A Single Submanifold: $B = A \times \{q\}$ See itoh2001lipschitz for potential exposition.

We now consider the variational problem dealing with minimizing geodesics connecting a submanifold to a single a point. To this end, let (M, g) be a complete Riemannian manifold. Suppose $A \subset M$ is an immersed submanifold. Then (A, g) can be treated as a Riemannian submanifold of (M, g), where we denote the induced metric on A identically, as it's the pullback of g via the inclusion, that is, $i : A \hookrightarrow M$ and $g = i^*g$ (and i is an isometric immersion). Let $q \in M \setminus A$, and let

$$B = A \times \{q\} \subset M \times M,$$

be our boundary condition in this setting. Then $C(B)$ is the space of all piecewise regular curves $\gamma : [a, b] \to M$ with $\gamma(a) \in A$ and $\gamma(b) = q$. Moreover, any variation field $V \in T_\gamma C(B)$ satisfies $V(a) \in T_{\gamma(a)}A$ and $V(b) = 0$. In this setting, our B-geodesics are the geodesics γ which are normal to A, that is,

$$\gamma'(a) \in T_{\gamma(a)}A^\perp.$$

Our variation formulas then simplify as follows.

thm[First Variation of Energy] Let $\gamma \in C(B)$ and $V \in T_\gamma C(B)$ with associated variation $\Gamma : I_\epsilon \times [a, b] \to M$. If $\{t_j : 0 \le j \le k\}$ is an admissible partition for Γ, then align* $\hat{E}'(s) = -\int_a^b g(\partial_s\Gamma, D_t\partial_t\Gamma)dt + \sum_{j=1}^{k-1} g(\partial_s\Gamma(s, t_j), \Delta\partial_t\Gamma(s, t_j))$

In particular, when $s = 0$, align* $\mathrm{dE}_\gamma(V) = -\int_a^b g(V(t), D_t\gamma'(t))dt + \sum_{j=1}^{k-1} g(V(t_j), \Delta\gamma'(t_j))$

thm[Second Variation of Energy] Let $\gamma \in C(B)$ be a B-geodesic and $V, W \in T_\gamma C(B)$ with associated two-parameter variation $\Gamma : I_\epsilon \times I_\epsilon \times [a, b] \to M$. If $\{t_j : 0 \le j \le k\}$ is an admissible partition for Γ, then align* $(\mathrm{E})_\gamma(V, W) = \int_a^b (g(D_tV, D_tW) - g(R_{V\gamma'}\gamma', W))dt$

A-Jacobi Fields

Let (M, g) be an n-dimensional, complete Riemannian manifold with k-dimensional, Riemannian submanifold $A \subset M$ with shape operator S, and boundary condition $B = A \times \{q\}$. Suppose $\gamma \in C(B)$ is a B-geodesic. Then a Jacobi field $J \in \gamma$ is called an A-Jacobi field if J satisfies the initial conditions

$$J(a) \in T_{\gamma(a)}A, \qquad D_tJ(a) - S_{\gamma'(a)}(J(a)) \in T_{\gamma(a)}A^\perp.$$

Notice that since the first initial condition is a restriction that $(n - k)$ equations be 0, and the second initial condition is a restriction that k equations be zero, the space of all A-Jacobi equations along γ, denoted by $J^A(\gamma)$ is n-dimensional.

Let NA denote the normal bundle of A in TM_A. That is,

$$TM_A = TA \oplus NA$$

as a Whitney sum, and let $\exp^\perp : D \subset NA \to M$ denote the restriction of the exponential map $\exp : TM \to M$.

thm Let $(x, \xi) \in NA$ and $\gamma = \gamma_{x,\xi} : [0, b] \to M$ be a geodesic segment normal to A, and let $J \in \gamma$. Then J is an A-Jacobi field if and only if J is the variation field for a smooth variation $\Gamma : I_\epsilon \times [0, b] \to M$ such that each curve $\Gamma(s, \cdot)$ is a geodesic normal to A at $t = 0$.