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Assignment 7

December 2, 2012

EXERCISE 4.3

Show that the function

$$\boldsymbol{f(x)} = \begin{cases} \boldsymbol{x^2 sin\frac{1}{x^2}} & \textbf{if } \boldsymbol{x \neq 0} \\ \boldsymbol{0} & \textbf{if } \boldsymbol{x = 0} \end{cases}$$

is not of a bounded variation over the interval $[-1,1]$.

Proof. Select the partition as follows:
For all $n \in \mathbb{N}$,

$$P_n = \frac{1}{\sqrt{j^\pi}} : j = \pm 1 \pm 2 \cdots \pm n\} \cup \{\pm 1\}.$$

Then,

$$t_{P_n} \geq 2 \sum_{j=1}^{n-1} \frac{1}{\sqrt{j^\pi}} 2 \to_{n\to\infty} \infty,$$

so that $T_{-1}^{+1} = \infty$. □

EXERCISE 4.4

Show that the function

$$\boldsymbol{f(x)} = \begin{cases} \boldsymbol{x^2 cos\frac{1}{x}} & \textbf{if } \boldsymbol{x \neq 0} \\ \boldsymbol{0} & \textbf{if } \boldsymbol{x = 0} \end{cases}$$

is a bounded variation over the interval $[0,1]$.

Proof. Not sure. □

EXERCISE 4.5

Let f be a BV-function on $[a,b]$ and $[c,d]$ be a nontrivial subinterval of $[a,b]$. Then f is a BV-function over $[c,d]$.

Proof. Consider a partition P of $[c,d]$ where $P = \{x_i = 1 \leq i \leq n\}$. Assume that $f(x_i) - f(x_{i-1})$ is bounded with regard to P. Then, the variation of f over $[c,d]$, $V(f,[c,d]_)$ is the supremum of the set S where $S = \sum_{i=1}^{n} |\{f(x_i) - f(x_{i-1})|$. If this supremum exists and is finite, then we say that the function is a BV function over $[c,d]$. □

EXERCISE 4.6

Show that the sum, difference and product of two BV-functions is a BV-function.

Proof. Let f, g, be two BV functions over $[a,b]$. Define a partition, $P = \{x_i = 1 \leq i \leq n\}$. Then ,

$$\begin{aligned}
\sum_{i=1}^{n} |(f+g)(x_i) - (f+g)(x_{i-1})| &= \sum_{i=1}^{n} |\{f(x_i) + g(x_i)\} - \{f(x_{i-1} + g(x_{i-1})\}| \\
&\leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1}| + \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| \\
&\leq V(f,P) + V(g,P)
\end{aligned}$$

$\Rightarrow V(f+g,P) \leq V(f,P) + V(g,P) \leq V(f,P) + V(g,P) \Rightarrow V(f+g,P) \leq V(f,P) + V(g,P)$. Thus, $(f+g)$ is a function of bounded variation.
To show that $(f-g)$ is of bounded variation, the proof is the same and $V(f-g) \leq V(f) + V(g)$
To show that the product of two functions of bounded variation is also of bounded variation, notice that both f and g are bounded, and thus there exists $K \in \mathbb{N}$ such that $|f(x)| \leq k, |g(x)| \leq k, \forall x \in P$.
Thus,

$$\begin{aligned}
\sum_{i=1}^{n} |(fg)(x_i) - (fg)(x_{i-1}| &= \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1}g(xi-1)| \\
&= \sum_{i=1}^{n} |f(x_i)\{g(x_i) - g(x_{i-1}\} + g(x_{i-1}\{f(x_i) - f(x_{i-1})\}| \\
&\leq \sum_{i=1}^{n} \{|f(x_i)||g(x_i) - g(x_{i-1}| + |g(x_{i-1}||f(x_i) - f(x_{i-1})|\} \\
&\leq k\sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| + k\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \\
&\leq kV(g) + kV(f).
\end{aligned}$$

the product of two functions of bounded variation is also of bounded variation. □

EXERCISE 4.7

Let f and g be two BV-functions on $[a,b]$. Show that f/g is a BV-function provided that $g(x) \geq \alpha > 0$ on $[a,b]$.

Proof. Let $h = \frac{1}{g}$. Then, for any partition P over $[a,b]$,

$$\begin{aligned}
\sum \left| \frac{1}{f}(x_i) - \frac{1}{f}(x_{i-1}) \right| &= \sum \left| \frac{1}{f(x_i)} - \frac{1}{f(x_{i-1}} \right| \\
&= \sum \left| \frac{f(x_{i-1}) - f(x_i)}{f(x_i)f(x_{i-1})} \right| \\
&\leq \frac{1}{k^2} \sum \left| f(x_{i-1} - f(x_i) \right| \\
&\leq \frac{1}{k^2} V(f,P).
\end{aligned}$$

This shows that $\frac{1}{g} = h$ is a BV-function. Then, by the previous question, $f \cdot h = \frac{f}{g}$ is a BV-function. □

EXERCISE 4.8

Show that the Dirichlet's function

$$\boldsymbol{f(n)} = \begin{cases} \mathbf{1}, & \textbf{if } \boldsymbol{x} \textbf{ is a rational number} \\ \mathbf{0}, & \textbf{otherwise} \end{cases}$$

on $[0,1]$ is not of a bounded variation.

Proof. We start by construction a partition $P = \{a = x_0, x_1, \ldots, x_{n+2}\}$.
Let x_1 be an irrational number inbetween a and b. Then let the next number in the partition, x_2, be a rational number inbetween x_1 and b. Since by Theorem 1.18 in the textbook by Wade from last semester, there is a rational number and an irrational number inbetween any two real numbers. So we construct the partition in that manner until between we have an interval that alternates between rationals and irrationals.
Now we consider the sum $\sum_{i=1} n+2 \left| f(x_i) - f(x_{i-1}) \right|$. This sum is bounded by the variation of f over $[a,b]$. Using this sum we state the following inequality:

$$\begin{aligned}
V(f,P) &\geq \sum_{i=1}^{n+2} \left| f(x_i) - f(x_{i-1}) \right| \\
&\geq \sum_{i=2}^{n+1} \left| f(x_i) - f(x_{i-1}) \right| \\
&= \left| f(x_2) - f(x_1) \right| + \cdots + \left| f(x_{n+1} - f(x_n) \right| \\
&= 1 + 1 + 1 + \cdots + 1 \\
&= n.
\end{aligned}$$

Since $V(f,P) = n$, where we can choose an n of any size. Thus $V(f,P) = \infty$. □

EXERCISE 4.9

Show that if f has a bounded derivative on $[a,b]$, then f is of bounded variation.

Proof. Since f has a bounded derivative on $[a,b]$ we write $\left|f'\right| \leq M$ on $[a,b]$. Now, consider any two points $x_{k-1}, x_k \in [a,b]$. By the Mean Value Theorem, we have $\left|f(x_k) - f(x_{k-1})\right| \leq M\sum_{k=1} n|x_k - x_{k-1}|$. Since this inequality is for any two arbitrary points, the for any partition we can write

$$\sum_{k=1}^{n} \left|f(x_k) - f(x_{k-1})\right| \leq M \sum_{k=1}^{n} |x_k - x_{k-1}| = M(b-a) \Rightarrow V_a^b f \leq M(b-a).$$

□