MATT O'BRIEN

Assignment V

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EXERCISE 1

Show from first principles that if f^1, f^2,...,f^k : $\mathbb{R} \to \mathbb{R}$ are convex (concave) functions with the same domain, and if ω_1,...,ω_k are non-negative scalars, then the function $\omega_1 f^1$+...+$\omega_k f^k$ is also convex (concave).

Proof. Given f^1, f^2,...,f^k as convex functions and ω_1,...,ω_k as non-negative scalars, we need to show that $\omega_1 f^1$+...+$\omega_k f^k$ is also convex; that is,

$$\sum_{i=1}^{k} \omega_i f_i$$

is a convex function.
Using first principles, we consider a linear combination $\lambda x + (1-\lambda) y$ where $\lambda \in [0,1]$, $x, y \in$ the domain of all f^k.

$$\begin{aligned}
f(\lambda x + (1-\lambda) y &= \sum_{i=1}^{k} \omega_i f_i (\lambda x + (1-\lambda) y) \\
&\leq \sum_{i=1}^{k} \omega_i (\lambda f_i(x) + (1-\lambda) f_i(y)) \\
&= \lambda \sum_{i=1}^{k} \omega_i f_i(x) = (1-\lambda) \sum_{i=1}^{k} \omega f_i(y) \\
&= \lambda f(x) + (1-\lambda) f(x).
\end{aligned}$$

Thus we have shown that convexity holds under this operation. □

EXERCISE 2

A) Let $f(x):\mathbb{R}^n \to \mathbb{R}$ be a convex function. Show that for any $\alpha \in \mathbb{R}$ the set $\{x \in \mathbb{R} : f(x) \leq \alpha\}$ is a convex set.

Proof. Let $f(x)$ be convex, where $f(x):\mathbb{R} \to \mathbb{R}$. Let $\alpha \in \mathbb{R}$. Let $\lambda \in [0,1]$. Consider the set $\{x \in \mathbb{R} : f(x) \leq \alpha\}$.
Take two vectors from this set: let **a**$= (x_1,\ldots,x_n)$, let **b**$= (w_1,\ldots,w_n) \in \{x \in \mathbb{R} : f(x) \leq \alpha\}$, such that $f(x_1,\ldots,x_n) < y$, and $f(w_1,\ldots,w_n) < v$.
We NTS that the convex combination of these two vectors is still within the set; we check directly by considering $\lambda(x_1,\ldots,x_n) + (1-\lambda)(w_1,\ldots,w_n)$. By distributing and grouping we get the vector $t = \lambda x_1 + (1-\lambda)w_1 + \cdots + x_n + (1-\lambda)(w_1,\ldots,w_n)$.
Recall from lecture the general form for convex functions:

$$f(\lambda(a) + (1-\lambda b) \leq \lambda f(a) = (1-\lambda)f(b).$$

In our instance, and since $f(a) \leq y$ and $f(b) \leq v$, we write

$$f(\lambda x_1 + (1-\lambda)w_1 + \ldots \lambda x_n + (1-\lambda)w_n) \leq \lambda f(x_1,\ldots,x_n) + (1-\lambda)f(w_1,\ldots,w_n). \leq \lambda y + (1-\lambda)v.$$

□

B) Show that the converse is not true. Give an example of a non-convex function with the above sets for every α convex.

Proof. As a counterexample consider the non-concave function $f(x) = -e^x$, of which the epigraph is convex. □

EXERCISE

The geometric mean function is $f(x_1,x_2,...,x_n) = \left(\Pi_{i=1}^n\right)^{\frac{1}{n}}$ defined on the positive orthant $\mathbb{R}^n_{>0} = (x_1,...,x_n) : x_i > 0, i = 1,...,n$. Show that this function is concave by computing its Hessian and using similar methods as in the case of log-sum-exp function.

Proof. The Hessian of the geometric mean function is $\triangledown^2 f(x) = \partial^2 f(x)$ □