

Points, Lines, and Vectors

CS 491 CAP

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Objectives

- ▶ Identify some of the corner cases computational geometry problems have.
- ▶ Develop a strategy for dealing with geometry problems in a contest.
- ▶ Review basic formula for
 - ▶ Points
 - ▶ Lines
 - ▶ Vectors
- ▶ Most code samples from Competitive Programming 3.

Contest Strategy

- ▶ These problems can be tricky
 - ▶ Tedious coding
 - ▶ High probability of WA initially
- ▶ Edge cases!
 - ▶ What is lines are parallel?
 - ▶ Can the polygons be concave?
 - ▶ Check your assumptions!
- ▶ Strategy
 - ▶ Usually solve these last
 - ▶ Bring library code to the contest

Representing Integer Points

```
struct point_i {  
    int x, y;  
    point_i() { x = y = 0; }  
    point_i(int _x, int _y) : x(_x), y(_y) {}  
    bool operator==(point_i & other) const {  
        return x == other.x && y == other.y;  
    }  
    bool operator<(point_i & other) const {  
        if (x == other.x)  
            return y < other.y;  
        else return x < other.x;  
    }  
};
```

Representing Floating Points

```
#include <math.h>
```

```
#define EPS 1E-9
```

```
struct point {  
    double x, y;  
    point() { x = y = 0; }  
    point(double _x, double _y) : x(_x), y(_y) {}  
    bool operator==(point & other) const {  
        return fabs(x - other.x) < EPS && fabs(y - other.y) < EPS;  
    }  
    bool operator<(point & other) const {  
        if (fabs(x - other.x) < EPS)  
            return y < other.y;  
        else return x < other.x;  
    }  
};
```

Formulae

- ▶ Distance between two points

$$\sqrt{(a_x - b_x)^2 + (a_y - b_y)^2}$$

- ▶ Counter-clockwise rotation by θ

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

```
double hypot(point &a, point &b) const {  
    double dx=a.x-b.x;  
    double dy=a.y-b.y;  
  
    return sqrt(dx * dx + dy * dy);  
}  
  
point ccw(point &a, double &theta) const {  
    double st = sin(theta);  
    double ct = cos(theta);
```

Formula for a line

- ▶ You all know the formula for a line, right?

$$y = mx + b$$

- ▶ What is wrong with using this formula?
- ▶ Hint: remember that the problem setters are trying to ruin you.

Representation

$$ax + by + c = 0$$

- ▶ if $b = 0$ the line is vertical.

// From Competitive Programming 3

```
struct line {  
    point a, b, c;  
};
```

```
void pointsToLine(point p1, point p2, line &l) {  
    if (fabs(p1.x - p2.x) < EPS) {// vertical line  
        l.a = 1.0; l.b = 0.0; l.c = -p1.x;  
    } else {  
        l.a = -(double)(p1.y - p2.y) / (p1.x - p2.x);  
        l.b = 1.0; // IMPORTANT: we fix the value of b to 1.0  
        l.c = -(double)(l.a * p1.x) - p1.y; // c = -ax - b  
    } }  
}
```


Parallel Lines

- ▶ Given lines $a_1x + b_1y + c_1$ and $a_2x + b_2y + c_2$
 - ▶ If $a_1 = a_2 \wedge b_1 = b_2$ the lines are parallel.
 - ▶ If also $c_1 = c_2$ the lines are identical.

```
bool areParallel(line l1, line l2) {  
    return fabs(l1.a-l2.a) < EPS &&  
           fabs(l1.b-l2.b) < EPS;  
}  
  
bool areSame(line l1, line l2) {  
    return areParallel(l1 ,l2) &&  
           fabs(l1.c - l2.c) < EPS;  
}
```

Intersections

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

// returns true (+ intersection point) if two lines are intersecting

```
bool areIntersect(line l1, line l2, point &p) {  
    if (areParallel(l1, l2)) return false; // no intersection  
    // solve system: 2 equations with 2 unknowns  
    p.x = (l2.b * l1.c - l1.b * l2.c) / (l2.a * l1.b - l1.a * l2.b);  
    // special case: test for vertical  
    if (fabs(l1.b) > EPS)  
        p.y = -(l1.a * p.x + l1.c);  
    else p.y = -(l2.a * p.x + l2.c);  
    return true;  
}
```

Representation

- ▶ A vector represents a direction. Similar to a point, but different interpretation.

```
struct vec {  
    double x, y;  
    vec(double _x, double _y) : x(_x), y(_y) {}  
};
```

```
// convert 2 points to vector a->b  
vec toVec(point a, point b) {  
    return vec(b.x - a.x, b.y - a.y);  
}
```

```
// scale v  
vec scale(vec v, double s) {  
    return vec(v.x * s, v.y * s);  
}
```

Norm and Dot Product

- ▶ The norm of vector A:

$$|A| = \sqrt{\sum_i A_i^2}$$

- ▶ How is this related to the hypotenuse of a triangle?
- ▶ The dot product “multiplies” vectors.

$$A \cdot B = |A||B| \cos(\theta)$$

- ▶ You don't have to compute $\cos(\theta)$!

$$\sum_i A_i B_i$$

- ▶ If zero, then the vectors are at right angles.

```
double dot(vec a, vec b) {  
    return (a.x * b.x + a.y * b.y);  
}
```

```
double norm_sq(vec v) {  
    return v.x * v.x + v.y * v.y;
```

Distance from point to line

- ▶ Given: point p and line ab .

$$ap \cdot ab = |ap||ab| \cos(\theta)$$

- ▶ Calculate u , the fraction of the line where an intersection occurs.

$$u = ap \cdot ab / |ab|^2 = |ap| \cos(\theta) / |ab|$$

- ▶ The variable u will be between 0 to 1 if the intersection is between the points given. It's still valid even if not.
- ▶ Intersection point is $c = a + |ab|u$

// returns the distance from p to the line ab

```
double distToLine(point p, point a, point b, point &c) {  
    vec ap = toVec(a, p), ab = toVec(a, b);  
    double u = dot(ap, ab) / norm_sq(ab);  
    c = translate(a, scale(ab, u));  
    return dist(p, c);  
}
```

Shortest Distance: Line Segment

```
double distToLineSeg(point p, point a, point b, point &c) {
    vec ap = toVec(a, p), ab = toVec(a, b);
    double u = dot(ap, ab) / norm_sq(ab);
    if (u < 0.0) {
        c = point(a.x, a.y); // closer to a
        return dist(p, a);
    }
    // Euclidean distance between p and a
    if (u > 1.0) {
        c = point(b.x, b.y); // closer to b
        return dist(p, b);
    }
    // Otherwise, do the normal thing
    c = translate(a, scale(ab, u));
    return dist(p, c);
}
```

Angles

- ▶ The angle between two lines induced by three points aob
- ▶ Dot product $oa \cdot ob = |oa| \times |ob| \times \cos(\theta)$
- ▶ Solve for θ to get $\theta = \arccos(oa \cdot ob / (|oa| \times |ob|))$

// returns angle aob in radians

```
double angle(point a, point o, point b) {  
    vec oa = toVector(o, a), ob = toVector(o, b);  
    return acos(dot(oa, ob) /  
                sqrt(norm_sq(oa) * norm_sq(ob)));  
}
```

Cross Products

- ▶ Cross product:

$$a \times b = a.x \times b.y - a.y \times b.x$$

- ▶ Given a line p, q and point r
- ▶ Let a be the vector pq and b be the vector pr
 - ▶ Magnitude $|a \times b| = |a||b| \sin(\theta)$ is area of parallelogram.
 - ▶ Positive means $p \rightarrow q \rightarrow r$ is a left turn.
 - ▶ Zero means p, q, r are colinear.
 - ▶ Negative means $p \rightarrow q \rightarrow r$ is a right turn.

// returns true if point r is on the left side of line pq

```
bool ccw(point p, point q, point r) {  
    return cross(toVec(p, q), toVec(p, r)) > EPS;  
}
```

// returns true if point r is on the same line as the line

```
bool collinear(point p, point q, point r) {  
    return fabs(cross(toVec(p, q), toVec(p, r))) < EPS;  
}
```