HW 2

Matt Krueger

Question 1 ***(2.5 points)***:

If $z = 3e^{j\frac{\pi}{6}}$, find the value of e^z in the complex number form $A + jB$.

Equations:

$e^{j\theta} = \cos\theta + j\sin\theta$

$z = 3e^{j\pi/6}$

apply Euler's on inside

$= 3(\cos(\pi/6) + j\sin(\pi/6))$

$= 3(\sqrt{3}/2 + \frac{j}{2})$

$= \frac{3\sqrt{3}}{2} + \frac{3j}{2}$

$e^z = e^a(\cos b + j\sin b)$

$= e^{\frac{3\sqrt{3}}{2}}\left(\frac{3\sqrt{3}}{2} + \frac{3j}{2}\right)$

$13.44(0.07 + 0.997j)$

$e^z = \boxed{0.906 + j\,13.404}$

Question 2 ***(4 points; 1 point each)***:

Find the instantaneous time sinusoidal functions corresponding to the following phasors:

a. $\tilde{V} = -5e^{j\frac{\pi}{3}}$ [V]

b. $\tilde{V} = j6e^{-j\frac{\pi}{4}}$ [V]

c. $\tilde{I} = -3 + j2$ [A]

d. $\tilde{I} = j$ [A]

Equations:

$\tilde{Y} = Be^{j\theta}$

$y(t) = \mathrm{Re}(Be^{j\theta} \cdot e^{j\omega t})$

$= B\cos(\omega t + \theta)$

a) $\tilde{V} = -5e^{j\pi/3}$

$= -5\cos(\omega t + \pi/3)$

$= -5\sin(\omega t + \pi/3 + \frac{\pi}{2})$ — offset cos → sin $+\pi/2$

$= -5\sin(\omega t + \frac{5\pi}{6})$ — remove leading sign $-\pi$

$= 5\sin(\omega t + 5\pi/6 - \pi)$

$= \boxed{5\sin(\omega t - \pi/6)}$ V

b) $\tilde{V} = j6e^{-j\pi/4}$

$= e^{j\pi/2}\,6e^{-j\pi/4}$

$= 6e^{j\pi/4}$ — apply Euler

$= 6\cos(\omega t + \pi/4)$ — offset cos → sin $\pi/2$

$= \boxed{6\sin(\omega t + 3\pi/4)}$ V

c) $\tilde{I} = -3 + j2$

$= \sqrt{(-3)^2 + (2)^2}$

$\tan^{-1}(\frac{2}{-3}) = 146°$

$= \sqrt{13}\,e^{146°}$

$= \sqrt{13}\cos(\omega t + 146)$

$= \sqrt{13}\sin(\omega t + 146 + \pi/2)$

$= \boxed{\sqrt{13}\sin(\omega t + 4.124)\,A}$

$146 \cdot \frac{\pi}{180} = 2.55$

d) $\tilde{I} = j$ A

$= e^{j\pi/2}$

$= \cos(\omega t + \frac{\pi}{2})$

$= \cos(\omega t + \pi/2 + \pi/2)$ — sin → cos $+\pi/2$

$= \boxed{\sin(\omega t - \pi)}$ A

Question 3 ***(2.5 points)*****:**

Given the following vectors, show that $\vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{B}$.

$$\vec{A} = 2\hat{x} - 3\hat{y} + \hat{z}, \vec{B} = 2\hat{x} - \hat{y} + 3\hat{z}, \vec{C} = 4\hat{x} + 2\hat{y} - 2\hat{z}$$

Equations:

$A \cdot B = AB\cos\theta$

$\perp$ if $A \cdot B = 0$

$\rightleftarrows$ if $A \cdot B = 1$

$\vec{A} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$\vec{B} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

$\vec{C} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$

$\vec{C} \cdot \vec{A} = (8 - 6 - 2) = 0 \quad \perp$

$\vec{C} \cdot \vec{B} = (8 - 2 - 6) = 0 \quad \perp$

Question 4 ***(5 points; 2.5 points each)*****:**

Given $\vec{A} = 2\hat{x} - 3\hat{y} + \hat{z}$ and $\vec{B} = B_x\hat{x} + 2\hat{y} + B_z\hat{z}$, answer the following:

a. Find B_x and B_z if $\vec{A}$ is parallel to $\vec{B}$.
b. Find a relation between B_x and B_z if $\vec{A}$ is perpendicular to $\vec{B}$.

Equations:

$A \times B = \hat{n} AB\sin(\theta_{AB})$

$\rightleftarrows$ if $A \times B = 0$

a) $\vec{A} \times \vec{B} = \hat{x}(A_yB_z - A_zB_y) + \hat{y}(A_zB_x - A_xB_z) + \hat{z}(A_xB_y - A_yB_x) = \hat{x}(-3B_z - 2) - \hat{y}(2B_z - B_x) + \hat{z}(4 + 3B_x)$

$B_z = \frac{-2}{3}$

$B_x = \frac{-4}{3}$

$\frac{-2}{3} = \frac{-2(2)}{3}$

$\therefore$ yes $\rightleftarrows$

$A \cdot B = AB\cos(\theta)$

$\perp$ if $A \cdot B = 0$

b) $\vec{A} \cdot \vec{B} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} B_x \\ 2 \\ B_z \end{pmatrix} = 2B_x - 6 + B_z$

if $\perp$ then B_x = RHS

$\boxed{B_x = \frac{6 - B_z}{2}}$

Question 5 ***(8 points)*****:**

Given the plane $2x + 3y + 4z = 16$, find the unit normal vector to the plane, pointing in the direction away from the origin.

Equations:

$|n| = \sqrt{i^2 + j^2 + k^2}$

$\hat{n} = \frac{n}{|n|}$

$n = (2, 3, 4)$

$|n| = \sqrt{(2)^2 + (3)^2 + (4)^2}$

$= \sqrt{4 + 9 + 16}$

$= \sqrt{29}$

$$\boxed{\hat{n} = \left(\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right)}$$

Question 6 ***(7 points)*****:**

Consider the line $y = x - 1$. $\vec{A}$ starts at point $P_1 = (0,2)$ and ends at point P_2 on the line such that $\vec{A}$ is orthogonal to the line. Find $\vec{A}$.

Equations:

$\vec{A} = P_2 - P_1$

ortho $= \perp$

i.e. $\vec{A} \cdot \vec{d} = 0$

$y = x - 1 \Rightarrow \vec{d} = (1, 1)$

$P_2 = (x, x-1)$

$\vec{A} = P_2 - P_1$

$= (x, x-1) - (0, 2)$

$= (x, x-3)$

$\vec{A} \cdot \vec{d} = (x, x-3) \cdot (1, 1)$

$= 2x - 3$

$2x - 3 = 0$

$x = \frac{3}{2}$

P_2 such that $x = \frac{3}{2}$

$y = \left(\frac{3}{2}\right) - 1 = \frac{1}{2}$

$\therefore P_2 = \left(\frac{3}{2}, \frac{1}{2}\right)$

$\vec{A} = P_2 - P_1$

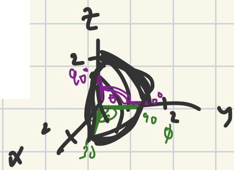
$$= \boxed{\vec{A} = \left(\frac{3}{2}, -\frac{3}{2}\right)}$$

Question 7 ***(6 points; 2 points each)*****:**

A section of a sphere is described by the following: $0 \le R \le 2$, $0 \le \theta \le 90°$, and $30° \le \phi \le 90°$. Complete the following:

a. Find the surface area of the spherical section.
b. Find the enclosed volume.
c. Sketch the outline of the section.

c)



Equations:

(R, θ, ϕ)

$dl_R = dR$

$dl_\theta = R\, d\theta$

$dl_\phi = R \sin\theta\, d\phi$

$S = R^2 \int \sin\theta\, d\theta \int d\phi$

a) surface area: R^2

$S = R^2 \int \sin\theta\, d\theta \int d\phi$

$= 2^2 \int_0^{\pi/2} \sin\theta\, d\theta \int_{\pi/6}^{\pi/2} d\phi = 4\left(-\cos\theta \Big|_0^{\pi/2}\right)\left(\pi/2 - \pi/6\right)$

$= -\cos(\pi/2) + \cos(0)$

$0 + 1$

$4(1)(\pi/3)$

$$\boxed{S = \frac{4\pi}{3} \text{ units}^2}$$

b) $v = R^2 \sin\theta\, dR\, d\theta\, d\phi = \int_0^2 R^2\, dR \int_0^{\pi/2} \sin\theta\, d\theta \int_{\pi/6}^{\pi/2} d\phi$

$= \left(\frac{R^3}{3}\Big|_0^2\right)\left(-\cos\theta \Big|_0^{\pi/2}\right)\left(\phi \Big|_{\pi/6}^{\pi/2}\right) = \boxed{\frac{8\pi}{9} \text{ units}^3}$

Question 8 ***(6 points; 2 points each)*****:**

Convert the coordinates of the following points from cylindrical coordinates to Cartesian coordinates: $(r, \phi, z) \longrightarrow (x, y, z)$

a. $P_1 = \left(2, \frac{\pi}{4}, -3\right)$
b. $P_2 = (3, 0, -2)$
c. $P_3 = (4, \pi, 5)$

Equations:

$x = r\cos\phi$
$y = r\sin\phi$
$z = z$

$P = (r, \phi, z)$

a) $P_1 = (2, \frac{\pi}{4}, -3)$ $\Longrightarrow$ $\boxed{P_1 = (\sqrt{2}, \sqrt{2}, -3)}$

$x = 2\cos(\frac{\pi}{4}) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$
$y = 2\sin(\frac{\pi}{4}) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$
$z = -3$

b) $P_2 = (3, 0, -2)$ $\Longrightarrow$ $\boxed{P_2 = (3, 0, -2)}$

$x = 3\cos(0) = 3 \cdot 1 = 3$
$y = 3\sin(0) = 3 \cdot 0 = 0$
$z = -2$

c) $P_3 = (4, \pi, 5)$ $\Longrightarrow$ $\boxed{P_3 = (-4, 0, 5)}$

$x = 4\cos(\pi) = 4 \cdot -1 = -4$
$y = 4\sin(\pi) = 4 \cdot 0 = 0$
$z = 5$

Question 9 ***(3 points)*****:**

Point $P = \left(2\sqrt{3}, \frac{\pi}{3}, -2\right)$ is given in cylindrical coordinates. Express point P in spherical coordinates. $(r, \phi, z) \rightarrow (R, \theta, \phi)$

Equations:

$R = \sqrt{r^2 + z^2}$
$\theta = \tan^{-1}\left(\frac{r}{z}\right)$
$\cos(\theta) = \frac{z}{R}$
$\sin(\theta) = \frac{r}{R}$
$\phi = \phi$

$P = (2\sqrt{3}, \frac{\pi}{3}, -2)$ $\Longrightarrow$ $\boxed{P = (4, \frac{2\pi}{3}, -2)}$

- $R = \sqrt{(2\sqrt{3})^2 + (-2)^2}$
$= \sqrt{12 + 4}$
$= \sqrt{16}$
$= 4$

- $\cos\theta = \frac{z}{R} = \frac{-2}{4} = \frac{-1}{2}$

$\theta = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

$\phi = \phi \rightarrow -2 = -2$

Question 10 ***(3 points)*:** $(x, y, z) \longrightarrow (r, \phi, z)$

Transform the following vector from Cartesian coordinates to cylindrical coordinates:

$$\vec{A} = (x+y)\hat{\mathbf{x}} + (y-x)\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

Equations:

$r = \sqrt{x^2+y^2}$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$

$x = r\cos(\phi)$

$y = r\sin(\phi)$

$z = z$

$A_r = A_x\cos\phi + A_y\sin\phi$

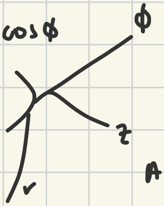
$A_\phi = -A_x\sin\phi + A_y\cos\phi$

$A_z = A_z$

$A_x = r(\cos\phi + \sin\phi)$

$A_y = r(\sin\phi - \cos\phi)$

$A_z = z$



$A_\phi = -A_x\sin\phi + A_y\cos\phi$

$= -r(\cos\phi + \sin\phi)\sin\phi + r(\sin\phi - \cos\phi)\cos\phi$

$= \underline{-r}$

$\underline{A_z = z}$

$A_r = A_x\cos\phi + A_y\sin\phi = r(\cos\phi + \sin\phi)\cos\phi + r(\sin\phi - \cos\phi)\sin\phi$

$= r(\cos^2\phi + \sin\phi\cos\phi + \sin^2\phi - \sin\phi\cos\phi)$

$= \underline{r}$

$A = r\hat{r} - r\hat{\phi} + z\hat{z}$

Question 11 ***(3 points; 1 point each)*:**

Using the appropriate differential surface element ds, determine the areas of the following surfaces and sketch the surfaces.

a. $r = 3; 0 \leq \phi \leq \frac{\pi}{3}; -2 \leq z \leq 2$ cylindrical (r, ϕ, z)

b. $R = 2; 0 \leq \theta \leq \frac{\pi}{3}; 0 \leq \phi \leq \pi$ spherical (R, θ, ϕ)

c. $0 \leq R \leq 5; \theta = \frac{\pi}{3}; 0 \leq \phi \leq 2\pi$ conical

a) $ds = r\,d\phi\,dz$

$A = r\int_0^{\pi/3} d\phi \int_{-2}^{2} dz = 3\left(\frac{\pi}{3}\right)\cdot 4 = \boxed{4\pi \text{ units}^2}$

b) $ds = R^2\sin\theta\,d\theta\,d\phi$

$A = R^2\int_0^{\pi/3}\sin\theta\,d\theta\int_0^{\pi}d\phi = 4\left[-\cos(\theta)\big|_0^{\pi/3}\right]\left[\phi\big|_0^{\pi}\right] = 4\left(1-\frac{1}{2}\right)\pi = \boxed{2\pi \text{ units}^2}$

c) $ds = R\sin(\theta)\,dR\,d\phi$

$A = \sin(\theta)\int_0^5 R\,dR\int_0^{2\pi}d\phi = \sin\left(\frac{\pi}{3}\right)\left(\frac{R^2}{2}\Big|_0^5\right)(2\pi)$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{25}{2}\right)(2\pi) = \boxed{\frac{25\sqrt{3}}{2}\pi \text{ units}^2}$