Assignment 8 | Math 710

December 6, 2012

EXERCISE 4.11

Show that a BV function is bounded.

Proof. WLOG, consider a function of BV over the interval $[a,b]$ with the partition defined by $P = \{a, x, b\}$. We NTS that $|f(x)| \leq \infty$. By Triangle Inequality,

$$|f(x)| = |f(x) - f(a) + f(a)| \leq |f(a)| + |f(x) - f(a)| + |f(b) - f(x)| = |f(a)| + V_a^b(P, f) \leq |f(a)| + V_a^b(f).$$

Since $V_a^b(f)$ is finite, $|f(a)| + V_a^b(f)$ is clearly finite as well. Thus $f(x)$ is bounded. □

EXERCISE 4.18

Let $\{I_1, \ldots, I_n\}$ be a set of bounded intervals such that $m(I_1) \geq m(I_i)$ and $I_i \cap I_1 \neq 0$ for all $1 \leq i \leq n$. Show that:
a) $\bigcup_{i=1}^{n} I_n$ is a bounded interval.
b) $m(I_i) \geq \frac{1}{3} m\left(\bigcup_{i=1}^{n} I_1\right)$

Proof. a) To show that the union is bounded, we need to show that $\inf \bigcup_{i=1}^{n} I_n$ and $\sup \bigcup_{i=1}^{n} I_n$ exist.
Since every interval is bounded, supremum exists for each I_i; it follows that $\sup \bigcup_{i=1}^{n} I_n$ is the supremum of this set of supremum. For $\inf \bigcup_{i=1}^{n} I_n$, the process follows the same reasoning. □

Proof. b) Given the set of bounded intervals $\{I_1, \ldots, I_n\}$ and the fact that $m(I_1) \geq m(I_i)$, we can select a nonempty subset S such that $m(I_1) \geq m(I_i)$ for $i \in S$. Then by Lemma 4.21, $m(I_1) \geq \frac{1}{3} m\left(\bigcup_{i=1}^{n} I_i\right)$. □

EXERCISE 4.19

Show that a bounded set E that is not of measure zero has a positive outer measure, $m^*(E) > 0$.

Proof. By hypothesis E has measure; thus $m(E)$ must be positive by definition of measure. By Definition 2.28, $m(E) = m^*(E) > 0$ and we are done. ☐

EXERCISE 4.20

Let A be a finite or countable subset of a bounded set E. Show that E is a set of measure zero if and only if $E\backslash A$ is a set of measure zero.

Proof. ($\Rightarrow$)
Let the bounded set E be the union of an at most countable family of sets $\{E_i\}_{i\in J}$. Since A is a finite or countable subset of E, by Lemma 4.22, $m(E\backslash A) = 0$.
($\Leftarrow$) Let $m(E\backslash A) = 0$. Recall A is a finite or countable subset of E, so $m(A) = 0$, and by Lemma 4.22, $m(E) = 0$. ☐