Matt O'Brien

MIDTERM #2 MATH 710

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Exercise 1

Show that the function $f(x) = \frac{1}{\sqrt[3]{(1-x)}}$ is Lebesgue integrable on the interval $(0,1)$ and evaluate the integral $\int_0^1 \frac{1}{\sqrt[3]{(1-x)}}$

Proof. First, note that the function is not Riemann integrable since the function is not bounded over $E = (0,1)$. To demonstrate Lebesgue integrability, first we let E be the union of a countable family of pairwise disjoint measurable sets. Next, we define an increasing sequence of functions, f_n, each of which are non negative and measurable over E. Let f_n converge to f and note that f_n must converge pointwise almost everywhere to f since f is unbounded at a single point.
We will show f is Lebesgue integrable directly.
First observe that $f_n \leq f$ over E, for all f_n.
Thus, $\int_E f_n \leq \int_E f$. Since f is not bounded, we consider the lim sup of f_n. We know that lim sup$\int_E f_n \leq \int_E f$. Recall that the convergence is pointwise almost everywhere. By Fatou's Lemma, $\int_E f \leq$lim sup$\int_E f_n \Rightarrow$lim inf$\int_E f_n =$lim sup$\int_E f_n$. Therefore, lim $\int_E f_n = \int_E$lim$f_n = \int_E f$, which shows that the integral exists.
Notice that lim sup$\int_E f_n$ equals $\frac{3}{2}$, which equals $\int_E f$ and we are done. □

Exercise 2

Let $E = \cup_{k=0}^{\infty} E_k$, where $E_k = (2^{-k-1}, 2^{-k}]$, and let $f(x) = (-1)^k$ for $x \in E_k$. Evaluate the integral $\int f$ or show that the function f is not integrable.

Proof. Observe that for $E_k = (2^{-k-1}, 2^{-k}]$, each subinterval is pairwise disjoint. By the additive property of the integral (Theorem 3.7), and Exercise 3.1a,

$$\int_E f = \sum_{k=0}^{\infty} \int_{E_k} f = \sum_{k=0}^{\infty} c \cdot m(E_K) = \sum_{k=0}^{\infty} (-1)^k m(E_k).$$

Further,

$$\sum_{k=0}^{\infty}(-1)^k m(E_k) = (-1)^0 m(E_0) + (-1)^1 m(E_1) + (-1)^2 m(E_2)...$$
$$= m(E_0) - m(E_1) + m(E_2) + \dots$$

Now, observe that $E_0 = \left(\frac{1}{2}, 1\right]$, $E_1 = \left(\frac{1}{4}, \frac{1}{2}\right]$, $E_2 = \left(\frac{1}{8}, \frac{1}{4}\right]$.
Thus,

$$\sum_{k=0}^{\infty}(-1)^k m(E_k) = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} + \dots$$

This is a geometric series whose first term is $\frac{1}{2}$ and whose common ration is $-\frac{1}{2}$, so it's sum is $\frac{1}{3}$; further, this is an alternating series and the convergence is absolute. □

EXERCISE 3

Let $E = \cup_{k=1}^{\infty}$ where all sets E_k's are measurable and $mE < \infty$. Suppose that f is bounded and integrable on every E_k.
True or False: f is integrable on E. Justify your answer.

Proof. False: to show that f is not integrable on E, we furnish a counterexample:
Suppose that $E = \cup_{k=1}^{\infty}$, where $E_k = \left(\frac{1}{k+1}, \frac{1}{k}\right]$.
Consider $f = \frac{1}{x}$. Notice that f is continuous and bounded on each interval E_n. But since f is not integrable at $x = 0$, f is not integrable over E. □