Catalan Numbers

CS 491 – Competitive Programming

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Objectives

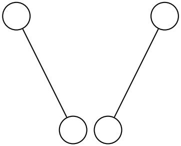
- Compute the nth Catalan number
- Map the Catalan numbers to various isomorphisms.

Example 1: Parenthesis

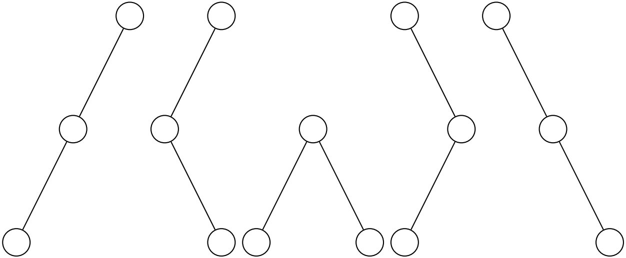
- Suppose you have a $2n$ parenthesis characters, half open, half closed. How many "algebraically legal" sequences are there?
  - Every open parenthesis followed by a matching close parenthesis.
  - No close parenthesis before its corresponding open.
- $n = 2$
  - ()()
  - (())
- $n = 3$
  - ((()))
  - ()(())
  - ()()()
  - (())()
  - (()())
- For $n = 4$ there are 14.

Example 2: Trees

- Given n indistinguishable nodes, how many binary trees can you make?
- $n = 2$



- $n = 3$

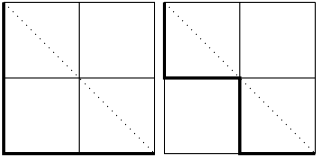


Example 3: Integer Sequences of +1 and -1

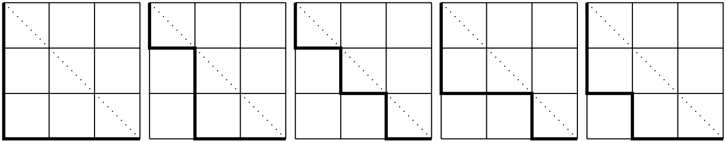
- Suppose you have n copies of 1 and n copies of $-$.
- How many sequences of $a_0, a_1, ...a_{2n}$ are there such that for all $k < 2n$, $0 \leq \Sigma_{i=0}^{2k} a_i$?
- $n = 2$
  - 1,1,-1,-1
  - 1,-1,1,-1
- $n = 3$
  - 1,-1,1,-1,1,-1
  - 1,-1,1,1,-1,-1
  - 1,1,-1,-1,1,-1
  - 1,1,-1,1,-1,-1
  - 1,1,1,-1,-1,-1

Example 4: Matrix Walk

- Suppose you have a $n \times n$ matrix. Start from the top left and reach the bottom right without crossing the diagonal.
- $n = 2$



- $n = 3$



The Formula

- Recursively: $Can(n) = \frac{(2n+1)(2n)}{(n+1)n} Cat(n-1)$
  - $Cat(0) = 1, Cat(1) = 1, Cat(2) = 2, Cat(3) = 5, Cat(4) = 14, \ldots$
- Others things:
  - number of ways to triangulate a polygon
  - Number of ways to count a tie vote so that candidate A never passes candedate B