Lecture notes

June 23, 2023

Contents

0.1 (Review) Lecture 1: Some category theory

Several people said they wanted to learn basic category theory. We already needed to review some category theory for this seminar, so we'll take the opportunity to review enough theory to hopefully help everyone.

The short-term plan is to review some category theory, Grothendieck sites, and sheaves. Then we will start actually talking about condensed things. I only reluctantly call this "review," since we are *not* assuming everyone is familiar with all the material moving forward. The goal is only to catch folks up to a working knowledge.

Probably everything we say here is said better somewhere in Emily Riehl's *Category Theory in Context*. That book also has many examples. You should read it.

Definition 0.1.1. A *category* C consists of the following data.

(1) A collection $Ob(\mathsf{C})$ we call *objects*.

(2) A collection $Mor(\mathsf{C})$ we call *morphisms*.

(3) For each morphism f, a *source* and *target* object. (We write $f : X \to Y$ to express that f is a morphism with source X and target Y.)

(4) For each object X, a distinguished morphism $\mathrm{id}_X : X \to X$ we call the *identity morphism*.

(5) For each pair of morphisms $f : X \to Y, g : Y \to Z$ such that $target(f) = source(g)$, a distinguished morphism $gf : X \to Z$ we call the *composite morphism*.

And this data must satisfy the following properties.

- Given a morphism $f : X \to Y$, we have $f = \mathrm{id}_Y f = f\mathrm{id}_G$.
- Given morphisms f, g, h, we have $(fg)h = f(gh)$ (when the source/targets match appropriately).

In practice, we think of categories as "like a collection of objects and maps between them, with all the structure that should accompany the word *maps*—identity self-maps, composites, associativity."

0.1.1 Examples of categories

(See Riehl for more)

Example 0.1.2. The category of sets Set has sets as objects and functions as morphisms. The rest of the structure is "obvious": source/target is domain/range, the composition of morphisms is defined as the composition of functions, and identity morphisms are the identity functions.

Example 0.1.3. The category of spaces Top has topological spaces as objects and continuous maps as morphisms. Again, the rest of the structure is "obvious."

Example 0.1.4. Define the category Top_* to have *based spaces*[1] as objects and *based maps*[2] as morphisms.

Example 0.1.5. Define the category Grp to have groups as objects and homomorphisms as morphisms.

Example 0.1.6. Define the category Ab to have abelian groups as objects and homomorphisms as morphisms.

Example 0.1.7. Denote by k a field (e.g., $k = \mathbb{R}$). Define the category $Vect_k$ to have k-vector fields as objects and k-linear maps as morphisms.

Example 0.1.8 (Morphisms do not have to be functions!)**.** Define a category $\mathsf{Naturals}$ to have

- As objects, the natural numbers $Ob(\mathsf{Naturals}) := \{0, 1, 2, \dots\}$; and

[1]A based space is pair (X, x) where X is a space and $x \in X$.
[2]A based map $f : (X, x) \to (Y, y)$ is a continuous map $f : X \to Y$ such that $f(x) = y$.

- A morphism $a \to b$ for each pair of numbers (a, b) such that $a \le b$.

Thus, given objects $a, b \in \{0, 1, 2, \dots\}$, there is at most one arrow $a \to b$, and it exists iff $a \le b$.

Example 0.1.9 (Morphisms do not have to be functions!)**.** Define a category $\mathsf{Skel}(\mathsf{FinSet}_{\mathsf{iso}})$ to have

- Natural numbers n as objects; and
- A morphism $n \to n$ for each element of the symmetric group Σ_n, and no morphisms $m \to n$ when $m \neq n$.

Example 0.1.10 (Example from Peter's talk)**.** A *poset* is a set S with a relation $\le$ that is reflexive, transitive, and antisymmetric. A *morphism of posets* $f : (S, \le) \to (S', \le')$ is a function $f : S \to S'$ that respects the partial orderings, i.e. $x \le y \iff f(x) \le' f(y)$. We denote by $Poset$ the category of posets and morphisms of posets.

0.1.2 Isomorphisms

All sorts of objects—groups, rings, sets, spaces—have a notion of "sameness." In an arbitrary category, we formalize this notion as *isomorphisms*.

Definition 0.1.11. Let C be a category. Suppose that $f : c \to c'$ is a morphism and that there exists a morphism $g : c' \to c$ such that $fg = \mathrm{id}_{c'}$ and $gf = \mathrm{id}_c$. Then we say f and g are *isomorphisms* and we say that c, c' are *isomorphic*.

Example 0.1.12. Isomorphisms in Set are bijections. Isomorphisms in Top are homeomorphisms. Isomorphisms in Top_* are based homeomorphisms. Isomorphisms in Grp and Ab are group isomorphisms.

Exercise: Figure out what the isomorphisms in $Poset$ are.

0.1.3 Functors

Definition 0.1.13. Let C, D be categories. A *functor from* C *to* D (which we write $F : \mathsf{C} \to \mathsf{D}$) is "a map of objects and morphisms that preserves categorical structure, i.e. sources, targets, composites, and identities." Formally, it consists of the following data.

(1) An object $FX \in \mathsf{D}$ for each object $X \in \mathsf{C}$.

(2) For each morphism $f : X \to Y$ in C, a morphism $Ff : FX \to FY$ in D.

And this data must satisfy the following properties.

- For any pair of composable morphisms f, g in C, we have $F(gf) = F(g)F(f)$.
- For any identity morphism id_X in C, we have $F(\mathrm{id}_X) = \mathrm{id}_{FX}$.

Very often, we say "(one type of object) are the same thing as (another type of objects)." Categories give us a great, concrete way to talk about "types of objects." Functors give us a way to "modify and compare" objects of different types. Can functors tell us when (one type of object) are "the same thing as" (another type)? Yes, and this is a very useful notion.

For the following definition, we will denote by $Mor(X, Y)$ the set of morphisms $X \to Y$ between objects $X, Y \in \mathsf{C}$.

Definition 0.1.14. Let C, D be categories. An *equivalence of categories* is a functor $F : \mathsf{C} \to \mathsf{D}$ that is

1. *Full*: for every pair of objects $X, Y \in \mathsf{C}$, the mapping $f \mapsto F(f)$ defines a surjection $Mor(X, Y) \to Mor(FX, FY)$;
2. *Faithful*: for every pair of objects $X, Y \in \mathsf{C}$, the mapping $f \mapsto F(f)$ defines an injection $Mor(X, Y) \to Mor(FX, FY)$; and

3. *Essentially surjective*: for every object $d \in \mathsf{D}$, there exists some $c \in \mathsf{C}$ such that $Fc \cong d$.

Remark 0.1.15. The analogy is, “full and faithful is like injectivity" and “essentially surjective is like surjectivity." If you have both, you have an isomorphism. Note that a full and faithful functor need not actually be surjective *on objects*.

Remark 0.1.16. There are other common, equivalent definitions of an equivalence of categories.

0.1.4 Examples of functors

(see also Riehl, p. 13)

Example 0.1.17. Let C denote one of the categories $\mathsf{Top}, Grp, \mathsf{Ab}$, or $Vect_k$. We can define a functor $U : \mathsf{C} \to \mathsf{Set}$ by mapping objects to their underlying sets and morphisms to their underlying set-maps. We call U the *forgetful functor*. (We generally refer to any functor that “tosses out" structure, e.g. a topology on a set, as a *forgetful functor*.)

Example 0.1.18. If C is any category, then we can form its *opposite category* C^{op} to have the same objects but with “flipped arrows," i.e. swapped source/targets of C's morphisms. There is a functor $\mathsf{C} \to \mathsf{C}^{\mathrm{op}}$ that takes objects to themselves and morphisms to their “flip."

Exercise: prove that $(\mathsf{C}^{\mathrm{op}})^{\mathrm{op}}$ is equivalent to C.

Example 0.1.19. For $V \in Vect_k$, recall that its *dual* is defined as the vector space $V^* := \{\text{linear maps } V \to k\}$. Given a linear map $f : V \to W$, there is induced a map $f^* : W^* \to V^*$ that sends $v : W \to k$ to $v \circ f : V \to k$. The mapping $V \mapsto V^*, f \mapsto f^*$ defines the *dualization functor* $(-)^* : Vect_k \to Vect_k^{\mathrm{op}}$.

Remark 0.1.20. You have heard probably heard that we have an isomorphism $V \cong V^{**}$ that is “canonical" or “natural" or “very nice," but that we do not have such an isomorphism $V \cong V^*$. (Although the two are isomorphic.) This can be expressed very concretely as a statement about the functor $(-)^*$ and its self-composite $(-)^{**}$. We do not yet have the language for this (natural transformations); the non-categorical reason is that an isomorphism $V \cong V^*$ requires a choice of basis, but there is an isomorphism $V \cong V^{**}$ that does not need any choice.

Many—and historically, the motivating—examples of functors come from algebraic topology.

Example 0.1.21. Let (X, x) be a based space (i.e., X is a space and $x \in X$). We define the *fundamental group* $\pi_1(X, x)$ as the set of based continuous maps $\ell : [0, 1] \to X$ such that $\ell(0) = \ell(1) = x$, modulo homotopy equivalence. The group structure is loop concatenation: given $\ell, \ell' : [0, 1] \to X$, define $\ell'\ell : [0, 1] \to X$ to do one loop over $[0.5]$ then the other over $[0.5, 1]$. Given a based map $f : (X, x) \to (Y, y)$, there is induced a map $f_* : \pi_1 X \to \pi_1 Y$ given by $\ell \mapsto f \circ \ell$. This defines a functor

$$\pi_1(-) : \mathsf{Top}_* \to \mathsf{Grp}.$$

Example 0.1.22. For each n, singular homology defines a functor $H_n(-) : \mathsf{Top} \to \mathsf{Ab}$. Similarly, singular cohomology defines a functor $H^n(-) : \mathsf{Top}^{\mathrm{op}} \to \mathsf{Ab}$.

0.1.5 Natural transformations

We often want to compare *functors*. This will help us explain why e.g. “an arbitrary vector space is not *naturally* isomorphic to its dual" but “an arbitrary vector space *is* naturally isomorphic to its double-dual."

There are more serious examples where we *really* care about comparisons between functors. For example, the *Hurewicz homomorphism* from algebraic topology is a comparison $h_X : \pi_n(X) \to H_n(X)$ for every space X. But in fact, more can be said—for any continuous based map $f : X \to Y$, the Hurewicz homomorphism satisfies $h_Y \circ \pi_n(f) = H_n(f) \circ h_X$. (Here, $\pi_n(f)$ and $H_n(f)$ are the maps induced by f on pi_n and H_n.) This is a seriously useful fact that is not “formally guaranteed" to be true. One might phrase this as, “the Hurewicz homomorphism compares objects $\pi_n(X) \to H_n(X)$ in a way that respects how maps induce homomorphisms via the functors $\pi_n(-), H_n(-)$."

Natural transformations give a simple way to express this.

Definition 0.1.23. Let $F, G : \mathsf{C} \to \mathsf{D}$ be two functors. A *natural transformation from F to G*, which we denote as $\alpha : F \implies G$, is the data of

- For each object $c \in \mathsf{C}$, a morphism $\alpha_c : F(c) \to G(c)$ in D

such that for every morphism $f : c \to c'$ in C, one has $G(f) \circ \alpha_c = \alpha_{c'} \circ F(f)$. In other words, the following diagram commutes.

$$\begin{tikzcd} F(c) \arrow[r, "F(f)"] \arrow[d, "\alpha_c"] & F(c') \arrow[d, "\alpha'_c"] \\ G(c) \arrow[r, "G(f)"] & G(c') \end{tikzcd}$$

It turns out, natural transformations can be composed!

Definition 0.1.24. Let $F, G, H : \mathsf{C} \to \mathsf{D}$ be three functors, and suppose we are given two natural transformations $\eta : F \Rightarrow G$ and $\varepsilon : G \Rightarrow H$. Define the *composition* $\varepsilon \circ \eta$ whose component at some $c \in \mathsf{C}$ is given by $(\varepsilon \circ \eta)_c := \varepsilon_c \circ \eta_c$.

Exercise 0.1.25. Show that given natural transformations $\eta : F \Rightarrow G$ and $\varepsilon : G \to H$, that the composition $\varepsilon \circ \eta$ as defined above is actually a natural transformation. In other words, verify that the naturality condition is satisfied.

Exercise 0.1.26. Show that given two categories C and D, you can define the **functor category** $\mathrm{Fun}(\mathsf{C}, \mathsf{D})$ (also sometimes denoted as $[\mathsf{C}, \mathsf{D}]$ or D^{C}) whose objects are functors $\mathsf{C} \to \mathsf{D}$ and whose morphisms are natural transformations $\eta : F \Rightarrow G$.

You will need to check that:

- Given any functor F in $\mathrm{Fun}(\mathsf{C}, \mathsf{D})$, there exists an **identity natural transformation** $\mathrm{Id}_F : F \Rightarrow F$.
- Composition of natural transformations is both associative and unital with respect to your identity transformations.

0.2 (Review) Lecture 2: More category theory

The first part of the talk on limits and colimits (sections 0.2.1 – 0.2.3 below) will be given by Isaiah (who is also writing the notes for these sections). If you have any questions about anything below, please do not hesitate to reach out to me on discord, my username there is isaiahtx.

0.2.1 Reminder about natural transformations

First, I will review the definition of a natural transformation. If I have time, at some point during the presentation I will introduce the idea of a *category associated to a preorder*:

Definition 0.2.1. A **preorder** is pair $(P, \leq)$ where P is a set and $\leq$ is a reflexive and transitive relation on P, i.e., given $x \in P$, $x \leq x$, and if $x, y, z \in P$ satisfy $x \leq y$ and $y \leq z$, then $x \leq z$.

Definition 0.2.2. A **preorder** is a category such that there is at most one morphism between any two objects.

Exercise 0.2.3. Understand that the above two definitions are equivalent. In other words, show that given a preorder defined as in Definition 0.2.1 gives rise to a preorder as defined in Definition 0.2.2, and vice-versa.

0.2.2 Limits and colimits

Definition 0.2.4. Let c be an object in a category C.

We say c is *initial* if, given any object c' in C, there is a unique morphism $c \to c'$.

Conversely, we say c is *terminal* if, given an object c' in C, there is a unique morphism $c' \to c$.

Definition 0.2.5. Recall: an arrow $f : x \to y$ in a category C is called an **isomorphism** if there exists an arrow $g : y \to x$ such that $f \circ g = \mathrm{id}_y$ and $g \circ f = \mathrm{id}_x$. We say two objects are **isomorphic** if there exists an isomorphism between them.

Exercise 0.2.6. Show that in a category C, given two initial objects c and c', there is a unique isomorphism $c \to c'$. Similarly, terminal objects in C are unique up to unique isomorphism. Thus, it makes sense to talk about *the* initial/terminal object in a category.

Hint: If c and c' are initial objects, there is a unique arrow $c \to c'$ and a unique arrow $c' \to c$ (why?). What can you say the compositions of these arrows?

Example 0.2.7. In the category Set, the initial object is the empty set and the terminal object is the singleton set.

Example 0.2.8. In the categories Grp and Ab, the trivial group is both initial and terminal.

Example 0.2.9. Given a preorder P, the terminal object, if it exists, is called the *top object*. The initial object is called the *bottom object*.

The top object is greater than or equal to every other object in the preorder. The bottom object is less than or equal to every other object in the preorder.

Given categories J and C, we often call a functor $F : \mathsf{J} \to \mathsf{C}$ a *diagram of shape* J *in* C.

Definition 0.2.10. Given two categories J and C and an object c in C, let $\underline{c} : \mathsf{J} \to \mathsf{C}$ denote the *constant functor on* c which sends every object in J to c, and every morphism in J to the identity morphism id_c on c.

Definition 0.2.11. Let J be a small category, and $F : \mathsf{J} \to \mathsf{C}$ be a functor.

A *cone under* F is a pair (λ, c), where c is an object in C and λ is a natural transformation $\lambda : F \Rightarrow \underline{c}$. We call c the **nadir** of the cone.

A *cone over* F is a pair (c, λ), where c is an object in C and λ is a natural transformation $\lambda : \underline{c} \Rightarrow F$. We call c the **summit** or **apex** of the cone.

Explicitly, the data of a cone λ under $F : \mathsf{J} \to \mathsf{C}$ with nadir c is a collection of morphisms $\lambda : F(j) \to c$, indexed by the objects j in J, such that for any morphism $f : j \to k$ in J, the following triangle commutes in C

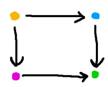
Oftentimes, you will see the word “cocone” instead of “cone under F”, and in this context usually the word “cone” will refer explicitly to cones over F.

$$\begin{tikzcd} F(j) \arrow[rr, "F(f)"] \arrow[dr, "\lambda_j"'] & & F(k) \arrow[dl, "\lambda_k"] \\ & c & \end{tikzcd}$$

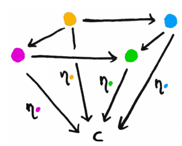
Dually, the data of a cone λ over $F : \mathsf{J} \to \mathsf{C}$ with apex c is a collection of morphisms $\lambda_j : c \to F(j)$, indexed by objects j in J, such that for any morphism $f : j \to k$ in J, the following triangle commutes in C

$$\begin{tikzcd} & c \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ F(j) \arrow[rr, "F(f)"] & & F(k) \end{tikzcd}$$

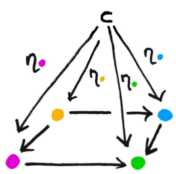
Typically, we think of limits and colimits of functors $F : \mathsf{J} \to \mathsf{C}$ when J is a relatively “small” or “simple” category. Maybe J looks something like this



Then if (c, η) is a cone under F, we have the following image in C:

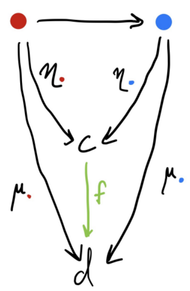


And if (η, c) is a cone over F, we have the following image in C:



Definition 0.2.12. Given a diagram F of shape J in a category C (so a functor $F : \mathsf{J} \to \mathsf{C}$) and two cones (η, c) and (μ, d) under F, a *morphism of cones under* F is a morphism $f \in \mathrm{Mor}(c, d)$ such that for all objects j in J, $\mu_j = f \circ \eta_j$.

Pictorally, a morphism of cones under F connects the nadirs of the cones.



Of course, we have a dual definition for cones over F, which connect the apexes of cones. Can you draw a picture?

Definition 0.2.13. Given a diagram F of shape J in a category C (so a functor $F : \mathsf{J} \to \mathsf{C}$) and two cones (c, η) and (d, μ) over F, a *morphism of cones over* F is a morphism $f \in \mathrm{Mor}(c, d)$ such that for all objects j in J, $\mu_j \circ f = \eta_j$.

Definition 0.2.14. Let F be a diagram of shape J in a category C.

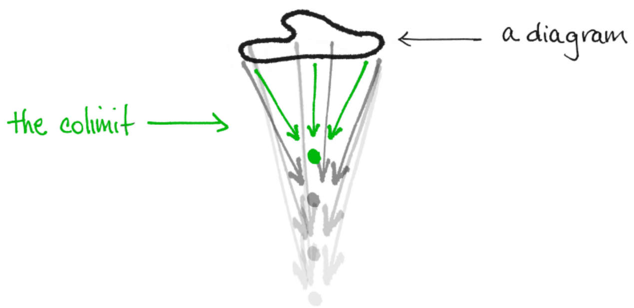
Define $\mathrm{Cone}_{\mathsf{C}}(F)$ to be the category whose objects are cones under F, and morphisms are morphisms of cones under F.

Conversely, define $\mathrm{Cone}^{\mathsf{C}}(F)$ to be the category whose objects are cones over F, and morphisms are morphisms of cones over F.

Definition 0.2.15. Given a diagram F of shape J in a category C, the *colimit cone* for F is the initial object in $\mathrm{Cone}^{\mathsf{C}}(F)$ (if it exists).

If (η, c) is a colimit cone for F, then we call the object c the *colimit* of F, and denote it by any of the following:

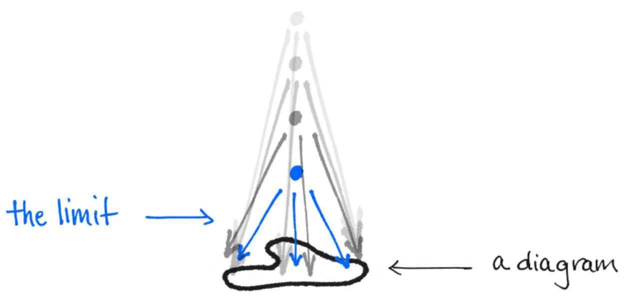
$$\operatorname*{colim}_{\mathsf{J}} F, \qquad \operatorname*{colim}_{\longrightarrow} F, \quad \text{or} \quad \varinjlim F$$



Definition 0.2.16. Given a diagram F of shape J in a category C, the *limit cone* for F is the terminal object in $\mathrm{Cone}^{\mathsf{C}}(F)$ (if it exists).

If (c, η) is a limit cone for F, then we call the object c the *limit* of F, and denote it by any of the following:

$$\lim_{\mathsf{J}} F \quad \text{or} \quad \varprojlim F$$



0.2.3 Examples of limits and colimits

I'll take a moment to say something about category theory: the above definitions were quite technical and abstract. Like most definitions in category theory, the definition of the (co)limit cannot be internalized or understood by reading it. You need to work through examples, preferably as many as possible. Thankfully, you already know lots of examples of limits and colimits!

I am going to explicitly give an example of computing a colimit and a limit in Set.

Example 0.2.17. Let J be a category with $n > 0$ objects and no non-identity morphisms. Then a functor $F : \mathsf{J} \to \mathsf{C}$ is the data of a choice of n objects $X_1, \ldots, X_n$ in C. Then the (co)limit of F is called the **(co)product** of the X_i's.

In Set, coproducts are disjoint unions and products are cartesian products of sets.

In Ab, the coproduct is given by direct sum, and the product is given by the product of groups.

In Top, the coproduct is given by the disjoint union of spaces with the disjoint union topology, while the product is given by the cartesian product with the product topology.

In Top_*, the coproduct is given by the wedge product (join spaces at their basepoint), while the product is the regular product in Top.

Example 0.2.18. An **equalizer** is a limit of a diagram indexed by the **parallel pair**, the category $\bullet \rightrightarrows \bullet$ with two objects and two parallel non-identity morphisms between them.

Example 0.2.19 (Limits in Set)**.** In general, limits in Set can be described completely as follows:

Given a diagram $S : \mathsf{J} \to \mathsf{Set}$, define the set

$$\varprojlim S := \{(s_i)_i \in \prod_{\mathsf{J}} Si : \forall \phi : i \to i', (S\phi)(s_i) = s_{i'}\}.$$

One can check that this is a limit of F.

Example 0.2.20 (Colimits in Set)**.** Given a diagram $S : \mathsf{J} \to \mathsf{Set}$, define the set

$$\operatorname{colim}_{\longrightarrow} S := \left(\coprod_{\mathsf{J}} Si \right) / (s_i \sim s_{i'} \text{ if } \exists \phi : (S\phi)(s_i) = s_{i'}).$$

One can check that this is a colimit of F.

Example 0.2.21 ((Co)limits in Top)**.** Like Set, the category Top is also complete and cocomplete. A limit in Top is formed by taking the limit of underlying sets and endowing it with the subspace topology. Likewise, a colimit in Top is formed by taking the colimit of underlying sets and endowing it with the quotient topology.

0.2.4 The Hom functor

Let X, Y be objects in C. We consider the set

$$\mathrm{Hom}_{\mathsf{C}}(X, Y)$$

of all morphisms $X \to Y$ in C. (When C is understood, we just write $\mathrm{Hom}(X, Y)$.) Given a morphism $f : Y \to Y'$, post-composition defines a function $f \circ - : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y')$. Given a morphism $g : X \to X'$, pre-composition defines a function $- \circ g : \mathrm{Hom}(X', Y) \to \mathrm{Hom}(X, Y)$. Notice that this map goes the “other way."

This describes two important functors.

Definition 0.2.22. Let C be a category. For each $c \in \mathsf{C}$, we define the *covariant* Hom *functor*

$$\mathrm{Hom}_{\mathsf{C}}(c, -) : \mathsf{C} \to \mathsf{Set}.$$

Definition 0.2.23. Let C be a category. For each $c \in \mathsf{C}$, we define the *contravariant* Hom *functor*

$$\mathrm{Hom}_{\mathsf{C}}(-, c) : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}.$$

Exercise 0.2.24. Think about the contravariant Hom functor.

Exercise 0.2.25. In a category C, prove that a morphism $f : X \to Y$ is an isomorphism $\iff$ for every object $Z \in \mathsf{C}$, the morphism $\mathrm{Hom}(Z, f) : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$ is an isomorphism. (I.e., iff the function $f \circ -$ is a bijection.)

(Not totally essential part.) The two Hom functors fit together in such a way that we can turn the co/contravariant Hom functors into a single Hom functor. We mean the following.

Proposition 0.2.26. *If $f : X \to Y$, $g : X' \to Y'$ are morphisms in C, then the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(X, Y) \arrow[r, "\mathrm{Hom}(X,g)"] \arrow[d, "\mathrm{Hom}(f,Y)"'] & \mathrm{Hom}_{\mathsf{C}}(X, Y') \arrow[d, "\mathrm{Hom}(f,Y')"] \\
\mathrm{Hom}_{\mathsf{C}}(X', Y) \arrow[r, "\mathrm{Hom}(X',g)"] & \mathrm{Hom}_{\mathsf{C}}(X', Y')
\end{tikzcd}$$

Thus, we have a functor $\mathrm{Hom}_{\mathsf{C}}(-, -) : \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$.

0.2.5 Adjunctions

There is a notion of adjunctions. The slogan is, "The slogan is, *adjunctions are everywhere.*" There are several equivalent definitions. None is "the best," they are very much so all useful. The idea is that two functors $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ may "undo each other on the level of hom sets;" we make this precise.

Definition 0.2.27. A *hom-set adjunction* is a pair of functors $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ together with a natural isomorphism

$$\Phi : \mathrm{Hom}_{\mathsf{D}}(F-, -) \cong \mathrm{Hom}_{\mathsf{C}}(-, G-).$$

Definition 0.2.28. A *unit-counit adjunction* is a pair of functors $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ together with natural transformations $\eta : \mathrm{id}_{\mathsf{C}} \implies GF$ and $\epsilon : FG \implies \mathrm{id}_{\mathsf{D}}$ such that the following diagrams commute (we call these the *triangle identities*).

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "\mathrm{id}_F"'] & FGF \arrow[d, "\epsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "\mathrm{id}_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

Proposition 0.2.29. *Let (F, G, Φ) be a hom-set adjunction. Define its canonical unit-counit structure as follows.*

- *For $c \in \mathsf{C}$, we define the morphism $\eta_c := \Phi_{c,Fc}(\mathrm{id}_{Fc})$; and*
- *For $d \in \mathsf{D}$, we define the morphism $\epsilon_d := \Phi^{-1}_{Gd,d}(\mathrm{id}_{Gd})$.*

The claim is that (F, G, η, ϵ) is a unit-counit adjunction, i.e. the η_c, ϵ_d's assemble to natural transformations satisfying the triangle identities.

Proposition 0.2.30. *Let (F, G, η, ϵ) be a unit-counit adjunction. Define its canonical hom-set adjunction structure as follows. For each $c \in \mathsf{C}$, $d \in \mathsf{D}$, and $f \in \mathrm{Hom}_{\mathsf{D}}(Fc, d)$, define $\Phi(f) := \eta_c \circ Gf \in \mathrm{Hom}_{\mathsf{C}}(c, Gd)$. The claim is that (F, G, Φ) is a hom-set adjunction.*

The last two propositions say that every hom-set adjunction gives rise to a unit-counit adjunction and vice-versa.

Proposition 0.2.31. *Hom-set adjunctions are "the same thing as" unit-counit adjunctions.*

Example 0.2.32 (Cool example from Peter's talk)**.** We work in Top^*, the category of based spaces.[3] Given based spaces X, Y, we may ask for some based space $G(X, Y)$ such that for every Z,

$$\mathrm{Hom}(G(X, Y), Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)). \tag{33}$$

In this sense, $G(X, Y)$ "undoes" the functor "take the space of based maps out of Y" on the level of hom-sets. In fact, such a based space $G(X, Y)$ exists for every X and Y, and the isomorphisms (32) are such that they describe an adjunction. (In the language above, a *hom-set adjunction*.) The space $G(X, Y)$ is the *smash product of based spaces*, defined as

$$G(X, Y) = X \wedge Y := (X \times Y)/(X \vee Y).$$

Precisely, the functors $- \wedge Y : \mathsf{Top}^* \to \mathsf{Top}^*$ and $\mathrm{Hom}(Y, -) : \mathsf{Top}^* \to \mathsf{Top}^*$ are adjoint. Smashing with Y is left adjoint to homming out of Y.

This matters because if we take $Y = S^1$, our adjoint functors specialize to important constructions: given a space X, one has

$$Y \wedge X = \Sigma X \quad \text{and} \quad \mathrm{Hom}(Y, X) = \Omega X.$$

The spaces $\Sigma X, \Omega X$ are called the *suspension* and *loop space* of X, respectively. They are essential to doing algebraic topology and homotopy theory. Then, being instances of functors which are adjoint, the adjunction tells us that for any space W we get

$$\mathrm{Hom}(\Sigma X, W) \cong \mathrm{Hom}(X, \Omega W)$$

and these isomorphisms are *natural*. I could go on about why this is great.

[3] Objects are spaces with a chosen point (X, x_0) and morphisms $(X, x_0) \to (Y, y_0)$ are continuous maps $f : X \to Y$ such that $f(x_0) = y_0$.

0.2.6 More Hom functor

Are hom-sets between (co)limits the (co)limits of hom-sets? The important answer is yes, and we will be thinking about questions like this more later.

Proposition 0.2.34. *For every object X, the functors $\mathrm{Hom}(X, -)$ and $\mathrm{Hom}(-, X)$ commute with limits. That is, if $\mathsf{I} \to \mathsf{C}$ is a diagram in C and $\varprojlim_{\mathsf{I}} c_i$ exists, then for any object X one has*

$$\mathrm{Hom}_{\mathsf{C}}(X, \varprojlim_{\mathsf{I}} c_i) \cong \varprojlim_{\mathsf{I}} \mathrm{Hom}_{\mathsf{C}}(X, c_i) \qquad \textit{and}$$
$$\mathrm{Hom}_{\mathsf{C}}(\varprojlim_{\mathsf{I}} c_i, X) \cong \underset{\overrightarrow{\mathsf{I}}}{\mathrm{colim}}\, \mathrm{Hom}_{\mathsf{C}}(c_i, X).$$

Proposition 0.2.35. *For every object X, the functors $\mathrm{Hom}(X, -)$ and $\mathrm{Hom}(-, X)$ commute with colimits. (The precise statement of this is dual to that in the previous proposition.)*

0.3 (Review) Lecture 3: Sheaves on spaces

0.3.1 What is a sheaf?

Let X be a topological space.

Definition 0.3.1. We let $\mathrm{Open}(X)$ be the category whose objects are open subsets $U \subseteq X$, and where morphisms $U \to V$ are precisely inclusions $U \subseteq V$.

In particular, between any two objects U and V, there are either exactly 0 maps between them (if U does not contain V), or exactly 1 map $U \to V$ (if $U \subseteq V$).

As a category, this is not so interesting. It is introduced only to make the following notions easier to define.

Definition 0.3.2 (Presheaves)**.** A *presheaf* on a topological space X is a contravariant functor

$$F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathsf{Ab},$$

for Ab the category of abelian groups.

Remark 0.3.3. You can replace Ab by any category you like, and get a different notion of presheaf; we will focus primarily on presheaves valued in abelian groups today, but other common use cases are presheaves valued in the category of sets, or presheaves valued in the category of rings.

Example 0.3.4. As one example of a presheaf, we define the *sheaf of continuous functions on a space X* (this presheaf, as the name suggests, will turn out to also be a sheaf).

It is the presheaf $\mathcal{F}$ on X given by

$$\mathcal{F}(U) := \{\text{continuous functions } \phi : U \to \mathbb{R}\}.$$

If $U \subseteq V$, then the map $\mathcal{F}(V) \to \mathcal{F}(U)$ (recall that, in addition to specifying abelian groups $\mathcal{F}(U)$, we also need to specify where our functor maps morphisms!) is defined to be the restriction map

$$\phi \mapsto \phi|_U,$$

where for $\phi : V \to \mathbb{R}$, the function $\phi|_U$ is just its restriction to U.

For a general presheaf $\mathcal{F}$, we often call the map $\mathcal{F}(V) \to \mathcal{F}(U)$ the *restriction* from V to U, and we call elements of $\mathcal{F}(U)$ *sections of $\mathcal{F}$ defined over U*. We often abbreviate the restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ by just writing $s|_U$, where $s \in \mathcal{F}(V)$.

The definition of sheaf is intended to capture two very important general properties of the sheaf of continuous functions.

Definition 0.3.5. A *sheaf* on a space X is a presheaf $\mathcal{F}$ on X so that for any open subset U of X, any open cover $\{U_i\}_{i \in I}$ of U_i, and any collection of $s_i \in \mathcal{F}(U_i)$ so that

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j},$$

there exists a unique $s \in \mathcal{F}(U)$ so that, for each i,

$$s|_{U_i} = s_i.$$

Remark 0.3.6. There are two parts to this notion: *existence* of glueings, and *uniqueness* of glueings.

0.3.2 The categories of sheaves and presheaves

Definition 0.3.7. A *morphism of presheaves* is just a natural transformation of functors. A *morphism of sheaves* is just a morphism of the underlying presheaves.

We write $\mathsf{Psh}(X), \mathsf{Sh}(X)$ to denote the categories of presheaves and sheaves on X.

There is a *fully faithful* forgetful functor

$$I : \mathsf{Sh}(X) \to \mathsf{Psh}(X),$$

since every sheaf is a presheaf.

There are some useful properties of the category of sheaves and presheaves.

Theorem 0.3.8. *The category* $\mathsf{Psh}(X)$ *of presheaves on* X *is complete and cocomplete.*

In fact, if $\mathcal{F}_i$ *is a diagram of presheaves, then* $\varprojlim_i \mathcal{F}_i$ *is the presheaf*

$$\left(\varprojlim_i \mathcal{F}_i\right)(U) := \varprojlim_i \mathcal{F}_i(U),$$

where the limit on the right is taken in the category of abelian groups. A similar formula holds for colimits.

Proof. Left to the reader; as a hint, construct a map

$$\mathrm{Hom}(\mathcal{G}, \varprojlim_i \mathcal{F}_i) \to \varprojlim_i \mathrm{Hom}(\mathcal{G}, \mathcal{F}_i)$$

(for $\varprojlim_i \mathcal{F}_i$ the sheaf defined in the theorem statement) and check it is both injective and surjective; this proves our formula is actually the categorical limit. □

Our next step will be to prove that $\mathsf{Sh}(X)$ is complete and cocomplete. First, we show completeness.

Theorem 0.3.9. *Let* $\mathcal{F}_i$ *be a diagram of* sheaves*. Then the presheaf limit*

$$\left(\varprojlim_i \mathcal{F}_i\right)(U) := \varprojlim_i \mathcal{F}_i(U)$$

is a sheaf.

Proof. For this, it is useful to give a reformulation of the definition of sheaf. Let $\mathcal{F}$ be a presheaf. Then $\mathcal{F}$ is a sheaf if and only if, for any open set $U \subseteq X$, and any open cover $\{U_\alpha\}_{\alpha \in A}$ of U, the sequence of maps

$$0 \to \mathcal{F}(U) \to \prod_{\alpha \in A} \mathcal{F}(U_\alpha) \to \prod_{\alpha, \beta \in A} \mathcal{F}(U_\alpha \cap U_\beta)$$

is left exact, where the first map is just the product of the restriction maps, and the second map sends $(s_\alpha)_{\alpha \in A}$ to

$$(s_\alpha|_{U_\alpha \cap U_\beta} - s_\beta|_{U_\alpha \cap U_\beta})_{\alpha, \beta \in A}.$$

Or equivalently, $\mathcal{F}$ is a sheaf if and only if the map

$$\mathcal{F}(U) \to \ker\left(\prod_{\alpha \in A} \mathcal{F}(U_\alpha) \to \prod_{\alpha, \beta \in A} \mathcal{F}(U_\alpha \cap U_\beta)\right)$$

is always an isomorphism.

Since each $\mathcal{F}_i$ is a sheaf, the maps

$$\mathcal{F}_i(U) \to \ker\left(\prod_{\alpha \in A} \mathcal{F}_i(U_\alpha) \to \prod_{\alpha, \beta \in A} \mathcal{F}_i(U_\alpha \cap U_\beta)\right)$$

are all isomorphisms. Take the limit of these diagrams over i, and so the map

$$\varprojlim_i \mathcal{F}_i(U) \to \varprojlim_i \ker\left(\prod_{\alpha \in A} \mathcal{F}_i(U_\alpha) \to \prod_{\alpha, \beta \in A} \mathcal{F}_i(U_\alpha \cap U_\beta)\right)$$

is an isomorphism. Limits commute with limits; the kernel is a limit, and so it commutes with $\varprojlim_i$. It also commutes with all the products, and hence the above isomorphism can be rewritten as an isomorphism

$$\varprojlim_i \mathcal{F}_i(U) \to \ker\left(\prod_{\alpha\in A} \varprojlim_i \mathcal{F}_i(U_\alpha) \to \prod_{\alpha,\beta\in A} \varprojlim_i \mathcal{F}_i(U_\alpha \cap U_\beta)\right).$$

Thus $\varprojlim_i \mathcal{F}_i$ is a sheaf. $\square$

Corollary 0.3.10. *The category* $\mathsf{Sh}(X)$ *is complete.*

Cocompleteness is harder. We state, without proof, a useful theorem.

Theorem 0.3.11 (Sheafification)**.** *The fully faithful forgetful functor*

$$I : \mathsf{Sh}(X) \to \mathsf{Psh}(X)$$

has a left adjoint functor

$$\# : (X) \to \mathsf{Sh}(X),$$

called sheafification*.*

Since I is fully faithful, this adjunction automatically obeys the property that the morphism

$$\mathcal{F} \to \mathcal{F}^{\#}$$

from a *sheaf* $\mathcal{F}$ into its sheafification is an isomorphism.

Theorem 0.3.12. *The category* $\mathsf{Sh}(X)$ *is cocomplete.*

Proof. Let $\mathcal{F}_i$ be a diagram of sheaves, and let $\mathcal{G}$ be the *presheaf* colimit. Then we claim that $\mathcal{G}^{\#}$ is the colimit of the $\mathcal{F}_i$ in the category of sheaves. Indeed, for any sheaf $\mathcal{F}'$,

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Sh}(X)}(\mathcal{G}^{\#}, \mathcal{F}') &= \mathrm{Hom}_{\mathsf{Psh}(X)}(\mathcal{G}, \mathcal{F}') \\ &= \varprojlim_i \mathrm{Hom}_{\mathsf{Psh}(X)}(\mathcal{F}_i, \mathcal{F}') \\ &= \varprojlim_i \mathrm{Hom}_{\mathsf{Sh}(X)}(\mathcal{F}_i, \mathcal{F}'),\end{aligned}$$

and so $\mathcal{G}^{\#}$ is the colimit. $\square$

Warning 0.3.13. The colimit of a family of presheaves is *not* computed as

$$\left(\operatorname{co}\varinjlim_i \mathcal{F}_i\right)(U) := \operatorname{co}\varinjlim_i \mathcal{F}_i(U),$$

in contrast to the case of limits. You have to use a sheafification procedure.

0.3.3 The inverse and direct image functors

Definition 0.3.14. Let $f : X \to Y$ be a continuous function of topological spaces X, Y. Then we define the *direct image functor*

$$f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$$

and the *inverse image functor*

$$f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$$

as follows. The functor f_* sends a sheaf $\mathcal{F}$ on X to

$$(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V)),$$

and f^{-1} sends a sheaf $\mathcal{G}$ on Y to the *sheafification* of the presheaf

$$(f^{-1}\mathcal{G})(U) := \operatorname{co}\varinjlim_{V\subseteq f(U)} \mathcal{G}(V).$$

Theorem 0.3.15. *The functors f^{-1}, f_* are an adjoint pair. Thus, for any $\mathcal{F} \in \mathsf{Sh}(X), \mathcal{G} \in \mathsf{Sh}(Y)$, we have*

$$\mathrm{Hom}_{\mathsf{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathsf{Sh}(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

In particular, f^{-1} is right exact and f_* is left exact.

Proof. Left to readers for time. □

We now look at some particularly important examples of this construction.

0.3.4 Sections and constant sheaves

For any topological space X, there is a unique continuous map $f : X \to \{*\}$, for $\{*\}$ denoting the one point topological space.

The category $\mathsf{Sh}(\{*\})$ is equivalent to to Ab, since a sheaf on a point is just the value of its global sections on the entire space. Thus f^{-1}, f_* give us functors

$$f^{-1} : \mathsf{Ab} \to \mathsf{Sh}(X)$$

and

$$f_* : \mathsf{Sh}(X) \to \mathsf{Ab}.$$

The direct image is simple:

$$(f_*\mathcal{F})(\{*\}) = \mathcal{F}(f^{-1}(\{*\})) = \mathcal{F}(X).$$

It has a special name and notation.

Definition 0.3.16. The *global sections functor* is the functor $\Gamma_X : \mathsf{Sh}(X) \to \mathsf{Ab}$ given by $\Gamma_X(\mathcal{F}) = \mathcal{F}(X)$.

The global sections functor has an adjoint f^{-1}, which is called the constant sheaf functor.

Definition 0.3.17. Let A be an abelian group. Then *the constant sheaf on* A is the sheaf $\underline{A} := f^{-1}(A)$.

Remark 0.3.18. As an exercise, prove that $\underline{A}(U) = A$ if U is connected. What happens if U is disconnected?

Remark 0.3.19. Since the global sections functor is a right adjoint, it commutes with all limits. This gives a category theoretic explanation for why the formula

$$\left(\varprojlim_i \mathcal{F}_i\right)(X) = \mathcal{F}_i(X)$$

holds at the level of sheaves.

The global sections functor in general does not commute with colimits, though, which is why it is difficult to describe the colimit of a diagram of sheaves directly.

0.3.5 Restriction and extension

Let X be a topological space, U an open subset, and $j : U \to X$ the inclusion.

Definition 0.3.20. The functor $j^{-1} : \mathsf{Sh}(X) \to \mathsf{Sh}(U)$ is *restriction to* U. Sometimes $j^{-1}\mathcal{F}$ is written $\mathcal{F}|_U$.

It is easy to verify that

$$(j^{-1}\mathcal{F})(V) = \mathcal{F}(V).$$

0.3.6 Skyscraper sheaves and stalks

For any point $x \in X$, there is a continuous inclusion $i_x : \{x\} \to X$. This gives us two functors

$$i_{x,*} : \mathsf{Ab} \to \mathsf{Sh}(X),$$

$$i_x^{-1} : \mathsf{Sh}(X) \to \mathsf{Ab}.$$

Definition 0.3.21. Let $\mathcal{F}$ be a sheaf on a space X. The *stalk of $\mathcal{F}$ at x* is the abelian group

$$\mathcal{F}_x = i_x^{-1}(\mathcal{F}) = \operatorname{co}\varinjlim_{U \ni x} \mathcal{F}(U).$$

The stalk is an incredibly useful construction. Here are some properties one should verify.

Definition 0.3.22.

Check that the colimit definition of a stalk makes sense for presheaves, so that it makes sense to talk about stalks.

A presheaf $\mathcal{F}$ and its sheafification $\mathcal{F}^{\#}$ have the same stalk at every point x. If you know the construction of the sheafification functor, you can check this directly – but you don't need to know the construction of sheafification! Try to prove this purely using categorical properties of stalks and sheafification.

If $U \subseteq X$ is open and $x \in U$, then there is a natural map $\mathcal{F}(U) \to \mathcal{F}_x$. If $s \in \mathcal{F}(U)$, we write s_x for the image of s under this natural map. (Hint: remember the definition of colimit!)

Let $\mathcal{F}$ be a presheaf. Then the axiom of uniqueness of glueings in the definition of sheaf is equivalent to saying that, for every section $s \in \mathcal{F}(U)$, if $s_x = 0$ for every $x \in U$, then $s = 0$.

Taking stalks commutes with limits and colimits.

If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a sequence of sheaves, then it is short exact if and only if for every $x \in X$, the sequence $0 \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to 0$ is.

Use the previous point to prove that sheafification is exact.

Definition 0.3.23. The *skyscraper sheaf* at a point $x \in X$ with stalk A is the sheaf $i_{x,*}A$.

0.4 (Review) Lecture 4: Sheaves on sites

In 1949, Andre Weil (while he was a UChicago professor) proposed the Weil conjectures, which propose a way of taking a system of equations defined over $\mathbb{Z}$, and relating the number of solutions to that system over a finite field $\mathbb{F}_q$ to the topology of the complex solution set. Weil was able to prove the Weil conjectures in a few special cases; most notably, Weil algebraized a large amount of the modern theory of algebraic geometry of curves, so that classical theorems for curves over $\mathbb{R}$ or $\mathbb{C}$ would apply to fields like $\mathbb{F}_q$, allowing him to give a proof of the Weil conjectures for curves.

As Weil proposed from the start, and as Serre made more precise in 1960 with his proof of the Kahler analogues of the Weil conjectures, the ultimate difficulty in proving the Weil conjectures stems from being able to define "Weil cohomology theories" – basically, a theory of cohomology for algebraic varieties defined over fields like $\mathbb{F}_q$.

Ultimately, Grothendieck defined etale cohomology to fulfill this dream.

0.4.1 Sites

The central difficulty with giving a cohomology theory on algebraic varieties over finite fields is that a finite topological space doesn't have that many open sets, and so if you just take an algebraic topologist's cohomology then you won't get anything useful.

Grothendieck resolved this by extending the definition of cohomology. We won't talk about this generalized cohomology (yet! but later on in condensed math, we will need to), but the starting point is a generalization of topological space, and a corresponding generalization of the notion of sheaf.

Definition 0.4.1. Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is a collection of distinguished families of morphisms sharing a common target, called *covering families* (so, a covering family is the data of an object $U \in \mathcal{C}$, and a collection of morphisms $\{U_i \to U\}_{i \in I}$), obeying the following axioms, analogous to the axioms for a topology.

Firstly, any isomorphism is always a covering family. This is in analogue to how the entire set of a topological space is open.

Secondly, analogously to how an arbitrary union of opens is open, if $\{U_i \to U\}_{i \in I}$ is a covering family, and if for each $i \in I$ we have a covering family $\{U_{j_i} \to U_i\}_{j_i \in J_i}$, then the family

$$\{U_{j_i} \to U\}_{i \in I, j_i \in J_i}$$

is a covering family of U.

Lastly, analogously to how finite intersections of open sets are open, we need one more axiom. Firstly, the categorical analogue of the intersection is something called the *fiber product*; if the reader doesn't know what it is, then it is given in the definition just below this one.

So, we require that $\mathcal{C}$ has all fiber products, and that if $V \to U$ is any morphism, and $\{U_i \to U\}_{i \in I}$ is any covering family, then $\{U_i \times_U V \to V\}_{i \in I}$ is a covering family as well.

Definition 0.4.2. If $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$ are two morphisms, then the *fiber product* of f_1, f_2 is the limit $X_1 \times_S X_2$ of the diagram

$$\begin{tikzcd} X_1 \times_S X_2 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & S \end{tikzcd}$$

0.4.2 Sheaves on a site

Definition 0.4.3. Let $\mathcal{C}$ be a site (that is, a category with a Grothendieck topology). Then the category $\mathsf{Psh}(\mathcal{C})$ of *presheaves* on $\mathcal{C}$ is the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$.

The category $\mathsf{Sh}(\mathcal{C})$ of *sheaves* on $\mathcal{C}$ is the full subcategory of $\mathcal{C}$ consisting of presheaves $\mathcal{F}$ to that, for every covering family $\{U_i \to U\}_{i \in I}$, $\mathcal{F}(U)$ is the equalizer of the two morphisms

$$\prod_{i \in I} \mathcal{F}(U_i) \to \prod_{i,j} \in \mathcal{F}(U_i \times_U U_j).$$

(The two morphisms are

$$(s_i)_{i \in I} \mapsto (s_i|_{U_i \times_U U_j})_{i,j \in I},$$
$$(s_i)_{i \in I} \mapsto (s_j|_{U_i \times_U U_j})_{i,j \in I}$$

obtained via the two natural maps $U_i \times_U U_j \to U_i$ and $U_i \times_U U_j \to U_j$.)

The results of sheaf theory we established last time essentially go through for the exact same abstract nonsense categorical reasons.

0.4.3 Weil conjectures

Instead of reproving the same theorems for sites, I want to spend some time talking about why they were originally introduced: to solve the Weil conjectures.

What are the Weil conjectures? I don't want to be too precise, but basically they are a series of conjectures of the following form (although, the actual ones are a little more general than what I am about to say).

Imagine we have a system of polynomial equations in some number of variables, where the coefficients in the equations are all algebraic numbers, so something like

$$y - x^2 = \sqrt{2}.$$

This system of equations defines a complex algebraic variety, which has (singular) cohomology groups, a very concrete topological invariant.

But over certain finite fields, the equation

$$y - x^2 = \sqrt{2}$$

still makes sense (for instance, for p odd, $\sqrt{2}$ exists over either $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$, and so this equation still makes sense).

Over a finite field, there's only some finite number of solutions to this equation. You could take singular cohomology of this finite set (with its Zariski topology), but the cohomology groups you get are essentially meaningless.

Is there any sense in which the finite field remembers the cohomology of the complex variety?

Weil found a miracle: it does! I want to look at an example which will be very easy to count points for, but which might not be so convincing. There was much more numerical evidence known to Weil (including a few special cases of the Weil conjectures that Weil proved before he even formulated the Weil conjectures, and the case of elliptic curves which was proven by Hasse in 1933).

Look at $\mathbb{P}^n_{\mathbb{F}_q}$, the projective space over $\mathbb{F}_q$. Recall that the cohomology of projective n-space over the complex numbers is given

$$H^*(\mathbb{P}^n_{\mathbb{C}}) = \mathbb{Z}[\alpha]$$

where $|\alpha| = 2$. In other words, in even degrees, $H^{2i}(\mathbb{P}^n_{\mathbb{C}}) = \mathbb{Z}$ for $0 \leq i \leq n$, and $H^{2i-1}(\mathbb{P}^n_{\mathbb{C}}) = 0$ in all odd degrees. There are many ways to compute this, but probably the easiest is to note that $\mathbb{P}^n_{\mathbb{C}}$ can be built up as a CW complex built out of a one 0-cell, one 2-cell, ..., one $2n$-cell. (Do the example of a disk glued along its boundary is a sphere.)

How many points are on $\mathbb{P}^n_{\mathbb{F}_q}$? This is easy: it's just

$$1 + q + \cdots + q^n.$$

Look at the analogy between how $\mathbb{P}^n_{\mathbb{C}}$ is built out of one 0-cell, one 2-cell, etc.

The Weil conjectures predict that in general, the number of points defined over $\mathbb{F}_q$ should be intimately related to the cohomology groups of the corresponding complex algebraic variety. Specifically, Weil defines the *zeta function* of a variety X over $\mathbb{F}_q$ as a certain exponential generating function of the sequence $|X(\mathbb{F}_q)|, |X(\mathbb{F}_{q^2})|, \ldots$, and then conjectures that this generating function ζ should obey certain properties related to the topology of the corresponding complex variety.

As another example, we consider the case of elliptic curves. An elliptic curve X recall is a certain type of equation of the form

$$y^2 = x^3 + ax + b.$$

Over the complex numbers, such equations correspond to genus 1 surfaces.

Over a finite field $\mathbb{F}_q$, Hasse proved that the number of points on an elliptic curve was of the form

$$|\#X(\mathbb{F}_q) - (q+1)| \leq 2\sqrt{q}.$$

How does this relate to topology? Well, over the complex numbers,

$$H^0(X;\mathbb{Z}) = \mathbb{Z},$$

$$H^1(X;\mathbb{Z}) = \mathbb{Z}^2.$$

$$H^2(X;\mathbb{Z}) = \mathbb{Z},$$

The $\mathbb{Z}, \mathbb{Z}$ in the even terms give us the $1+q$, just like in the projective space case the even cohomology gave us $1 + q + \cdots + q^n$. The $\mathbb{Z}^2$ in the odd degree terms give us that

$$\#X(\mathbb{F}_q) = 1 + q + \alpha + \overline{\alpha},$$

where α is a complex number of absolute value $\sqrt{q}$. It's easy to deduce the Hasse bound from this.

0.4.4 Etale cohomology

So, it seems that varieties over finite fields can see shadows of the cohomology of the corresponding complex variety. Can one make this precise?

It turns out yes, via etale cohomology. Grothendieck had the insight that the Zariski topology on a variety over a finite field failed to give enough cohomology because it failed to have enough open sets. Grothendieck replaced it with the etale topology.

Basically, an etale morphism of varieties is the algebraic geometer's analogue of a 'local diffeomorphism' in differential topology. Some examples of etale maps are covering maps and open inclusions, for instance.

Grothendieck defined the (small) etale site over a base scheme S to be the category of all etale morphisms of schemes $X \to S$ (morphisms being commuting triangles), endowed with the etale topology: an open cover in this category is then just any collection of morphisms $\{\phi_i : X_i \to X\}$ which are jointly surjective.

Topoi give rise to notions of sheaves and cohomology theories. One of the really cool useful results is the following comparison theorem: if X is a smooth complex variety, then for any *finite* ring Λ,

$$H^i_{\text{et}}(X, \underline{\Lambda}) \cong H^i(X; \Lambda),$$

where the left hand side is etale cohomology with coefficients in the constant sheaf associated to Λ, and the right hand side is singular cohomology.

So, the etale cohomology remembers the complex cohomoloy, at least for finite coefficient rings! One can proceed to define the ℓ-adic cohomology of X to be

$$H^i_{\text{et}}(X, \mathbb{Z}_\ell) := \varprojlim_n H^i_{\text{et}}(X, \mathbb{Z}/\ell^n\mathbb{Z}).$$

We remark that in general, the etale cohomology in the constant sheaf $\mathbb{Z}_\ell$ is NOT equal to the ℓ-adic cohomology we defined above. Very sad.

It would be nice if we could have etale cohomology with coefficients in $\mathbb{Z}$ or , but it seems like the ℓ-adic ones are the best we can do.