Density of Visible Points from Origin

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Table of Contents

High Level Overview

Our main goal is to prove the following theorem.

Theorem

The set of lattice points visible from the origin has density $6/\pi^2$.

Definition

Two lattice points P and Q are said to be mutually visible if the line segment which joins them contains no lattice points other than the endpoints P and Q.

Definition

Let $N(r)$ denote the number of lattice points in the square defined by $|x| \leq r$ and $|y| \leq r$ and let $N'(r)$ be the number of points visible from the origin in the same square. The density is computed by $\lim_{r\to\infty} \frac{N'(r)}{N(r)}$.

Important Results

Lemma

Two lattice points (a, b) and (m, n) are mutually visible iff $\gcd(a - m, b - n) = 1$.

Thus, our main theorem can also be reworded as “the probability that two randomly chosen integers are coprime is $\frac{6}{\pi^2} \approx 0.6079$”.

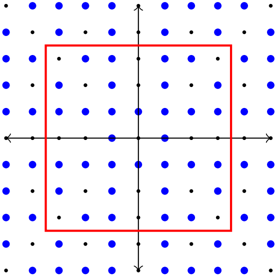
Proposition

For $r \geq 1$, we have $N'(r) = 8 \sum_{1 \leq n \leq r} \varphi(n)$.

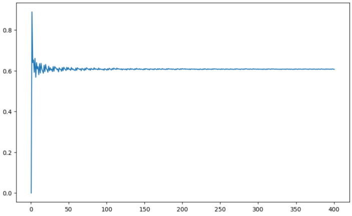
Proposition

For $r \geq 1$, we have $N(r) = (2\lfloor r \rfloor + 1)^2$.

Empirical Results



r	$N'(r)$	$N(r)$	$N'(r)/N(r)$
0.5	0	1	0
1.5	8	9	0.8889
2.5	16	25	0.64
3.5	32	49	0.6531
⋮	⋮	⋮	⋮
100.5	24352	40401	0.6028
200.5	97856	160801	0.6086



Mobius and Euler Totient

Definition

Let $n = p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_k}$. Then, the Mobius function μ satisfies

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } e_1 = \cdots = e_k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Definition

The Euler totient function φ is the number of positive integers less than or equal to n which are coprime to n.

Dirichlet Convolutions

Definition

If f and g are arithmetic functions, then the Dirichlet convolution $f * g$ is a new arithmetic function defined by

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

The following properties hold:

- $f * g = g * f$
- $(f * g) * h = f * (g * h)$
- $f * (g + h) = f * g + f * h$
- $f * \epsilon = \epsilon * f = f$

Sum of Mobius Function

The last property implies there exists an identity element ϵ. Namely, it is the function defined by

$$\epsilon(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Proposition

We have

$$(\mu * 1)(n) = \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} = \epsilon(n).$$

Mobius Inversion Formula

Theorem

If $g(n) = \sum_{d|n} f(d)$ then $f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right)$.

In terms of Dirichlet convolutions, this is equivalent to saying if $g = f * 1$, then $f = \mu * g$.

Corollary

If $n \geq 1$ we have

$$\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}.$$

Riemann Zeta

Definition

The Riemann zeta function ζ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ if } s > 1$$

Theorem

We have

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

Suffices to show $\sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1$.

Big Oh Notation

Big O notation is used to asymptotically approximate functions.

Definition

If $g(x) > 0$ for all $x \geq a$, then $f(x) = O(g(x))$ if there exists a constant $M > 0$ such that $|f(x)| \leq Mg(x)$ for all $x \geq a$.

Euler's Summation Formula

Our main goal is to prove the following theorem:

Theorem

If some function f has a continuous derivative f' on the interval $[y, x]$, where $0 < y < x$, then

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)\,dt + \int_y^x (t - \lfloor t \rfloor) f'(t)\,dt + f(x)(\lfloor x \rfloor - x) - f(y)(\lfloor y \rfloor - y) \tag{1}$$

Proof

We start by letting some $m = \lfloor y \rfloor$ and $k = \lfloor x \rfloor$. Then, for integers n and $n-1$ s.t. $y \leq n-1 < n \leq x$, we have:

$$\int_{n-1}^{n} \lfloor t \rfloor f'(t)\, dt = \int_{n-1}^{n} (n-1) f'(t)\, dt. \tag{2}$$

Then, bringing out the $n-1$ and integrating gives:

$$\begin{aligned}(n-1)\int_{n-1}^{n} f'(t)\, dt &= \int_{n-1}^{n} (n-1)(f(n) - f(n-1)) \\ &= \int_{n-1}^{n} (nf(n) - (n-1)f(n-1)) - f(n)\end{aligned} \tag{3}$$

Proof

Next, we will integrate from $n = m + 1$ to $n = k$. We get:

$$\begin{aligned} \int_m^k \lfloor t \rfloor f'(t)\,dt &= \int_m^k (nf(n) - (n-1)f(n-1)) - f(n) \\ &= \sum_{n=m+1}^{k} (nf(n) - (n-1)f(n-1)) - f(n) \end{aligned} \tag{4}$$

Distributing the summation notation and cancelling terms gives

$$\begin{aligned} &\sum_{n=m+1}^{k} (nf(n) - (n-1)f(n-1)) - f(n) \\ &= \sum_{n=m+1}^{k} (nf(n) - (n-1)f(n-1)) - \sum_{m<n\le k} f(n) \\ &= kf(k) - mf(m) - \sum_{m<n\le k} f(n) \end{aligned} \tag{5}$$

Proof

Putting everything together gives:

$$\int_m^k \lfloor t \rfloor f'(t)\,dt = kf(k) - mf(m) - \sum_{m<n\leq k} f(n) \tag{6}$$

Thus, solving for $\sum_{m<n\leq k} f(n)$ gives

$$\sum_{m<n\leq k} f(n) = -\int_m^k \lfloor t \rfloor f'(t)\,dt + kf(k) - mf(m) \tag{7}$$

Finally, integration over x and y instead of m and k gives

$$\sum_{y<n\leq x} f(n) = -\int_y^x \lfloor t \rfloor f'(t)\,dt + kf(x) - mf(y) \tag{8}$$

Proof

Finally, using integration by parts, we can deduce that

$$\int_y^x f(t)\,dt = xf(x) - yf(y) - \int_y^x tf'(t)\,dt \tag{9}$$

Now, subtracting equation (9) from (8) and grouping like terms gives Euler's summation formula, as follows,

$$\sum_{y \le n \le x} f(n) = \int_y^x f(t)\,dt + \int_y^x (t - \lfloor t \rfloor) f'(t)\,dt + f(x)(\lfloor x \rfloor - x) - f(y)(\lfloor y \rfloor - y) \tag{10}$$

and we are done.

Basic Asymptotic Formulas

We will need the following results for the future.

Theorem

1. $$\sum_{n\leq x}\frac{1}{n}=\log x+C+\mathcal{O}\left(\frac{1}{x}\right)$$

2. $$\sum_{n\leq x}\frac{1}{n^s}=\frac{x^{1-s}}{1-s}+\zeta(s)+\mathcal{O}(x^{-s}) \text{ if } s>0, s\neq 1$$

3. $$\sum_{n>x}\frac{1}{n^s}=O(x^{1-s}) \text{ if } s>1$$

4. $$\sum_{n\leq x}x^{\alpha}\frac{x^{\alpha+1}}{\alpha+1}+O(x^{\alpha})=O(x^{1-s}) \text{ if } \alpha\geq 0$$

Theorem 1

We will begin by proving Theorem 1. Referring to Euler's summation formula, we plug in $f(t) = \frac{1}{t}$ to get:

$$1 + \sum_{1<n\leq x} \frac{1}{n} = 1 + \int_1^x \frac{1}{t}\, dt - \int_1^x \frac{t - \lfloor t \rfloor}{t^2}\, dt - \frac{x - \lfloor x \rfloor}{x} \tag{11}$$

Combining 1 and $\sum_{1<n\leq x} \frac{1}{n}$ gives:

$$\sum_{n\leq x} \frac{1}{n} = 1 + \int_1^x \frac{1}{t}\, dt - \int_1^x \frac{t - \lfloor t \rfloor}{t^2}\, dt - \frac{x - \lfloor x \rfloor}{x} \tag{12}$$

Next, we note that $O(\frac{x-\lfloor x \rfloor}{x}) = \frac{1}{x}$, so we write:

$$\sum_{n\leq x} \frac{1}{n} = 1 + log(x) - \int_1^x \frac{t - \lfloor t \rfloor}{t^2}\, dt + O\left(\frac{1}{x}\right) \tag{13}$$

Theorem 1

Now, we look at $\int_1^x \frac{t-\lfloor t \rfloor}{t^2}\, dt$. Note that:

$$-\int_1^x \frac{t-\lfloor t \rfloor}{t^2}\, dt = -\int_1^\infty \frac{t-\lfloor t \rfloor}{t^2}\, dt + \int_x^\infty \frac{t-\lfloor t \rfloor}{t^2}\, dt \tag{14}$$

Furthermore, since the maximum value of $t-\lfloor t \rfloor = 1$, and the minimum value is 0 we can see that

$$0 \leq \int_x^\infty \frac{t-\lfloor t \rfloor}{t^2}\, dt \leq \int_x^\infty \frac{1}{t^2}\, dt = \frac{1}{x} \tag{15}$$

This means that $O\left(\int_x^\infty \frac{t-\lfloor t \rfloor}{t^2}\, dt\right) = \frac{1}{x}$ as well, so we can combine this with equation (13) to get:

$$\sum_{n \leq x} \frac{1}{n} = 1 + log(x) - \int_1^\infty \frac{t-\lfloor t \rfloor}{t^2}\, dt + O\left(\frac{1}{x}\right) \tag{16}$$

Theorem 1

Finally, based on the definition of C, if we let $x \to \infty$ we get:

$$\lim_{x\to\infty}\left(\sum_{n\leq x}\frac{1}{n} - log(x)\right) = 1 - \int_1^{\infty}\frac{t-\lfloor t\rfloor}{t^2}\,dt \tag{17}$$

we can substitute into equation (16) to get:

$$\sum_{n\leq x}\frac{1}{n} = log(x) + C + O\left(\frac{1}{x}\right) \tag{18}$$

Theorem 2

Now prove Theorem 2. Referring to Euler's summation formula again, we plug in $f(t) = \frac{1}{t^s}$ to get:

$$\sum_{n \leq x} \frac{1}{n^s} = \int_1^x \frac{dt}{t^s} - s \int_1^x \frac{t - \lfloor t \rfloor}{t^{s+1}} + 1 - \frac{x - \lfloor x \rfloor}{x^s} \tag{19}$$

Noting that $O\left(\frac{x - \lfloor x \rfloor}{x^s}\right) = O(x^{-s})$, we get:

$$\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{s-1}}{1-s} - \frac{1}{1-s} - s \int_1^\infty \frac{t - \lfloor t \rfloor}{t^{s+1}} + 1 + s \int_x^\infty \frac{t - \lfloor t \rfloor}{t^{s+1}} + O(x^{-s}) \tag{20}$$

Theorem 2

Now, we again note that $0 \leq s\int_x^\infty \frac{t-\lfloor t\rfloor}{t^{s+1}} \leq s\int_x^\infty \frac{1}{t^{s+1}} = \frac{1}{x^s}$. Thus, $O\left(\int_x^\infty \frac{t-\lfloor t\rfloor}{t^{s+1}}\right) = x^{-s}$, and we combine big O notation to get:

$$\sum_{n\leq x}\frac{1}{n^s} = \frac{x^{s-1}}{1-s} - \frac{1}{1-s} + 1 - s\int_1^\infty \frac{t-\lfloor t\rfloor}{t^{s+1}} + O(x^{-s}) \tag{21}$$

Finally, we let some constant $C(s)$ be s.t.

$$C(s) = 1 - \frac{1}{1-s} - s\int_1^\infty \frac{t-\lfloor t\rfloor}{t^{s+1}} \tag{22}$$

Furthermore, it can be shown that $C(s) = \zeta(s)$, so we get:

$$\sum_{n\leq x}\frac{1}{n^s} = \frac{x^{s-1}}{1-s} + C(s) + O(x^{-s}) = \frac{x^{s-1}}{1-s} + \zeta(s) + O(x^{-s}) \tag{23}$$

Theorem 3

Now, we prove theorem 3. Theorem 3 follows from Theorem 2 for the specific case when $s > 1$.

Theorem 4

Now prove Theorem 4. Referring to Euler's summation formula again, we plug in $f(t) = t^\alpha$ to get:

$$\sum_{n \leq x} n^\alpha = 1 + \int_1^x t^\alpha \, dt - \int_1^x \alpha t^{\alpha-1}(t - \lfloor t \rfloor) \, dt - x^\alpha(x - \lfloor x \rfloor) \qquad (24)$$

Next, we note that $O(x^\alpha(x - \lfloor x \rfloor)) = x^\alpha$ and

$$0 \leq \int_1^x \alpha t^{\alpha-1}(t - \lfloor t \rfloor) \, dt \leq \int_1^x \alpha t^{\alpha-1} \, dt = x^\alpha - 1 \qquad (25)$$

so $O(\int_1^x \alpha t^{\alpha-1}(t - \lfloor t \rfloor) \, dt) = x^\alpha$ as well.

Theorem 4

Finally, combining equation (19) and simplifying gives:

$$\sum_{n \leq x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha+1} + O(x^{\alpha}) + 1 = \frac{x^{\alpha+1}}{\alpha+1} + O(x^{\alpha}) \qquad (26)$$

Note that all constants can be absorbed the $O(x^{\alpha})$, as they make no contribution to the big O notation. As a result, we absorbed the constants $\frac{1}{a+1}$ and 1 into $O(x^{\alpha})$, which is why they are gone.

Asymptotic Behavior

Theorem

For $x > 1$ we have $\sum_{n\leq x} \varphi(n) = \frac{3}{\pi^2}x^2 + \mathcal{O}(x\log x)$.

We may write

$$\sum_{n\leq x} \varphi(n) = \sum_{n\leq x}\sum_{d|n} \mu(d)\left(\frac{n}{d}\right) = \sum_{q,d,qd\leq x} \mu(d)q$$

$$= \sum_{d\leq x} \mu(d) \sum_{q\leq x/d} q = \sum_{d\leq x} \mu(d)\left(\frac{1}{2}\left(\frac{x}{d}\right)^2 + \mathcal{O}\left(\frac{x}{d}\right)\right)$$

$$= \frac{x^2}{2}\sum_{d\leq x}\frac{\mu(d)}{d^2} + \mathcal{O}\left(x\sum_{d\leq x}\frac{1}{d}\right) = \frac{x^2}{2}\sum_{d\leq x}\frac{\mu(d)}{d^2} + \mathcal{O}(x\log x)$$

More Asymptotics

Note that

$$\sum_{d \le x} \frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - \sum_{d > x} \frac{\mu(d)}{d^2}$$

$$= \frac{6}{\pi^2} + \mathcal{O}\left(\sum_{d > x} \frac{1}{d^2}\right) = \frac{6}{\pi^2} + \mathcal{O}\left(\frac{1}{x}\right)$$

Now let's solve our original problem. Recall we want to find

$$\lim_{r \to \infty} \frac{8 \sum_{1 \le n \le r} \varphi(n)}{(2\lfloor r \rfloor + 1)^2} = \lim_{r \to \infty} \frac{\frac{24}{\pi^2} r^2 + \mathcal{O}(r \log r)}{4r^2 + \mathcal{O}(r)} = \lim_{r \to \infty} \frac{\frac{24}{\pi^2} + \mathcal{O}\left(\frac{\log r}{r}\right)}{4 + \mathcal{O}\left(\frac{1}{r}\right)} = \frac{6}{\pi^2}$$