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Assignment 7

Problem 1. First-order logical inference

Part a. Let $M(x)$ represent "x is a member of the Alpine Club."

Let $S(x)$ represent "x is a skiier."

Let $C(x)$ represent "x is a mountain climber."

Let $L(x, y)$ represent "x likes y."

The knowledge base can be represented in first-order logic as follows:

$$M(Tony) \land M(Mike) \land M(John)$$
$$\forall x\,[M(x) \implies (S(x) \lor C(x) \lor [S(x) \land C(x)])]$$
$$\neg\exists x\,[C(x) \land L(x, Rain)] \land \forall x\,[S(x) \implies L(x, Snow)]$$
$$\forall x\,([L(Tony, x) \implies \neg L(Mike, x)] \land \forall x\,[\neg L(Tony, x) \implies L(Mike, x)])$$
$$L(Tony, Rain) \land L(Tony, Snow)$$

Part b. Then the knowledge base can be converted to conjunctive normal form:

Proof.

1. $M(Tony) \land M(Mike) \land M(John)$ (Given)

2. $\forall x\,[M(x) \implies (S(x) \lor C(x) \lor [S(x) \land C(x)])]$ (Given)
3. $\forall a\,[M(a) \implies (S(a) \lor C(a) \lor [S(a) \land C(a)])]$ (Standardize, from 2)
4. $M(a) \implies (S(a) \lor C(a) \lor [S(a) \land C(a)])$ (Universal elimination, from 3)
5. $\neg M(a) \lor S(a) \lor C(a) \lor [S(a) \land C(a)]$ (Conditional equivalence, from 4)
6. $[\neg M(a) \lor S(a) \lor C(a) \lor S(a)] \land$
   $[\neg M(a) \lor S(a) \lor C(a) \lor C(a)]$ (Distributive property, from 5)
7. $[\neg M(a) \lor S(a) \lor C(a) \lor S(a)] \land [\neg M(a) \lor S(a) \lor C(a)]$ (Idempotence, from 6)
8. $[\neg M(a) \lor S(a) \lor C(a)] \land [\neg M(a) \lor S(a) \lor C(a)]$ (Idempotence, from 7)
9. $[\neg M(a) \lor S(a) \lor C(a)]$ (Idempotence, from 8)

10. $\neg\exists x\,[C(x) \land L(x, Rain)] \land \forall x\,[S(x) \implies L(x, Snow)]$ (Given)
11. $\neg\exists x\,[C(x) \land L(x, Rain)]$ (Conjunction elimination, from 10)
12. $\neg\exists b\,[C(b) \land L(b, Rain)]$ (Standardize, from 11)
13. $\forall b\,\neg[C(b) \land L(b, Rain)]$ (Universal equivalence, from 12)
14. $\neg[C(b) \land L(b, Rain)]$ (Universal elimination, from 13)
15. $[\neg C(b) \lor \neg L(b, Rain)]$ (De Morgan's law, from 14)

16. $\forall x\,[S(x) \implies L(x, Snow)]$ (Conjunction elimination, from 10)
17. $\forall c\,[S(c) \implies L(c, Snow)]$ (Standardize, from 16)
18. $[S(c) \implies L(c, Snow)]$ (Universal elimination, from 17)
19. $[\neg S(c) \lor L(c, Snow)]$ (Conditional equivalence, from 18)

20. $\forall x\,[L(Tony, x) \implies \neg L(Mike, x)] \land$
    $\forall x\,[\neg L(Tony, x) \implies L(Mike, x)]$ (Given)
21. $\forall x\,[L(Tony, x) \implies \neg L(Mike, x)]$ (Conjunction elimination, from 20)
22. $\forall d\,[L(Tony, d) \implies \neg L(Mike, d)]$ (Standardize, from 21)
23. $[L(Tony, d) \implies \neg L(Mike, d)]$ (Universal elimination, from 22)
24. $[\neg L(Tony, d) \lor \neg L(Mike, d)]$ (Conditional equivalence, from 23)

25. $\forall x\,[\neg L(Tony, x) \implies L(Mike, x)]$ (Conjunction elimination, from 20)
26. $\forall e\,[\neg L(Tony, e) \implies L(Mike, e)]$ (Standardize, from 25)
27. $[\neg L(Tony, e) \implies L(Mike, e)]$ (Universal elimination, from 26)
28. $[\neg\neg L(Tony, e) \lor L(Mike, e)]$ (Conditional equivalence, from 27)
29. $[L(Tony, e) \lor L(Mike, e)]$ (Double negation, from 28)

30. $L(Tony, Rain) \land L(Tony, Snow)$ (Given)

31. $M(Tony) \land M(Mike) \land M(John) \land$
    $[\neg M(a) \lor S(a) \lor C(a)] \land$
    $[\neg C(b) \lor \neg L(b, Rain)] \land$
    $[\neg S(c) \lor L(c, Snow)] \land$
    $[\neg L(Tony, d) \lor \neg L(Mike, d)] \land$
    $[L(Tony, e) \lor L(Mike, e)] \land$
    $L(Tony, Rain) \land L(Tony, Snow)$ (Conjunction intro., from 1, 9, 15, 19, 24, 29)

$\square$

Part c. The theorem can be translated to first-order logic as:

$$\exists x\,[M(x) \land C(x) \land \neg S(x)]$$

Its negation can be converted to conjunctive normal form:

Proof.

1. $\neg\exists x\,[M(x) \land C(x) \land \neg S(x)]$ (Given)
2. $\forall x\,\neg[M(x) \land C(x) \land \neg S(x)]$ (Universal equivalence)
3. $\neg[M(x) \land C(x) \land \neg S(x)]$ (Universal elimination)
4. $[\neg M(x) \lor \neg C(x) \lor \neg\neg S(x)]$ (De Morgan's law)
5. $[\neg M(x) \lor \neg C(x) \lor S(x)]$ (Double negation)

□

We can prove that the knowledge base entails the theorem by resolution refutation:

Proof.

1. $M(Tony) \land M(Mike) \land M(John) \land$
   $[\neg M(a) \lor S(a) \lor C(a)] \land$
   $[\neg C(b) \lor \neg L(b, Rain)] \land$
   $[\neg S(c) \lor L(c, Snow)] \land$
   $[\neg L(Tony, d) \lor \neg L(Mike, d)] \land$
   $[L(Tony, e) \lor L(Mike, e)] \land$
   $L(Tony, Rain) \land L(Tony, Snow)$ (Given)
2. $[\neg M(x) \lor \neg C(x) \lor S(x)]$ (Assumption)
3. $M(Mike)$ (Conjunction elimination, from 1)
4. $[\neg C(Mike) \lor S(Mike)]$ (Generalized resolution, from 2, 3)
5. $[\neg M(a) \lor S(a) \lor C(a)]$ (Conjunction elimination, from 1)
6. $[S(Mike) \lor C(Mike)]$ (Generalized resolution, 3, 5)
7. $L(Tony, Snow)$ (Conjunction elimination, from 1)
8. $[\neg L(Tony, d) \lor \neg L(Mike, d)]$ (Conjunction elimination, from 1)
9. $\neg L(Mike, Snow)$ (Generalized resolution, from 7, 8)
10. $[\neg S(c) \lor L(c, Snow)]$ (Conjunction elimination, from 1)
11. $\neg S(Mike)$ (Generalized resolution, from 9, 10)
12. $\neg C(Mike)$ (Generalized resolution, from 4, 11)
13. $C(Mike)$ (Generalized resolution, from 6, 11)
14. $\perp$ (Contradiction, from 12, 13)

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