Riemannian Geodesics

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1 Variations

In this section we borrow freely from many of the standard sources of Riemannian geometry of [1], [3], [4], [5], [9], [10] [11], [13], [14], [15], [16], with an emphasis on [18].

Let M be a connected smooth manifold. A smooth curve segment $\gamma : [a,b] \to M$ is said to be *regular* if $\gamma'(t) \neq 0$ for all $t \in [a,b]$. A continuous curve segment $\gamma : [a,b] \to M$ is said to be *piecewise regular* if there exits a partition

$$a = t_0 < t_1 < \cdots < t_k = b,$$

such that $\gamma|_{[t_j,t_{j-1}]}$ is regular for each $1 \leq j \leq k$. A curve $\tilde{\gamma} : [c,d] \to M$ is a reparametrization of γ if $\tilde{\gamma} = \gamma \circ \phi$, where $\phi : [c,d] \to [a,b]$ is a homeomorphism, which is piecewise smooth.

Suppose $B \subset M \times M$ is a (topologically) closed, immersed submanifold. Let $\mathcal{C}(B)$ (or $\mathcal{C}([a,b]; B)$ when the domain needs to be specified) denote the space of piecewise regular curves $\gamma : [a,b] \to M$ such that

$$(\gamma(a), \gamma(b)) \in B.$$

This submanifold B will give us a boundary condition on our curves, and so we shall say such a B is a *boundary condition*.

Given $\gamma \in \mathcal{C}(B)$, we say $\Gamma : I_\epsilon \times [a,b] \to M$ is *variation of* γ, if

i. Γ is continuous, and there exists a subdivision $a = t_0 < t_1 < \cdots < t_k = b$, such that $\Gamma|_{I_\epsilon \times [t_{j-1},t_j]}$ is smooth,

ii. $\Gamma(s, \cdot) \in \mathcal{C}(B)$ for each $s \in I_\epsilon$,

iii. and $\Gamma(0, \cdot) = \gamma$.

Given a continuous family of curve $\Gamma(s,t) : I_\epsilon \times [a,b] \to M$ such that $(i.)$ holds, we say that Γ is an *admissible family*, and the subdivision is an *admissible partition*.

Let's now treat $\mathcal{C}(B)$ as a manifold itself[1]. Let Γ be a variation of $\gamma \in \mathcal{C}(B)$, then the *longitudinal curves* $s \mapsto \Gamma(s,t) = \Gamma_s(t)$ is a point in $\mathcal{C}(B)$ and the *transverse curves* $s \mapsto \Gamma(s,t) = \Gamma_t(s)$ is a curve in $\mathcal{C}(B)$ starting at γ.

[1]See *Notes on Riemannian Geometry of Manifolds of Maps* in "PDF: Other", as well as check out Hilbert, Banach and Fréchet manifolds.

Consider the vector field $V(t)$ along γ defined by

$$V(t) = \partial_s \Gamma(0,t) = d\Gamma\left(\left.\frac{\partial}{\partial s}\right|_{s=0} \right),$$

which we call the *variation field* of Γ. Since the initial velocity of a curve is its linear description at the starting point, we see that the initial velocity of Γ is the variational field V. Since $(\Gamma(s,a), \Gamma(s,b)) \in B$ for all $s \in I_\epsilon$, we have that $(V(a), V(b)) \in T_{(\gamma(a),\gamma(b))}B$.

The tangent vectors to a manifold at a point are exactly all initial velocities of all curves starting at that point. This leads us to characterize the tangent space $T_\gamma \mathcal{C}(B)$ of $\mathcal{C}(B)$ at γ at the space of all piecewise-smooth vector fields V on γ such that $(V(a), V(b)) \in T_{(\gamma(a),\gamma(b))}B$.

Note that when M is endowed with a semi-Riemannian metric g, for any such $V(t) \in T_\gamma \mathcal{C}(B)$, the variation given by

$$\Gamma(s,t) = \exp_{\gamma(t)} sV(t), \tag{1.1}$$ {eq:genVariation}

is the curve $s \mapsto \Gamma_s(t) \in \mathcal{C}(B)$ with "initial velocity" $V(t)$. This leads to a complete characterization of the tangent spaces $T_\gamma \mathcal{C}(B)$. Moreover, note that for any $(X,Y) \in T_{(\gamma(a),\gamma(b))}B$, fix any $c \in [a,b]$, parallel transport X along γ from $\gamma(a)$ to $\gamma(c)$, and piecewise connect it to the parallel transport of Y along γ from $\gamma(b)$ to $\gamma(c)$, and we obtain a vector $V \in T_\gamma \mathcal{C}(B)$.

We shall use the following auxiliary lemmata dealing with changing the order of differentiation of variations.

thm:symmetryLemma **Lemma 1.1** (Symmetry Lemma). *Let ∇ denote a connection on M and let $\Gamma : I_\epsilon \times [a,b] \to M$ be an admissible family of curves, then on any rectangle $I_\epsilon \times [t_j, t_{j-1}]$ for which Γ is smooth,*

$$D_s \partial_t \Gamma = D_t \partial_s \Gamma.$$

Proof: As this is local, we write in coordinates

$$\Gamma(s,t) = (x^j(s,t)),$$

and so

$$\partial_t \Gamma(s,t) = \frac{\partial x^j}{\partial t}\partial_j, \qquad \partial_s \Gamma(s,t) = \frac{\partial x^j}{\partial s}\partial_j.$$

Hence

$$D_s \partial_t \Gamma = \left(\frac{\partial^2 x^k}{\partial s \partial t} + \frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial s}\Gamma^k_{ij} \right)\partial_k,$$

and

$$D_t \partial_s \Gamma = \left(\frac{\partial^2 x^k}{\partial t \partial s} + \frac{\partial x^i}{\partial s} \frac{\partial x^j}{\partial t} \Gamma^k_{ij} \right) \partial_k,$$

and the result follows.

Note that since ∇ is torsion-free, we have the coordinate-free expression

$$\begin{aligned} D_s \partial_t \Gamma - D_t \partial_s \Gamma &= \left[d\Gamma \left(\frac{\partial}{\partial s} \right), d\Gamma \left(\frac{\partial}{\partial t} \right) \right] \\ &= d\Gamma \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] \\ &= 0. \end{aligned}$$

$\square$

thm:curvatureLemma **Lemma 1.2** (Curvature Lemma). *Let ∇ denote a connection on M and let $\Gamma : I_\epsilon \times [a, b] \to M$ be an admissible family of curves and let V be a piecewise-smooth vector field on Γ, where V is smooth on the same admissible partition. Then on any rectangle $I_\epsilon \times [t_j, t_{j-1}]$ for which Γ and V are smooth,*

$$[D_s, D_t]V = R_{\partial_s \Gamma \partial_t \Gamma} V.$$

Proof: As this is local, we write in coordinates

$$\Gamma(s, t) = (x^j(s, t)), \qquad V(s, t) = V^j(s, t) \partial_j.$$

Then

$$D_t V = \frac{\partial V^j}{\partial t} \partial_j + V^j D_t \partial_j,$$

and

$$D_s D_t V = \frac{\partial^2 V^j}{\partial s \partial t} \partial_j + \frac{\partial V^j}{\partial t} D_s \partial_j + \frac{\partial V^j}{\partial s} D_t \partial_j + V^j D_s D_t \partial_j.$$

Interchanging s and t, we then see that

$$\begin{aligned} [D_s, D_t]V &= D_s D_t V - D_t D_s V \\ &= V^j (D_s D_t \partial_j - D_t D s \partial_j) \\ &= V^j [D_s, D_t] \partial_j. \end{aligned}$$

Computing this commutator, we first see that

$$D_t \partial_j = \nabla_{\partial_t \Gamma} \partial_j = \frac{\partial x^i}{\partial t} \nabla_{\partial_i} \partial_j,$$

and then

$$\begin{aligned}
D_s D_t \partial_j &= D_s \left(\frac{\partial x^i}{\partial t} \nabla_{\partial_i} \partial_j \right) \\
&= \frac{\partial^2 x^i}{\partial s \partial t} \nabla_{\partial_i} \partial_j + \frac{\partial x^i}{\partial t} D_s \nabla_{\partial_i} \partial_j \\
&= \frac{\partial^2 x^i}{\partial s \partial t} \nabla_{\partial_i} \partial_j + \frac{\partial x^i}{\partial t} \frac{\partial x^k}{\partial s} \nabla_{\partial_k} \nabla_{\partial_i} \partial_j .
\end{aligned}$$

Again interchaing s and t, we then see that

$$\begin{aligned}
[D_s, D_t]\partial_j &= D_s D_t \partial_j - D_t D_s \partial_j \\
&= \frac{\partial x^i}{\partial s} \frac{\partial x^k}{\partial t} \nabla_{\partial_i} \nabla_{\partial_k} \partial_j - \frac{\partial x^i}{\partial s} \frac{\partial x^k}{\partial t} \nabla_{\partial_k} \nabla_{\partial_i} \partial_j \\
&= \frac{\partial x^i}{\partial s} \frac{\partial x^k}{\partial t} [\nabla_{\partial_i}, \nabla_{\partial_k}] \partial_j \\
&= \frac{\partial x^i}{\partial s} \frac{\partial x^k}{\partial t} R_{\partial_i \partial_k} \partial_j \\
&= R_{\partial_s \Gamma \partial_t \Gamma} \partial_j .
\end{aligned}$$

Thus

$$\begin{aligned}
[D_s, D_t]V &= V^j [D_s, D_t] \partial_j \\
&= V^j R_{\partial_s \Gamma \partial_t \Gamma} \partial_j \\
&= R_{\partial_s \Gamma \partial_t \Gamma} V
\end{aligned}$$

□

1.1 The Energy and Length Functionals

Let (M, g) be a connected Riemannian manifold with boundary condition B. Consider the Lagrangian $TM \to \mathbb{R}$ given by

$$(x, v) \mapsto \frac{1}{2} g_x(v, v),$$

then we obtain the associated action, called the *energy functional*, $E : \mathcal{C}(B) \to \mathbb{R}$ given by

$$E(\gamma) = \frac{1}{2} \int_a^b g_{\gamma(t)}(\gamma'(t), \gamma'(t)) dt.$$

Consider the related Lagrangian $TM \to \mathbb{R}$ given by

$$(x, v) \mapsto \sqrt{g_x(v, v)},$$

and its associated action, called the *length functional*, $L : \mathcal{C}(B) \to \mathbb{R}$ given by

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt.$$

By Hölder's inequality, we see that

$$\begin{aligned}
(L(\gamma))^2 &= \left\{ \int_a^b |\gamma'(t)|_g dt \right\}^2 \\
&\leq \left(\int_a^b 1^2 dt \right) \left(\int_a^b |\gamma'(t)|_g^2 dt \right) \\
&= 2(b-a)E(\gamma),
\end{aligned}$$

where we have equality if and only if $|\gamma'(t)|_g$ is constant.

Proposition 1.3. *A curve γ minimizes E if and only if γ minimizes L and $|\gamma'|_g$ is constant. Moreover, as any regular curve can be reparametrized to have constant speed, we see that the minimization of either functional is equivalent.*

1.2 The First Variation

Let (M, g) be a complete Riemannian manifold with boundary condition B. Let $\gamma \in \mathcal{C}(B)$ and $V \in T_\gamma \mathcal{C}(B)$, then as the smooth map[2] $E : \mathcal{C}(B) \to \mathbb{R}$, we have the exterior differential $dE_\gamma : T_\gamma \mathcal{C}(B) \to \mathbb{R}$. Alternatively, let $\Gamma : I_\epsilon \times [a, b] \to M$ be a variation of γ with variation field V. Then we can consider the function

$$\hat{E}(s) := E \circ \Gamma(s, \cdot), \qquad \hat{E} : I_\epsilon \to \mathbb{R},$$

and hence the derivative $\hat{E}'(s)$. In particular, we have that

$$\begin{aligned}
\hat{E}'(0) &= dE_\gamma \circ \frac{\partial \Gamma}{\partial s}(0, \cdot) \\
&= dE_\gamma(V).
\end{aligned}$$

[2]Show E is smooth once we have Hilbert manifold knowledge.

thm:firstVarEnergy **Theorem 1.4** (First Variation of Energy)**.** *Let* $\gamma \in \mathcal{C}(B)$ *and* $V \in T_\gamma \mathcal{C}(B)$ *with associated variation* $\Gamma : I_\epsilon \times [a,b] \to M$*. If* $\{t_j : 0 \leq j \leq k\}$ *is an admissible partition for* Γ*, then*

$$\begin{aligned}\hat{E}'(s) = -\int_a^b g(\partial_s \Gamma, D_t \partial_t \Gamma) dt + \sum_{j=1}^{k-1} g(\partial_s \Gamma(s,t_j), \Delta \partial_t \Gamma(s,t_j)) \\ + g(\partial_s \Gamma(s,b), \partial_t \Gamma(s,b)) - g(\partial_s \Gamma(s,a), \partial_t \Gamma(s,a)).\end{aligned}$$

In particular, when $s = 0$*,*

$$\begin{aligned}dE_\gamma(V) = -\int_a^b g(V(t), D_t \gamma'(t)) dt + \sum_{j=1}^{k-1} g(V(t_j), \Delta \gamma'(t_j)) \\ + g(V(b), \gamma'(b)) - g(V(a), \gamma'(a)).\end{aligned}$$

Note that we're using the notation

$$\Delta f(t) = \lim_{h \to 0^+} (f(t+h) - f(t-h)).$$

Proof: Since everything is smooth on the compact set $[t_{j-1}, t_j]$, we may differentiate under the integral sign,[3] and hence

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\int_{t_{j-1}}^{t_j} g(\partial_t \Gamma, \partial_t \Gamma) dt &= \frac{1}{2}\int_{t_{j-1}}^{t_j} \frac{\partial}{\partial s} g(\partial_t \Gamma, \partial_t \Gamma) dt \\ &= \int_{t_{j-1}}^{t_j} g(D_s \partial_t \Gamma, \partial_t \Gamma) dt \\ &= \int_{t_{j-1}}^{t_j} g(D_t \partial_s \Gamma, \partial_t \Gamma) dt \\ &= \int_{t_{j-1}}^{t_j} \frac{\partial}{\partial t} g(\partial_s \Gamma, \partial_t \Gamma) dt - \int_{t_{j-1}}^{t_j} g(\partial_s \Gamma, D_t \partial_t \Gamma) dt \\ &= g(\partial_s \Gamma, \partial_t \Gamma)\big|_{(s,t_{j-1})}^{(s,t_j)} - \int_{t_{j-1}}^{t_j} g(\partial_s \Gamma, D_t \partial_t \Gamma) dt.\end{aligned}$$

Noting that $\partial_s \Gamma$ is continuous by construction from the exponential map,

[3] Make a file for Leibniz integral rule, and include the various Fubini-Tonelli, and usual convergence.

and summing up $1 \leq j \leq k$, we see that

$$\begin{aligned}\hat{E}'(s) &= \sum_{j=1}^{k} \frac{1}{2}\frac{d}{ds}\int_{t_{j-1}}^{t_j} g(\partial_t\Gamma, \partial_t\Gamma)dt \\ &= -\int_a^b g(\partial_s\Gamma, D_t\partial_t\Gamma)dt + \sum_{j=1}^{k-1} g(\partial_s\Gamma(s,t_j), \Delta\partial_t\Gamma(s,t_j)) \\ &\qquad + g(\partial_s\Gamma(s,b), \partial_t\Gamma(s,b)) - g(\partial_s\Gamma(s,a), \partial_t\Gamma(s,a)),\end{aligned}$$

as desired. □

We can now characterize the critical points of the energy (and length functional). A curve $\gamma \in \mathcal{C}(B)$ is a *critical point* of E if

$$dE_\gamma(V) = 0,$$

for all $V \in T_\gamma\mathcal{C}(B)$. To this end, suppose γ is a geodesic that satisfies

$$(\gamma'(a), -\gamma'(b)) \in T_{(\gamma(a),\gamma(b))}B^{\perp},$$

then we say γ is a *B-geodesic*. Since all geodesics are smooth (by definition), we see that all B-geodesics are critical points for E.

Corollary 1.5 (Regularity Theorem[4]). *$\gamma \in \mathcal{C}(B)$ is a critical point for the energy functional if and only γ is a B-geodesic.*

Proof: By preceding remarks, if γ is a B-geodesic, it's a critical point. Conversely, suppose γ is a critical point. Let $\{t_j : 0 \leq j \leq k\}$ be an admissible partition for γ. Fix one such subinterval $[t_{j-1}, t_j]$, and let $\phi \in C^\infty(\mathbb{R})$ be a bump function with $\phi > 0$ on (t_{j-1}, t_j) and 0 elsewhere. Then $\phi D_t\gamma' \in T_\gamma\mathcal{C}(B)$, and hence

$$\begin{aligned}0 &= dE_\gamma(\phi D_t\gamma') \\ &= -\int_{t_{j-1}}^{t_j} g(\phi D_t\gamma', D_t\gamma')dt \\ &= -\int_{t_{j-1}}^{t_j} \phi|D_t\gamma'|_g^2 dt,\end{aligned}$$

[4]Criticality implies smoothness. This is a feature of Elliptic Regularity, see Chapter 11 in [14]

and hence $D_t\gamma' = 0$ on (t_{j-1}, t_j). Since $j \in \{1, ..., k\}$ was arbitrary, we see also see that

$$D_t\gamma'(t_j^+) = 0 = D_t\gamma'(t_j^-).$$

and hence γ is a broken geodesic on $[a, b]$.

Fix one t_j, $1 \leq j \leq k-1$ and let U be a coordinate chart about $\gamma(t_j)$. Let V be the constant vector field equal to $\Delta\gamma'(t_j)$ in U, and extend via another bump function to all γ such that $V(t_i) = 0$ for all $i \neq j$. Again, we then see

$$\begin{aligned} 0 &= dE_\gamma(V) \\ &= g(V(t_j), \Delta\gamma'(t_j)) \\ &= |\Delta\gamma'(t_j)|_g^2, \end{aligned}$$

and hence $\Delta\gamma'(t_j) = 0$. Since the two one-sided velocities are equal, by the uniqueness and existence theorem for geodesics, we see all the segments are extensions of each other and hence γ is smooth, and thus a geodesic.

Finally, for any $(X, Y) \in T_{(\gamma(a), \gamma(b))}B$, let $V \in T_\gamma\mathcal{C}(B)$ be such that $V(a) = X$ and $V(b) = Y$. Then

$$\begin{aligned} 0 &= dE_\gamma(V) \\ &= g(V(b), \gamma'(b)) - g(V(a), \gamma'(a)) \\ &= g(Y, \gamma'(b)) - g(X, \gamma'(a)) \\ &= -(g \times g)((X, Y), (\gamma'(a), -\gamma'(b)), \end{aligned}$$

concluding the proof. □

Moreover, we note that if $\gamma \in \mathcal{C}(B)$ is parametrized to have constant speed $c = |\gamma'(t)|_g$, and let Γ be a variation of γ. Then

$$\frac{\partial}{\partial s}\bigg|_{s=0} \sqrt{g(\partial_t\Gamma(s,t), \partial_t\Gamma(s,t))} = \frac{1}{2c}\frac{\partial}{\partial s}\bigg|_{s=0} g(\partial_t\Gamma(s,t), \partial_t\Gamma(s,t)).$$

Thus for any $V \in T_\gamma\mathcal{C}(B)$, we have that

$$dL_\gamma(V) = \frac{1}{c}dE_\gamma(V).$$

Since any curve $\gamma \in \mathcal{C}(B)$ can be reparametrized to have constant speed, we see that the critical points of the length functional are "essentially" the same as the critical points for the energy functional.

We call a curve γ, a *pregeodesic* if there is a parametrization $\phi : [a, b] \to [a, b]$ such that $\gamma \circ \phi$ is a geodesic.

Corollary 1.6. *$\gamma \in \mathcal{C}(B)$ is a critical points for the length functional if and only if γ is a pregeodesic.*

1.3 The Second Variation

Suppose M is a smooth manifold and $f \in C^\infty(M)$ and p is a critical point of f, that is, $df_p = 0$. Define the bilinear form $\text{Hess}(f)_p : T_pM \times T_pM \to \mathbb{R}$ as follows. Given $X, Y \in T_pM$, and consider a map $(s_1, s_2) \mapsto \alpha(s_1, s_2)$ such that $\alpha(0,0) = p$, and

$$\frac{\partial \alpha}{\partial s_1}(0,0) = X, \qquad \frac{\partial \alpha}{\partial s_2}(0,0) = Y.$$

Then define

$$\text{Hess}(f)_p(X,Y) = \frac{\partial^2 (f \circ \alpha)}{\partial s_1 \partial s_2}(0,0).$$

Since p is critical for f, we will see that $\text{Hess}(f)_p$ is independent of map α and symmetric.

Alternatively, if (M,g) is a semi-Riemannian manifold with Levi-Civita connection ∇, then we can define symmetric bilinear from as

$$\text{Hess}(f)(X,Y) = \nabla^2 f(X,Y) = Y[X[f]] - (\nabla_Y X)[f].$$

In particular if p is a critical point for f, then

$$\text{Hess}(f)_p(X,Y) = Y_p[X[f]].$$

Lemma 1.7. *The two definitions for Hess$(f)_p$ for a critical point p are equivalent.*

Proof: Suppose $p \in M$ is a critical point for f. Let $X, Y \in T_pM$, and $(U, (x^j))$ be a coordinate neighborhood about p, and suppose $\alpha : I_\epsilon^2 \to U$ is a smooth map such that $\alpha(0,0) = p$, $\frac{\partial \alpha}{\partial s_1}(0,0) = X$, and $\frac{\partial \alpha}{\partial s_2}(0,0) = Y$. Extend Y to any smooth vector field $Y^j \frac{\partial}{\partial x^j}$ about p, and consider

$$\begin{aligned}
X_p[Y[f]] &= X_p\left[Y^j \frac{\partial f}{\partial x^j}\right] \\
&= X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p \left[Y^j \frac{\partial f}{\partial x^j}\right] \\
&= X^i(p) \frac{\partial Y^j}{\partial x^i}(p) \frac{\partial f}{\partial x^j}(p) + X^i(p) Y^j(p) \frac{\partial^2 f}{\partial x^i \partial x^j}(p) \\
&= X^i(p) Y^j(p) \frac{\partial^2 f}{\partial x^i \partial x^j}(p) && \text{since } df_p = 0.
\end{aligned}$$

Conversely, we now consider

$$
\begin{aligned}
\partial_1\partial_2(f\circ\alpha)(0,0) &= \partial_1|_{(0,0)}\left(\partial_2(f\circ\alpha)\right)\\
&= \partial_1|_{0,0)}\left(\frac{\partial f}{\partial x^j}(\alpha(s_1,s_2))\frac{\partial \alpha}{\partial s_2}(s_1,s_2)\right)\\
&= \frac{\partial^2 f}{\partial x^2\partial x^j}(p)\frac{\partial \alpha^i}{\partial s_1}(0,0)\frac{\partial \alpha}{\partial s_2}(0,0)+\frac{\partial f}{\partial x^j}(p)\frac{\partial^2 \alpha^j}{\partial s_1\partial s_2}(0,0)\\
&= \frac{\partial^2 f}{\partial x^i\partial x^j}(p)X^i(p)Y^j(p),
\end{aligned}
$$

again sine $df_p = 0$. Thus showing the two expressions are equal. □

Using the Levi-Civita definition for the Hessian, it's clear that $\mathrm{Hess}(f)_p)$ is a bilinear and symmetric, and independent of of map α. However, in practice, using such a map α is what allows us compute our second variation of energy.

Returning to our setup of (M,g) being a complete Riemannian manifold with boundary condition B, let $\gamma\in\mathcal{C}(B)$ and $V,W\in T_\gamma\mathcal{C}(B)$. Then we construct a two-parameter variation $\Gamma : I_\epsilon\times I_\epsilon\times[a,b]\to M$ by

$$\Gamma(s_1,s_2,t)=\exp_{\gamma(t)}(s_1V+s_2W),$$

and have $\partial_{s_1}\Gamma(0,0,t)=V(t)$, $\partial_{s_2}\Gamma(0,0,t)=W(t)$, and $\Gamma(0,0,t)=\gamma(t)$.

We now consider the Hessian of our energy functional. To this end, let $\gamma\in\mathcal{C}(B)$ be a B-geodesic, which is a critical point for E. Let $V,W\in T_\gamma\mathcal{C}(B)$ with associated two-parameter variation Γ. Then

$$\mathrm{Hess}(E)_\gamma(V,W)=\frac{\partial^2\hat{E}}{\partial s_1\partial s_2}(0,0).$$

hm:secondVarEnergy **Theorem 1.8** (Second Variation of Energy)**.** *Let $\gamma\in\mathcal{C}(B)$ be a B-geodesic and $V,W\in T_\gamma\mathcal{C}(B)$ with associated two-parameter variation $\Gamma : I_\epsilon\times I_\epsilon\times[a,b]\to M$. If $\{t_j : 0\le j\le k\}$ is an admissible partition for Γ, then*

$$
\begin{aligned}
Hess(E)_\gamma(V,W) = &\int_a^b (g(D_tV,D_tW)-g(R_{V\gamma'}\gamma',W))dt\\
&+(g\times g)(S_{\gamma'(a),-\gamma'(b)}(V(a),V(b)),(W(a),W(b))),
\end{aligned}
\tag{1.2}
$$

{eq:indexForm}

or alternatively,

$$
\begin{aligned}
Hess(E)_\gamma(V,W) = &-\int_a^b (g(D_t^2 V + R_{V\gamma'}\gamma', W)dt \\
&+ (g\times g)(S_{(\gamma'(a),-\gamma'(b))}(V(a),V(b)) + (-D_tV(a), D_tV(b)), (W(a),W(b))) \\
&\quad + \sum_{j=1}^{k-1} g(\Delta(D_tV)(t_j), W(t_j)),
\end{aligned}
\tag{1.3}
$$

{eq:jacobiForm}

where S denotes the shape operator of the submanifold $B \subset M \times M$ with respect to the normal vector $(\gamma'(a), -\gamma'(b))$, that is, given a normal vector $\xi \in T_xB^\perp$ and vectors $X, Y \in T_xB$, we have that

$$(g\times g)(S_\xi(X), Y) = -(g\times g)(\mathrm{I\!I}(X,Y),\xi).$$

Proof: For notational convenience, let $T = \partial_t\Gamma$, $Z_1 = \partial_{s_1}\Gamma$, and $Z_2 = \partial_{s_2}\Gamma$; and let $D_1 = D_{s_1}$ and $D_2 = D_{s_2}$ denote the covariant derivatives in the s_1 and s_2 directions. Let $\{t_0, ..., t_k\}$ denote an admissible partition for Γ. Let $\hat{E}_j$ denote the restriction to interval $[t_{j-1}, t_j]$ on which all maps and vector fields are smooth.

Then by our first variation formula Theorem 1.4, we have that

$$\frac{\partial \hat{E}_j}{\partial s_2}(s_1,s_2) = g(Z_2, T)\Big|_{t_{j-1}}^{t_j} - \int_{t_{j-1}}^{t_j} g(Z_2, D_tT)dt.$$

Then

$$\begin{aligned}
\frac{\partial^2 \hat{E}_j}{\partial s_1 \partial s_2}(s_1, s_2) &= \frac{\partial}{\partial s_1}\, g(Z_2, T)\big|_{t_{j-1}}^{t_j} - \int_{t_{j-1}}^{t_j} \frac{\partial}{\partial s_1} g(Z_2, D_t T)dt \\
&= g(D_1 Z_2, T)\big|_{t_j-1}^{t_j} + g(Z_2, D_1 T)\big|_{t_j-1}^{t_j} \\
&\quad - \int_{t_{j-1}}^{t_j} (g(D_1 Z_2, D_t T) + g(Z_2, D_1 D_t T))dt \\
&= g(D_1 Z_2, T)\big|_{t_j-1}^{t_j} + g(Z_2, D_t Z_1)\big|_{t_j-1}^{t_j} \\
&\quad - \int_{t_{j-1}}^{t_j} * + g(D_t D_t Z_1 + R_{Z_1 T} T, Z_2)dt \\
&= g(D_1 Z_2, T)\big|_{t_j-1}^{t_j} + g(Z_2, D_t Z_1)\big|_{t_j-1}^{t_j} - g(D_t Z_1, Z_2)\big|_{t_{j-1}}^{t_j} \\
&\quad + \int_{t_{j-1}}^{t_j} (-*) - g(D_t Z_1, D_t Z_2) + g(R_{Z_1 T} T, Z_2)dt \\
&= g(D_1 Z_2, T)\big|_{t_j-1}^{t_j} + 0 \\
&\quad + \int_{t_{j-1}}^{t_j} (-*) + g(D_t Z_1, D_t Z_2) - g(R_{Z_1 T} T, Z_2)dt.
\end{aligned}$$

Notice that since $D_1 Z_2 = D_{\partial_{s_1}} \partial_{s_2} \Gamma$ is smooth away from the plane $t = t_j$, and only depends on the values of Γ when $t = t_j$, it follows $D_1 Z_2$ is continuous, and that $\Delta D_1 Z_2(t_j) = 0$ for $1 \leq j \leq k-1$. Summing from $j = 1$ to $j - k$, we see that

$$\begin{aligned}
\frac{\partial^2 \hat{E}}{\partial s_1 \partial s_2}(s_1, s_2) &= g(D_1 Z_2, T)\big|_{(s_1, s_2, a)}^{(s_1, s_2, b)} \\
&\quad + \int_a^b (g(D_t Z_1, D_t Z_2) - g(R_{Z_1 T} T, Z_2))dt - \int_a^b g(D_1 Z_2, D_t T)dt.
\end{aligned}$$

Evaluating at $(s_1, s_2) = (0, 0)$, we get that

$$\begin{aligned}
\mathrm{Hess}(E)_\gamma(V, W) &= g(D_1 W(b), \gamma'(b)) - g(D_1 W(a), \gamma'(a)) \\
&\quad + \int_a^b (g(D_t V, D_t W) - g(R_{V\gamma'} \gamma', W))dt \\
&= +(g \times g)(S_{\gamma'(a), -\gamma'(b)}(V(a), V(b)), (W(a), W(b))) \\
&\quad + \int_a^b (g(D_t V, D_t W) - g(R_{V\gamma'} \gamma', W))dt.
\end{aligned}$$

Moreover, noting that

$$
\begin{aligned}
\int_a^b g(D_tV, D_tW)dt &= \int_a^b \left(\frac{\partial}{\partial t} g(D_tV, W) - g(D_t^2V, W)\right) dt \\
&= -\int_a^b g(D_t^2, W)dt + \sum_{j=1}^{k-1} g(\Delta D_tV(t_j), W(t_j)) \\
&\quad + (g \times g)((D_tV(b), D_tV(a)), (W(a), W(b))),
\end{aligned}
$$

we get that

$$
\begin{aligned}
\mathrm{Hess}(E)_\gamma(V, W) = &-\int_a^b (g(D_t^2V + R_{V\gamma'}\gamma', W)dt \\
&+ (g \times g)(S_{(\gamma'(a), -\gamma'(b))}(V(a), V(b)) + (-D_tV(a), D_tV(b)), (W(a), W(b))) \\
&+ \sum_{j=1}^{k-1} g(\Delta(D_tV)(t_j), W(t_j)),
\end{aligned}
$$

completing the proof. ☐

We remark that when $\mathrm{Hess}(E)_\gamma$, when expressed in the form of Equation (1.2) is clearly symmetric and bilinear. Moreover, if γ is minimizing for all nearby variations, then $\mathrm{Hess}(E)_\gamma$ is positive semi-definite.

Moreover, depending on certain structures the boundary condition $B \subset M \times M$ posses, we can simplify the expression.

Corollary 1.9. *Suppose $\gamma \in \mathcal{C}(N)$ is a B-geodesic and $V, W \in T_\gamma\mathcal{C}(B)$.*

i. If B is totally geodesic (i.e., $\mathrm{I\!I} \equiv 0$), for example if $B = \{(p, q)\}$, or if $B = \Delta(M)$, the diagonal of $M \times M$, then

$$
HessE_\gamma(V, W) = \int_a^b (g(D_tV, D_tW) - g(R_{V\gamma'}\gamma', W))dt.
$$

ii. If $A_1, A_2 \subseteq M$ are immersed submanifolds of M with respective shape operators S^1, S^2, and $B = A_1 \times A_2$, then

$$
\begin{aligned}
Hess(E)_\gamma(V, W) = &\int_a^b (g(D_tV, D_tW) - g(R_{V\gamma'}\gamma', W))dt \\
&+ g(S^1_{\gamma'(a)}(V(a)), W(a)) - g(S^2_{\gamma'(b)}(V(b)), W(b))
\end{aligned}
$$

2 Jacobi Fields

This sections follows largely from [11] and [18]. For Jacobi fields of a distance function and the Riccati equation see [12] and the Sphere Theorem with [17].

Let (M, g) be a connected Riemannian manifold. If γ is any geodesic on M, then a smooth vector field $J \in \mathfrak{X}(\gamma)$ is called a *Jacobi field* if J satisfies the *Jacobi equation*

$$D_t^2 J + R_{J\gamma'}\gamma' = 0.$$

Theorem 2.1. *Let $\gamma : [0, b] \to M$ be a geodesic segment in M. Then J is a Jacobi field along γ if and only if J is a variation field through geodesics.*

Proof: Suppose $J(t) = \partial_s\Gamma(0, t)$ is the variation field for the geodesic variation Γ. Then

$$D_t\partial_t\Gamma(s, t) = 0,$$

and hence by Lemma 1.1 and Lemma 1.2,

$$\begin{aligned} 0 &= D_s D_t \partial_t \Gamma \\ &= D_t D_s \partial_t \Gamma + R_{\partial_s\Gamma\partial_t\Gamma}\partial_t\Gamma \\ &= D_t D_t \partial_s \Gamma + R_{\partial_s\Gamma\partial_t\Gamma}\partial_t\Gamma. \end{aligned}$$

Evaluating at $s = 0$, we then see that

$$0 = D_t^2 J + R_{J\gamma'}\gamma',$$

showing that J is a Jacobi field.

Conversely, suppose $J(t)$ is a Jacobi field along γ. Let $x = \gamma(0)$ and $\xi = \gamma'(0)$, and choose any smooth curve $\sigma : I_\epsilon \to M$ and smooth vector field V along σ such that

$$\sigma(0) = x, \qquad \sigma'(0) = J(0);$$

$$V(0) = \xi, \qquad D_s V(0) = D_t J(0).$$

Since the domain of the exponential map is an open subset of TM that contains the compact set $\{(x, t\xi) : 0 \leq t \leq b\}$, there exists $0 < \delta < \epsilon$ for which the function

$$\Gamma(s, t) = \exp_{\sigma(s)}(tV(s)),$$

is well-defined for all $(s,t) \in (-\delta,\delta) \times [0,b]$. Moreover, by the uniqueness of geodesics, we have that

$$\Gamma(0,t) = \exp_x(t\xi) = \gamma(t),$$

showing that $\Gamma(s,t)$ is variation of γ. and similarly is a geodesic variation.

Let $W(t)$ denote this variation field. The first part of this proof showed that W is a Jacobi field along γ. First note that since $\Gamma(s,0) = \sigma(s)$,

$$W(0) = \partial_s \Gamma(0,t) = \sigma'(0) = J(0).$$

Moreover, we have that

$$\partial_t \Gamma(s,0) = d(\exp_{\sigma(s)})_0(V(s)) = V(s),$$

and hence

$$D_t W(0) = D_t \partial_s \Gamma(0,0) = D_s \partial_t \Gamma(0,0) = D_s V(0) = D_t J(0),$$

thus showing $W \equiv J$ by the uniqueness of Jacobi fields.

□

Let $\mathfrak{J}(\gamma)$ denote the space of Jacobi fields along γ. Since the Jacobi equation is equivalent to a linear system of $2n$ first order, ordinary differential equations, we see that from the existence and uniqueness theorem for Jacobi fields that $\dim \mathfrak{J}(\gamma) = 2n$ as a linear subspace of $\mathfrak{X}(\gamma)$, with a bijection $\mathfrak{J} \to T_{\gamma(t_0)}M \oplus T_{\gamma(t_0)}M$, $J \mapsto (J(t_0), D_t J(t_0))$.

Note that since tangential Jacobi fields are completely characterized by $J(t) = (a + tb)\gamma'(t)$, which correspond to the initial value problem

$$D_t^2 J + R_{J\gamma'}\gamma' = 0, \qquad J(0) = a\gamma'(0), D_t J(0) = b\gamma'(0),$$

we see that $J(t)$ is a *normal Jacobi field* (i.e., $J(t) \perp \gamma'(t)$ for all t) if both J and $D_t J$ are orthogonal to γ' at one point, or if J is orthogonal to γ' at two distinct points. Thus the space of normal Jacobi fields along γ is a $(2n-2)$-dimensional subspace of $\mathfrak{J}(\gamma)$.

We say that a point $y = \gamma_{x,\xi}(t)$ is a *conjugate point at x in the direction ξ* if

$$d(\exp_x)_{t\xi} : T_{t\xi}T_x M \to T_y M$$

is degenerate.

hm:conjugateJacobi **Lemma 2.2** (Proposition 10.10 in [11]). *If $\xi \in T_xM$ and $y = \gamma_{x,\xi}(1)$ is a conjugate point along $\gamma_{x,\xi}$ if and only if there exists a nontrivial Jacobi field $J(t)$ along $\gamma_{x,\xi}([0,1])$ with the Dirichlet boundary conditions $J(0) = J(1) = 0$.*

Proof: Suppose ξ is critical for $\exp_x$, then there exists $\eta \in T_xM$ such that $d(\exp_x)_\xi(\eta) = 0$. Define the variation

$$\Gamma_\eta(s,t) = \exp_x(t(\xi + s\eta)).$$

This is clearly a variation of geodesics, and let

$$J(t) = \partial_s \Gamma_\eta(0,t).$$

Then J is a Jacobi field with $J(0) = 0$. Moreover,

$$\begin{aligned} J(1) &= \frac{\partial \Gamma_\eta}{\partial s}(0,1) \\ &= \frac{\partial}{\partial s}\bigg|_{s=0} \exp_x(\xi + s\eta) \\ &= d(\exp_x)_\xi(\eta) \\ &= 0. \end{aligned}$$

Conversely, suppose $J(t)$ is a nontrivial Jacobi field along $\gamma_{x,\xi}(t)$ with $J(0) = J(1) = 0$. Let $\eta = D_tJ(0) \in T_xM$. Note that if $\eta = 0$, then $J(t) = \gamma'_{x,\xi}(t)$, and so $0 = J(0) = \xi$, and our Jacobi field would be trivial. Thus $\eta \neq 0$ and defining a variation $\Gamma_\eta(s,t)$ as above, we then see that $J(t) = \partial_s\Gamma_\eta(0,t)$, and hence

$$d(\exp_x)_\xi(\eta) = J(1) = 0,$$

thus showing that ξ is a critical point for $\exp_x$ and thus y is conjugate along $\gamma_{x,\xi}$. □

In the above, the dimension of the space of all Jacobi fields vanishing at x and y along $\gamma_{x,\xi}$ is called the *multiplicity* of the conjugate point y. That is,

$$\text{multiplicity} = \dim\left(\ker d(\exp_x)_\xi\right).$$

In particular, at conjugate points, the Jacobi fields are *not* unique.

From the existence and uniqueness theorem, there is an n-dimensional subspace of $\mathfrak{J}(\gamma)$ which vanish at 0. Since tangential Jacobi fields can vanish at most one points, we see that the multiplicity of a conjugate points can be at most $n-1$.

It's important to note that conjugate points are precisely the obstruction to the uniqueness of the Jacobi equation under the Dirichlet boundary conditions.

Corollary 2.3. *If $y = \gamma_{x,\xi}(t_0)$ is not conjugate, then for any $\eta \in T_xM$, $\zeta \in T_yM$, there exists a unique $J \in \mathfrak{J}(\gamma)$ such that*

$$J(0) = \eta, \qquad J(t_0) = \zeta.$$

Proof: Follows immediately, since $d(\exp_x)_{t_0\xi}$ is bijective, and

$$J(t) = d(\exp_x)_{t\xi}(tD_tJ(0)).$$

□

Theorem 2.4 (Regularity Theorem)**.** *Let (M, g) be a complete Riemannian manifold with boundary condition B. Suppose $\gamma \in \mathcal{C}(B)$ is a B-geodesic, and let $V \in T_\gamma\mathcal{C}(B)$. Then V is in the null space of Hess$(E)_\gamma$ if and only if V is a Jacobi field and satisfies the boundary condition*

$$(-D_tV(a), D_tV(b)) + S_{(\gamma'(a),-\gamma'(b))}(V(a), V(b)) \in T_{(\gamma(a),\gamma(b))}B^\perp. \tag{2.1}$$ {eq:jacobiBC}

Proof: Suppose V is Jacobi field satisfying the boundary condition, then using the alternative form Equation (1.3) for Hess$(E)_\gamma$, we see immediately that Hess$(E)_\gamma(V, W) = 0$ for all $W \in T_\gamma\mathcal{C}(p, q)$.

Conversely, suppose Hess$(E)_\gamma(V, W) = 0$ for all $W \in T_\gamma\mathcal{C}(p, q)$. Let $\{t_j : 0 \le j \le k\}$ be an admissible partition for V. Fix one such subinterval $[t_{j-1}, t_j]$ and let $\phi \in C^\infty(R)$ be a bump function with $\phi > 0$ on (t_{j-1}, t_j) and 0 elsewhere. Define the vector field

$$W = \phi(D_t^2V + R_{V,\gamma'}\gamma',$$

which is in $T_\gamma\mathcal{C}(B)$. Moreover,

$$\begin{aligned} 0 &= \text{Hess}_\gamma(V, W) \\ &= -\int_{t_{j-1}}^{t_j} \phi|D_t^2V + R_{V\gamma'}\gamma'|_g^2 dt, \end{aligned}$$

and since $\phi > 0$, we conclude that

$$D_t^2V + R_{V\gamma'}\gamma' = 0,$$

on (t_{j-1}, t_j). Since $j \in \{1, ..., k\}$ was arbitrary, we have that

$$D_t^2 V(t_j^-) = -R_{V(t_j)\gamma'(t_j)}\gamma'(t_j) = D_t^2 V(t_j^+),$$

for $j = 1, .., k-1$, and hence J is a broken Jacobi field on $[a, b]$.

Next choose $W \in T_\gamma \mathcal{C}(B)$ so that $W(a) = W(b) = 0$ and for each $j = 1, ..., k-1$, W satisfies

$$W(t_j) = \Delta D_t V(t_j).$$

Then

$$\begin{aligned} 0 &= \mathrm{Hess}(E)_\gamma(V, W) \\ &= \sum_{j=1}^{k-1} |\Delta D_t V(t_j)|_g^2, \end{aligned}$$

and hence $\Delta D_t V(t_j) = 0$ for each $1 \leq j \leq k-1$, thus showing that V is a Jacobi field.

Finally for any $(X, Y) \in T_{(\gamma(a),\gamma(b))}B$, let $W \in T_\gamma \mathcal{C}(B)$ which satisfies

$$W(a) = X, \qquad W(b) = Y,$$

then

$$\begin{aligned} 0 &= \mathrm{Hess}(E)_\gamma(V, W) \\ &= (g \times g)(S_{\gamma'(a),-\gamma'(b)}(V(a, V(b)) + (-D_t V(a), D_t V(B)), (X, Y)), \end{aligned}$$

thus showing the desired boundary condition and concluding the proof.

□

thm:gaussLemma **Lemma 2.5** (Global Gauss Lemma). *Let $x \in M$, $\xi, \eta \in T_p M$ and let $\gamma(t) = \exp_x(t\xi)$. Then for any t in the domain,*

$$g_{\gamma(t)}(d(\exp_x)_{t\xi}(\xi), d(\exp_x)_{t\xi}(\eta)) = g_x(\xi, \eta).$$

Proof: Note that the above equality is trivially true for $t = 0$, so we assume $t > 0$. Let J be the Jacobi field along γ with $J(0) = 0, D_t J(0) = \eta$. Then

$$\gamma'(t) = d(\exp_x)_{t\xi}(\xi)$$

and

$$\frac{1}{t} J(t) = d(\exp_x)_{t\xi}(\eta).$$

Decompose $\eta = \lambda\xi + \eta_1$, where $\eta_1 \perp \xi$. Now, let J_0, J_1 be the Jacobi fields along γ that vanish at x satisfying

$$D_t J_0(0) = \lambda\xi, \qquad D_t J_1(0) = \eta_1.$$

Note that $J_0(0) = \lambda t\gamma'(t)$ for all t, J_1 is a normal Jacobi field, and so

$$J(t) = \lambda t\gamma'(t) + J_1(t).$$

Hence

$$\begin{aligned}
g_{\gamma(t)}(d(\exp_x)_{t\xi}(\xi), d(\exp_x)_{t\xi}(\eta)) &= g_{\gamma(t)}\left(\gamma'(t), \frac{1}{t}J(t)\right) \\
&= \lambda g_{\gamma(t)}(\gamma'(t), \gamma'(t)) + \frac{1}{t}g_{\gamma(t)}(\gamma'(t), J_1(t)) \\
&= \lambda g_x(\xi, \xi) \\
&= g_x(\xi, \lambda\xi) \\
&= g_x(\xi, \eta).
\end{aligned}$$

□

Note that the usual Gauss lemma is just an application of the above with $t = 1$.

thm:paramProp

Proposition 2.6. *Let $x \in M$, $\xi \in T_xM$ and $\phi : [0,1] \to T_xM$ be any arbitrary piecewise smooth curve joining 0 to ξ. Let $\gamma(t) = \exp_x(t\xi)$ and $\tilde{\gamma}(t) = \exp_x(\phi(t))$. Then*

$$L_g(\tilde{\gamma}) \geq L_g(\gamma) = |\xi|_g.$$

Moreover, if $\exp_x$ is nonsingular on the line $[0,1]\xi$, then

$$L_g(\tilde{\gamma}) > L_g(\gamma),$$

unless γ is a reparametrization of $\tilde{\gamma}$.

Proof: Without loss of generality, assume $\phi(t) \neq 0$ for all $t \in (0,1]$. For $t \in (0,1]$, write $\phi(t) = r(t)u(t)$, where $r : (0,1] \to (0,\infty)$ and $u : (0,1] \to S_xM$ are smooth. Then

$$\phi'(t) = r'(t)u(t) + r(t)u'(t), \qquad u(t) \perp u'(t).$$

Furthermore, we see

$$\begin{aligned}
|\tilde{\gamma}'(t)|_g^2 &= |(\exp_x \circ \phi)'(t)|_g^2 \\
&= |d(\exp_x)_{\phi(t)}(\phi'(t))|_g^2 \\
&= |r'(t)|^2 |d(\exp_x)_{\phi(t)}(u(t))|_g^2 \\
&\qquad + 2r'(t)r(t)g(d(\exp_x)_{\phi(t)}(u(t)), d(\exp_x)_{\phi(t)}(u'(t))) \\
&\qquad\qquad + |d(\exp_x)_{\phi(t)}(r(t)u'(t))|_g^2 \\
&= |r'(t)|^2 |u(t)|_g^2 + 2r'(t)r(t)g_x(u(t), u'(t)) \qquad \text{by Lemma 2.5} \\
&\qquad\quad + |d(\exp_x)_{\phi(t)}(r(t)u'(t))|_g^2 \\
&= |r'(t)|^2 + 0 + |d(\exp_x)_{\phi(t)}(r(t)u'(t))|_g^2 \\
&\geq |r'(t)|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
L_g(\tilde{\gamma}) &= \int_0^1 |\tilde{\gamma}'(t)|_g dt \\
&\geq \int_0^1 |r'(t)| dt \\
&\geq \left| r(1) - \lim_{t \to 0^+} r(t) \right| \\
&= |r(1)| \\
&= |\phi(1)|_g \\
&= |\xi|_g \\
&= L_g(\gamma).
\end{aligned}$$

Moreover, if $\exp_x$ is nonsingular on $[0,1]\xi$, then there is some $\epsilon > 0$ neighborhood of $[0,1]\xi$ in T_xM for which $\exp_x$ is nonsingular. Suppose $L_g(\gamma) = L_g(\tilde{\gamma})$, then $|\tilde{\gamma}'(t)|_g = |r'(t)|$ for all t, and so

$$d(\exp_x)_{\phi(t)}(u'(t)) = 0,$$

for all t. Since $\exp_x$ is nonsingular near $[0,1]\xi$, we have that $u'(t) = 0$ whenever ϕ is ϵ-close to $[0,1]\xi$, and hence $u(t)$ is constant there. As $\phi(t)$ starts in such a neighborhood, and ends in such a neighborhood, it must stay in the neighborhood, thus showing $u(t) = u_0$ for all $t \in [0,1]$.

Thus $\phi(t) = r(t)u_0$, where $u_0 = \frac{\xi}{|\xi|_g}$. Moreover, since $r'(t)$ cannot change sign, we have that

$$\tilde{\gamma}(t) = \exp_x\left(\frac{r(t)}{|\xi|_g}\xi\right),$$

is just a reparametrization of γ. $\square$

3 The Classical Problem: $B = \{(p, q)\}$

Most results are blending of [9], [11], [18].

We consider the classical variational problem dealing with minimizing geodesics connecting two points. To this end, let (M, g) be a complete Riemannian manifold, and we require the boundary condition to be two distinct points. That is, for $p, q \in M, p \neq q$,

$$B = \{(p, q)\} \subset M \times M.$$

Then $\mathcal{C}(B)$ is the space of all piecewise regular curves $\gamma : [a, b] \to M$ such that $\gamma(a) = p$ and $\gamma(b) = q$, and we write $\mathcal{C}(p, q)$. All variations of γ are then proper, and all variation fields $V \in T_\gamma \mathcal{C}(p, q)$ are proper, i.e., $V(a) = V(b) = 0$. Moreover, in this setting our B-geodesics are simply just (Riemannian) geodesics. Our variation formulas then simplify as follows.

Theorem 3.1 (First Variation of Energy). *Let $\gamma \in \mathcal{C}(p, q)$ and $V \in T_\gamma \mathcal{C}(p, q)$ with associated variation $\Gamma : I_\epsilon \times [a, b] \to M$. If $\{t_j : 0 \leq j \leq k\}$ is an admissible partition for Γ, then*

$$\hat{E}'(s) = -\int_a^b g(\partial_s \Gamma, D_t \partial_t \Gamma) dt + \sum_{j=1}^{k-1} g(\partial_s \Gamma(s, t_j), \Delta \partial_t \Gamma(s, t_j)).$$

In particular, when $s = 0$,

$$dE_\gamma(V) = -\int_a^b g(V(t), D_t \gamma'(t)) dt + \sum_{j=1}^{k-1} g(V(t_j), \Delta \gamma'(t_j)).$$

Theorem 3.2 (Second Variation of Energy). *Let $\gamma \in \mathcal{C}(p, q)$ be a geodesic and $V, W \in T_\gamma \mathcal{C}(p, q)$ with associated two-parameter variation $\Gamma : I_\epsilon \times I_\epsilon \times [a, b] \to M$. If $\{t_j : 0 \leq j \leq k\}$ is an admissible partition for Γ, then*

$$Hess(E)_\gamma(V, W) = \int_a^b (g(D_t V, D_t W) - g(R_{V\gamma'}\gamma', W)) dt,$$

or alternatively,

$$Hess(E)_\gamma(V, W) = -\int_a^b (g(D_t^2 V + R_{V\gamma'}\gamma', W) dt + \sum_{j=1}^{k-1} g(\Delta(D_t V)(t_j), W(t_j)).$$

3.1 Minimizing Geodesics and Conjugate Points

Let (M,g) be a complete Riemannian manifold and suppose $\gamma \in \mathcal{C}(p,q)$ with $\gamma = \gamma|_{[a,b]}$ a geodesic segment. Then we define the *index form* of γ to be the Hessian of the energy, that is,

$$I(V,W) = \mathrm{Hess}(E)_\gamma(V,W),$$

for $V, W \in T_\gamma\mathcal{C}(p,q)$.

Corollary 3.3. *If γ is a minimizing geodesic, then I is positive semi-definite on $T_\gamma\mathcal{C}(p,q)$, that is, $I(V,V) \geq 0$ for all $V \in T_\gamma\mathcal{C}(p,q)$.*

We say a geodesic segment, $\gamma : [a,b] \to M$ has an *interior conjugate point* if there exists some $t \in (a,b)$ such that $\gamma(t)$ is conjugate to $\gamma(a)$ along γ.

:interiorConjugate **Theorem 3.4** (Theorem 10.15 in [11]). *Suppose $\gamma \in \mathcal{C}(p,q)$ is a geodesic segment. If γ has an interior conjugate point, then $I(V,V) < 0$ for some $V \in T_\gamma\mathcal{C}(p,q)$. In particular, if γ has an interior conjugate point, then γ is not minimal.*

Proof: Suppose $q_0 = \gamma(t_0)$ is an interior conjugate point for γ. Let J be a nontrivial Jacobi field along γ vanishing at $t = a, t = t_0$. Extend J to the vector field

$$\tilde{J}(t) = \begin{cases} J(t) & a \leq t \leq t_0 \\ 0 & t_0 \leq t \leq b. \end{cases}$$

Then $\tilde{J}$ is a piecewise smooth Jacobi field along γ. Let $Z \in T_\gamma\mathcal{C}(p,q)$ be such that $Z = -D_tJ(t_0) = \Delta D_t\tilde{J}(t_0)$, which nonzero since J is nontrivial (take a bump function and frame along γ to extend $-D_tJ(t_0)$). For $\epsilon > 0$ small, define $V_\epsilon = \tilde{J} + \epsilon Z$ which is in $T_\gamma\mathcal{C}(p,q)$. Then

$$I(V_\epsilon, V_\epsilon) = I(\tilde{J},\tilde{J}) + 2\epsilon I(\tilde{J}, Z) + \epsilon^2 I(Z,Z).$$

Since $\tilde{J}$ is a Jacobi field,

$$I(\tilde{J},\tilde{J}) = -g(\Delta D_t\tilde{J}(t_0), \tilde{J}(t_0)) = 0.$$

Similarly,

$$\begin{aligned} I(\tilde{J}, Z) &= -g(\Delta D_t\tilde{J}(t_0), Z(t_0)) \\ &= g-(\Delta D_t\tilde{J}(t_0), -D_tJ(t_0)) \\ &= -|D_tJ(t_0)|_g^2. \end{aligned}$$

Thus

$$I(V_\epsilon, V_\epsilon) = -2\epsilon |D_t J(t_0)|_g^2 + \epsilon^2 I(Z, Z),$$

taking ϵ sufficiently small yields

$$I(V_\epsilon, V_\epsilon) < 0.$$

$\square$

Theorem 3.5. *Suppose $\gamma \in \mathcal{C}(p,q)$ is a geodesic segment. If p has no conjugate point along γ, then there exists $\epsilon > 0$ such that for any piecewise smooth curve $\tilde{\gamma} \in \mathcal{C}(p,q)$ satisfying*

$$\max_{t \in [a,b]} dist_g(\gamma(t), \tilde{\gamma}(t)) < \epsilon,$$

we have that

$$L_g(\tilde{\gamma}) \geq L_g(\gamma),$$

with equality if and only if $\tilde{\gamma}$ is a reparametrization of γ.

Proof: Letting $\ell = b - a$, we have that $\exp_p$ is nonsingular at $t\gamma'(a)$ for all $t \in [0, \ell]$. Hence there exists a subdivision

$$a = t_1 < t_2 < \cdots < t_n < t_{n+1} = b,$$

and open neighborhoods V_j of the line segments $[t_j, t_{j+1}]\gamma'(a)$ in T_pM, $1 \leq j \leq n$ such that $\exp_p\big|_{V_j}$ is a diffeomorphism. Let $U_j = \exp_p(V_j)$. Then for $\epsilon > 0$ small enough, $\gamma([t_j, t_{j+1}]) \subset U_j$ for $1 \leq j \leq n$. Define

$$\phi(t) = \left(\exp_p\big|_{V_j}\right)^{-1}(\tilde{\gamma}(t)), \qquad t \in [t_j, t_{j+1}].$$

Then ϕ is a piecewise smooth curve connecting 0 to $\ell\gamma'(a)$, so that $\tilde{\gamma}(t) = \exp_p(\phi(t))$.

Thus by Proposition 2.6, $L_g(\tilde{\gamma}) \geq L_g(\gamma)$ with equality if and only if $\tilde{\gamma}$ is a reparametrization of γ. $\square$

thm:jacobi **Theorem 3.6** (Jacobi's Theorem)**.** *Suppose $\gamma \in \mathcal{C}(p,q)$ is a geodesic segment.*

a. p has no conjugate along γ if and only if I is positive definite on $T_\gamma\mathcal{C}(p,q)$.

b. q is the first conjugate point along γ if and only if I is positive semi-definite, but not positive definite on $T_\gamma\mathcal{C}(p,q)$.

c. *γ has an interior conjugate point if and only if there exists some $V \in T_\gamma \mathcal{C}(p,q)$ such that $I(V,V) < 0$.*

Proof:

a. Suppose p has no conjugate point along γ. Then I is positive semi-definite. Indeed, suppose there exists $V \in T_\gamma \mathcal{C}(p,q)$ such that $I(V,V) < 0$, then γ is not minimal. Let $\Gamma(s,t)$ be a variation such that $\partial_s \Gamma(0,t) = V(t)$. Then for sufficiently small s, the curves $t \mapsto \Gamma(s,t)$ have strictly bigger length than γ, a contradiction to the above theorem. Thus I is positive semi-definite. If $I(V,V) = 0$ for some $V \in T_\gamma \mathcal{C}(p,q)$, then V is a Jacobi field, and since p as no conjugate points, it must be trivial. Thus $V = 0$.

   Conversely, if p has a conjugate point at $\gamma(t_0)$ for some $t_0 \in (a,b]$. Let J be the nontrivial Jacobi field such that $J(a) = J(t_0) = 0$. Extend J trivially to $\tilde{J} \in T_\gamma \mathcal{C}(p,q)$. As this is still a nontrivial (broken) Jacobi field, we have that $I(\tilde{J},\tilde{J}) = 0$, and hence I is not positive definite.

b. Suppose q is the first conjugate point along γ. In particular, γ has no interior conjugate point. For any $V \in T_\gamma \mathcal{C}(p,q)$, we write $V(t) = V^j(t) E_j(t)$ where $\{E_j(t)\}$ is an orthonormal frame, parallel along γ. For any $a < s < b$, define

$$V_s(t) = V^j \left(a + \frac{b-a}{s-a}(t-a) \right) E_j(t), \qquad a \le t \le s.$$

   Then $V_s(a) = V_s(s) = 0$, and since $\gamma|_{[a,c]}$ is a geodesic segment with no conjugate points, we know the from part (a.) that

$$I_s(V_s, V_s) > 0,$$

   where I_s is the truncation of I to $[a,s]$. Hence by (any) convergence theorem, we have that

$$I(V,V) = \lim_{s \to b} I_s(V_s, V_s) \ge 0.$$

   So I is positive semi-definite. It's not positive definite, since $I(J,J) = 0$ for any Jacobi field $J \in T_\gamma \mathcal{C}(p,q)$.

   Conversely, if I was positive definite, then by (a.), q would not be a conjugate point along γ. If there exists $V \in T_\gamma \mathcal{C}(p,q)$ such that $I(V,V) < 0$, then by the following (c.), γ has an interior conjugate point, and so q is not the first conjugate point along γ.

c. The only if direction is exactly Theorem 3.4. Conversely, if $I(V,V) < 0$ for some $V \in T_\gamma \mathcal{C}(p,q)$, then I is not positive semi-definite. Hence p must have a conjugate point along γ, and it must come before q, that is, γ has an interior conjugate point.

□

Let's unravel Jacobi's theorem.

Corollary 3.7. *Let (M,g) be a Riemannian manifold, and $\gamma : [0,l] \to M$ a unit-speed geodesic.*

a. If γ has an interior conjugate point, then γ is not minimizing between $\gamma(0)$ and $\gamma(l)$.

b. IF $\gamma(0)$ has no conjugate point along γ, then γ is locally minimizing, that is for any proper variation $\Gamma(s,t)$, we have that $L_g(\gamma) < L_g(\Gamma(s,\cdot))$ for all sufficiently small s.

c. If $\gamma(l)$ is the first conjugate point to $\gamma(0)$ along γ, then for any proper variation $\Gamma(s,t)$, we have that $L_g(\gamma) < L_g(\Gamma(s,\cdot)$ for all sufficiently small s, unless $\partial_s\Gamma(0,t)$ is a Jacobi field.

3.2 Cut Points

Most definitions and results are a blending of [8] [9], [18]. See also [2], [19], [20].

Let (M,g) be a complete Riemannian manifold. Then we define various critical distances related to our exponential map. Let $(x,\xi) \in SM$ and let $\gamma_{x,\xi}$ denote the unit-speed geodesic emanating from x in the direction ξ, that is,

$$\gamma_{x,\xi} = \exp_x(t\xi),$$

where

$$\exp_x : T_xM \equiv S_xM \times [0,\infty) \to M,$$

is our *Riemannian exponential map.*

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and define the function $\tau : SM \to \overline{\mathbb{R}}$,

$$\tau(x,\xi) = \sup\{t > 0 : \mathrm{dist}_g(x, \gamma_{x,\xi}(t)) = t\}.$$

$\tau(x,\xi)$ is called the *cut locus distance function* along $\gamma_{x,\xi}$. This leads us to define the *cut locus of a point* x to be the set

$$\mathrm{Cut}(x) = \{\gamma_{x,\xi}(\tau(x,\xi)) : \xi \in S_xM, \tau(x,\xi) < \infty\}.$$

Such a $y \in \mathrm{Cut}(x)$ with $y = \gamma_{x,\xi}(\tau(x,\xi))$ is called a *cut point of* x *in the direction* ξ.

The *injectivity radius at* x *of* M is the supremum in $\overline{\mathbb{R}}$ given by

$$\mathrm{inj}_x(M) = \sup\{c > 0 : \exp_x|_{B(0,c)} \text{ is injective}\}.$$

Alternatively, using the cut locus distance function, we could equivalently define

$$\mathrm{inj}_x(M) = \min\{\tau(x,\xi) : \xi \in S_xM\}.$$

The *injectivity radius of* M is then given by

$$\begin{aligned}\mathrm{inj}(M) &= \inf_{x\in M}(\mathrm{inj}_x(M)) \\ &= \inf_{(x,\xi)\in SM} \tau(x,\xi).\end{aligned}$$

Recall that we say a point $y = \gamma_{x,\xi}(t)$ is a *conjugate point at* x *in the direction* ξ if

$$d(\exp_x)_{t\xi} : T_xM \to T_yM$$

is degenerate. This leads us to define the *conjugate distance function along* $\gamma_{x,\xi}$, $\tau_c : SM \to \overline{\mathbb{R}}$,

$$\tau_c(x,\xi) = \inf\{t > 0 : d(\exp_x)_{t\xi} \text{ is degenerate}\}.$$

We say that a sequence $\{\gamma_j\}$ of geodesics *converge* to a geodesic $\gamma_{x,\xi}$ if $x_j = \gamma_j(0) \to x$ and $\xi_j = \gamma_j'(0) \to \xi \in T_xM$.

Lemma 3.8. *Suppose* $\gamma_j \to \gamma_{x,\xi}$. *Then* $\gamma_j(t_j) \to \gamma_{x,\xi}(t)$ *whenever* $t_j \to t$. *Furthermore, if* γ_j *are minimizing geodesics joining* $x_j = \gamma_j(0)$ *to* $y_j = \gamma_j(t_j)$ *and* $\gamma_j \to \gamma_{x,\xi}$ *and* $t_j \to \ell$, *then* $\gamma_{x,\xi}$ *is a minimal geodesic joining* x *to* $y = \gamma_{x,\xi}(\ell)$.

Proof: Note that the first assertion is true because of continuous dependence of on initial conditions (x,ξ) and parameter t. For the second assertion, note that

$$\begin{aligned}\operatorname{dist}_g(x,y) &= \lim_{j\to\infty} \operatorname{dist}_g(x_j,y_j)\\ &= \lim_{j\to\infty} t_j\\ &= \ell\\ &= L_g(\gamma_{x,\xi}([0,\ell])).\end{aligned}$$

□

Lemma 3.9. *Let (M,g) be a complete Riemannian manifold.*

i. The cut locus distance function $\tau : SM \to \overline{\mathbb{R}}$ is continuous.

ii. The conjugate distance function $\tau_c : SM \to \overline{\mathbb{R}}$ is continuous.

iii. The mapping $x \to inj_x(M)$ is continuous.

Proof:

i. Suppose $(x_j,\xi_j),(x,\xi) \in SM$ is such that $(x_j,\xi_j) \to (x,\xi)$. We first wish to show that $\tau(x_j,\xi_j) \to \tau(x,\xi)$. Let T be any limit point of $\{\tau(x_j,\xi_j)\}$ in $\overline{\mathbb{R}}$. Letting $\gamma_j = \gamma_{x_j,\xi_j}$, we see that $\gamma_j \to \gamma_{x,\xi}$, and since γ_j are minimal on $[0,\tau(x_j,\xi_j)]$, we conclude by the above lemma that $\gamma_{x,\xi}|_{[0,T]}$ is minimal as well. By definition of $\tau(x,\xi)$, this implies

$$T \le \tau(x,\xi).$$

If $T = +\infty$, then we're done. Assume $T < +\infty$, and by possibly passing to a subsequence, we shall write $T = \lim_{j\to\infty} \tau(x_j,\xi_j)$.

Assume $T < \tau(x,\xi)$, and let $\epsilon > 0$ be such that $T+\epsilon < \tau(x,\xi)$. Thus for infinitely many j, we have that $\gamma_j|_{[0,T+\epsilon]}$ will not be minimizing between x_j and $y_j = \gamma_j(T+\epsilon)$. Hence for each such j, there exists $\eta_j \in S_{x_j}M$, $\eta_j \neq \xi_j$ such that $\gamma_{x_j,\eta_j}\big|_{[0,T+\epsilon]}$ is minimizing between x_j and y_j. Since $x_j \to x$ is convergent and $|\eta_j| = 1$, by possibly passing to a subsequence, we may assume $\lim_{j\to\infty} \eta_j = \eta \in S_xM$ and $y = \lim_{j\to\infty} y_j$.

We claim that $\xi \neq \eta$. To this end, suppose $\xi = \eta$. Then $\gamma_{x,\xi}(T)$ is a conjugate point to x along $\gamma_{x,\xi}$. Indeed, assume $\exp_x$ is regular at $T\xi \in T_xM$, and define the map $E : TM \to M \times M$, by

$$E = \pi \times \exp, \qquad E(p,v) = (p, \exp_p(v)).$$

Then $dE_{(x,T\xi)}$ has maximal rank and hence hence a diffeomorphism on some neighborhood U of $(x, T\xi)$ in TM. Thus for sufficiently large j, we have that $(x_j, \tau(x_j, \xi_j)\xi_j), (x_j, \tau(x_j, \xi_j)\eta_j) \in U$, but

$$E(x_j, \tau(x_j, \xi_j)\xi_j) = (x_j, y_j) = E(x_j, \tau(x_j, \xi_j)\eta_j),$$

and hence by injectivity, we have that $\eta_j = \xi_j$, a contradiction our choice of each η_j. Thus $\gamma_{x,\xi}(T)$ is conjugate to x. Since $T < \tau(x, \xi)$, this is in fact a contradiction, as no geodesic can be minimizing past its first conjugate point by Jacobi's theorem (Theorem 3.6). Thus $\xi \neq \eta$, and $\gamma_{x,\xi}$ and $\gamma_{x,\eta}$ are two distinct minimizing geodesics between x and y, a contradiction to $T < \tau(x, \xi)$ and the exponential map being injective here. Thus $T = \tau(x, \xi)$ as desired.

ii. In similarity to above, suppose $\tau_c(x_j, \xi_j) \to T$ in $\overline{\mathbb{R}}$. Then there exists $\eta_j \in \ker\left(d(\exp_{x_j})_{\tau_c(x_j,\xi_j)\xi_j}\right)$, $|\eta_j| = 1$. Noting that since $x_j \to x$ converges, $\{\eta_j\}$ is contained in a compact subset of SM and hence we may assume converges to some $\eta \in S_xM$. Then by continuity of the exponential, we see that

$$\eta \in \ker\left(d(\exp_x)_{T\xi}\right),$$

and hence $\gamma_{x,\xi}(T)$ is conjugate to x, showing that $T = \tau_c(x, \xi)$.

iii. Suppose $x_j \to x$ and choose $\xi_j \in S_{x_j}M$ such that $\mathrm{inj}_{x_j}(M) = \tau(x_j, \xi_j)$. Assume $\lim_{j\to\infty} \xi_j = \xi \in S_xM$. By continuity of τ, we then have that

$$\mathrm{inj}_x(M) \leq \tau(x, \xi).$$

Assume $\mathrm{inj}_x(M) < \tau(x, \xi)$ and let $\eta \in S_xM$ be such that $\mathrm{inj}_x(M) = \tau(x, \eta)$. Let $\eta_j \in S_{x_j}M$ be any sequence such that $\eta_j \to \eta$. But since

$$\lim_{j\to\infty} \tau(x_j, \eta_j) = \tau(x, \eta) < \tau(x, \xi) = \lim_{j\to\infty} \tau(x_j, \xi_j),$$

it follows that $\tau(x_j, \eta_j) < \tau(x_j, \xi_j)$ for all sufficiently large j, a contradiction to $inj_{x_j}(M) = \tau(x_j, \xi_j)$.

□

Proposition 3.10. *Suppose (M, g) is a complete Riemannian manifold. Then the following are equivalent:*

i. M is compact.

ii. There exists $x \in M$ such that $\tau(x, \xi) < \infty$ for all $\xi \in S_x M$.

iii. $\tau(x, \xi) < \infty$ for all $(x, \xi) \in SM$.

Proof:

(i.)$\Rightarrow$(iii.) Suppose M is compact, then since τ is continuous on the compact space SM, it's bounded above.

(iii.)$\Rightarrow$(ii.) Trivial.

(ii.)$\Rightarrow$(i.) Suppose $\tau(x, \xi) < \infty$ for all $\xi \in S_x M$, then the map $\xi \mapsto \tau(x, \xi)$ is continuous on $S_x M$, and hence there exists $C > 0$ such that $\tau(x, \xi) < C$ for all $\xi \in S_x M$, since M is complete we have that $\exp_x(\overline{B(0, C)}) = M$, and thus as the continuous image of a compact set, M must be compact.

□

Consider the surface of revolution of $z = \frac{1}{x}$, $x > 0$ for a complete (non-compact) Riemannian manifold with $\text{inj}(M) = 0$.

Proposition 3.11. *Let (M, g) be a complete Riemannian manifold, and suppose $x, y \in M$ with $y \notin Cut(x)$. Let $\gamma : [0, b] \to M$ be the unique unit-speed, minimizing geodesic from x to y, and let $p \in M$. Then $p \in \gamma([0, b])$ if and only if*

$$dist(x, p) + dist(p, y) + dist(x, y).$$

Proof: Suppose $\gamma(s) = p$ for some $s \in [0, b]$. Then

$$\begin{aligned} \text{dist}(x, y) &= L(\gamma) \\ &= L\left((\gamma|_{[0,s]}\right) + L\left(\gamma|_{[s,b]}\right) \\ &= \text{dist}(x, p) + \text{dist}(p, y). \end{aligned}$$

Conversely, suppose

$$\text{dist}(x, p) + \text{dist}(p, y) = \text{dist}(x, y).$$

Let α be a minimizing geodesic from x to p and let β be a minimizing geodesic from p to y. Let $\tilde{\gamma}$ denote the concatenation of α and β, that is,

$\tilde{\gamma} = \alpha \cdot \beta$. Then

$$\begin{aligned} L(\tilde{\gamma}) &= L(\alpha) + L(\beta) \\ &= \mathrm{dist}(x, p) + \mathrm{dist}(p, y) \\ &= \mathrm{dist}(x, y). \end{aligned}$$

That is, γ is a minimizing broken geodesic from x to y, and hence a minimizing geodesic from x to y. That is $\tilde{\gamma} = \gamma$ up to reparametrization and hence $p \in \gamma([0, b])$. □

Lemma 3.12. *Let $(x, \xi) \in SM$, then $\gamma_{x,\xi}|_{[0,\tau(x,\xi))}$ contains no conjugate points. That is, the cut point on a geodesic appears before or coincides with the first conjugate point, or rather*

$$\tau(x, \xi) \leq \tau_c(x, \xi),$$

for all $(x, \xi) \in SM$.

Proof: Assume $\tau_c(x, \xi) < \tau(x, \xi)$. Then $\gamma_{x,\xi}$ is minimizing past its first conjugate point $\gamma_{x,\xi}(\tau_c(x, \xi))$ a contradiction to Jacobi's Theorem (Theorem 3.6). □

We say $y \in \mathrm{Cut}(x)$ is an *ordinary cut point* if there exists $\xi, \eta \in S_xM$, $\xi \neq \eta$ such that

$$\tau(x, \xi) = \tau(x, \eta),$$

and

$$\gamma_{x,\xi}(\tau(x, \xi)) = y = \gamma_{x,\eta}(\tau(x, \xi)).$$

We say $y \in \mathrm{Cut}(x)$ is a *singular cut point* if there exists exactly one minimal geodesic going x to y.

thm:klingLemma

Theorem 3.13 (Klingenberg Lemma)**.** *(Lemma 2.1.11 in [9], Proposition 4.1 in [18], Lemma 9.2.16 in [16])*

Suppose $\tau(x, \xi) < \infty$, $(x, \xi) \in SM$, and let $y = \gamma_{x,\xi}(T)$. Then $T = \tau(x, \xi)$ if and only if $\gamma = \gamma_{x,\xi}|_{[0,T]}$ is a minimizing geodesic segment, and either

i. y is an ordinary cut point of x, or

ii. y is the first conjugate point along γ from x to y.

Proof: Suppose γ is minimal. If y is the first conjugate point along γ, then by Jacobi's theorem, γ cannot be minimal past y. If y is an ordinary cut point of x, then for any $\epsilon > 0$, the geodesic segment $\gamma|_{[0,T+\epsilon]}$ is not minimal. Indeed, assume to the contrary, that $\gamma|_{[0,T+\epsilon]}$ is minimal, and let $\eta \in S_M$, $\eta \neq \xi$ be such that $\gamma_\eta(T) = \gamma(T)$. Then for $\epsilon > 0$ sufficiently small, let α denote the unique, unit-speed, minimal geodesic connecting $\gamma_\eta(T-\epsilon)$ to $\gamma(T+\epsilon)$. Then

$$\begin{aligned}2\epsilon &= \mathrm{dist}(\gamma_\eta(T-\epsilon), \gamma_\eta(T)) + \mathrm{dist}(\gamma(T), \gamma(T+_\epsilon)) \\ &> \mathrm{dist}(\gamma_\eta(T-\epsilon), \gamma(T+\epsilon)),\end{aligned}$$

where the strict inequality, follows because $\gamma_\eta'(T) \neq \gamma'(T)$. Thus

$$\begin{aligned}L\left(\gamma_\eta|_{[0,T-\epsilon]} \cup \beta\right) &= (T-\epsilon) + \mathrm{dist}(\gamma_\eta(T-\epsilon), \gamma(T+\epsilon)) \\ &< (T-\epsilon) + 2\epsilon \\ &= T+\epsilon \\ &= L\left(\gamma|_{[0,T+\epsilon]}\right),\end{aligned}$$

a contradiction. Hence $T = \tau(x,\xi)$.

Conversely, suppose $T = \tau(x,\xi)$. Then γ is minimal since,

$$\mathrm{dist}(x, \gamma(T)) = \lim_{t\to T^-} \mathrm{dist}(x, \gamma(t)) = \lim_{t \to T^-} t = T.$$

If y if the first conjugate point, then we're done, so assume x has no conjugate point along γ from x to y. In particular, $\exp_x$ is nonsingular at $T\xi$.

Let $y_j = \gamma(T + 1/j)$, and let γ_j denote the minimal geodesic connecting x to y_j. That is,

$$y_j(t) = \exp_x(t\eta_j),$$

for some $\eta_j \in S_x M$. Let

$$s_j = \mathrm{dist}(x, y_j).$$

So $\gamma_j(s_j) = y_j = \gamma(T + 1/j)$. Since $\gamma|_{[0,T+1/j]}$ is not minimal, we have that

$$s_j < T + \frac{1}{j}.$$

Let $\eta \in S_x M$ be a limit point of $\{\eta_j\}$. Note that

$$\begin{aligned}\lim_{j\to\infty} s_j &= \lim_{j\to\infty} \mathrm{dist}(x, y_j) \\ &= \mathrm{dist}(x, y) \\ &= T,\end{aligned}$$

and so

$$\begin{aligned}
\exp_x(T\eta) &= \lim_{j\to\infty} \exp_x(s_j\eta_j) \\
&= \lim_{j\to\infty} \gamma_j(s_j) \\
&= \lim_{j\to\infty} \gamma(T+1/j) \\
&= \gamma(T) \\
&= y.
\end{aligned}$$

That is,

$$\exp_x(T\xi) = \exp_x(T\eta).$$

If $\eta \neq \xi$, then we're done. Suppose $\eta = \xi$. We've already seen that

$$\exp_x(s_j\eta_j) = \gamma(T+1/j) = \exp_x((T+1/j)\xi),$$

and $s_j\eta_j \to T_\eta = T\xi$ and $(T+1/j)\xi \to T\xi$. Moreover, since $s_j < T+1/j$, we have that $s_j\eta_j \neq (T+1/j)\xi$. Thus $\exp_x$ is not locally injective near $T\xi$, and hence that $T\xi$ is a singular point for $\exp_x$, a contradiction. Hence $\eta \neq \xi$, and the result follows. □

As both conditions (i.) (trivially) and (ii.) (by Jacobi's Theorem) are symmetric in x and y, we have the following corollary:

Corollary 3.14. *$y \in Cut(x)$ if and only if $x \in Cut(y)$.*

Moreover, we have that the Klingenberg Lemma implies that all singular cut points are first conjugate points.

We can strengthen the Klingenberg Lemma to when the cut point $y \in \mathrm{Cut}(x)$ is the closest point in the cut locus to x.

thm:klingTheorem

Theorem 3.15 (Klingenberg Theorem)**.** *If $y \in Cut(x)$ is such that $dist(x,y) = dist(x, Cut(x))$, that is, $dist(x,y) = inj_x(M)$, and none of the minimizing geodesics from x to y posses a conjugate point, then there exists exactly two minimizing geodesics $\gamma_1, \gamma_2 : [0,1] \to M$ from x to y, and they meet at y with opposite directions, $\gamma_1'(1) = -\gamma_2'(1)$. If $dist(x,y) = inj(M)$, then γ_1 and γ_2 form a closed geodesic loop, i.e., $\gamma_1'(0) = -\gamma_2'(0)$.*

Proof: Let $\gamma_1, \gamma_2[0,1] \to M$ be two distinct minimizing geodesics from x to y with initial velocity $\xi_1, \xi_2 \in T_xM$, $\xi_1 \neq \xi_2$. This is guaranteed, since y is not a conjugate point along any any minimal geodesic connecting x to y.

Suppose $\gamma_1'(1) \neq -\gamma_2'(1)$. Let $\eta \in T_yM$ be such that

$$g_y(\eta, \gamma_1'(1)) < 0 \qquad \text{and} \qquad g_y(\eta, \gamma_2'(1)) < 0,$$

i.e., η forms an angle greater than $\frac{\pi}{2}$ with both $\gamma_1'(1)$ and $\gamma_2'(1)$.

Let $V_1, V_2 \subset T_xM$ be neighborhoods of ξ_1, ξ_2 respectively, such that $\exp_x|_{V_j}$ is a diffeomorphism. Let $U_j = \exp_x(V_j$. Since $\exp_x(\xi_1) = y = \exp_x(\xi_2)$, we have that $U_1 \cap U_2 \neq \emptyset$ and is hence open. By possibly shrinking V_j, we may assume $V_1 \cap V_2 = \emptyset$.

For $\epsilon > 0$ sufficiently small and $s \in I_\epsilon$, the curve $\sigma(s) = \exp_y(s\eta) \in U_1 \cap U_2$. Define the curves

$$\phi_j(s) = \left(\exp_x|_{V_j}\right)^{-1}(\sigma(s)), \qquad s \in I_\epsilon$$

in T_xM. Note that $\phi_j(0) = \xi_j$. Now define the variations $\Gamma_j : I_\epsilon \times [0,1]$ of γ_j by

$$\Gamma_j(s,t) = \exp_x(t\phi_j(s)).$$

Moreover, we have variation field

$$V_j(t) = d(\exp_x)_{t\xi_j}(t\phi_j'(0)),$$

and so $V_j(0) = 0$ and $V_j(1) = \eta$ (and hence this is not a proper variation). Then by our first variation formula, we have that

$$\frac{dE}{ds}(\gamma_j) = g(V_j(1), \gamma_j'(1)) = g(\eta, \gamma_j'(1)) < 0.$$

Hence

$$\frac{dL}{ds}(\gamma_j) < 0.$$

That is, $s \mapsto L(\Gamma_j(s,\cdot))$ is a decreasing function near $s = 0$. Hence for $\delta > 0$ sufficiently small, we have that

$$L(\Gamma_j(\delta,\cdot)) < L(\gamma_j).$$

Moreover, we know that since $\exp_x$ is nonsingular for each V_j, that

$$L(\Gamma_1(\delta,\cdot)) = \mathrm{dist}(x, \sigma(\delta)) = L(\Gamma_2(\delta,\cdot)).$$

Thus $\mathrm{dist}(x,\sigma(\delta)) < \mathrm{dist}(x,y)$, and $\sigma(\delta) \in \mathrm{Cut}(x)$, and contradiction to our choice of y. Hence $\gamma_1'(1) = -\gamma_2'(1)$.

Furthermore, if $\mathrm{dist}(x,y) = \mathrm{inj}(M)$, then applying the above result to $-\gamma_1$ and $-\gamma_2$, the desired result follows. $\square$

As seen in [19], for any compact manifold M which is not homeomorphic to S^2, there exists a Riemannian metric g on M such that there exists $p \in M$ for which $\mathrm{Cut}(p)$ contains no conjugate point.

We now fix $p \in M$, and let $\tau(\xi) = \tau(p,\xi)$ for $\xi \in S_pM$, that is, we're considering the restriction of our cut distance function to single tangent space. We let

$$\widetilde{\mathrm{Cut}}(p) := \{\tau(\xi)\xi : \xi \in S_pM, \tau(\xi) < \infty\},$$

and hence

$$\mathrm{Cut}(p) = \exp_p(\widetilde{\mathrm{Cut}}(p)).$$

We then define the interior sets

$$\tilde{\mathcal{I}}_p := \{t\xi : 0 \leq t < \tau(\xi), \xi \in S_pM\}$$

and

$$\mathcal{I}_p := \exp_p(\tilde{\mathcal{I}}_p).$$

Lemma 3.16 (Cf. Theorem 2.1.14 in [9] and Lemma 4.4 in [18])**.** *Our cut locus and interiors sets satisfy the following:*

i. $\mathcal{I}_p \cap \mathit{Cut}(p) = \emptyset$*,* $M = \mathcal{I}_p \cup \mathit{Cut}(p)$*, and* $\overline{\mathcal{I}_p} = M$*,*

ii. $\tilde{\mathcal{I}}_p$ *is the maximal domain containing* $0_p \in T_pM$ *on which* $\exp_p$ *is a diffeomorphism.*

iii. $\mathit{Cut}(p)$ *is a null set in* M *and* $\dim \mathit{Cut}(p) \leq n-1$*.*

Proof:

1. Let $q \in M$, then since M is complete, there exists a minimizing geodesic from p to q. Let $\gamma_{p,\xi}$ denote the geodesic, and let $\gamma_{p,\xi}(s) = \mathrm{dist}(p,q)$. Then
$$s = \mathrm{dist}(p,q),$$
and hence $s \leq \tau(\xi)$ which shows both that $q \in \mathcal{I}_p \cup \mathrm{Cut}(p)$ and $q \in \overline{\mathcal{I}_p}$.

   Now, suppose $\exp_p(u) = \exp_p(v)$ for some $u \in \tilde{\mathcal{I}}_p$ and some $v \in \widetilde{\mathrm{Cut}}(p)$. That is,
$$\gamma_{p,\frac{u}{|u|_g}}(|u|_g) = \gamma_{p,\frac{v}{|v|_g}}(|v|_g),$$

and hence, in particular

$$|u|_g = \tau(u/|u|_g),$$

which contradicts $u \in \tilde{\mathcal{I}}_p$. Hence $\mathcal{I}_p \cap \mathrm{Cut}(p) = \emptyset$.

2. We note that $\tilde{\mathcal{I}}_p$ is a connected (since it's star-shaped about $0_p \in T_pM$) open subset of T_pM. Since it contains no conjugate tangent vectors by Klingenberg's lemma, and it's injective by the an identical argument to previous part ((.i)), we conclude $\exp_p\big|_{\tilde{\mathcal{I}}_p}$ is a diffeomorphism. Moreover, suppose the set is not maximal. Then there exists $u \in \widetilde{\mathrm{Cut}}(p)$ for which $\exp_p$ is regular on some open neighborhood U of $u \in T_pM$. Hence $\exp_p$ is a diffeomorphism on $\tilde{\mathcal{I}}_p \cup U$, and since $\exp_p(U \setminus \tilde{\mathcal{I}}_p) \subseteq \mathrm{Cut}(p)$, we obtain a contradiction since by the following assertion of $\dim \mathrm{Cut}(p) \leq n-1$ holds, but $\exp_p$ has full rank U.

3. Since $\tau(\xi)$ is continuous, it follows that $\widetilde{\mathrm{Cut}}(p)$ is null in T_pM and $\dim \widetilde{\mathrm{Cut}}(p) = n-1$ if it's nonempty. Since $\exp_p$ is smooth, it then follows that $\dim \mathrm{Cut}(p) \leq n-1$ as desired.

□

3.3 The Gradient of the Distance Function

[5]

Suppose (M, g) is a complete Riemannian manifold. Fix $p \in M$, and let $r : M \to \mathbb{R}$,

$$r(q) = \mathrm{dist}_g(p, q).$$

Lemma 3.17 (Lemma 4.4 in [18])**.** *$(\exp_p)^{-1} : M \setminus Cut(p) \to T_pM$ is a diffeomorphism onto its image.*

Proof: This follows immediately from our characterization of $\tilde{\mathcal{I}}_p$. □

On $M \setminus (\mathrm{Cut}(p) \cup \{p\})$, we have that

$$r(q) = |(\exp_p)^{-1}(q)|_g.$$

[5] See [12] for an exposition on the Taylor expansion.

Theorem 3.18 (Proposition 4.8 in [18]).

$$grad\,(r)_q = \gamma'_{pq}(r(q)),$$

where γ_{pq} denotes the unique minimal unit-speed geodesic from p to q. In particular,

$$|grad\,(r)_q|_g = 1.$$

Proof: Let $X \in T_qM$ and $\alpha : (-\epsilon, \epsilon) \to M$ be the integral curve with $\alpha(0) = q$, $\alpha'(0) = X$. By possibly shrinking $\epsilon > 0$, we may assume $\alpha(s) \notin (\mathrm{Cut}(p) \cup \{p\}$ for all $-\epsilon < s < \epsilon$. Let $\Gamma : (-\epsilon, \epsilon) \times [0, r(q)] \to M$ be the smooth variation of geodesics

This should be able to be reworded cleaner using $\mathcal{C}(B)$ I think.

$$\Gamma(s,t) = \gamma_{p\alpha(s)}(t).$$

Then by the first variation formula, we obtain

$$\begin{aligned}
dr_q(X) &= \left.\frac{\partial}{\partial s}\right|_{s=0} r \circ \alpha(s) \\
&= \left.\frac{\partial}{\partial s}\right|_{s=0} d(p, \alpha(s)) \\
&= \left.\frac{\partial}{\partial s}\right|_{s=0} L(\Gamma(s,\cdot)) \\
&= \left.\frac{\partial}{\partial s}\right|_{s=0} \int_0^{r(q)} g(\partial_t\Gamma(s,t), \partial_t\Gamma(s,t))^{1/2} dt \\
&= \int_0^{r(q)} \frac{1}{2}\frac{1}{|\partial_t\Gamma(0,t)|_g} 2g(\partial_s|_{s=0}\, \partial_t\Gamma(s,t), \partial_t\Gamma(0,t))dt \\
&= \int_0^{r(q)} \frac{g(\partial_s|_{s=0}\, \partial_t\Gamma(s,t), \gamma'_{pq}(t))}{|\gamma'_{pq}(t)|_g} dt \\
&= -\int_0^{r(q)} g(\partial_s|_{s=0}\, \Gamma(s,t), \nabla_{\gamma'_{pq}(t)}\gamma'_{pq}(t))dt + g(\partial_s|_{s=0}\, \Gamma(s,t), \gamma'_{pq}(t)\big|_0^{r(q)}) \\
&= g(\partial_s|_{s=0}\, \Gamma(s, r(q)), \gamma'_{pq}(r(q))) - g(\partial_s|_{s=0}\, p, \gamma'_{pq}(0)) \\
&= g(\partial_s|_{s=0}\, \alpha(s), \gamma'_{pq}(r(q))) \\
&= g(\gamma'_{pq}(r(q)), X),
\end{aligned}$$

and hence

$$\mathrm{grad}\,(r)_q = \gamma'_{pq}(r(q)),$$

as desired. $\square$

Corollary 3.19. *If $r(p) = d(p, q)$, then*

$$grad\,(r)_p = \gamma'_{qp}(r(p)).$$

Lemma 3.20. *r is smooth near q if and only if $q \notin Cut(p) \cup \{p\}$.*

Proof: By the previous Lemma, we know if $q \notin \mathrm{Cut}(p) \cup \{p\}$ then r is smooth. Moreover, if $q = p$, then our norm is not smooth at 0 as usual. Finally, suppose $q \in \mathrm{Cut}(p)$. Then by the Klingenberg lemma, q is either an ordinary cut point or a first conjugate point. Suppose q is an ordinary cut point, that is, there exists distinct $\xi, \eta \in S_qM$ such that

$$\gamma_{q,\xi}(s) = q = \gamma_{q,\eta}(s)$$

for some $s > 0$. We note for all $t < s$, that

$$\mathrm{grad}\,(\mathrm{dist}(q, \cdot))_{\gamma_{q,\xi}(t)} = \gamma'_{q,\xi}(t),$$

and similarly at $\gamma_{q,\eta}(t)$. Thus

$$\begin{aligned}
\lim_{t\to 0^+} \mathrm{grad}\,(\mathrm{dist}(q, \cdot))_{\gamma_{q,\xi}(t)} &= \lim_{t\to 0^+} \gamma'_{q,\xi}(t) \\
&= \xi \\
&\neq \eta \\
&= \lim_{t\to 0^+} \gamma'_{q,\eta}(t) \\
&= \lim_{t\to 0^+} \mathrm{grad}\,(\mathrm{dist}(q, \cdot))_{\gamma_{q,\eta}(t)},
\end{aligned}$$

thus showing r cannot be smooth at q.

If q is a first conjugate point, then q is a limit of ordinary cut points, and hence r cannot be smooth at q. $\square$

4 A Single Submanifold: $B = A \times \{q\}$

See [6] for potential exposition.

We now consider the variational problem dealing with minimizing geodesics connecting a submanifold to a single a point. To this end, let (M, g) be a complete Riemannian manifold. Suppose $A \subset M$ is an immersed submanifold. Then (A, g) can be treated as a Riemannian submanifold of (M, g), where we denote the induced metric on A identically, as it's the pullback of g via the inclusion, that is, $i : A \hookrightarrow M$ and $g = i^*g$ (and i is an isometric immersion). Let $q \in M \setminus A$, and let

$$B = A \times \{q\} \subset M \times M,$$

be our boundary condition in this setting. Then $\mathcal{C}(B)$ is the space of all piecewise regular curves $\gamma : [a, b] \to M$ with $\gamma(a) \in A$ and $\gamma(b) = q$. Moreover, any variation field $V \in T_\gamma\mathcal{C}(B)$ satisfies $V(a) \in T_{\gamma(a)}A$ and $V(b) = 0$. In this setting, our B-geodesics are the geodesics γ which are normal to A, that is,

$$\gamma'(a) \in T_{\gamma(a)}A^\perp.$$

Our variation formulas then simplify as follows.

Theorem 4.1 (First Variation of Energy)**.** *Let $\gamma \in \mathcal{C}(B)$ and $V \in T_\gamma\mathcal{C}(B)$ with associated variation $\Gamma : I_\epsilon \times [a, b] \to M$. If $\{t_j : 0 \leq j \leq k\}$ is an admissible partition for Γ, then*

$$\begin{aligned}\hat{E}'(s) = -\int_a^b g(\partial_s\Gamma, D_t\partial_t\Gamma)dt + \sum_{j=1}^{k-1} g(\partial_s\Gamma(s, t_j), \Delta\partial_t\Gamma(s, t_j)) \\ - g(\partial_s\Gamma(s, a), \partial_t\Gamma(s, a)).\end{aligned}$$

In particular, when $s = 0$,

$$dE_\gamma(V) = -\int_a^b g(V(t), D_t\gamma'(t))dt + \sum_{j=1}^{k-1} g(V(t_j), \Delta\gamma'(t_j))$$

Theorem 4.2 (Second Variation of Energy)**.** *Let $\gamma \in \mathcal{C}(B)$ be a B-geodesic and $V, W \in T_\gamma\mathcal{C}(B)$ with associated two-parameter variation $\Gamma : I_\epsilon \times I_\epsilon \times$*

$[a, b] \to M$. *If* $\{t_j : 0 \leq j \leq k\}$ *is an admissible partition for* Γ*, then*

$$Hess(E)_\gamma(V, W) = \int_a^b (g(D_t V, D_t W) - g(R_{V\gamma'}\gamma', W))dt \\ + g(S_{\gamma'(a)}(V(a)), (W(a)),$$

or alternatively,

$$Hess(E)_\gamma(V, W) = -\int_a^b (g(D_t^2 V + R_{V\gamma'}\gamma', W)dt \\ + g(S_{\gamma'(a)}(V(a)) - D_t V(a), W(a)) \\ + \sum_{j=1}^{k-1} g(\Delta(D_t V)(t_j), W(t_j)),$$

where S *denotes the shape operator of the submanifold* $A \subset M$ *with respect to the normal vector* $\gamma'(a)$*, that is, given a normal vector* $\xi \in T_x A^\perp$ *and vectors* $X, Y \in T_x A$*, we have that*

$$g(S_\xi(X), Y) = -g(\mathrm{II}(X, Y), \xi).$$

4.1 A-Jacobi Fields

Let (M, g) be an n-dimensional, complete Riemannian manifold with k-dimensional, Riemannian submanifold $A \subset M$ with shape operator S, and boundary condition $B = A \times \{q\}$. Suppose $\gamma \in \mathcal{C}(B)$ is a B-geodesic. Then a Jacobi field $J \in \mathfrak{X}(\gamma)$ is called an *A-Jacobi field* if J satisfies the initial conditions

$$J(a) \in T_{\gamma(a)} A, \qquad D_t J(a) - S_{\gamma'(a)}(J(a)) \in T_{\gamma(a)} A^\perp.$$

Notice that since the first initial condition is a restriction that $(n - k)$ equations be 0, and the second initial condition is a restriction that k equations be zero, the space of all A-Jacobi equations along γ, denoted by $\mathfrak{J}^A(\gamma)$ is n-dimensional.

Let NA denote the normal bundle of A in $TM|_A$. That is,

$$TM|_A = TA \oplus NA$$

as a Whitney sum, and let $\exp^\perp : \mathcal{D} \subset NA \to M$ denote the restriction of the exponential map $\exp : TM \to M$.

Theorem 4.3. *Let $(x, \xi) \in NA$ and $\gamma = \gamma_{x,\xi} : [0, b] \to M$ be a geodesic segment normal to A, and let $J \in \mathfrak{X}(\gamma)$. Then J is an A-Jacobi field if and only if J is the variation field for a smooth variation $\Gamma : I_\epsilon \times [0, b] \to M$ such that each curve $\Gamma(s, \cdot)$ is a geodesic normal to A at $t = 0$.*

Proof: Suppose $J(t) = \partial_s\Gamma(0, t)$ is the variation field for such a variation Γ. Then as before J is a Jacobi field. Moreover, by assumption we have that $\Gamma(s, 0) \in A$ for all $s \in I_\epsilon$, hence $J(0) = \partial_s\Gamma(0, 0) \in T_xA$. Let $\xi(s) = \partial_t\Gamma(s, 0)$. Then $\xi = \xi(0)$, and

$$\begin{aligned}
D_tJ(0) &= D_t\partial_s\Gamma(0, 0)\\
&= D_s\partial_t\Gamma(0, 0)\\
&= \tan(\nabla_{J(0)}\xi(s)\big|_{s=0} + \operatorname{nor}\nabla_{J(0)}\xi(s)\big|_{s=0}\\
&= S_\xi(J(0)) + \operatorname{nor}\nabla_{J(0)}\xi(s)\big|_{s=0}.
\end{aligned}$$

That is, $D_tJ(0) - S_\xi(J(0)) \in T_xA^\perp$, and J is an A-Jacobi field.

Conversely, suppose J is an A-Jacobi field. Let

$$\eta = (J(0), D_tJ(0) - S_\xi(J(0))) \in T_xA \oplus T_xA^\perp = T_\xi(NA).$$

Take a curve $s \mapsto \xi(s) \in NA$ such that $\xi(0) = \xi$ and $\xi'(0) = \eta$. Define the variation

$$\Gamma(s, t) = \exp^\perp(t\xi(s)).$$

Clearly, $\Gamma(0, t) = \exp^\perp(t\xi) = \exp_x(t\xi) = \gamma(t)$, so this is a normal geodesic variation of γ. Finally, let $\alpha(s) = \pi(\xi(s))$, and we see that the variation field $V(t)$ is a Jacobi field satisfying the initial conditions

$$\begin{aligned}
Y(0) &= \partial_s\Gamma(0, 0)\\
&= \alpha'(0) = d\pi_\xi(\eta)\\
&= J(0),
\end{aligned}$$

and

$$\begin{aligned}
D_tY(0) &= D_s\partial_t\Gamma(0, 0)\\
&= \tan(\nabla_{Y(0)}\xi(s))\big|_{s=0} + \operatorname{nor}\nabla_{Y(0)}\xi(0)\big|_{s=0}\\
&= \tan(\nabla_{J(0)}\xi(s))\big|_{s=0} + \operatorname{nor}\nabla_{J(0)}\xi(0)\big|_{s=0}\\
&= S_\xi(J(0)) + K^\perp\xi'(0)\\
&= S_\xi(J(0)) + D_tJ(0) - S_\xi(J(0))\\
&= D_tJ(0),
\end{aligned}$$

thus showing that $Y \equiv J$. $\square$

Corollary 4.4. *J is an A-Jacobi field along γ_ξ if and only if there exists $(X, Y) \in T_\xi NA$ such that*

$$J(t) = d(\exp^\perp)_{t\xi}(X, tY),$$

where

$$X = J(0), \qquad Y = D_t J(0) - S_\xi(J(0)).$$

Proof: □ Fill in this proof.

4.1.1 Focal Points

For $\gamma = \gamma_\xi : [0, b] \to M$ a normal geodesic to A, if there exists a nontrivial A-Jacobi field J along γ for which $J(t_0) = 0$, $t_0 > 0$, we call $\gamma(t_0) = \exp^\perp(t_0\xi)$ a *focal point* of A along γ.

Proposition 4.5. *Let $\gamma_{x,\xi}$ be a normal geodesic to A. Then $\gamma_{x,\xi}(t_0)$ is a focal point of A if and only if $d(\exp^\perp)_{t_0\xi}$ is degenerate.*

Proof: Proof should be similar to Lemma 2.2, but requirers more care since $\exp^\perp$ has the critical point at (x, ξ), and hence we need a better way to identify $T_{(x,\xi)}(NA)$.

Figure out proof.

□

Corollary 4.6. *Let $\gamma_{x,\xi} : [0, b] \to M$ be a nontrivial normal geodesic to A. If $\gamma(b)$ is not a focal point of A, then for any $\eta \in T_{\gamma(b)}M$ there exists a unique A-Jacobi field J along γ with $J(b) = \eta$.*

Proof: Should follow after understanding the previous proposition better. □

Lemma 4.7 (Gauss Lemma)**.** *Let $\xi \in T_xA^\perp$ and under the usual identification let $(X, Y) \in T_\xi NA$. Then we may identify $(0, \xi) \in T_\xi(T_xA^\perp)$ with ξ via the isomorphism $k : T_xA^\perp \to T_\xi(T_xA^\perp)$. Then*

i.

$$d(\exp^\perp)_{t\xi}(0, t\xi) = t\gamma'_\xi(t),$$

and in particular

$$|d(\exp^\perp)_\xi(0, \xi)|_g = |\xi|_g,$$

ii.

$$g(d(\exp^\perp)_{t\xi}(X, tY), \gamma'_\xi(t)) = tg(Y, \xi).$$

Proof: Should follow after understanding the previous proposition better. Also see Kling Lemma 1.12.17 p. 121 □

4.2 A-Cut Points

Let (M, g) be an n-dimensional, complete Riemannian manifold with k-dimensional, compact (and hence embedded) Riemannian submanifold $A \subset M$ with shape operator S. Let NA denote the normal bundle of A in $TM|_A$, and let $N^1A = SM|_A \cap NA$ the unit-normal bundle of A in NA.

Lemma 4.8. *For any $y \in M$, there exists $(x, \xi) \in N^1A$ such that*

$$y = \gamma_{x,\xi}(s),$$

where

$$s = dist_g(x, y) = dist_g(A, y).$$

Proof: Letting $r(x) = \mathrm{dist}_g(x, y)$, we have that r is continuous on M. Moreover, as M is complete, $r(x) < \infty$, and since A is compact, we have that

$$s = \min_{x \in A} r(x),$$

attains its minimum at some $x \in A$. Hence

$$y = \gamma_{x,\xi}(s),$$

for some $\xi \in S_xM$.

Consider the boundary condition $B = A \times \{y\} \subset M \times M$. Then $\gamma \in \mathcal{C}([0, s]; B)$ is a B-geodesic, and by our first variation formula Theorem 1.4, for any $X \in T_xA$, let $V \in T_\gamma \mathcal{C}(B)$ be such that $V(0) = X$, we have that

$$\begin{aligned} 0 &= dE_{\gamma_{x,\xi}}(V) \\ &= -g(V(0), \gamma'_{x,\xi}(0)) \\ &= -g(X, \xi), \end{aligned}$$

and hence $\xi \in N_xA$ as desired. □

Since A is compact by the Tubular Neighborhood Theorem[6] there exists $\rho > 0$ such that

$$\exp^\perp : V^\rho \to A^\rho,$$

is a diffeomorphism, where

$$V^\rho = \{\xi \in NA : |\xi|_g < \rho\},$$

[6] Cf. Vector Bundle Notes

and $A^\rho = \exp^\perp(V^\rho)$ is an open neighborhood of A in M. In particular, if $y \in A^\rho$, then there exists a unique $(x, \xi) \in N^1A$ such that

$$\exp^\perp(x, s\xi) = y,$$

for $s < \rho$.

Although $\gamma_{x,\xi}([0, s])$ is the unique shortest geodesic to A when s is small, it fails to be so when s is large. This leads to defining our critical distance function for our normal exponential map. That is, *A-cut locus distance function* $\tau_A : N^1A \to \mathbb{R}$,

$$\tau_A(x, \xi) = \max\{s > 0 : \mathrm{dist}_g(\gamma_{x,\xi}(s), A) = s\}.$$

Since A is compact, $\tau_A(x, \xi) < \infty$ for any $(x, \xi) \in N^1A$.

We define the *A-cut locus*

$$\mathrm{Cut}(A) = \{y \in M : y = \gamma_{x,\xi}(\tau_A(x, \xi)), (x, \xi) \in N^1A\}.$$

Recall a point $y = \gamma_{x,\xi}(t_0)$ is a *focal point* of A if

$$d(\exp^\perp)_{t_0\xi} : T_{t_0\xi}NA \to T_yM$$

is degenerate. We can then define the *focal distance*, $\tau_f = \tau_{f,A} : N^1A \to \overline{\mathbb{R}}$,

$$\tau_f(x, \xi) = \inf\{s > 0 : d(\exp^\perp)_{s\xi} \text{ is degenerate}\}.$$

Recall $\gamma_{x,\xi}(\tau_f(x, \xi))$ is a focal point if and only if there exists an A-Jacobi field J along $\gamma_{x,\xi}$ such that $J(\tau_f(x, \xi)) = 0$.

We say a point $y \in \mathrm{Cut}(A)$ is an *ordinary A-cut point* if there exists $(x, \xi), (z, \zeta) \in N^1A$, $x \neq z$ such that

$$t_0 = \tau_A(x, \xi) = \tau_A(z, \zeta),$$

and

$$\gamma_{x,\xi}(t_0) = y = \gamma_{z,\zeta}(t_0).$$

thm:klingLemSubman **Theorem 4.9** (A Klingenberg Lemma). *(See Theorem 2.12 in [7] and Lemma 2.11 (p.98) in [18].)*

Let $A \subset M$ be a compact Riemannian submanifold.

1. *$\tau_A : N^1A \to \mathbb{R}$ is continuous,*

2. *$\tau_f : N^1A \to \overline{R}$ is continuous.*

3. *Suppose* $(x,\xi) \in N^1A$*, and let* $y = \gamma_{x,\xi}(T)$*. Then* $T = \tau_A(x,\xi)$ *if and only if* $\gamma = \gamma_{x,\xi}|_{[0,T]}$ *is a minimizing geodesic segment, and either*

   i. y *is an ordinary* A*-cut point of* x*, or*

   ii. y *is the first focal point.*

   Moreover, for $t < \tau_A(x,\xi)$*, we have that* $d(\exp^\perp)_{t\xi}$ *is nondegenerate.*

4. $\tau_A(x,\xi) < \tau(x,\xi)$ *for all* $(x,\xi) \in N^1A$*.*

Proof: This is going to be a long, hard, and detailed proof. Take the time to do it right because it's important and I can't find it in the literature anywhere. $\square$

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