetric monoidal category and ask about group completions. We produced $X_0(M)$ by Construction II.1, and this straightforwardly generalizes.

Definition II.1. Suppose that $(\mathsf{C}, \oplus, 0)$ is a symmetric monoidal category. We define its *Quillen completion* $S^{-1}S(\mathsf{C})$ to be the category whose...

- Objects are pairs (A, B).
- Morphisms $(A, B) \to (C, D)$ are triples $(K, A \oplus K \to C, B \oplus K \to D)$, modulo the equivalence relation identifying morphisms when there exists $K \cong K'$ making the obvious four triangles commute.

Remark II.2 (Structure of $S^{-1}S(\mathsf{C})$)**.** Here is some basic structure on the Quillen completion.

- The identity morphism $\text{id}_{(A,B)}$ is $(0, 0 \to A, 0 \to B)$.
- The monoidal structure on C induces one on $S^{-1}S(\mathsf{C})$. It is computed "coordinate-wise," i.e. $(A, B) \oplus (C, D) = (A \oplus C, B \oplus D)$ and likewise for sums of morphisms. If $s_{A \oplus B} : A \oplus B \to B \oplus A$ denotes the swap maps in C, then $(0, s_{AC}, s_{BD}) : (A, B) \oplus (C, D) \to (C, D) \oplus (A, B)$ provide swap maps for $S^{-1}S(\mathsf{C})$.
- There is a *transposition* functor $t : S^{-1}S(\mathsf{C}) \to S^{-1}S(\mathsf{C})$ given by

$$(A, B) \mapsto (B, A) \quad \text{and} \quad (K, \alpha, \beta) \mapsto (K, \beta, \alpha).$$

- There are inclusion functors $i, -i : \mathsf{C} \to S^{-1}S(\mathsf{C})$ given on objects by $i(C) = (0, C)$ and $-i(C) = (C, 0)$. We want to think of the object $(A, B) \in S^{-1}S(\mathsf{C})$ as a formal difference $B - A$,[10] whence this notation makes sense.

[10]This signage is decided by the direction of edges in $S^{-1}S(\mathsf{C})$.

We would like to understand $S^{-1}S(\mathsf{C})$ as a "categorical group completion." For this, consider the following three properties that $(\mathsf{C}, \oplus, 0)$ might possess.

(I) C is a groupoid.

(II) C is "cancellative:" For every $A \in \mathsf{C}$, the functor $A \oplus - : \mathsf{C} \to \mathsf{C}$ is faithful.

(III) C has "object-level inverses:" there exists a natural transformation $0 \to \mathrm{id} \oplus t$.

Proposition II.1. If C satisfies (I) and (II) above, then the inclusion $i : \mathsf{C} \to S^{-1}S\mathsf{C}$ given by $A \mapsto (0, A)$ induces a group completion on classifying spaces. That is to say

$$\pi_0(BS^{-1}S(\mathsf{C})) = \pi_0(B\mathsf{C})^{\mathrm{gp}} \quad \text{and} \quad H_0(BS^{-1}S(\mathsf{C})) = H_0(B\mathsf{C})[\pi_0 B\mathsf{C}^{-1}].$$

Suppose that (I) and (II) hold. Since $S^{-1}S\mathsf{C}$ is symmetric monoidal, its classifying space is an H-space, and by the proposition it is an H-group.[11] We regard $(A, B) \in S^{-1}S(\mathsf{C})$ as the formal difference $B - A$, and (A, B) represents this difference in $\pi_0 BS^{-1}S(\mathsf{C})$. Also on π_0, the transposition functor t induces the inverse map. The symmetric monoidal structure gives $BS^{-1}S\mathsf{C}$ its monoidal structure, and we would like to say that t induces its homotopy inverse $g \mapsto g^{-1}$.

Proposition II.2. If C satisfies (I), (II), and (III) above, then the transposition t for the Quillen completion $S^{-1}S(\mathsf{C})$ induces a homotopy inverse for the H-group $BS^{-1}S(\mathsf{C})$.

Proof. Write $Z = BS^{-1}S(\mathsf{C})$. By (III), we may choose a transformation $\eta : 0 \to \mathrm{id} \oplus t$. We get a map $B\eta : B0 \to B(\mathrm{id} \oplus t)$, and this represents a homotopy inducing (in $[Z, Z]$) $0 = [B0] = [B\mathrm{id}] + [Bt] \in [Z, Z]$, which begets $[B\mathrm{id}] = -[Bt]$. Hence, Bt is a homotopy inverse for the H-space Z. □

Thomason's essential observation is that (III) is secretly a very strong condition, unfulfilled in even the most standard cases. For example, neither $\mathsf{C} = \mathsf{Mod}_R^{\mathrm{fg},proj}$ nor its maximal subgroupoid have this property unless $R = 0$. Fortunately, Proposition II.2 still holds when you do not assume (III)! But results implementing the homotopy inverse for $BS^{-1}S(\mathsf{C})$ via *t and utilizing the naturality of η in an essential way* still fail (since η does not even exist).

The functor $t \oplus \mathrm{id}$ acts on objects as $(A, B) \mapsto (B \oplus A, A \oplus B)$. Hence, a transformation $\eta : 0 \to t \oplus \mathrm{id}$ amounts to a natural system of morphisms $\eta_{AB} = \{K, 0 \oplus K \to B \oplus A, 0 \oplus K \to A \oplus B\}$. One candidate is

$$\eta_{AB} := \{A \oplus B, s_{A \oplus B}, \mathrm{id}_{A \oplus B}\}.$$

Proposition II.3. The system of morphisms $\{\eta_{AB} : (0,0) \to [t \oplus \mathrm{id}](A, B)\}_{(A,B) \in \mathsf{C}}$ is natural in C if and only if the swap isomorphisms $s_{S \oplus S}$ are *equalities* for all $S \in \mathsf{C}$.

Proof. Choose arbitrary $A, B, C, D \in S^{-1}S(\mathsf{C})$. If we denote by f an arbitrary morphism $(S, \alpha, \beta) : (A, B) \to (C, D)$, then $(t \oplus \mathrm{id})f$ is the morphism $(B \oplus A, A \oplus B) \to (D \oplus C, C \oplus D)$ consisting of the following two arrows.

$$B \oplus A \oplus S \oplus S \xrightarrow{\mathrm{id}_B \oplus s_{AS} \oplus \mathrm{id}_S} B \oplus S \oplus A \oplus S \xrightarrow{\beta \oplus \alpha} D \oplus C$$

$$A \oplus B \oplus S \oplus S \xrightarrow{\mathrm{id}_A \oplus s_{BS} \oplus \mathrm{id}_S} A \oplus S \oplus B \oplus S \xrightarrow{\alpha \oplus \beta} C \oplus D$$

Naturality amounts to the commutativity of the following diagram, for all A, B, C, D, and f.

$$\begin{tikzcd}
(0,0) \arrow[r, "\eta_{AB}"] \arrow[d, "0(f)=0"'] & (B \oplus A, A \oplus B) \arrow[d, "(t \oplus \mathrm{id})f"] \\
(0,0) \arrow[r, "\eta_{CD}"] & (D \oplus C, C \oplus D)
\end{tikzcd}$$

[11] If C is an exact category, you could also argue that the H-space is an H-group by proving that the map $K_0() \to \pi_0(BS^{-1}S)$ is an isomorphism and that the former is a group. I read this in **??**.

Writing out η_{CD} and the composite $(t \oplus \mathrm{id}) f \circ \eta_{AB}$, Thomason finds commutativity to be equivalent to that of

$$\begin{tikzcd} BASS \arrow[r, "s_{AS}"] & BSAS \\ ABSS \arrow[u, "s_{AB}"] \arrow[r, "s_{BS}"] & ASBS \arrow[u, "s_{(AS)(BS)}"'] \end{tikzcd}$$

Note that this diagram is in C. Observe that up-right does not reverse the order of the two S's. However, right-up does (because $s_{(AS)(BS)}$ does). Hence, this diagram commutes if and only if the swap map $s_{S \oplus S}$ is a *strict* equality. As S was arbitrary, the result follows. $\square$

The swap isomorphisms are rarely equalities, and although it is possibly to "strictify" a symmetric monoidal category into a permutative one (in which associativity and unitality are strict), commutativity cannot be strictified except in trivial cases. Hence, the result outs (III) a tricky condition to satisfy, for the obvious natural transformation is *not* a natural transformation! A tragedy.

Let me quote some of Peter's commentary [May80]:

> Thomason [...] has given an amusing illustration of the sort of mistake that can arise from a too cavalier attitude towards this kind of categorical distinction when studying pairings of categories, and one of my concerns is to correct a similar mistake of my own.
>
> In [...], I developed a coherence theory of higher homotopies for ring spaces up to homotopy and for pairings of H-spaces. That theory is entirely correct. I also discussed the analogous categorical coherence, proving some results and asserting others. That theory too is entirely correct, my unproven assertions having been carefully proven by Laplaza [unpublished]. However, my translations from the categorical to the homotopical theories in [...], that of course being the part I thought to be obvious, are quite wrong.
>
> The moral is that to treat the transition from categorical coherence to homotopical coherence smoothly and rigorously, one should take advantage of the definitional framework established by the category theorists.

There's more to say. But I have been writing some time now, and have generated me thoughts I need to lay bare before the crystal ball (ask Charles about). Let me just say something about the first great victim of Thomason's observation.

Recall that if we put a semiring structure on a monoid M (i.e., a unital operation $\times$ distributing over $+$), for example that already present on the underlying monoid of any unital associative ring, then this straightforwardly extends to a ring structure on the group $K_0(M) = M^{\mathrm{gp}}$. In fact, the full K-theory $K_*(A)$ (of rings, schemes, whatever) has a ring structure. Back in the day, Quillen invented higher algebraic K-theory

Finish this. Talk about ring structure on $K(R)$, relation of Quillen completion to Q-construction, Fiederowicz unwell-defined avatar for the inverse.

II.2 (2/7)

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