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Assignment 10

Problem 1. Inference efficiency

Let B = "Battery," R = "Radio," L = "Lights," I = "Ignition," G = "Gas," E = "Engine starts," and C = "Car moves." Assume the domain of all random variables is $\mathbb{B} = \{1, 0\}$.

a. The "blind approach" expression for the marginal probability $P(C = 0)$ is defined as:

$$P(C=0) = \sum_{B\in\mathbb{B}} \sum_{R\in\mathbb{B}} \sum_{L\in\mathbb{B}} \sum_{I\in\mathbb{B}} \sum_{G\in\mathbb{B}} \sum_{E\in\mathbb{B}} P(C=0|E)P(E|I,G)P(G)P(I|B)P(L|B)P(R|B)P(B)$$

There are 6 uninstantiated variables, each with a domain of size 2, so there are a total of $2^6 = 64$ summation terms. Each summation term contains 7 factors, so there are $7-1=6$ multiplications per term. Therefore, the computational cost of this equation is $64-1=63$ additions and $64 \times 6 = 384$ multiplications.

b. A more efficient expression obtained by interleaving sums and products can be written:

$$P(C=0) = \sum_{E\in\mathbb{B}} P(C=0|E) \left[\sum_{I\in\mathbb{B}} \left[\sum_{G\in\mathbb{B}} P(E|I,G)P(G) \right] \left[\sum_{B\in\mathbb{B}} P(I|B) \left[\sum_{L\in\mathbb{B}} P(L|B) \right] \left[\sum_{R\in\mathbb{B}} P(R|B) \right] P(B) \right] \right]$$

Again we will express the computational cost in terms of the number of additions and multiplications. The number of additions n_+ and multiplications $n_\times$ can be derived:

$$\begin{aligned} n_+ &= 1 + 2[1 + 2(1 + [1 + 2(1+1)])] \\ &= 1 + 2[1 + 2(1 + [1 + 2(2)])] \\ &= 1 + 2[1 + 2(1 + [1 + 4])] \\ &= 1 + 2[1 + 2(1 + 5)] \\ &= 1 + 2[1 + 2(6)] \\ &= 1 + 2[1 + 12] \\ &= 1 + 2[13] \\ &= 1 + 26 \\ &= 27 \end{aligned}$$

$$\begin{aligned} n_\times &= 2(1 + 2[1 + 2(1) + 2(3)]) \\ &= 2(1 + 2[1 + 2 + 6]) \\ &= 2(1 + 2[9]) \\ &= 2(1 + 18) \\ &= 2(19) \\ &= 38 \end{aligned}$$

Problem 2. Variable elimination

Let B = "Burglary," E = "Earthquake," A = "Alarm," R = "RadioReport," J = "JohnCalls," and M = "MaryCalls." Assume the domain of all random variables is $\mathbb{B} = \{1, 0\}$.

a. The expression for the marginal probability $P(B = 1, R = 0)$ is defined as:

$$P(B=1,R=0) = \sum_{E\in\mathbb{B}} \sum_{A\in\mathbb{B}} \sum_{J\in\mathbb{B}} \sum_{M\in\mathbb{B}} P(M|A)P(J|A)P(A|B=1,E)P(R=0|E)P(B=1)P(E)$$

b. If we define the following factors:

$$f_1(M,A) = P(M|A) \qquad f_4(E) = P(R=0|E)$$
$$f_2(J,A) = P(J|A) \qquad f_5() = P(B=1)$$
$$f_3(A,E) = P(A|B=1,E) \qquad f_6(E) = P(E)$$

We can then rewrite the probability expression in terms of those factors:

$$P(B=1,R=0) = \sum_{E\in\mathbb{B}}\sum_{A\in\mathbb{B}}\sum_{J\in\mathbb{B}}\sum_{M\in\mathbb{B}} f_1(M,A)f_2(J,A)f_3(A,E)f_4(E)f_5()f_6(E)$$

The factor values are listed in the following tables:

M	A	$f_1(M,A)$
1	1	0.70
1	0	0.01
0	1	0.30
0	0	0.99

J	A	$f_2(J,A)$
1	1	0.90
1	0	0.05
0	1	0.10
0	0	0.95

A	E	$f_3(A,E)$
1	1	0.95
1	0	0.94
0	1	0.05
0	0	0.06

E	$f_4(E)$
1	0.10
0	0.95

$f_5()$
0.001

E	$f_6(E)$
1	0.002
0	0.998

c. We will now eliminate variables in the order M, J, A, E.

$$\begin{aligned}
P(B=1,R=0) &= \sum_{E\in\mathbb{B}}\sum_{A\in\mathbb{B}}\sum_{J\in\mathbb{B}}\sum_{M\in\mathbb{B}} f_1(M,A)f_2(J,A)f_3(A,E)f_4(E)f_5()f_6(E) \\
&= \sum_{E\in\mathbb{B}}\sum_{A\in\mathbb{B}}\sum_{J\in\mathbb{B}}\left[\sum_{M\in\mathbb{B}} f_1(M,A)\right] f_2(J,A)f_3(A,E)f_4(E)f_5()f_6(E) \\
&= \sum_{E\in\mathbb{B}}\sum_{A\in\mathbb{B}}\sum_{J\in\mathbb{B}} 1f_2(J,A)f_3(A,E)f_4(E)f_5()f_6(E) \\
&= \sum_{E\in\mathbb{B}}\sum_{A\in\mathbb{B}}\left[\sum_{J\in\mathbb{B}} f_2(J,A)\right] f_3(A,E)f_4(E)f_5()f_6(E) \\
&= \sum_{E\in\mathbb{B}}\sum_{A\in\mathbb{B}} 1f_3(A,E)f_4(E)f_5()f_6(E) \\
&= \sum_{E\in\mathbb{B}}\left[\sum_{A\in\mathbb{B}} f_3(A,E)\right] f_4(E)f_5()f_6(E) \\
&= \sum_{E\in\mathbb{B}} 1f_4(E)f_5()f_6(E) \\
&= f_5()\left[\sum_{E\in\mathbb{B}} f_4(E)f_6(E)\right] \\
&= f_5()\sum_{E\in\mathbb{B}} \tau_1(E) = f_5()\tau_2() \\
&= (0.001)(0.9483) \\
&= 0.0009483
\end{aligned}$$

Through the above variable elimination procedure, additional factors were introduced:

$$\tau_1(E) = f_4(E)f_6(E)$$
$$\tau_2(E) = \sum_{E\in\mathbb{B}} \tau_1(E)$$

E	$\tau_1(E)$
1	0.0002
0	0.9481

$\tau_2()$
0.9483

d. We can instead eliminate variables in the order A, M, E, J.

$$\begin{aligned}
P(B=1, R=0) &= \sum_{E\in\mathbb{B}}\sum_{A\in\mathbb{B}}\sum_{J\in\mathbb{B}}\sum_{M\in\mathbb{B}} f_1(M,A)f_2(J,A)f_3(A,E)f_4(E)f_5()f_6(E) \\
&= \sum_{E\in\mathbb{B}}\sum_{J\in\mathbb{B}}\sum_{M\in\mathbb{B}} \left[\sum_{A\in\mathbb{B}} f_1(M,A)f_2(J,A)f_3(A,E)\right] f_4(E)f_5() \\
&= \sum_{E\in\mathbb{B}}\sum_{J\in\mathbb{B}}\sum_{M\in\mathbb{B}} \left[\sum_{A\in\mathbb{B}} \tau_3(M,J,A)f_3(A,E)\right] f_4(E)f_5()f_6(E) \\
&= \sum_{E\in\mathbb{B}}\sum_{J\in\mathbb{B}}\sum_{M\in\mathbb{B}} \left[\sum_{A\in\mathbb{B}} \tau_4(M,J,A,E)\right] f_4(E)f_5()f_6(E) \\
&= \sum_{E\in\mathbb{B}}\sum_{J\in\mathbb{B}}\sum_{M\in\mathbb{B}} \tau_5(M,J,E)f_4(E)f_5()f_6(E) \\
&= \sum_{E\in\mathbb{B}}\sum_{J\in\mathbb{B}} \left[\sum_{M\in\mathbb{B}} \tau_5(M,J,E)\right] f_4(E)f_5()f_6(E) \\
&= \sum_{E\in\mathbb{B}}\sum_{J\in\mathbb{B}} \tau_6(J,E)f_4(E)f_5()f_6(E) \\
&= \sum_{J\in\mathbb{B}} f_5() \left[\sum_{E\in\mathbb{B}} \tau_6(J,E)f_4(E)f_6(E)\right] \\
&= \sum_{J\in\mathbb{B}} f_5() \left[\sum_{E\in\mathbb{B}} \tau_7(J,E)\right] \\
&= \sum_{J\in\mathbb{B}} f_5()\tau_8(J) = f_5()\left[\sum_{J\in\mathbb{B}} \tau_8(J)\right] \\
&= f_5()\tau_9(J) \\
&= (0.001)(0.9483) \\
&= 0.0009483
\end{aligned}$$

Through the above variable elimination procedure, additional factors were introduced:

$$\begin{aligned}
\tau_3(M,J,A) &= f_1(M,A)f_2(J,A) \\
\tau_4(M,J,A,E) &= \tau_3(M,J,A)f_3(A,E) \\
\tau_5(M,J,E) &= \sum_{A\in\mathbb{B}} \tau_4(M,J,A,E) \\
\tau_6(J,E) &= \sum_{M\in\mathbb{B}} \tau_5(M,J,E)
\end{aligned}$$

$$\begin{aligned}
\tau_7(J,E) &= \tau_6(J,E)f_4(E)f_6(E) \\
\tau_8(J) &= \sum_{E\in\mathbb{B}} \tau_7(J,E) \\
\tau_9(J) &= \sum_{J\in\mathbb{B}} \tau_8(J)
\end{aligned}$$

M	J	A	$\tau_3(M,J,A)$
1	1	1	0.6300
1	1	0	0.0005
1	0	1	0.0700
1	0	0	0.0095
0	1	1	0.2700
0	1	0	0.0495
0	0	1	0.0300
0	0	0	0.9405

M	J	E	$\tau_5(M,J,E)$
1	1	1	0.598525
1	1	0	0.592230
1	0	1	0.066975
1	0	0	0.066370
0	1	1	0.258975
0	1	0	0.256770
0	0	1	0.075525
0	0	0	0.084630

M	J	A	E	$\tau_4(M,J,A,E)$
1	1	1	1	0.598500
1	1	1	0	0.592200
1	1	0	1	0.000025
1	1	0	0	0.000030
1	0	1	1	0.066500
1	0	1	0	0.065800
1	0	0	1	0.000475
1	0	0	0	0.000570
0	1	1	1	0.256500
0	1	1	0	0.253800
0	1	0	1	0.002475
0	1	0	0	0.002970
0	0	1	1	0.028500
0	0	1	0	0.028200
0	0	0	1	0.047025
0	0	0	0	0.056430

J	E	$\tau_6(J,E)$
1	1	0.8575
1	0	0.8490
0	1	0.1425
0	0	0.1510

J	E	$\tau_7(J,E)$
1	1	0.000171
1	0	0.804937
0	1	0.000029
0	0	0.143163

J	$\tau_8(J)$
1	0.805108
0	0.143192

$\tau_9()$
0.9483

e. To analyze and compare the complexity of the two variable elimination orders, we can consider the scopes of the factors. For the variable order in part c, the maximum scope size encountered by the algorithm was 2 variables (if we include the initial factors). In part d, the maximum scope size that was introduced was 4 variables. Since the size of the domain of the variables is 2, the worst-case complexity of variable elimination is 2^k, where k is the maximum factor scope size. Therefore, the variable order from part d was $2^4/2^2 = 4$ times as computationally complex as the variable order from part c.

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Assignment 10

Problem 3. Likelihood weighting

a. When sampling $P(B=1|J=1,R=0)$, you should use the following likelihood weights.

For the sample $B=0,E=0,R=0,J=1,M=0,A=0$:

$$\begin{aligned}
&P(J=1,R=0|B=0,E=0,M=0,A=0)\\
&=P(J=1|A=0)P(R=0|E=0)\\
&=(0.05)(0.95)\\
&=0.0475
\end{aligned}$$

For the sample $B=1,E=0,R=0,J=1,M=1,A=1$:

$$\begin{aligned}
&P(J=1,R=0|B=1,E=0,M=1,A=1)\\
&=P(J=1|A=1)P(R=0|E=0)\\
&=(0.90)(0.95)\\
&=0.855
\end{aligned}$$

b. When sampling $P(B=0|M=0,E=0)$, you should use the following likelihood weights.

For the sample $B=0,E=0,R=0,J=1,M=0,A=0$:

$$\begin{aligned}
&P(M=0,E=0|B=0,R=0,J=1,A=0)\\
&=P(M=0|A=0)P(E=0)\\
&=(0.99)(0.998)\\
&=0.98802
\end{aligned}$$

For the sample $B=1,E=0,R=1,J=1,M=0,A=1$:

$$\begin{aligned}
&P(M=0,E=0|B=1,R=1,J=1,A=1)\\
&=P(M=0|A=1)P(E=0)\\
&=(0.3)(0.998)\\
&=0.2994
\end{aligned}$$