Fiber Bundles

Matt Robinson

Contents

1 Vector Bundles

Introductory definitions follow from [5], [6], [10].
Check out [3]

Let M be a topological space. A *real vector bundle of rank k over M* is a topological space E together with surjective map $\pi : E \to M$ satisfying the following conditions

a. For each $p \in M$, the fiber $E_p := \pi^{-1}(p)$ over p is endowed with the structure of a k-dimensional, real vector space.

b. For each $p \in M$, there exists a neighborhood $U \subseteq M$ of p, and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ (called the *local trivialization of E over U*) satisfying the following conditions:

   i. $\pi_U \circ \Phi = \pi$ (where $\pi_U : U \times \mathbb{R}^k \to U$ is the projection onto U), and

   ii. for each $q \in U$, the restriction of Φ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

If M and E are smooth manifolds (with or without boundary), $\pi : E \to M$ is smooth, and the local trivialization can be chosen to be diffeomorphisms, then $\pi : E \to M$ is a *smooth vector bundle*. We also call any local trivialization that is a diffeomorphism onto its image a *smooth local trivialization*. We write $\mathcal{E}$ or (E, π, M) to denote smooth vector bundles.

We say that E is the *total space*, M is the *base space*, and π is the *projection*. If there exists a local trivialization of E over all M, then we that E is the *trivial bundle*, and have that E is homeomorphic to $M \times \mathbb{R}^k$. If E is *smoothly trivial*, then E is diffeomorphic to $M \times \mathbb{R}^k$.

Alternative Definition: Perhaps a better working definition for vector bundles can be described as follows. First note, that since we're mostly interested in the smooth category, we shall ignore topological vector bundles.

Let $\pi : E \to M$ be a smooth mapping between smooth manifolds. A *vector bundle chart* on (E, π, M) is a pair (U, ψ), where $U \subseteq M$ is an open subset and ψ is a fiber-respecting diffeomorphism so the following diagram commutes

$$\begin{tikzcd}
E|_U := \pi^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & U \times V \arrow[dl, "\pi_U"] \\
& U &
\end{tikzcd}$$

where $V = \pi^{-1}(p)$ is some fixed (real) k-dimensional vector space called the *standard fiber* and $\pi_U : U \times V \to U$ is the projection onto the first factor.

Given two vector bundle charts (U_α, ψ_α) and (U_β, ψ_β), we say they are *compatible* if $\psi_\alpha \circ \psi_\beta^{-1}$ is fiber-wise linear isomorphism, i.e.,

$$\psi_\alpha \circ \psi_\beta^{-1}(p, v) = (p, \psi_{\alpha\beta}(p)v),$$

for some mapping

$$\psi_{\alpha\beta} : U_{\alpha\beta} := U_\alpha \cap U_\beta \to GL(V).$$

The mapping $\psi_{\alpha\beta}$ is then unique and smooth, and is called the *transition function* between the two bundle charts.

A *vector bundle atlas* $\{(U_\alpha, \psi_\alpha)\}$ for (E, π, M) is a set of pairwise compatible bundle charts such that $\{U_\alpha\}$ is an open cover of M. Two vector bundle atlases are *equivalent* if their union is again a vector bundle atlas.

Thus a vector bundle (E, π, M) consists of the smooth mapping $\pi : E \to M$ between smooth manifolds with an equivalence class of vector bundle atlases.

Corollary 1.1. *Given a vector bundle $\pi : E \to M$, it follows that π is a surjective submersion.*

A *local section* of a smooth vector bundle $\pi : E \to M$ is a smooth map $\xi : U \to E$ for some open $U \subset M$ such that $\pi \circ \xi = \mathbb{1}_U$. A *(global) section* is one where $U = M$. The *zero section* is the section $\iota : M \to E$ defined by

$$\iota(p) = 0_p \in E_p,$$

for each $p \in M$. We denote the space of global sections by $\Gamma(E)$, and the space of local sections by $\Gamma(E|_U)$ (this notation will become more clear once we define the restriction of a bundle).

Given two smooth bundles (E, π, M) and (E', π', M'), a smooth map $F : E \to E'$ is a *bundle homomorphism* if there exists $f : M \to M'$ such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "F"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd}$$

and F is a fiber-respecting, fiber-linear map, i.e., $F_p : E_p \to E'_{f(p)}$ is linear. Note that this implies f is smooth since $f = \pi' \circ F \circ \iota$. A bijective bundle homomorphism that's inverse is also a bundle homomorphism is called a *bundle isomorphism.* If E and E' are bundles over the same base space M, then we say a *bundle homomorphism over M*.

1.1 Operations on Vector Bundles

Whitney Sums Suppose $E^1 \to M$ and $E^2 \to M$ are smooth vector bundles of rank k_1 and k_2, respectively. Then for each $p \in M$, we have the vector space direct sum

$$E^1_p \oplus E^2_p,$$

and we define the space

$$E = E^1 \oplus E^2 := \coprod_{p \in M} \left(E^1_p \oplus E^2_p \right).$$

Letting $\pi : E \to M$ denote the obvious projection, we have a new smooth vector bundle $\pi : E \to M$ called the *Whitney sum of E^1 and E^2*.

Subbundles A *vector subbundle* (F, π, M) of a vector bundle (E, π, M) is a vector bundle and vector bundle homomorphism $\phi : F \to E$ which coves $\mathbb{1}_M$ such that $\phi_p : F_p \to E_p$ is a linear embedding for each $p \in M$.

Lemma 1.2. *Let $\phi : (E, \pi, M) \to (E', \pi', M')$ be a bundle homomorphism such that rank(ϕ_x) is constant in $x \in M$. Then $\ker \phi$, given by $(\ker \phi)_x = \ker \phi_x$ is a vector subbundle of (E, π, M).*

Bundle Restrictions If $\pi : E \to M$ is a smooth vector bundle, and $A \subset M$ is any immersed submanifold of M, then we define the *restriction of E to A* to be the set

$$E|_A = \bigcup_{a \in A} E_a,$$

and we have a new smooth vector bundle

$$\pi|_A : E|_A \to A.$$

If $i : A \hookrightarrow M$ is an immersed submanifold, then the inclusion $i_\# : E|_A \hookrightarrow E$ is a bundle homomorphism covering i:

$$\begin{tikzcd} E|_A \arrow[r, "i_\#"] \arrow[d, "\pi|_A"'] & E \arrow[d, "\pi"] \\ A \arrow[r, "i"'] & M \end{tikzcd}$$

To further the understanding of the restriction of a bundle, we introduce the notion of a pullback bundle. Let $\pi : E \to M$ be a smooth vector bundle, and $f : A \to M$ a smooth map of smooth manifolds. The *pullback bundle of E along f* is the vector bundle $f^*\pi : f^*E \to A$ given by

$$f^*E = \{(a, (x, v)) \in A \times E : f(a) = \pi(x, v) = x\},$$

and the bundle projection $f^*\pi : f^*E \to A$ is the projection onto the first factor of $A \times E$. Then we have the vector space isomorphism

$$f^*E_a \cong E_{f(a)}.$$

Moreover, the projection onto the second factor, called the *pushforth of f*, $f_\# : f^*E \to E$ yields the commutative diagram

$$\begin{tikzcd}
f^*E \arrow[r, "f_\#"] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] \\
A \arrow[r, "f"'] & M
\end{tikzcd}$$

Let $X \in \Gamma(f^*E)$, then the pushforth of X along f is the map

$$\tilde{X} = f_\# X : A \to E$$

and the space of all sections along f is denoted $\Gamma_f(E) = f_\#\Gamma(f^*E)$.

In the setting of pullbacks, we let $i : A \hookrightarrow M$ be an immersed submanifold. Then the pullback bundle

$$\begin{aligned}
i^*E &= \{(a,(x,v)) \in A \times E : i(a) = \pi(x,v) = x\} \\
&= \{(a,(i(a),v)) \in A \times E\} \\
&\cong \{(i(a),v) \in E : a \in A\} \\
&= E|_A\,,
\end{aligned}$$

and hence leads to the notation of $i_\# : E|_A \hookrightarrow E$ being the inclusion.

Lifts Let $\pi : E \to M$ be a smooth vector bundle and let $\gamma : I \to M$ be curve with $0, 1 \in I$ and $\gamma(0) = p$, $\gamma(1) = q$. Let $c : I \to \gamma^*E$ be a section of the pullback bundle $\gamma^*\pi : \gamma^*E \to I$. Then the pushforth $\gamma_\# c : I \to E$ is a path in E that satisfies

$$\pi \circ \gamma_\# c = \gamma.$$

That is, for the curve $\tilde{\gamma} := \gamma_\# c$, the following diagram commutes

$$\begin{tikzcd}
\gamma^*E \arrow[r, "\gamma_\#"] & E \arrow[d, "\pi"] \\
I \arrow[u, "c"] \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & M
\end{tikzcd}$$

The path $\tilde{\gamma}$ is called a *lift of* γ. A lift of γ is dependent on choice of section $c \in \Gamma(\gamma^* E)$, and hence we denote the space of all lifts along γ as $\Gamma_\gamma(E) = \gamma_\# \Gamma(\gamma^* E)$. Note that if γ has a self-intersection, then the lift is not a section of E along γ (despite the notational similarities).

2 Examples of Vector Bundles

In this section we construct in detail many of the most common smooth vector bundles in connection to smooth manifolds. The most obvious first choice is the tangent bundle.

2.1 The Tangent Bundle

See [6]

Let M be a smooth n-dimensional manifold. Let

$$TM = \coprod_{p \in M} T_p M,$$

denote the "*tangent bundle*". Let $\pi : TM \to M$, $\pi(p, v) = p$. Then TM has a natural topology and smooth structure that makes TM into a smooth $2n$-dimensional manifold, and with respect to this structure, the map $\pi : TM \to M$ is smooth.

Let (ϕ, U) be a chart on M with coordinates (x^j). Define the map $\tilde{\phi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\tilde{\phi}\left(v^j \left. \frac{\partial}{\partial x^j} \right|_p \right) = (x^1(p), ..., x^n(p), v^1, ..., v^n).$$

Its image set is $\phi(U) \times \mathbb{R}^n$ and is a bijection, since it's inverse can explicitly be written as

$$\tilde{\phi}^{-1}(x^1, , , , x^n, v^1, ..., v^n) = v^j \left. \frac{\partial}{\partial x^j} \right|_{\phi^{-1}(x)}.$$

Suppose we are given two charts (ϕ, U) and (ψ, V) on M with corresponding $(\tilde{\phi}, \pi^{-1}(U))$ and $(\tilde{\psi}, \pi^{-1}(V))$ on TM. Then the sets

$$\tilde{\phi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \phi(U \cap V) \times \mathbb{R}^n$$

and

$$\tilde{\psi}(\pi^{-1}(U) \cap \pi^{-1}(V)) = \psi(U \cap V) \times \mathbb{R}^n$$

are open in $\mathbb{R}^{2n}$. The transition map

$$\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n,$$

can then be explicitly written as

$$\tilde{\psi} \circ \tilde{\phi}^{-1}(x^1, ..., x^n, v^1, ..., v^n) = \left(\tilde{x}^1(x), ..., \tilde{x}^n(x), v^j \frac{\partial \tilde{x}^1}{\partial x^j}(x), ..., v^j \frac{\partial \tilde{x}^n}{\partial x^j}(x) \right),$$

which is clearly smooth.

Let $\{U_j\}$ be a countable cover of charts for M, the $\{\pi^{-1}(U_j)\}$ is a countable cover of charts for TM, and hence TM is a smooth $2n$-dimensional manifold.

To show that $\pi : TM \to M$ is smooth, fix some local coordinates and observe that $\pi(x, v) = x$ which is clearly smooth. Moreover, we have that $\pi : TM \to M$ is a submersion. Since π is a submersion if and only if every point in TM is the image of a smooth local section, we fix $(p, v) \in TM$ and chart about p, and define the section $X : U \to TU$ by $X(q) = (q, v^j \frac{\partial}{\partial x^j}\big|_q)$.

We now show that $\pi : TM \to M$ is smooth vector bundle of rank n. Clearly each T_pM is an n-dimensional vector space by construction. Fix a coordinate chart (ϕ, U) on M and corresponding $(\tilde{\phi}, \pi^{-1}(U))$ on TM. Define the map $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by

$$\Phi \left(v^j \left. \frac{\partial}{\partial x^j} \right|_p \right) = (p, (v^1, ..., v^n)).$$

Then $\pi_U \circ \Phi = \pi$ and is clearly linear on each T_pM. Moreover, since

$$\phi \times \mathbb{1}_{\mathbb{R}^n} \circ \Phi : \pi^{-1}(U) \to \phi(U) \times \mathbb{R}^n,$$

is our usual chart map $\tilde{\phi}$ on TM, it's a diffeomorphism, and hence $\Phi = (\phi^{-1} \times \mathbb{1}_{\mathbb{R}^n}) \circ \tilde{\phi}$ is a diffeomorphism as well, and hence $\{(\pi^{-1}(U), \Phi)\}$ is our desired vector bundle atlas.

3 The Tangent Bundle of Vector Bundles

This section follows [5] and [8] closely.
See also

Let (E, π, M) be a vector bundle, and denote

$$+_E : E \times_M E \to E, \qquad +_E((x,u),(x,v)) = (x, u+v),$$

fiberwise addition and

$$m_t^E : E \to E, \qquad m_t^E(x,v) = (x, tv),$$

fiberwise scalar multiplication.

As usual, since E is a smooth manifold, we have the tangent bundle (TE, π_E, E) which is again a smooth vector bundle with fiberwise addition $+_T E$ and fiberwise scalar multiplication m_t^{TE}. We wish to see how the bundle charts of (TE, π_E, E) relate to the bundle charts of (E, π, M).

Suppose (E, π, M) is a vector bundle with bundle atlas $\{(U_\alpha, \psi_\alpha) : \psi_\alpha : E|_{U_\alpha} \to U_\alpha \times V\}$ such that $\{(U_\alpha, \phi_\alpha) : \phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n\}$ is a smooth atlas for M. Then $\{(E|_{U_\alpha}, \tilde{\phi}_\alpha) : \tilde{\phi}_\alpha : E|_{U_\alpha} \to \phi_\alpha(U_\alpha) \times V\}$ is a compatible smooth atlas for E, where

$$\tilde{\phi}_\alpha = (\phi_\alpha \times \mathbb{1}_V) \circ \psi_\alpha : E|_{U_\alpha} \to \phi_\alpha(U_\alpha) \times V.$$

Then our atlas for TE is given by $\{(T(E|_{U_\alpha}), d\tilde{\phi}_\alpha)\}$ where

$$d\tilde{\phi}_\alpha : T(E|_{U_\alpha}) \to T(\phi_\alpha(U_\alpha) \times V) = (\phi_\alpha(U_\alpha) \times V) \times (\mathbb{R}^n \times V).$$

For $p \in U_{\alpha\beta}$ and $x = \phi_\beta(p)$, and $v, w \in V$, $\xi \in \mathbb{R}^n$ we have the transition functions

$$\phi_\alpha \circ \phi_\beta^{-1}(x) = \phi_{\alpha\beta}(x),$$

$$\psi_\alpha \circ \psi_\beta^{-1}(p,v) = (p, \psi_{\alpha\beta}(p)v),$$

$$\begin{aligned}
\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1}(x,v) &= (\phi_\alpha \times \mathbb{1}_V) \circ \psi_\alpha \circ \psi_\beta^{-1} \circ (\phi_\beta^{-1} \times \mathbb{1}_V)(x,v) \\
&= (\phi_\alpha \times \mathbb{1}_V) \circ \psi_\alpha \circ \psi_\beta^{-1}(p,v) \\
&= (\phi_\alpha \times \mathbb{1}_V)(p, \psi_{\alpha\beta}(p)v) \\
&= (\phi_{\alpha\beta}(x), \psi_{\alpha\beta}(\phi_\beta^{-1}(x))v),
\end{aligned}$$

and

$$\begin{aligned} d\tilde{\phi}_\alpha \circ d\tilde{\phi}_\beta^{-1}(x,v;\xi,w) &= d(\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1})_{(x,v)}(\xi,w) \\ &= (\phi_{\alpha\beta}(x), \psi_{\alpha\beta}(\phi_\beta^{-1}(x))v; d(\phi_{\alpha\beta})_x(\xi), d(\psi_{\alpha\beta}\circ\phi_\beta^{-1})_x(\xi)v + \psi_{\alpha\beta}(\phi^{-1}(x))w). \end{aligned}$$

That is, letting $\tilde{\psi}_\alpha = d\tilde{\phi}_\alpha$, we have the transition maps

$$\tilde{\psi}_{\alpha\beta} : E|_{U_{\alpha\beta}} \to GL(\mathbb{R}^n \times V),$$

$$(p,v) \mapsto \begin{pmatrix} (d(\phi_\alpha)_p & 0 \\ v \cdot d(\psi_{\alpha\beta})_p & \psi_{\alpha\beta}(p) \end{pmatrix},$$

where we have identifies $d(\phi_\beta)_p : T_pM \xrightarrow{\cong} \mathbb{R}^n$. Thus we have described the vector bundle structure of (TE, π_E, E) in terms of our other structures.

Note from the above, when considering $(p,\xi) \in TU_{\alpha\beta}$, then we have a similar map

$$TU_{\alpha\beta} \to GL(V\times V), \qquad (p,\xi) \mapsto \begin{pmatrix} \psi_{\alpha\beta}(p) & 0 \\ \xi \cdot d(\psi_{\alpha\beta})_p & \psi_{\alpha\beta}(p) \end{pmatrix}.$$

This shows that $(TE, d\pi, TM)$ is a vector bundle as well with fiberwise addition $d(+_E)$ and fiberwise scalar multiplication $d(m_t^E)$.

Thus when (E,π,M) has an appropriately chosen vector bundle structure, we obtain the natural tangent bundle (TE,π_E,E) structure in correlation with the former structure, and we obtain another vector bundle structure $(TE, d\pi, TM)$ which can be considered as the differential of the original one.

Recall that an *exact sequence* is a sequence of the form

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\delta} D \xrightarrow{\gamma} E$$

where

$$\operatorname{im}\alpha = \ker\beta, \qquad \operatorname{im}\beta = \ker\delta, \qquad \operatorname{im}\delta = \ker\gamma.$$

And a *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

Proposition 3.1. *Let (E,π,M) be a smooth vector bundle. There exists a canonical short exact sequence*

$$0 \longrightarrow \pi^*E \xrightarrow{\alpha} TE \xrightarrow{\beta} \pi^*TM \longrightarrow 0$$

of vector bundles.

Proof: Let's first unravel notation:

Recall that

$$\pi^*E = \{(u,v) \in E \times E : \pi(u) = \pi(v)\} \cong E \oplus E,$$

and as a bundle $\pi_1 : E \oplus E \to E$ with projection onto the first factor. Post composing with π, we obtain a vector bundle $E \oplus E \to M$.

We have (TE, π_E, E) as the usual tangent bundle of E, and post composing with π we obtain a vector bundle $TE \to M$.

Let $\iota : TM \to TE$ be the zero section, and let $\pi^*TM = \iota(TM)$ denote the the subbundle of $d\pi : TE \to TM$ and then post composing with $\pi_M : TM \to M$ yields the vector bundle $\pi^*TM \to M$.

Define the map $\alpha : \pi^*E \to TE$ by

$$\alpha(u,v) = \left(p, u; 0, \left.\frac{d}{dt}\right|_{t=0} (u+tv)\right).$$

Define the map $\beta : TE \to \pi^*TM$ by

$$\beta(x, v; \xi, w) = (x, 0; \xi, 0),$$

i.e., $\beta = \iota \circ d\pi$. Then we see that

$$\begin{aligned} \beta \circ \alpha((x,v),(x,w)) &= \beta\left(x, v; 0, \left.\frac{d}{dt}\right|_{t=0} (v+tw)\right) \\ &= (x, 0; 0, 0). \end{aligned}$$

Hence $\operatorname{im}\alpha \subseteq \ker\beta$. Moreover, since π^*TM is a rank n vector bundle over M and TE is a rank $n+2k$ vector bundle over M, we have that $\ker\beta$ is $2k$-dimensional, as is $\operatorname{im}\alpha$ by construction. Thus by dimensionality, we have that $\operatorname{im}\alpha = \ker\beta$. Moreover, by construction α is injective and β is surjective thus showing exactness. $\square$

3.1 The Vertical Bundle

Let $\pi : E \to M$ and consider the associated bundle $d\pi : TE \to TM$. Thinking of $d\pi$ as a bundle homomorphism, we have the following commutative diagram

$$\begin{tikzcd} TE \arrow[r, "d\pi"] \arrow[d, "\pi_E"'] & TM \arrow[d, "\pi_M"] \\ E \arrow[r, "\pi"'] & M \end{tikzcd}$$

In local coordinates, we have that $d\pi(x, v; \xi, w) = (x, \xi)$. That is, for each $(x, v) \in E$, the map $d\pi_{(x,v)} : T_{(x,v)}E \to T_xM$ has constant rank n. Thus $\ker d\pi$ is a vector subbundle of (TE, π_E, E) where

$$(\ker d\pi)_\theta = \ker d\phi_\theta.$$

Moreover, since TE is a rank $n + k$ vector bundle over E, we have that $\ker d\pi$ is a rank k vector subbundle by dimensionality.

We define the *vertical bundle over* E to be

$$VE := \ker d\pi.$$

The local form of a vertical vector $X \in VE$ is given by

$$X = (x, v; 0, w).$$

Moreover, since the transition functions

$$d\tilde{\phi}_\alpha \circ d\tilde{\phi}_\beta^{-1}(x, v; 0, w) = (\phi_{\alpha\beta}(x), \psi_{\alpha\beta}(\phi_\beta^{-1}(x))v; 0, \psi_{\alpha\beta}(\phi_\beta^{-1}(x))w)$$

are linear in $(v, w) \in V \times V$ for fixed x, we see that VE is actually a vector bundle over M of rank $2k$.

Let $\iota : M \to TM$ denote the zero section of (TM, π_M, M). Then consider the pullback bundle $\iota^*(TE, d\pi, TM)$

$$\begin{tikzcd} \iota^*TE \arrow[r, "\iota_\#"] \arrow[d, "\iota^*d\pi"'] & TE \arrow[d, "d\pi"] \\ M \arrow[r, "\iota"'] & TM \end{tikzcd}$$

and recall that

$$\begin{aligned}\iota^* TE &= \{(x,(y,v;\xi,w)) \in M \times TE : \iota(x) = d\pi(y,v;\xi,w)\} \\ &= \{(x,(y,v;\xi,w)) \in M \times TE : x = y \text{ and } \xi = 0\} \\ &\cong VE.\end{aligned}$$

Thus we may think of the vertical bundle as the pullback via the zero section. Moreover, we have a canonical isomorphism $\mathtt{vl} : E \oplus E \to VE$ given by

$$\mathtt{vl}(u_p, v_p) = \left.\frac{d}{dt}\right|_{t=0} (u_p + tv_p).$$

In local coordinates

$$\mathtt{vl}((x,u),(x,v)) = (x,u;0,v).$$

The map $\mathtt{vl}$ is called the *vertical lift*. We define the *vertical projection* to be the map $\mathtt{vpr} : VE \to E$ given by

$$\mathtt{vpr} := \pi_2 \circ \mathtt{vl}^{-1},$$

where $\pi_2 : E \oplus E \to E$, $\pi_2(u,v) = v$. Note that

$$\pi_1 \circ \mathtt{vl}^{-1} = \pi_E|_{VE}.$$

3.2 The Double Tangent Bundle

Consider the tangent bundle $\pi : TM \to M$ as a vector bundle in its own right. Then from preceding remarks, we have two canonical vector bundle structures on the double tangent bundle $TTM \to TM$. The first is as the tangent bundle to the tangent bundle given by $\pi_{TM} : TTM \to TM$. The second is as the differential $d\pi : TTM \to TM$.

We wish to see how $(TTM, d\pi, TM)$ and (TTM, π_{TM}, TM) are related. To this end, define the *canonical flip* $\kappa : TTM \to TTM$ given in local coordinates by

$$\kappa(x,v;\xi,\eta) = (x,\xi;v,\eta).$$

Then

$$d\pi \circ \kappa = \pi_{TM}$$

and

$$\pi_{TM} \circ \kappa = d\pi.$$

Moreover, it's clear that $\kappa^{-1} = \kappa$.

Proposition 3.2. *$\kappa : TTM \to TTM$ is the unique smooth mapping such that*

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\Gamma(s,t) = \kappa(\frac{\partial}{\partial t}\frac{\partial}{\partial s}\Gamma(s,t))$$

for any smooth $\Gamma : \mathbb{R}^2 \to M$.

Recall that for a vector field $X \in \mathfrak{X}(M)$, we may treat X as smooth map $X : M \to TM$, and hence we have the differential $dX : TM \to TTM$.

Proposition 3.3. *For $X, Y \in \mathfrak{X}(M)$, we have that*

$$[X,Y] = \mathtt{vpr} \circ (dY \circ X - \kappa \circ dX \circ Y)$$

and

$$dY \circ X - \kappa \circ dX \circ Y = \mathtt{vl}(Y, [X,Y]).$$

Proof: Recall that in local coordinates, we have that

$$\begin{aligned}
[X,Y] &= X[Y^j]\partial_j - Y[X^j]\partial_j \\
&= X^i \frac{\partial Y^j}{\partial x^i}\partial_j - Y^i \frac{\partial X^j}{\partial x^i}\partial_j \\
&= dY^j(X)\partial_j - dX^j(Y)\partial_j.
\end{aligned}$$

Now, treating $X, Y : M \to TM$ as a smooth map, we have in local coordinates that

$$X(x) = (x, \overline{X}(x)), \qquad Y(x) = (x, \overline{Y}(x)).$$

Then as a map $dX_p : T_pM \to T_{X(p)}TM$ we have the local expression

$$dX_p = \begin{pmatrix} \mathbb{1}_{T_pM} \\ d\overline{X}_p \end{pmatrix},$$

and hence for $Y \in TM$, we see that that

$$dX \circ Y(x) = (x, \overline{X}(x); \overline{Y}(x), d\overline{X}_x(\overline{Y}_x)).$$

It's clear from this that the first expression follows, as does the second from our local expression of vertical lift. $\square$

4 Sprays

Let M be a smooth manifold with tangent bundle (TM, π, M). Let $X : TM \to TTM$ be a smooth vector field. We say that X is a *differential equation of second order* or a *vector field of second order* if

$$d\pi \circ \xi = \mathbb{1}_{TM}.$$

That is, in particular, ξ is a section of (TTM, π_{TM}, TM) and of $(TTM, d\pi, TM)$.

A differential equation of second order X is called a *spray of M* if

$$X(sv) = ds(s(X(v)))$$

for all $s \in \mathbb{R}, v \in TM$, where $s : TM \to TM$, $s(x, v) = (x, sv)$, and similarly, $s : TTM \to TTM$, $s(v, \theta) = (sv, s\theta)$.

Let (U, x) be local coordinates on M which trivialize TM and TTM. Then under the usual identifications, we have that

$$TU = U \times \mathbb{R}^n, \qquad TTU = (U \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n).$$

In these local coordinates, a vector field $X : TU \to TTU$ is given by

$$X(x, v) = (x, v; f(x, v), g(x, v)).$$

Now X is a vector field of second order if and only if $f(x, v) = v$.

Lemma 4.1. *$X(x, v) = (x, v; v, g(x, v))$ is a spray if and only if $g(x, sv) = s^2 g(x, v)$ for all $s \in \mathbb{R}$ and $(x, v) \in U$.*

Proof: Noting that as a map $s : TU \to TU$, $s(x, v) = (x, sv)$, we have that $ds : TTU \to TTU$ given by

$$ds(x, v; u, w) = (x, v; u, sw).$$

Hence

$$\begin{aligned} ds(s(X(x, v))) &= ds(s(x, v; v, g(x, v))) \\ &= ds(x, sv; sv, sg(x, v)) \\ &= (x, sv; sv, s^2 g(x, v)) \end{aligned}$$

and since

$$X(x, sv) = (x, sv; sv, g(x, sv)),$$

the result then follows. $\square$

Note that that the above Lemma is trivially satisfied if $g = 0$. Thus sprays exist local, and hence via a partition of unity sprays exist globally.

Proposition 4.2. *Let $X \in \mathfrak{X}(TM)$. Then X is a vector field of second order if and only if each maximal integral curve $\beta_v : (a_v, b_v) \to TM$ with initial point $v \in TM$ and projection $\alpha_v := \pi \circ \beta_v$ satisfies $\alpha_v' = \beta_v$.*

Proof: Fix $v \in TM$ and let β_v be a maximal integral curve for X, and in particular, $X(v) = \beta_v'(0) = \beta_{v,*}(\frac{d}{dt}\big|_{t=0})$ and suppose $\alpha_v' = \beta_v$. Then

$$\begin{aligned} d\pi \circ X(v) &= d\pi \circ \beta_{v,*}\left(\frac{d}{dt}\bigg|_{t=0}\right) \\ &= d(\pi \circ \beta_v)\left(\frac{d}{dt}\bigg|_{t=0}\right) \\ &= d(\alpha_v)\left(\frac{d}{dt}\bigg|_{t=0}\right) \\ &= \alpha_v'(0) \\ &= \beta_v(0) \\ &= v, \end{aligned}$$

and thus X is of second order.

Conversely, suppose X is of second order, and let β_v be a maximal integral curve of X. Then

$$\begin{aligned} \alpha_v'(t) &= (\pi \circ \beta_v)'(t) \\ &= d\pi \circ \beta_v'(t) \\ &= d\pi \circ X(\beta_v(t)) \\ &= \beta_v(t), \end{aligned}$$

as desired. □

Proposition 4.3. *Let X be a vector field of second order. Then X is a spray if and only if its integral curves (with the previous proposition's notation) satisfies the following two properties:*

i. For $s, t \in \mathbb{R}$ and $v \in TM$, then $st \in (a_v, b_v)$ if and only if $t \in (a_{sv}, b_{sv})$.

ii. For $s, t \in \mathbb{R}$ and $v \in TM$ with $st \in (a_v, b_v)$, then $\alpha_v(st) = \alpha_{sv}(t)$.

Proof: Suppose all of the integral curves of X have the desired properties, and let β_v be one such curve. Recall that for $\alpha_v = \pi \circ \beta_v$, since X is of second order, $\alpha_v' = \beta_v$. Since $\alpha_v(st) = \alpha_{sv}(t)$ for $st \in (a_v, b_v)$, differentiating with respect to t, we obtain

$$s(\alpha_v'(st)) = \alpha_{sv}'(t),$$

or rather

$$s(\beta_v(st)) = \beta_{sv}(t).$$

Differentiating once more and using the chain rule for $s : TM \to TM$, we then obtain

$$\beta'_{sv}(t) = ds(s(\beta'_v(st))),$$

and hence at $t = 0$,

$$X(sv) = \beta'_{sv}(0) = ds(s(\beta'_v(0))) = ds(s(X(v))),$$

thus showing that X is a spray.

Conversely, let $\beta_v : (a_v, b_b) \to TM$ be a maximal integral curve of X. Fix $s \in \mathbb{R}$ and define the curve $t \mapsto \gamma_v(t)$, where $\gamma_v(t) = s\beta_v(st)$ and t is such that $st \in (a_v, b_v)$. Then

$$\begin{aligned}
\gamma'_v(t) &= ds(s(\beta'_v(st))) \\
&= ds(s(X(\beta_v(st)))) \\
&= X(s\beta_v(st)) \\
&= X(\gamma_v(t)).
\end{aligned}$$

Thus γ_v is an integral curve of X with initial condition $\gamma_v(0) = sv$. By the uniqueness of integral curves, we conclude that $\gamma_v(t) = \beta_{sv}(t)$ and that $t \in (a_{sv}, b_{sv})$. When $s \neq 0$, we obtain the reverse inclusion replacing s by $\frac{1}{s}$. Moreover, if $s = 0$ and $t \in (a_0, b_0)$, then we trivially have that $0 \in (a_v, b_v)$ for any $v \in TM$.

Finally, for any such s, t we have that

$$\beta_{sv}(t) = s\beta_v(st),$$

and taking the projection, we see that

$$\alpha_{sv}(t) = \pi \circ \beta_{sv}(T) = \pi(s(\beta_v)(st))) = \alpha_v(st).$$

□

4.1 The Exponential Map

Let X be a spray on M. Then define the *domain of the exponential map* to be the set

$$\mathcal{O}^X := \{v \in TM : 1 \in (a_v, b_v)\}.$$

We then define the *exponential map with respect to the spray* X to be the map $\exp : \mathcal{O}^X \to M$ given by

$$\exp(v) = \alpha_v(1),$$

where $\alpha_v := \pi \circ \beta_v$ and $\beta_v : (a_v, b_v) \to TM$ is the maximal integral curve for X with initial condition $v \in TM$. For each $p \in M$, we also define the *restricted exponential map* to be $\exp_p : \mathcal{O}_p^X \to M$ given by $\exp_p = \exp|_{\mathcal{O}_p^X}$, where $\mathcal{O}_p^X = \mathcal{O}^X \cap T_pM$. When the spray X is understand, the dependence is typically suppressed in the notation.

Recall that subset S of a vector space V is *star-shaped with respect to* $x \in V$ if for all $y \in S$, the line segment from x to y is contained in S, i.e., the curve $\psi(t) := ty + (1-t)x$ satisfies $\psi(t) \in S$ for all $t \in [0, 1]$.

Proposition 4.4 (Properties of the Exponential Map)**.** *Let M be a smooth manifold and let X be a spray on M.*

a. $\mathcal{O}$ is an open neighborhood of TM containing the zero section $\iota(M)$, and each $\mathcal{O}_p$ is a star-shaped region with respect to 0 in T_pM.

b. For each $v \in \mathcal{O}$,

$$\alpha_v(t) = \exp(tv)$$

for all $t \in (a_v, b_v)$.

c. $\exp : \mathcal{O} \to M$ is smooth.

d. For each $p \in M$, the differential $d(\exp_p)_0 : T_0T_pM \to T_pM$ is the identity map on T_pM under the usual identification.

Proof: By our rescaling properties for sprays, (b.) is immediately shown. Moreover, since for any $v \in \mathcal{O}_p$, we have that $[0, 1] \subset (a_v, b_b)$, we conclude that $[0, 1]v \subset \mathcal{O}_p$ and that $\mathcal{O}_p$ is a star-shaped region with respect to 0 in T_pM for each $p \in M$.

To show that $\mathcal{O}$ is open, recall that by the Fundamental Theorem on Flows of Vector Fields, there exists an open set $\mathcal{D} \subseteq \mathbb{R} \times TM$ and smooth map $\theta : \mathcal{D} \to TM$ such that $(0, v) \in \mathcal{D}$ for all $v \in TM$ and $\theta(t, v) = \beta_v(t)$.

Let $v \in \mathcal{O}$, then since $(1, v) \in \mathcal{D}$ by definition, there exists an open neighborhood of $(1, v)$ in $\mathbb{R} \times TM$ such that θ is defined. Therefore, there exists a neighborhood about $v \in TM$ for which $\beta_v(t)$ exists for all $t \in [0, 1]$. Thus $\mathcal{O}$ is open in TM. Moreover, since $\exp = \pi \circ \theta(1, \cdot)|_{\mathcal{O}}$, we conclude that exp is smooth on $\mathcal{O}$.

Finally, let $v \in T_pM$, so we have by the corresponding isomorphism

$$k(v) = \left.\frac{d}{dt}\right|_{t=0} (tv) \in T_0(T_pM).$$

Then

$$\begin{aligned} d(\exp_p)_0(k(v)) &= \left.\frac{d}{dt}\right|_{t=0} (\exp_p(tv)) \\ &= \left.\frac{d}{dt}\right|_{t=0} (\alpha_v(t)) \\ &= \alpha_v'(0) \\ &= \beta_v(0) \\ &= v, \end{aligned}$$

thus completing the proof. $\square$

5 Constructions via Sprays

Let $\hat{\pi} : E \to M$ be a smooth vector bundle. Let $\iota : M \to E$ denote the zero section, and consider the pullback $\iota^*(TE, \pi_E, E)$ bundle given by $(\iota^* TE, \iota^*\pi_E, M)$. We've seen that this is precisely the restriction $TE|_M$, and we let $\pi : TE|_M \to M$ denote this vector bundle.

It should be noted that this is not the vertical bundle since we're taking the zero section of $(E, \hat{\pi}, M)$ and not (TM, π_M, M). If $E = TM$, then $TTM|_M$ is the canonical involution of VTM.

As a map $\iota : M \to E$ and so $d\iota : TM \to TE$ with $\operatorname{im} d\iota \subseteq TE|_M$. Since ι is an embedding, $d\iota$ is a fiberwise injective bundle morphism, and hence $TM \cong \operatorname{im} d\iota$ is a subbundle of $TE|_M$. Moreover, in local coordinates, we have that

$$d\iota(x, \xi) = (x, 0; \xi, 0).$$

Now define the bundle morphism $k : E \to TE$ by

$$k(v) = \mathtt{vl}(0, v) = \left.\frac{d}{dt}\right|_{t=0} (tv),$$

and hence in local coordinates

$$k(x, v) = (x, 0; 0, v).$$

Thus $\operatorname{im} k \subseteq TE|_M$ and we similarly have that k is an embedding. Thus $E \cong \operatorname{im} k$ is a vector subbundle of $TE|_M$.

Finally, since $TE|_M$ consists of points in local coordinates given by $(x, 0; \xi, w)$ we conclude that the map $(k, d\iota) : E \oplus TM \to TE|_M$ is a vector bundle isomorphism.

With the above decomposition of the vector bundle $\pi : TE|_M \to M$, we see from construction that the vertical bundle

$$V(TE|_M) = \ker d\pi = \operatorname{im} k \cong E.$$

We call the other portion of this decomposition the *horizontal bundle*, that is, as subbundle $H(TE|_M) \leq TE|_M$, we have that

$$H(TE|_M) = \operatorname{im} d\iota \cong TM.$$

Now, when $E = TM$, we let VM and HM denote the the above vertical and horizontal bundles, and have that each $VM \cong TM$ and $HM \cong TM$. Thus by our above decomposition, we have that

$$TTM|_M = VM \oplus HM \cong TM \oplus TM.$$

Let X be a spray on M with exponential domain $\mathcal{O} \subseteq TM$. Since $\mathcal{O}$ is open, we have an identical splitting of

$$\begin{aligned} T\mathcal{O}|_M &= k(TM) \oplus d\iota(TM) \\ &= TM \oplus TM. \end{aligned}$$

Moreover, from construction $\exp \circ \iota = \mathbb{1}_M$.

Lemma 5.1. *The differential of the exponential map on the zero section,*

$$d(\exp)_{\iota(x)} : T_{\iota(x)}\mathcal{O} = k(T_xM) \oplus d\iota_x(T_xM) \to T_xM$$

is given by the map

$$(v, w) \mapsto v + w.$$

Proof: Fix $x \in M$, $u, v \in T_xM$, and so $d\iota_x(w) = (0, w)$ and $k(v) = (v, 0)$. Since $\exp \circ i = \mathbb{1}_M$, for $x \in M$, we have that

$$\begin{aligned} d(\exp)_{\iota(x)}(0, w) &= d(\exp)_{\iota(x)}(d\iota_x(w)) \\ &= d(\exp \circ \iota)_x(w) \\ &= d(\mathbb{1}_M)_x(w) \\ &= \mathbb{1}_{T_xM}(w) \\ &= w. \end{aligned}$$

On the other hand, note that $k(v) = \big(\frac{d}{dt}\big|_{t=0}(tv), 0\big) = (v, 0)$. Then

$$\begin{aligned} d(\exp)_{\iota(x)}(v, 0) &= d(\exp)_{\iota(x)}\left(\frac{d}{dt}\bigg|_{t=0}(tv)\right) \\ &= \frac{d}{dt}\bigg|_{t=0} \exp(tv) \\ &= \frac{d}{dt}\bigg|_{t=0} \alpha_{tv}(1) \\ &= \frac{d}{dt}\bigg|_{t=0} \alpha_v(t) \\ &= \alpha_v'(0) \\ &= v. \end{aligned}$$

That is, for $(v, w) = k(v) + d\iota_x(w)$, we have that

$$d(\exp)_{\iota(x)}(v, w) = v + w,$$

as desired. □

Lemma 5.2. *The differential of the map* $(\pi, \exp) : \mathcal{O} \to M \times M$ *on the zero section*

$$d(\pi, \exp)_{\iota(x)} : T_{\iota(x)}\mathcal{O} = k(T_xM) \oplus d\iota_x(T_xM) \to T_xM \oplus T_xM$$

is given by

$$(v, w) \mapsto (w, v + w).$$

Proof: By the previous lemma, we need only show that $d\pi_{\iota(x)} : T_{\iota(x)}\mathcal{O} = k(T_xM) \oplus d\iota_x(T_xM) \to T_xM$ is given by

$$(v, w) \mapsto w.$$

Indeed,

$$\begin{aligned}
d\pi_{\iota(x)}(v, w) &= d\pi_{\iota(x)}(k(v)) + d\pi_{\iota(x)}(d\iota_x(w)) \\
&= d\pi_{\iota(x)}\left(\left.\frac{d}{dt}\right|_{t=0}(tv)\right) + d(\pi \circ \iota)_x(w) \\
&= \left.\frac{d}{dt}\right|_{t=0}(\pi(tv)) + d(\mathbb{1}_M)_x(w) \\
&= \left.\frac{d}{dt}\right|_{t=0}(x) + \mathbb{1}_{T_xM}(w) \\
&= 0 + w \\
&= w,
\end{aligned}$$

and the result follows. $\square$

Thus for all $x \in M$, by the inverse function theorem, there exists an open neighborhood $U_x \subseteq \mathcal{O}$ of $\iota(x)$ such that $(\pi, \exp)|_{U_x} : U_x \to W_x$ is a diffeomorphism, where W_x is an open neighborhood of (x, x) in $M \times M$.

5.1 A Metric Space Lemma

thm:metricSpaceLem **Lemma 5.3.** *Let* (Z, d) *be a metric space, and suppose* X, Y, D *are all subspaces of* Z *with* $Y \subseteq X$ *and* $Y \subseteq D$*. Suppose* $f : D \to X$ *is a continuous function such that* $f|_Y = \mathbb{1}_Y$*. Furthermore, assume that for each* $y \in Y$*, there exists* $\epsilon(y) > 0$ *such that* $f|_{B_D(y,\epsilon(y))}$ *is a homeomorphism onto an open subset of* X*. Then there exists an open subspace* $U \subseteq D$ *of* Y *for which* f *is injective.*

Proof: For each $y \in Y$, we have that $f(B_D(y, \epsilon(y)/2))$ is open in X. Hence there exists $\epsilon'(y) > 0$ such that

$$B_X(y, \epsilon'(y)) \subseteq f(B_D(y, \epsilon(y)/2)),$$

and $\epsilon'(y) < \frac{\epsilon(y)}{4}$. Thus for each $y \in Y$, define the open sets

$$U_y = \left(f|_{B_D(y,\epsilon(y)/2)}\right)^{-1}(B_X(y, \epsilon'(y)),$$

and let

$$U = \bigcup_{y \in Y} U_y.$$

Now f is injective on U. Indeed, assume $f(z_1) = f(z_2) = y_0$ with $z_1 \in U_{y_1}, z_2 \in U_{y_2}$. In particular, we have that

$$y_0 = f(z_j) \in f(U_j) \subseteq B_X(y_j, \epsilon'(y_j)) \subseteq B_X(y_j, \epsilon(y_j)/4).$$

Without loss of generality, assume that $\epsilon(y_1) \geq \epsilon(y_2)$. Hence

$$\begin{aligned} d(z_2, y_1) &\leq d(z_2, y_2) + d(y_2, y_0) + d(y_0, y_1) \\ &\leq \frac{\epsilon(y_2)}{2} + \frac{\epsilon(y_2)}{4} + \frac{\epsilon(y_1)}{4} \\ &\leq \frac{\epsilon(y_1)}{2} + \frac{\epsilon(y_1)}{4} + \frac{\epsilon(y_1)}{4} \\ &= \epsilon(y_1), \end{aligned}$$

and so both $z_1, z_2 \in B_D(y_1, \epsilon(y_1))$, and since f is a homeomorphism here, we conclude $z_1 = z_2$. □

Theorem 5.4. *Suppose (M, g) is a Riemannian manifold and let $\hat{g}$ denote the Sasaki-metric on TM. Let ξ be any spray on M with associated exponential map* exp*. Then there exists an open neighborhood U of the zero section M in TM and an open neighborhood V of the diagonal $\Delta(M)$ in $M \times M$ such that $(\pi, \exp) : U \to V$ is a diffeomorphism.*

Proof: We need to modify either Lemma 5.3 to f being just injective on Y and $f(Y) \subseteq X$ for a different metric space (X, d'), or compose with a new map to consider the diagonal and the zero section, the same set. □

6 The Normal Bundle

Following *Differential Manifolds* by Tammo tom Dieck. See also the "Flowout Theorem" in [6], as I'm fairly certain the Flowout Theorem is another way to characterize the integral curves of sprays.
Maybe let $I = \iota|_A = \iota \circ i$ for clarity? Idk, fix this at some point though.
[2]

Let (M, g) be a Riemannian manifold, and let $A \hookrightarrow M$ be a submanifold, then the differential $TA \to TM|_A$ is an injective bundle morphism, and we can regard T_aA as a subspace of T_aM. Let $N_aA = T_aA^\perp$ in T_aM. Then we have the orthogonal product

$$T_aM = N_aA \oplus T_aA.$$

Moreover, as we obtain a subbundle NA of $TM|_A$, we have the decomposition

$$TM|_A = NA \oplus TA.$$

Let ξ be a spray on M, and let $\exp : \mathcal{O} \subseteq TM \to M$ be its exponential map. Let $\mathcal{D} = \mathcal{O} \cap NA$, then $\mathcal{D}$ is an open neighborhood of the zero section

$$i|_A (A) \subset \mathcal{D} \subset NA.$$

Thus with respect to our decomposition of

$$T_{i(a)}\mathcal{O} = k(T_aM) \oplus di_a(T_aM),$$

into horizontal and vertical components, we then have

$$\begin{aligned} T_{i(a)}(\mathcal{D}) &= k(N_aM) \oplus di_a(T_aA) \\ &\cong N_aA \oplus T_aA. \end{aligned}$$

Let $\exp^\perp = \exp|_{\mathcal{D}} : \mathcal{D} \to M$. Then on the zero section $i(A)$, we have that

$$d(\exp^\perp)_{i(a)} : (v, w) = v + w,$$

but $T_aM = N_aA \oplus T_aA$, hence $d(\exp^\perp)_{i(a)}$ is the identity.

Thus for each $a \in A$, there exists a neighborhood $U_a \subseteq \mathcal{D}$ of $i(a)$ such that $\exp^\perp : U_a \to V_a$ is a diffeomorphism, where $V_a \subseteq M$ is a neighborhood of a.

thm:tublar **Theorem 6.1** (Tubular Neighborhood Theorem)**.** *Let (M, g) be a Riemannian manifold with Sasaki-metric $\hat{g}$ on TM. Let ξ be any spray on M with associated exponential map* exp*. Suppose $A \subset M$ is a submanifold. Then there exists an open neighborhood U of the zero section $i(A)$ in NA, and an open neighborhood V of A in M such that $\exp^{\perp}\big|_U : U \to V$ is a diffeomorphism.*

Proof: Let $\mathcal{D}$ denote the domain of $\exp^{\perp}$ in the normal bundle NA. Let d denote the induced distance on TM from the Sasaki metric $\hat{g}$, so that (TM, d) is a metric space. Then $\mathcal{D}$ is a subspace of TM containing the zero section $i(A)$. Moreover, $i(A)$ is a subspace of $i(M)$ which is a subspace of TM. Finally, we have that $i \circ \exp^{\perp} : \mathcal{D} \to i(M)$, where

$$i \circ \exp^{\perp}(i(a)) = i(a),$$

so the restriction to $i(A)$ is the identity, and for each $a \in A$, there exists a open neighborhood $U_a \subseteq \mathcal{D}$ of a such that $i \circ \exp^{\perp}\big|_{U_a}$ is a diffeomorphism.

Since TM is a metric space, for each $a \in A$, we can find $\epsilon(a) > 0$ so that $B_{\mathcal{D}}(i(a), \epsilon(a)) \subseteq U_a$. As this restriction is still a homeomorphism, we may apply Lemma 5.3 directly to conclude there exists there exists an open neighborhood $U \subseteq \mathcal{D}$ of $i(A)$ for which $i \circ \exp|_U$ is a diffeomorphism onto its image in $i(M)$. Since i is diffeomorphism onto it's image, post-compositing with i^{-1}, the result follows. $\square$

When ξ is the geodesic spray, and exp our Riemannian exponential map, we say the restriction of exp to the normal bundle, the *normal exponential map*.

Lemma 6.2. *There exists a smooth function $\epsilon : A \to \mathbb{R}$ such that the ϵ-neighborhood,*

$$U^{\epsilon} = \{(a, v) \in NA : |v|_g < \epsilon(a)\}$$

is contained in U.

Proof: Let $\{W_\beta : \beta \in B\}$ be a collection of locally-finite charts which cover A. Then due to the trivialization of the bundle $NA \to A$, we have that

$$i(W_\beta \cap A) = (W_\beta \cap A) \times \{0\}.$$

Since this is contained in the open set U, there exists $\epsilon_\beta > 0$ such that

$$(W_\beta \cap A) \times D(0, \epsilon_\beta) \subseteq U,$$

where

$$D(0, \epsilon_\beta) = \{v \in N_{a_0}A : |v|_{g(a_0)} < \epsilon_\beta\},$$

for some fixed $a_0 \in A$, since they're all equivalent. Let $\{\theta_\beta : \beta \in B\}$ be a partition of unity subordinate to $\{W_\beta : \beta \in B\}$. Define the function $\epsilon : A \to \mathbb{R}$ by

$$\epsilon(a) = \sum_{\beta \in B} \epsilon_\beta \theta_\beta(a).$$

Then ϵ is smooth and

$$\epsilon(a) \le \max\{\epsilon_\beta : a \in W_\beta\},$$

showing that

$$\{a\} \times D(0, \epsilon(a)) \subset U,$$

for each $a \in A$. Then

$$U^\epsilon = \bigcup_{a \in A} \{v \in N_a A : |v|_g < \epsilon(a)\} \subseteq U,$$

as desired. $\square$

Corollary 6.3. *When $A \subset M$ is compact, there exists $\epsilon > 0$ such that the U in the above theorem can be taken to be*

$$U = \{v \in NA : |v|_g < \epsilon\}.$$

Proof: Since A is compact, the continuously defined $\epsilon : A \to \mathbb{R}$ in the above proof attains a positive minimum value, and taking the constant ϵ to be this value gives the desired result. $\square$

7 General Fiber Bundles

This section will mostly follow [5], [8], and [10].

A *fiber bundle* is a quadruple (E, π, M, S) which consists of smooth manifolds E, M, and and S, and a smooth surjective submersion $\pi : E \to M$ with the requirement that for each $p \in M$, there exists an open neighborhood $U \subseteq M$ of p such that $E|_U := \pi^{-1}(U)$ is diffeomorphic to $U \times S$ via a fiber respecting diagram

$$\begin{tikzcd} E|_U \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times S \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

We say E is the *total space*, M is the *base manifold*, S is the *model fiber*, and π is the *bundle projection*. In practice, the fiber is usual understand from context, and so we typically denote a fiber a bundle as the mapping $\pi : E \to M$, or as s script lettering of the total space, e.g., $(E, \pi, M, S) = \mathcal{E}$. Moreover, (U, ϕ) as above is called a *fiber chart* or a *local trivialization of* E.

A collection of fiber charts $\{(U_\alpha, \phi_\alpha)\}$ such that $\{U_\alpha\}$ is an open cover of M is called *(fiber) bundle atlas*. If we fix such an atlas, then

$$\phi_\alpha \circ \phi_\beta^{-1}(x, p) = (x, \phi_{\alpha\beta}(p)),$$

where

$$\phi_{\alpha\beta} : U_{\alpha\beta} \times S \to S$$

is smooth and $U_{\alpha\beta} = U_\alpha \cap U_\beta$, moreover, for each $x \in U_{\alpha\beta}$, we have that $p \mapsto \phi_{\alpha\beta}(x, p)$ is a diffeomorphism of S. It is sometimes useful to then consider $\phi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Diff}(S)$, but its differentiability is a very subtle question.[1] In either form, the functions $\{\phi_{\alpha\beta}\}$ are called the *transition functions* and satisfy the *cocycle condition*:

$$\phi_{\alpha\beta} \circ \phi_{\beta\gamma}(x) = \phi_{\alpha\gamma}(x), \qquad x \in U_{\alpha\beta\gamma},$$

[1] See [7] for treatment of the subtlety.

and

$$\phi_{\alpha\alpha}(x) = \mathbb{1}_F.$$

Given an open cover $\{U_\alpha\}$ of M and a cocycle of transition functions $\{\phi_{\alpha\beta}\}$, we may construct a fiber bundle $\mathcal{E}$.

Lemma 7.1. *Let $\pi : E \to M$ be a surjective submersion. If π is proper and M is connected, then $\pi : E \to M$ is a fiber bundle.*

Given a fiber bundle (E, π, M, S), we consider the differential $d\pi : TE \to TM$, and define the *vertical bundle*

$$VE := \ker d\pi.$$

A *connection* on a fiber bundle (E, π, M, S) is a vector-valued 1-form $\Phi \in \Omega^1(E; VE)$ such that $\Phi \circ \Phi = \Phi$ and $\operatorname{im} \Phi = VE$. Note that such a $\Phi \in \Omega^1(E; VE)$ is a C^∞-linear map $\Phi : TE \to VE$, and since $VE \subseteq TE$ the composition makes sense.

7.0.1 Considerations

Recall, a topological fiber bundle is quadruple (E, π, M, F), where E, M, F are topological spaces, and $\pi : E \to M$ is a continuous surjection; along with an equivalence class of bundle atlases $\{(U_\alpha, \phi_\alpha)\}$, where U_α is an open cover of M, and ϕ_α satisfies the trivialization criteria.

Let G be a topological group, i.e., G is a group with a topology so that the multiplication and inversion operations are continuous. Let F be a topological space, then we say that G acts on F if

$$(g_1(g_2 v)) = (g_1 g_2)v,$$

for all $g_1, g_2 \in G$ and $v \in F$. We say that G acts faithfully if for every $g \in G \setminus \{e\}$, there exists $v \in F$ such that $gv \neq v$. We say G acts freely if $gv = v$ implies $g = e$. Note that freely acting groups (on any nonempty set) are faithful.

Let G act freely on F. Then G is (group) isomorphic to a subgroup of $\mathrm{Homeo}(F)$.

Given a bundle (E, π, M, F), a G-atlas $\{U_\alpha, \phi_\alpha\}$ is a bundle atlas for $\mathcal{E}$ such that our transitions maps for a trivialization

$$\phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F, \qquad \phi_\alpha \circ \phi_\beta^{-1}(x, v) = (x\phi_{\alpha\beta}(x)v)$$

are such that $\phi_{\alpha\beta} : U_{\alpha\beta} \to G$ is continuous. A G-bundle is a fiber bundle $\mathcal{E}$ with an equivalence class of G-atlases. The group G is called the structure group of the bundle $\mathcal{E}$.

The transition functions satisfy the following:

i. $\phi_{\alpha\alpha}(x) = e$,

ii. $\phi_{\beta\alpha}(x) = \phi_{\alpha\beta}(x)^{-1}$,

iii. $\phi_{\alpha\beta}(x)\phi_{\beta\gamma}(x) = \phi_{\alpha\gamma}(x)$,

where property (iii.) is called the cocycle condition. The cocycle condition allows the transition functions to determine the fiber bundle F.

A principle G-bundle is a G-bundle where G acts on F freely and transitively, and hence we may identify G with F.

8 Basics of Vector Bundles

Introductory definitions follow from [5], [6], [10].
Check out [3] and [2]

8.0.1 Ehresmann Connections

See [10].

Let's now generalize this notion beyond the zero section. To this end, let $\pi : E \to M$ be a smooth vector bundles, and $\pi_E : TE \to E$, $\pi_M : TM \to M$ denote the two tangent bundles, and the usual differential $d\pi : TE \to TM$ which commutes via

$$\begin{tikzcd}
TE \arrow[r, "d\pi"] \arrow[d, "\pi_E"'] & TM \arrow[d, "\pi_M"] \\
E \arrow[r, "\pi"'] & M
\end{tikzcd}$$

We define the vertical bundle to be $V = \ker d\pi$. That is, for each $\theta = (x, u) \in E$, we have

$$V_\theta = \ker d\pi_\theta.$$

Let $\pi_V : V \to E$ denote this smooth vector bundle. Since each V_θ is isomorphic to E_x, we get by considering the composition of bundles $\pi \circ \pi_V$ that

$$V \cong E \oplus E.$$

Consider now the pullback bundle $\pi^* E$, that is,

$$\begin{aligned}
\pi^* E &= \{(\theta, \eta) \in E \times E : \pi(\theta) = \pi(\eta)\} \\
&= \{((x, u), (x, v) : u, v \in E_x\} \\
&= E \oplus E.
\end{aligned}$$

This allows us to define the fiber-isomorphism $j : \pi^*E \to V$ via

$$j((x,u),(x,v)) \mapsto \left.\frac{d}{dt}\right|_{t=0} (u+tv),$$

and hence the fiber-isomorphism $k : V \to E$ via

$$k(J(x,u),(x,v)) = (x,v).$$

The horizontal bundle H is the subbundle of TE that is complementary to V, that is,

$$TE = H \oplus V,$$

and hence

$$T_\theta E = H_\theta \oplus V_\theta.$$

A horizontal bundle can be completely characterized by a *connection form* $\omega : TE \to TE$, as a bundle endomorphism (a $(0,2)$-tensor on E) and satisfies

i. $\omega^2 = \omega$, and

ii. $\mathrm{im}\,(\omega) = V$.

Then the horizontal bundle is given by

$$H = \ker \omega,$$

and this connection form ω can be thought as the projection onto the vertical space.

To describe such a connection form, we need the notion of a horizontal lift. To this end, let $\gamma : I \to M$ denote a path, and we say its lift $\tilde{\gamma} : I \to E$ is a *horizontal lift* if $\tilde{\gamma}'(t) \in H_{\tilde{\gamma}(t)}$ for all $t \in I$. A path $\gamma : I \to M$, $\gamma(0) = x$, $\gamma(1) = y$ *has horizontal lifts* if for every $v \in E_x$, there is a unique horizontal lift $\tilde{\gamma} : I \to E$ such that $\tilde{\gamma}(0) = v$ and $\tilde{\gamma}(1) \in E_y$. By rescaling, this definition is equivalent to $\tilde{\gamma}(t) \in E_{\gamma(t)}$ for all $t \in I$.

If every path $\gamma : I \to M$ has a horizontal lift, we say the horizontal bundle H has the *horizontal lifting property* and we call H an *Ehresmann connection on E*. We use the term connection here because horizontal lifts can be used to connect the fibers of E. Indeed, let $L_\gamma(v)$ denote the image of the horizontal lift of γ with initial points (x,v). Then we may consider $L_\gamma(v)$ as the image of a section in $\Gamma(\gamma^*E)$ (noting the difference if γ has self-intersection points), and hence the unique horizontal lift is given by

$$\tilde{\gamma} = \gamma_\# L_\gamma(v) : I \to E.$$

Hence we have the diffeomorphism (with slight abuse of notation)

$$L_\gamma : E_x \to E_y, \qquad v \mapsto (L_\gamma(v))(1).$$

8.0.2 Tangent Bundle - Revisited

This follows from [9], [10], [11].

Suppose now that (M, g) is a Riemannian manifold, and we have the tangent bundle $\pi : TM \to M$, and our double tangent bundle $d\pi : TTM \to TM$. Our vertical space V is defined as usual

$$V_\theta = \ker d\pi_\theta, \qquad \theta \in TM.$$

Since (M, g) is Riemannian, let ∇ denote the Levi-Civita connection, and for a smooth curve, let $P_t^\gamma : T_{\gamma(0)}M \to T_{\gamma(t)}M$ denote the linear isomorphism of parallel translation along γ. For $\theta = (x, v) \in TM$, define the *horizontal lift* $L_\theta : T_xM \to T_\theta TM$ as follows: Let $X \in T_xM$, let $\gamma : I_\epsilon \to M$ be any curve with $\gamma(0) = x$, $\gamma'(0) = X$, and consider the parallel translation $P_t^\gamma(v)$. Then we have a curve $\alpha : I_\epsilon \to TM$ given by

$$\alpha(t) = (\gamma(t), P_t^\gamma(v)).$$

With a slight abuse of notation, we can consider $t \mapsto P_t^\gamma(v)$ a section of $I \to \gamma^*TM$, and thus define

$$L_\theta(\gamma'(0)) = \left.\frac{d}{dt}\right|_{t=0} P_t^\gamma(v).$$

Note that L_θ is well-defined (i.e., independent of choice of γ). Indeed, in coordinates, let

$$X = X^i \frac{\partial}{\partial x^i}, \qquad \alpha(t) = \xi^i(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}.$$

Since α is parallel along γ, we have that

$$\nabla_{\gamma'(t)}\alpha(t) = 0.$$

In particular,

$$\begin{aligned}
0 &= (\nabla_{\gamma'(t)}\alpha(t))^k\big|_{t=0} \\
&= \left(\nabla_{\gamma'(t)}\left(\xi^j(t)\ \frac{\partial}{\partial x^j}\bigg|_{\gamma(t)}\right)\right)^k\Bigg|_{t=0} \\
&= \nabla_{\gamma'(t)}\xi^k\big|_{t=0} + \left(\xi^j(0)\nabla_{\gamma'(t)}\ \frac{\partial}{\partial x^j}\bigg|_{\gamma(t)}\right)^k\Bigg|_{t=0} \\
&= \dot{\xi}^k(0) + \xi^j(0)\dot{\gamma}^i(0)\Gamma^k_{ij} \\
&= \dot{\xi}^k(0) + \xi^j(0)X^i\Gamma^k_{ij},
\end{aligned}$$

and so

$$\dot{\xi}^k(0) = -\xi^j(0)X^i\Gamma^k_{ij}.$$

Letting $v^i = \xi^i(0)$, we get that

$$\begin{aligned}
\alpha'(0) &= (x^k, v^k, X^k, \cdot\xi^k(0)) \\
&= (x^k, v^k, X^k, -v^iX^j\Gamma^k_{ij}).
\end{aligned}$$

That is,

$$L_{(x,v)}(X) = (x^k, v^k, X^k, -v^iX^j\Gamma^k_{ij}),$$

independent of choice of curve.

Now for $\theta \in TM$, we can define the horizontal subspace

$$H(\theta) = L_\theta(T_xM).$$

Since

$$d\pi(x^i, v^i, X^i, \eta^i) = (x^i, X^i),$$

we clearly have that

$$d\pi_\theta \circ L_\theta = \mathbb{1}_{TpM},$$

and hence $V(\theta) \cap H(\theta) = \{0\}$. Since both are n-dimensional, we have the decomposition

$$T_\theta TM = H(\theta) \oplus V(\theta).$$

Thus in coordinates, if $(x^i, v^i, X^i, \eta^i) \in T_\theta TM$, we get the decomposition

$$(x^i, v^i, X^i, \eta^i) = (x^i, v^i, X^i, -v^jX^k\Gamma^i_{jk}) + (x^i, v^i, 0, \eta^i + v^jX^k\Gamma^i_{jk}).$$

Now, define the connection map $K : TTM \to TM$ as follows: For $\theta = (x, v) \in TM$ and $\eta = (X, \eta) \in T_\theta TM$, identify $I_\theta : T_\theta T_x M \xrightarrow{\cong} T_x M$, $I_\theta(x^i, v^i, 0, \eta^i) = (x^i, \eta^i)$ and define

$$K_\theta(\eta) = I_\theta(\eta_v) = (x^i, \eta^i + v^j X^k \Gamma^i_{jk}),$$

where $\eta = \eta_h + \eta_v$ in the dirct sum.

Note that

$$\begin{aligned} K_\theta \circ L_\theta(X) &= K_\theta(x^i, v^i, X^i, -v^j X^k \Gamma^i_{jk}) \\ &= I_\theta(x^i, v^i, 0, 0) \\ &= 0. \end{aligned}$$

An equivalent definition to $K : TTM \to TM$ is as follows: Fix $\theta \in TM$ and $\xi \in T_\theta TM$. Let $\alpha : I_\epsilon \to TM$ be a curve with $\alpha(0) = \theta$ and $\alpha'(0) = \xi$. Then $\alpha(t) = (\gamma(t), Z(t))$. Then define

$$K_\theta(\xi) = \left(\nabla_{\gamma'(t)} Z(t)\right)\Big|_{t=0}.$$

We know define the *Sasaki metric* $\hat{g}$ on TM. For $\theta \in TM$ and $\xi, \eta \in T_\theta TM$, define

$$\hat{g}_\theta(\xi, \eta) = g_{\pi(\theta)}(d\pi_\theta(\xi), d\pi_\theta(\eta)) + g_{\pi(\theta)}(K_\theta(\xi), K_\theta(\eta)).$$

9 Principle Bundles

See [1] Chapter 3 and 5. And [4].

References

[1] Richard L Bishop and Richard J Crittenden. *Geometry of manifolds*, volume 15. Academic press, 2011.

[2] AA Borisenko and AL Yampol'skii. On the sasaki metric of the tangent and normal bundle. 294(1):19–22, 1987.

[3] AA Borisenko and AL Yampol'skii. Riemannian geometry of fibre bundles. *Russian Mathematical Surveys*, 46(6):55–106, 1991.

[4] Shōshichi Kobayashi. Foundations of differential geometry, 1996.

[5] Ivan Kolár, Jan Slovák, and Peter W Michor. *Natural operations in differential geometry*. Springer-Verlag, Berlin ; New York, 1993.

[6] John M Lee. *Introduction to Smooth manifolds*. Springer, 2003.

[7] Peter W. Michor. Gauge theory for fiber bundles, 1991.

[8] Peter W Michor. *Topics in differential geometry*, volume 93. American Mathematical Soc., 2008.

[9] Gabriel Paternain. *Geodesic flows*, volume 180. Springer Science & Business Media, 2012.

[10] Justin M Ryan. *Geometry of horizontal bundles and connections*. PhD thesis, Wichita State University, 2014.

[11] Takashi Sakai. *Riemannian geometry*, volume 149. American Mathematical Soc., 1996.