

Sheaves and Sheaf Cohomology

Reading group notes

July 6, 2022

Abstract

Notes from a reading group at the University of Chicago 2022 REU. The broad goal is to understand some basics regarding sheaves and sheaf cohomology, and to see applications of these and any additional theory (derived functors) we develop. As to the audience's 'level': some of the participants have not seen singular homology of spaces, some have read Hartshorne, ... It will be interesting to see how this goes.

At the very least, everyone will have gotten biscuits and coffee out of this reading group.

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0.1 (Week 2) June 22 - Of Presheaves and Sheaves

Reading: my paper [1].

First meeting. Opened with some history. (History motivates; see paper.)

Q: What is a sheaf? Furthermore, what (and I cannot stress this enough) *is* a sheaf?

Let X be a topological space. Define $\text{Op}(X) :=$ category of its open sets, morphisms given by inclusions.

WANT: A systematic way to 'track' the 'constructions' possible on open sets. E.g., given an open set U , what is the set of cts functions $U \rightarrow \mathbb{R}$? How do these sets vary as U varies? How are the resulting sets related, w.r.t. the relation of their underlying open sets? (Try replacing 'cts function' with: C^∞ functions, k -forms, exact k -forms, sections of a vector bundle, solutions to a the PDE arising from a fixed vector field, ...)

Cts functions are the prototypical notion of a 'construction.' Among other things, they have a nice property: if $U \hookrightarrow V$ then a cts $f : V \rightarrow \mathbb{R}$ restricts to a cts $f|_U : U \rightarrow \mathbb{R}$.

(This answers ‘how are the resulting sets related?’—they are related by a *restriction map*.)

Any notion of ‘construction’ should have this restriction property. And this restriction property was just (contravariant) functoriality w.r.t. inclusions.

Definition. A *presheaf* is a contravariant functor $\mathcal{F} : \text{Op}(X) \rightarrow \mathbf{D}$, where \mathbf{D} is any (concrete) category. Usually it will be sets, rings, abelian groups, ... We call the image of an inclusion under \mathcal{F} a *restriction*.

So a presheaf describes ‘constructions’ that come with restrictions. Continuous functions have more than just restrictions, though. Two properties in particular stand out (or rather, are so obvious that the need to identify them stands out):

1. if $f_1 : U_1 \rightarrow R$ and $f_2 : U_2 \rightarrow R$ agree on intersection, then there exists a global $f : U_1 \cup U_2 \rightarrow R$ that restricts to f_1 or f_2 .
2. If $f, g : U_1 \cup U_2 \rightarrow R$ agree when restricted to U_1 or U_2 , then $f = g$.

Generalizing (1) and (2) to arbitrary covers leads us to define a sheaf.

Definition. A *sheaf* is a presheaf satisfying two axioms.

1. (Identity) Let $\{U_i\}$ be any open covering of any open set U . If $s, t \in \mathcal{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ for each U_i , then $s = t$.
2. (Gluing) Let $\{U_i\}$ be any open covering of any open set U . If one can choose a section s_i from each U_i such that s_i, s_j agree when restricted to $U_i \cap U_j$ for each i, j , then there is a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for each i .

EXAMPLES—important, see paper, end of §1.

Remark. (Pre)sheaves form categories $\text{PSh}(X)$, $\text{Sh}(X)$. Objects are (pre)sheaves, morphisms are natural transformations. The inclusion functor $\text{Sh}(X) \hookrightarrow \text{PSh}(X)$ has a left adjoint called *sheafification*. Thus, sheafifying a presheaf gives the ‘best’ sheaf approximate. The exact construction of sheafification is not widely useful; people (algebraic geometers) usually just need to know the universal property relating a presheaf to its sheafification. (The one described by adjunctions.)

0.2 (Week 2) June 26 - Abelian Categories I

Still reading [1].

Opened with historical refresher: homological algebra had just systematized our understanding of modules, at least in the context of their use in algebraic topology. This clearly worked for objects besides modules, but a rigorous abstraction was not made. In particular, *sheaves* had ‘homological algebra-like’ properties, e.g. kernels, quotients. And sheaves had JUST effected advances in complex/algebraic geometry (Serre, Cartan, ...) The big question: can we figure out how to do homological algebra generally, fully fit sheaves into their framework, then recruit the techniques of algebraic topology (cohomology!) to study sheaves? Short answer: yes.

Today, we want to identify what makes homological algebra work. Or rather, where it works.

The base case is Mod_R , the category of R -modules—this is where we first understood homological algebra. Let's specialize to Ab , the category of abelian groups. What are its properties?

1. Given two groups G and H , the set $\text{Hom}(G, H)$ is an abelian group.
2. With respect to the group structure above, morphisms compose bilinearly; $h \circ (f + g) = h \circ f + h \circ g$, and similarly with $(f + g) \circ h$.
3. It turns out that (1) and (2) imply that the (finite) product and coproduct of two groups are isomorphic, *if either exists*. (Here, I mean the categorical (co)product; in strictly group-theoretic terms, these are the direct sum/product.) Assuming they exist, we have a natural categorical description of the addition law on hom-sets (that's good!), see the reading p. 5-6, starting at "Getting categorical, . . ." And it turns out that Ab *does* have all finite products and coproducts. (That's the property.)
4. It has a *zero object*, to mean an object with a unique morphism to AND from all other objects. (It's the trivial group.) This gives rise to e.g. *zero morphisms*.

Then the Reg closed and some of us got dinner. We'll finish listing properties next time.

0.3 (Week 3) June 29 - Abelian Categories II

Reading: my paper [1]. (Supplementary: [Here's](#) a reference for the basics of singular homology. [Here's](#) a basic example of 'use singular homology to study a space.' Sheaf cohomology makes an appearance in the second link.)

Picking up where we left off—what are the good properties of Ab , insofar as homological algebra is concerned?

5. Kernels and cokernels of morphisms are (sub)groups.
6. The first isomorphism theorem: if $f : G \rightarrow H$ is a hom, then $G/\ker f \cong \text{im } f$. Another name for $G/\ker f$ is the *coimage* of f . This notation is good to know.

Facts (1)-(6) let us do 'homological algebra' with abelian groups—we can talk about chain complexes, quotients, kernels, etc. Now we reformulate these properties for arbitrary categories. This requires we 'categorify' certain terms above. Here's what we get:

Definition. Consider the following properties a category may have.

(PA1) Every hom-set has an abelian group structure.

(PA2) (PA1), and composition is bilinear with respect to hom-set addition law.

(A) (AB2), and it has all finitary products and coproducts. **Consequences:** we mentioned that this implies the (co)product of two objects coincide. We also mentioned that this incidence gives us a categorical construction of the addition law on hom-sets.

(AB0) It has a *zero object*.

(AB1) (AB0), and it has all *kernels* and *cokernels*. **Definitions:** the *kernel* of a morphism $f : X \rightarrow Y$ is the equalizer of f and the zero map $X \rightarrow Y$. Equivalently, it

is a map $k : K \rightarrow X$ such that (i) $f \circ k$ is the zero map, and (ii) for any map $k' : K' \rightarrow X$ such that $f \circ k'$ is zero, there is a unique map $K' \rightarrow K$ making that whole diagram commute. Dually, the *cokernel* of f is the coequalizer of f and the zero map $X \rightarrow Y$.

(AB2) *Images* and *coimages* are isomorphic. **Definitions:** Given f , the *image* of f is defined as the kernel of its cokernel. Dually, the *coimage* of f is the cokernel of its kernel. Actually, I think the isomorphism is ‘canonical’: by the definitions of (co)images, there appears a map $\text{coker } \ker f \rightarrow \ker \text{coker } f$, which gives the isomorphism(?)

Definitions: A category satisfying (PA2) is called *preadditive*. A category satisfying (A) is called *additive*. A category satisfying (A) and (AB2) is called *abelian*.

Abelian categories have homological algebra!

0.3.1 Digression: homological algebra?

At this point, I should stop saying ‘homological algebra’ without any indication of what it is. Especially since some of us might not know examples of (co)homology. I’ll introduce *singular (co)homology*. It is a good ‘toy example’ witnessing the emergence of algebra in the study of spaces.

Let X be a topological space. Define $C_n(X) := \text{FAb}\{\text{continuous } \sigma : \Delta^n \rightarrow X\}$.¹ The n -simplex Δ^n has faces, so $\sigma \in C_n(X)$ can be restricted to a face, which defines a map $\Delta^{n-1} \rightarrow X$. Define a homomorphism $\partial : C_n(X) \rightarrow C_{n-1}(X)$ by

$$\sigma \mapsto \sum (-1)^i \sigma_{\text{its } i\text{-th face}}.$$

That $(-1)^i$ accounts for orientation, which makes everything work. We can forget about it.

Now we look at the following ‘chain’:

$$\cdots \rightarrow C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} C_{n-2} \rightarrow \cdots \rightarrow C_1 \xrightarrow{\partial} C_0.$$

Note that $\partial^2 = 0$; in this case, we call the chain a *chain complex of abelian groups*. Since $\partial^2 = 0$, we know $\text{im } \partial \subseteq \ker \partial$. So we may define $H_n(X) := \ker \partial / \text{im } \partial$. These are the *homology groups* of X .

We wish to know things about $H_n(X)$. It’s definition is in terms of that chain complex—*it turns out we can just study abstract chain complexes and learn a lot about $H_n(X)$ as a result.* That’s homological algebra.

***Example:** one learns that given a pair of spaces $A \subset X$, there is a corresponding ‘long exact sequence’ (LES) of singular homology groups of A , X , and (A, X) . The proof is pure (homological) algebra, making no reference to how C_n is defined. It’s an example of a more general fact: given a SES of chain complexes, there is an induced LES in homology. In this case, $A \rightarrow X \rightarrow X/A$ is a SES of spaces, which we hit with $C_*(-)$ to get a SES of chain complexes, to get the LES in homology. The proof depends on the *snake lemma* and *zig-zag lemma* from hml. algebra.

¹FAb means ‘free abelian group on this set.’ And Δ^n is the n -simplex; Google its definition.

0.3.2 The furniture of abelian categories

Finally, we have defined abelian categories. They come with:

- A first isomorphism theorem (by assumption).
- Quotients (and ‘subobjects’).
- (Co)chain complexes. And since we have quotients, we can define the (co)homology of a chain (co)complex.
- Since we have (co)kernels, we can talk about *exactness*.
- A snake lemma and zig-zag lemma.

This exactly extends the language we used to define and study singular homology to work in other categories (e.g., modules). This corroborates our history lesson: they were coming up with algebra to help them do ‘combinatorial topology,’ then wanted to separate and systematize all that algebra.

0.3.3 Upshot

Here’s vaguely how we use singular homology:

1. Take a space, (e.g. X)
2. Construct algebraic things from that space that ‘knows’ something about the topology of X , (e.g. $C_n(X)$, ∂)
3. Use homological algebra to study those ‘things,’ (e.g. $H_n(X)$, derivation of LES induced by SES)
4. Leverage the relationship between X ’s topology and the $H_n(X)$ to learn about X . (E.g., show X is not simply connected by showing $H_1(X)$ is nontrivial; show that a 3-manifold has a non-contractible embedded curve if H_2 is nontrivial; [this](#).)

With abelian categories defined, we can be a bit more creative. We may replicate (3) for more interesting ‘algebraic things.’ As a pointed example, we may try to:

1. Take a space X ,
2. Construct a gadget that ‘knows’ when ‘local constructions we care about’ are possible and how they patch together. Note: this gadget ‘knows’ something about X ’s topology AND about the ‘constructions we care about.’
3. See if those gadgets form an abelian category. If they do, study them from the POV of homl. algebra, perhaps deriving algebraic data K from our gadgets.
4. Leverage the relationship between X and the K to learn about X , AND to learn about the ‘constructions.’

Of course, by ‘gadgets’ I mean *sheaves*! I hope the two set of steps I gave appear analogous; I tried to get across some of the intuition I developed while trying to learn this stuff the first time around.

Note: steps (3) and (4) are very similar. However, although X ’s topology tells us about $H_n(X)$, we don’t care about $H_n(X)$ past what it says about X ; what $H_n(X)$ concretely describes (maps of n -simplices) is not intrinsically significant. But such is not generally the case...

0.4 (Week 4) July 6 - Derived Functors

PREVIOUSLY: Defined abelian categories. They had homological algebra. To motivate homl. algebra, we defined and discussed the singular homology $H_n(X)$ of a space X . Pure homological algebra produced useful facts about $H_n(X)$.

0.4.1 A bit more thinking about chains

We should work out one example of ‘using homological algebra to do (co)homology.’ Let’s cover a particularly important property of $H_n(X)$. This is the ‘long exact sequence (LES)’ I talked about in the last talk.

Proposition 0.1. *Let X be a space, A a subspace. Then the following sequence is exact:*

$$\cdots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A) \rightarrow \cdots \rightarrow H_0(A) \rightarrow H_0(X) \rightarrow H_0(X, A) \rightarrow 0.$$

The maps $i_i : H_i(A) \rightarrow H_i(X)$ are induced by inclusion. The maps $j_i : H_i(X) \rightarrow H_i(X, A)$ are induced by ‘relative inclusion.’ The maps $\partial : H_i(X, A) \rightarrow H_{i-1}(A)$ are the **connecting homomorphisms**.

The maps i_* , j_* are God-given and trivial. From definitions it follows that $H_i(A) \rightarrow H_i(X) \rightarrow H_i(X, A)$ is exact. These constitute ‘pieces’ of our LES.

But where did ∂ come from, the maps that connect all ‘pieces’ together? Answer: ∂ arises from pure algebra. Here’s how:

Definition. A *chain complex of abelian groups* is a sequence of abelian groups

$$\cdots \xrightarrow{\partial_1} G_3 \xrightarrow{\partial_0} G_2 \xrightarrow{\partial_{-1}} G_1 \rightarrow 0$$

Such that $\partial_i \partial_{i-1} = 0$ for all i .

Definition. Let G_* be a chain complex, written as above. Its k -th *homology group* is defined to be $H_k(G_*) := \ker \partial_{k+1} / \text{im } \partial_k$.

Fact. Suppose $0 \rightarrow A_* \rightarrow B_* \rightarrow C_* \rightarrow 0$ is an exact sequence of chain complexes. Then there are **connecting morphisms** of abelian groups $\partial : H_i(C_*) \rightarrow H_{i-1}(A_*)$ making the following sequence exact:

$$\cdots \xrightarrow{\partial} H_n(A_*) \rightarrow H_n(B_*) \rightarrow H_n(C_*) \xrightarrow{\partial} H_{n-1}(A_*) \rightarrow \cdots$$

In the context of singular homology: a pair of spaces $A \subseteq X$ induces an exact sequence of spaces $pt \rightarrow A \hookrightarrow X \rightarrow X/A \rightarrow pt$, which gives rise to an exact sequence of chain complexes $0 \rightarrow C_*(A) \rightarrow C_*(X) \rightarrow C_*(X/A) \rightarrow 0$ where $C_*(Y)$ is the chain of groups of n -simplices mapping into Y . Then the fact in this case produces 0.1.

I wrote this out for two reasons:

1. To get us used to some of the language (exactness, kernels, ...)
2. The LES 0.1 is very important in algebraic topology. Given a pair of spaces $A \subseteq X$, many other (co)homology theories have an identical LES to 0.1, with H_i replaced. *In fact, this is so important that the axioms for a (co)homology theory demand that any pair $A \subseteq X$ give rise to a LES; see Axiom 5.* Soon, we are going to ‘look’ for a long exact sequence; I discussed them here so as to convince you that this is a natural thing to look for.

0.4.2 Derived functors

A fundamental problem in algebraic topology is to describe when and how complexes fail to be exact. We saw this concretely: the singular homology of a space $H_n(X)$ describes the extent to which n -simplices mapping into X did not arise as the boundaries of $(n + 1)$ -simplices. Many other (co)homologies are also just concrete exactness measurements, too: simplicial, cellular, de Rahm, ...

Philosophically, (co)homology describes 'obstruction' to exactness:

- If $H_1(X)$ is nontrivial (i.e. the singular chain is not exact at $C_1(X)$), then nontrivial topology of X is obstructing some 1-chain $\sigma \in C_1(X)$ from occurring in the image of $\partial : C_2(X) \rightarrow C_1(X)$. In particular, a '1-dimensional hole.'
- If X has trivial topology, then $H_{i>0}(X) = 0$.

We understand that homological algebra arose as a systemization of the algebra appearing in algebraic topology.

References

- [1] Matthew A. Niemi. *The derived functor approach to sheaf cohomology*. University of Chicago Mathematics REU. 2020.