Semi-Riemannian Representations

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Contents

1 Boundary Distance Representation of a Riemannian Manifold **2**

2 Reconstruction of (M, g) as a Lorentzian Manifold **6**

2.1 Construction of V as a Topological and Differentiable Manifold 9

3 Distance Different Representation **11**

3.1 M and $\mathcal{D}(M)$ are Homeomorphic 11

3.2 M and $\mathcal{D}(M)$ are Diffeomorphic 13

4 Constructing a Connection from Pregeodesics **17**

4.1 Pregeodesics 17

4.2 Uniqueness of Connection up to Gauge Freedom 20

1 Boundary Distance Representation of a Riemannian Manifold

See Section 3.8 in [8].

Let (M, g) be a compact, connected Riemannian manifold with nonempty boundary of dimension n. We will construct a differentiable and Riemannian structure on the set $R(M)$ of Riemannian distance functions and we will show that $(R(M), \tilde{g})$ is isometric to (M, g).

Let $d : M \times M \to \mathbb{R}$ denote the Riemannian distance function on M, and for $x \in M$, let $r_x : \partial M \to \mathbb{R}$ denote $r_x(y) = d(x, y)$. Since d is continuous, we have that r_x is continuous for each $x \in M$, that is, $r_x \in C(\partial M) \subset L^\infty(\partial M)$ for each $x \in M$. Let $R : M \to C(\partial M)$ denote this map, that is,

$$R(x) = r_x.$$

Then $R(M) \subset C(\partial M)$ and is a topological space with the inherited topology of $L^\infty(\partial M)$.

Lemma 1.1. *$R : M \to R(M) \subset C(\partial M) \subset L^\infty(M)$ is a homeomorphism.*

Proof: By the reverse triangle inequality, we get that

$$\begin{aligned}
\|r_x - r_y\|_{L^\infty(\partial M)} &= \sup_{z \in \partial M} |r_x(z) - r_y(z)| \\
&= \sup_{z \in \partial M} |(d(x, z) - d(y, z)| \\
&\leq d(x, y),
\end{aligned}$$

and so R is continuous. Next, suppose $r_x = r_y$ in $C(\partial M)$. Let

$$s = \min_{z' \in \partial M} r_x(z'),$$

and let $z \in \partial M$ be such that $r_x(z) = s$. But then x lies on the normal geodesic from the boundary $\gamma_{z,\nu}$ with $\gamma_{z,\nu}(s) = x$. As the same is true for r_y, we see that $x = y$. Hence R is injective. Since $R : M \to R(M)$ is by definition surjective, M is compact and $L^\infty(\partial M)$ is Hausdorff, we see by the Closed Map Lemma that $R : M \to R(M)$ is a homeomorphism. Indeed, we need only show that R is a closed map. To this end, suppose $K \subseteq M$ is closed, and hence compact. Since R is continuous, $R(K)$ is compact in $L^\infty(\partial M$, and since $L^\infty(\partial M)$ is Hausdorff, $R(K)$ is closed. $\square$

Remark 1.2. *Note that if (M, g) is geodesically regular, (i.e., any two points has a unique geodesic connecting them, and any geodesic can* *be continued to a geodesic whose endpoints lie on the boundary), then (M, d_g) is an isometric as metric spaces to $(R(M), d_\infty)$ via R, and hence via the Myers-Steenrod theorem is actually a Riemannian isometry.*

Proposition 1.3. *There is a differentiable structure on $R(M)$ making $R : M \to R(M)$ a diffeomorphism.*

Proof: Fix $r \in R(M)$. Let $S(r) \in [0, \infty)$ and $Z(r) \in \partial M$ be defined as

$$S(r) = \min_{z \in \partial M} r(z),$$

and

$$r(Z(r)) = S(r).$$

Note that $Z(r) \in \partial M$ may not be unique. Let

$$\Gamma_z = \{r \in R(M) : r(z) = S(r)\}, \quad z \in \partial M,$$

and

$$\tau_b(z) = \max_{r \in \Gamma_z} r(z),$$

is the boundary cut distance as defined in the previous section. Note that

$$\Gamma_z = R(\gamma_{z,\nu}([0, \tau_b(z)]),$$

is the image of the normal geodesic before the cut point. Letting $\mathrm{Cut}_{\partial M}$ denote the boundary cut locus as in the previous section, it's clear that we can now define $R(\mathrm{Cut}_{\partial M})$ from these terms. Indeed,

$$R(\mathrm{Cut}_{\partial M}) = \{r \in R(M) : S(r) = \tau_b(Z(r))\}.$$

We first construct coordinates on $R(M \backslash \mathrm{Cut}_{\partial M}) = R(M) \backslash R(\mathrm{Cut}_{\partial M})$. Since we have boundary normal coordinates given by

$$\exp_{\partial M} : \partial M \times [0, \rho) \to \{x \in M : d(x, \partial M) < \rho\},$$

that is,

$$(z, s)(x) = \exp_{\partial M}^{-1}(x).$$

We know that $s = s(x) \in C(M)$ and $z = z(x) \in C(M \setminus \mathrm{Cut}_{\partial M})$. Hence $S = s \circ R^{-1}$ is continuous on $R(M)$ and $Z = z \circ R^{-1}$ is continuous on $R(M \setminus \mathrm{Cut}_{\partial M})$.

Now let $r_0 \in R(M \setminus \mathrm{Cut}_{\partial M})$ and $z_0 = Z(r_0)$. Let $V \subseteq \partial M$ be a coordinate neighborhood of z_0 with coordinates $(z^1, ..., z^{n-1})$. Then $(z^1, ..., z^{n-1}, s)$ form coordinates on the open set

$$Z^{-1}(V) = \{r \in R(M \setminus \cup(\partial M)) : Z(r) \in V\}.$$

Thus, the pair $(Z(r), S(r))$ determine a system of smooth coordinates on $R(M \setminus \mathrm{Cut}_{\partial M})$ making

$$R : M \setminus \mathrm{Cut}_{\partial M} \to R(M \setminus \mathrm{Cut}_{\partial M})$$

a diffeomorphism.

Near $R(\mathrm{Cut}_{\partial M})$ we will use boundary distance coordinates instead of normal coordinates. Let $r_0 \in R(\mathrm{int}(M))$ and $x = R^{-1}(r_0)$. Then there are points $z^1, ..., z^n \in \partial M$ such that $\rho^j(x) = d(x, z^j)$ define local coordinates near x. For $z \in \partial M$, let $E_z : R(M) \to \mathbb{R}_+$ denote the evaluation functions, that is,

$$E_z(r) = r(z).$$

Then

$$\begin{aligned}
(E_{z^1}(r), ..., E_{z^n}(r)) &= (r(z^1), ..., r(z^n)) \\
&= (d(x, z^1), ..., d(x, z^n)) \\
&= (\rho^1(x), ..., \rho^n(x)) \\
&= (\rho^1(R^{-1}(r)), ..., \rho^n(R^{-1}(r))).
\end{aligned}$$

These two coordinate structures combine to make $R(M)$ a smooth manifold such that $R : M \to R(M)$ is a diffeomorphism.

□

Proposition 1.4. *There exists a Riemannian metric $\tilde{g}$ on $R(M)$ such that $R : (M, g) \to (R(M), \tilde{g})$ is a Riemannian isometry.*

Note that such a metric exists on $R(M)$ since R is a diffeomorphism, its pushforward R_* is an isomorphism, so defining $\tilde{g} = R_* g$ would work, but as we don't know g, we need to construct $\tilde{g}$ explicitly.

Fix $r_0 \in R(\mathrm{int}(M))$, and let $z^1, ..., z^n$ be points in ∂M such that

$$(\rho^1, ..., \rho^n) = (E_{z^1}(r), ..., E_{z^n}(r)),$$

define local coordinates near r_0. Consider the evaluation function $E_z(r)$ where z lies in some neighborhood V of $z_0 = Z(r_0)$ and r lies in some neighborhood $R(U)$, where U is a neighborhood of $x_0 = R^{-1}(r_0)$. Now

$$E_z(r) = d(x, z),$$

where $R(x) = r$. By possibly shrinking U and V, the distance function d is smooth on $U \times V$ and the collection of vectors

$$W = \{\mathrm{grad}_x(d(x, z))_{x_0} \in S_{x_0} M : z \in V\}$$

is open in the unit ball $S_{x_0}M \subset T_{x_0}M$. Since R_* is an isomorphism, it then follows that

$$\mathcal{W} = \{\mathrm{grad}\,(E_z)_{r_0} \in T_{r_0}R(M) : z \in V\},$$

is an $n-1$-dimensional submanifold of $T_{r_0}R(M)$. Moreover, for R to be an isometry, we actually have that $\mathcal{W} \subseteq S_{r_0}R(M)$. For the moment, due to not knowing $\tilde{g}$ as of yet, let's work with

$$\mathcal{W}^* = \{d(E_z)_{r_0} \in T^*_{r_0}R(M) : z \in V\}.$$

Then similarly, $\mathcal{W}^* \subseteq S^*_{r_0}R(M)$ is open.

This submanifold $\mathcal{W}^* \subseteq S^*_{r_0}R(M)$ determines the metric tensor $\tilde{g}^{jk}(r_0)$. Since $(\rho^1, ..., \rho^n)$ are local coordinates about r_0, we have that

$$\mathcal{W}^* = \{(\partial_{\rho^1}(E_z)_{r_0}, ..., \partial_{\rho^n}(E_z)_{r_0}) : z \in V\},$$

and so $\mathbb{R}_+\mathcal{W}$ is an open cone in $T^*_{r_0}R(M)$. Therefore, for any

$$\xi = \alpha(\partial_{\rho^1}(E_z)_{r_0}, ..., \partial_{\rho^n}(E_z)_{r_0}) \in \mathbb{R}_+\mathcal{W},$$

we have the function

$$F(\xi) = \tilde{g}(\xi, \xi) = \tilde{g}^{jk}(r_0)\xi_j\xi_k = \alpha^2,$$

is known. Since this is known on the open subset $\mathbb{R}_+\mathcal{W}^*$, we can compute the differentials to determine

$$\tilde{g}^{jk}(r_0) = \partial_{\xi_j}\partial_{\xi_k}F(\xi).$$

Thus we have determined $\tilde{g}^{jk}(r_0)$. Since $r_0 \in R(\mathrm{int}(M))$ was arbitrary, we've determine $\tilde{g}$ on all of $R(\mathrm{int}(M))$. Rewriting $\tilde{g}$ in the boundary normal coordinates and using the smoothness on $R(M)$ then recovers $\tilde{g}$ on all of $R(M)$, and hence concludes the reconstruction.

2 Reconstruction of (M, g) as a Lorentzian Manifold

See [9].

Let (M, g) be a globally hyperbolic spacetime of dimension $n+1$ with $n \geq 2$. Let $U \subseteq M$ be a domain and suppose p^- and p^+ are two points U such that there is a timelike path $\mu \subset U$ from p^- to p^+. Also suppose that $V \subset J^-(p^+) \setminus I^-(p^-)$ is a relatively compact open subset of M.

We will construct a differentiable and Lorentzian structure on the set $\mathcal{F}(V)$ of *observation time functions* which is conformally equivalent to $(V, g|_V)$.

Let g^+ be an arbitrary Riemannian metric placed on M. For $q \in V$, we define the light observation set of q as

$$\mathcal{P}_U(q) = \mathcal{L}_q^+ \cap U.$$

Then we have the unindexed collection of light observation sets

$$\mathcal{P}_U(V) = \{\mathcal{P}_U(q) : q \in V\} \subset 2^U.$$

Given such a μ, we can find a family $\{\mu_a : [-1, 1] \to U : a \in \mathcal{A}\}$ of future-pointing timelike paths indexed by $a \in \mathcal{A}$ with $\mathcal{A}$ a metric space and $\mu = \mu_{a_0}$ for some $a_0 \in \mathcal{A}$. Moreover, we may assume that $(a, s) \mapsto \mu_a(s)$ is an open and continuous map, and by possibly shrinking U that

$$U = \bigcup_{a \in \mathcal{A}} \mu_a([-1, 1]).$$

Theorem 2.1. *Suppose we know the differentiable manifold U, the conformal class of $g|_U$, the paths $\mu_a : [-1, 1] \to U$, $a \in \mathcal{A}$ with the above properties, and the set $\mathcal{P}_U(V)$. Then this data determines the unique topological and differentiable structure of V and the conformal class of $g|_V$.*

Let $-1 < s_- < s_+ < 1$ be such that $\mu(s_\pm) = p^\pm$. Furthermore, let $s_{-2} \in (-1, s_-)$ and $s_{+2} \in (s_+, 1)$ with $p^{\pm 2} = \mu(s_{\pm 2})$. By possibly shrinking $\mathcal{A}$, we further assume that for any $a \in \mathcal{A}$ that

$$\mu_a(s_{-2}) \in I(\mu(-1), p^-),$$

$$\mu_a(s_{+2}) \in I(p^+, \mu(1)).$$

Let $\mathfrak{t}$ denote the time-separation function, that is, for $x \leq y$,

$$\mathfrak{t}(x, y) = \sup_\gamma L(\gamma) = \sup_\gamma = \int_0^1 \sqrt{-g(\gamma'(s), \gamma'(s))} ds,$$

where γ is any piecewise smooth causal path $\gamma : [0,1] \to M$ from x to y. If $x \not\leq y$, then $\mathfrak{t}(x,y) = 0$. Since M is globally hyperbolic, we have that $\mathfrak{t} : M \times M \to [0,\infty)$ is continuous and $J^{\pm}(x)$ is closed for all $x \in M$. Moreover for $x < y$ there is a longest causal geodesic $\gamma : [0,1] \to M$ with $\gamma(0) = x, \gamma(1) = y$ and $\mathfrak{t}(x,y) = L(\gamma)$.

For a nonzero $(x,\xi) \in TM$, define $\mathcal{T}(x,\xi) \in (0,\infty]$ to be the maximal value for which $\gamma_{x,\xi} : [0,\mathcal{T}(x,\xi)) \to M$ is defined, that is, $\gamma_{x,\xi}$ on this interval is future-inextendible. Now, for $(x,\xi) \in L^+M$, $x \in J^-(p^+)$, define

$$T_{+2}(x,\xi) = \sup\{t \geq 0 : \gamma_{x,\xi}(t) \in J^-(p_{+2})\}.$$

Since $J^-(p_{+2})$ is closed, and $\gamma_{x,\xi}$ are future-pointing curves, $T_{+2} : L^+(J^-(p^+)) \to \mathbb{R}$ is upper-semicontinuous. Moreover, since the set

$$K := \{(x,\xi) \in L^+M : x \in \overline{V}, \|\xi\|_{g^+} = 1\},$$

is compact, there exists $c_0 \in \mathbb{R}_+$ such that $T_{+2}(x,\xi) \leq c_0$ for all $(x,\xi) \in K$.

For $(x,\xi) \in L^+M$, we define the null cut distance function

$$\rho(x,\xi) = \sup\{s \in [0,\mathcal{T}(x,\xi)) : \mathfrak{t}(x,\gamma_{x,\xi}(s)) = 0\}.$$

If $\rho(x,\xi) < \infty$, then the point $p(x,\xi) = \gamma_{x,\xi}(\rho(x,\xi))$ is called the (first) null cut point of the geodesic $\gamma_{x,\xi}$. Since (M,g) is globally hyperbolic $p(x,\xi)$ is either a first conjugate point (i.e., $\exp_x$ is degenerate at $\rho(x,\xi)\xi) \in T_xM$) or there exists another lightlike geodesic $\gamma_{x,\eta}$ from x to $p(x,\xi)$ with $\eta \neq c\xi$ for any $c \in \mathbb{R}$.

Definition 2.2. *Let $a \in \mathcal{A}$ and $q \in J^-(p^+) \setminus I^-(p^-)$. The observation time function $f_a : J^-(p^+) \setminus I^-(p^-) \to [-1,1]$ is defined by*

$$f_a(q) = \inf\left(\{s \in [-1,1] : \mu_a(s) \in J^+(q)\} \cup \{1\}\right).$$

Moreover, let $\mathcal{E}_a(q) = \mu_a(f_a(q))$. Then $\mathcal{E}_a(q)$ is the earliest point on μ_a as which light is observed from q.

Lemma 2.3. *Let $a \in \mathcal{A}$ and $q \in J^-(p^+) \setminus I^-(p^-)$.*

i. It holds that $s_{-2} \leq f_a(q) \leq s_{+2}$.

ii. We have that $\mathcal{E}_a(q) \in J^+(q)$ and $\mathfrak{t}(q,\mathcal{E}_a(q)) = 0$. Moreover, the function $s \mapsto \mathfrak{t}(q,\mu_a(s))$ is continuous, nondecreasing on $[-1,1]$ and strictly increasing on $[f_a(q),1]$.

iii. Assume that $p \in U$. Then $p = \mathcal{E}_a(q)$ for some $a \in \mathcal{A}$ if and only if $p \in \mathcal{P}_U(q)$ and $\mathfrak{t}(q,p) = 0$. Furthermore, this is equivalent to the fact that there exists $\xi \in L^+_qM$ and $t \in [0,\rho(q,\xi)]$ such that $p = \gamma_{q,\xi}(t)$.

iv. The function $q \mapsto f_a(q)$ is continuous on $J^-(p^+) \setminus I^-(p^-)$.

Definition 2.4. *Let $q \in J^-(p^+) \setminus I^-(p^-)$. Let*

$$\begin{aligned}\mathcal{D}_U(q) := \{(y,\eta) \in L^+U : y = \gamma_{q,\xi}(t) \in U, \eta = \gamma'_{q,\xi}(t),\\ \text{for some } \xi \in L_q^+M, 0 \le t \le \rho(q,\xi)\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_U^{reg}(q) := \{(y,\eta) \in L^+U : y = \gamma_{q,\xi}(t) \in U, \eta = \gamma'_{q,\xi}(t),\\ \text{for some } \xi \in L_q^+M, 0 < t < \rho(q,\xi)\}.\end{aligned}$$

We say that $\mathcal{D}_U(q)$ is the direction set of q and $\mathcal{D}_U^{reg}(q)$ is the regular direction set of q.

Let $\mathcal{E}_U(q) = \pi(\mathcal{D}_U(q))$ and $\mathcal{E}_U^{reg}(q) = \pi(\mathcal{D}_U^{reg}(q))$, where $\pi : TU \to U$ is the standard bundle projection. We say that $\mathcal{E}_U(q)$ is the set of earliest observations of q in U and $\mathcal{E}_U^{reg}(q)$ is the set of regular earliest observations of q in U. Denote

$$\mathcal{E}_U(V) = \{\mathcal{E}_U(q) \in 2^U : q \in V\}.$$

Note that $\mathcal{E}_U(q) = \{\mathcal{E}_a(q) : a \in \mathcal{A}\}$ and the lower-semicontinuity of ρ implies that $\mathcal{D}_U^{\mathrm{reg}}(q) \subset TU$ is a smooth $2n$-manifold and $\mathcal{E}_U^{\mathrm{reg}}(q) \subset U$ is a smooth n-manifold.

It's now easily seen that

$$\mathcal{E}_U(q) = \{x \in \mathcal{P}_U(q) : \text{there is no } y \in \mathcal{P}_U(q) \text{ such that } y \ll x\}.$$

By our Lemma, we have that

$$f_a(q) = \min\{s \in [-1,1] : \mu_a(s) \in J^+(q)\}, \qquad \mathcal{E}_a(q) = \mu_a(f_a(q)),$$

and by the above remark we have that

$$\mathcal{E}_U(q) = \{\mathcal{E}_a(q) : a \in \mathcal{A}\},$$

so we may now conclude that the data $\mathcal{P}_U(V)$ and $\{\mu_a : a \in \mathcal{A}\}$ determine $\mathcal{E}_U(V)$.

Remark. Given $\mathcal{E}_U(V)$, one can then determine the sets $\mathcal{D}_U(q)$, $\mathcal{D}_U^{\mathrm{reg}}(q)$, and $\mathcal{E}_U^{\mathrm{reg}}(q)$.

2.1 Construction of V as a Topological and Differentiable Manifold

Given $q \in J^-(p^+) \setminus I^-(p^-)$, define the function $F_q : \mathcal{A} \to \mathbb{R}$ by

$$F_q(a) = f_a(q).$$

We then can define the function $\mathcal{F} : J^-(p^+) \setminus I^-(p^-) \to \mathbb{R}^{\mathcal{A}}$, that maps q to the function $F_q : \mathcal{A} \to \mathbb{R}$, that is,

$$\mathcal{F}(q) = F_q.$$

We endow $\mathbb{R}^{\mathcal{A}}$ with the product topology (which is Hausdorff since $\mathbb{R}$ is Hausdorff). By considering the set $\mathcal{F}(V)$ we will construct our topological and differentiable structure on V, and by using $\mathcal{E}_U(V)$ our conformal class of the metric $g|_V$.

Lemma 2.5. *Let $V \subset J^-(p^+) \setminus I^-(p^-)$ be a relatively compact open set. Then the map $\mathcal{F} : V \to \mathcal{F}(V)$ is a homeomorphism.*

Proof: Since $\mathbb{R}^{\mathcal{A}}$ has the product topology, and

$$\pi_a \circ \mathcal{F} = f_a,$$

is continuous by the above, we see that $\mathcal{F} : V \to \mathcal{F}(V)$ is continuous.

Now we show that $\mathcal{F} : \overline{V} \to \mathcal{F}(\overline{V}) = \overline{\mathcal{F}(V)}$ is injective. Since $\mathcal{F}(q)$ uniquely determines the set $\mathcal{E}_U(q)$, it suffices to show that $\mathcal{E}_U : \overline{V} \to \mathcal{E}_U(\overline{V})$ is injective. To this end, suppose $q_1, q_2 \in \overline{V}$ with $q_1 \neq q_2$ and assume that $\mathcal{E}_U(q_1) = \mathcal{E}_U(q_2)$. By our remark, we then have that $\mathcal{D}_U(q_1) = \mathcal{D}_U(q_2)$. Choose $a \in \mathcal{A}$ such that $q_j \notin \mu_a$ for $j = 1, 2$. Let $(p, \eta) \in \mathcal{D}_U(q_j)$ with $p = \mathcal{E}_a(q_j)$. Then there exists $t_1, t_2 > 0$ such that $\gamma_{p,\eta}(-t_j) = q_j$, $j = 1, 2$. Since $q_1 \neq q_2$, we have that $t_1 \neq t_2$ and so without loss of generality assume that $t_2 > t_1$. Moreover, by definition of $\mathcal{D}_U(q_j)$, there exists $\xi_j \in L^+_{q_j}M$ such that

$$(p, \eta) = (\gamma_{q_j,\xi_j}(t_j), \gamma'_{q_j,\xi_j}(t_j)), \qquad (q_1, \xi_1) = (\gamma_{q_2,\xi_2}(t_2 - t_1), \gamma'_{q_2,\xi_2}(t_2 - t_1)),$$

with

$$0 \le t_1 \le \rho(q_1, \xi_1), \qquad 0 \le t_2 \le \rho(q_2, \xi_2).$$

Thus

$$t_2 - t_1 < t_2 \le \rho(q_2, \xi_2),$$

and so (q_1, ξ_1) is not a null cut point of γ_{q_2,ξ_2}. By lower-semicontinuity, for any $\delta_1 > 0$ there exists $\delta_2 > 0$ such that

$$\rho(q_2, \xi_2') > \rho(q_2, \xi_2)\delta_1,$$

whenever $\|\xi_2' - \xi_2\| < \delta_2$. Choose $\xi_2' \in T_{q_2}M$ with $\|\xi_2' - \xi_2\| < \delta_2$ and ξ_2' not parallel with ξ_2, and $t_2' \in (t_2 - 2\delta_1, t_2 - \delta_1)$ such that $p' = \gamma_{q_2,\xi_2'}(t_2') \in U$ and $p' \neq q_1$. Then

$$t_2' < t_2 - \delta_1 \leq \rho(q_2, \xi_2) - \delta_1 < \rho(q_2, \xi_2').$$

Let $\eta' = \gamma_{q_2,\xi_2'}(t_2')$ and so $(p', \eta') \in \mathcal{D}_U(q_2) = \mathcal{D}_U(q_1)$. Hence, there exists $t_1' > 0$ such that $q_1 = \gamma_{p',\eta'}(-t_1')$. Let $\xi_1' = \gamma_{p',\eta'}(-t_1')$, which is seen to not be parallel with ξ_1, otherwise ξ_2' would be parallel with ξ_2. Now, the union of the geodesic $\gamma_{q_2,\xi_2}([0, t_2 - t_1])$ and the geodesic $-\gamma_{p',\eta'}([-t_1', 0])$ is a causal curve from q_2 to p' that is not a lightlike pregeodesic, indeed, if it were a lightlike geodesic then p' would be an ordinary cut point of $\gamma_{q_2,\xi_2'}$, but $t_2' < \rho(q_2, \xi_2')$. Thus $\mathfrak{t}(q_2, p') > 0$ which is a contradiction since $p' \in \mathcal{E}_U(q_2)$. Thus $\mathcal{E}_U : \overline{V} \to \mathcal{E}_U(\overline{V})$ in injective, and hence so is $\mathcal{F} : \overline{V} \to \mathcal{F}(\overline{V})$.

Finally, since $\overline{V}$ is compact, $\mathbb{R}^{\mathcal{A}}$ is Hausdorff and $\mathcal{F} : \overline{V} \to \mathcal{F}(\overline{V})$ is bijective, it follows that $\mathcal{F} : \overline{V} \to \mathcal{F}(\overline{V})$ is a homeomorphism, and hence $\mathcal{F} : V \to \mathcal{F}(V)$ is a homeomorphism as desired.

□

We now introduce coordinates on $\mathcal{F}(V)$ to make it into a smooth manifold for which $\mathcal{F} : V \to \mathcal{F}(V)$ is a diffeomorphism.

Let

$$\mathcal{Z} := \{(q, p) \in V \times U : p \in \mathcal{E}_U^{\mathrm{reg}}(q)\}.$$

Then for every $(q, p) \in \mathcal{Z}$, there exists a unique $\xi \in L_q^+ M$ such that $\gamma_{q,\xi}(1) = p$ and $\rho(q, \xi) > 1$. We denote the aforementioned map via $\Theta(q, p) = (q, \xi)$ which maps $\Theta : \mathcal{Z} \to L^+V$. Given $(q, \xi) \in TM$, let $B_\epsilon(q, \xi)$ denote an ϵ-neighborhood about (q, ξ) in TM with respect the g^+-Sasaki metric on TM.

3 Distance Different Representation

See [7] and [11], and in particular [10].

Let (M, g) be an n-dimensional, compact Riemannian manifold without boundary with $n \geq 2$. Let $N \subseteq M$ be an open submanifold, and let $\Omega = M \setminus N$ with the crucial assumption that $\mathrm{int}(\Omega) \neq \emptyset$.

For a point $x \in M$, define the *(restricted) distance difference function* $D_x : \Omega \times \Omega \to \mathbb{R}$ by

$$D_x(p, q) = \mathrm{dist}_g(p, x) - \mathrm{dist}_g(q, x),$$

which yields the collection

$$\mathcal{D}(M) = \mathcal{D}_\Omega(M) = \{D_x : \Omega \times \Omega \to \mathbb{R} : x \in M\},$$

or rather as a representation

$$\mathcal{D} : M \to C(\Omega \times \Omega), \qquad x \mapsto D_x.$$

We then have the subcollection

$$\mathcal{D}(N) = \{D_y : y \in N\} \subset \mathcal{D}(M).$$

3.1 M and $\mathcal{D}(M)$ are Homeomorphic

We consider $\mathcal{D}(M) \subset C(\Omega \times \Omega)$ as a subspace with the supremum norm

$$\|f\|_{L^\infty} = \sup_{x,y \in \Omega} |f(x, y)|.$$

Theorem 3.1. *$\mathcal{D}(M) \subset C(\Omega \times \Omega)$ is a topological manifold homeomorphic to M. In particular, $\mathcal{D}(N)$ is homeomorphic to N.*

Uses both $n \geq 2$ and $\mathrm{int}(\Omega) \neq \emptyset$.

Proof: We first note that $\mathcal{D}$ is 2-Lipschitz and hence continuous. Indeed, let $x, y \in M$ and $p, q \in \Omega$, then

$$\begin{aligned}
|D_x(p, q) - D_y(p, q)| &= |\mathrm{dist}(p, x) - \mathrm{dist}(q, x) - \mathrm{dist}(p, y) + \mathrm{dist}(q, y)| \\
&\leq |\mathrm{dist}(p, x) - \mathrm{dist}(p, y)| + |\mathrm{dist}(q, x) - \mathrm{dist}(q, y)| \\
&\leq \mathrm{dist}(x, y) + \mathrm{dist}(x, y) \\
&= 2\mathrm{dist}(x, y),
\end{aligned}$$

independent of $p, q \in \Omega$. Thus

$$\|D_x - D_y\|_{L^\infty} \leq 2\mathrm{dist}(x, y),$$

as desired.

We need show $\mathcal{D}$ is injective.[1] To this end, let $x, y \in M$ and suppose $D_x = D_y$, but $x \neq y$. Let $q \in \operatorname{int}(\Omega)$ and $\ell_x = \operatorname{dist}(q, x)$, $\ell_y = \operatorname{dist}(q, y)$. Without loss of generality, assume $\ell_x \leq \ell_y$. Let $\eta \in S_q M$ be such that $\gamma_{q,\eta}([0, \ell_x])$ is a minimizing geodesic segment from q to x.

Let $0 < s < \ell_x$ be such that $\gamma_{q,\eta}([0, s]) \subset \operatorname{int}(\Omega)$ and let $p = \gamma_{q,\eta}(s)$. Then

$$\begin{aligned}(\operatorname{dist}(q,p) + \operatorname{dist}(p,y)) - \operatorname{dist}(q,y) &= \operatorname{dist}(q,p) + D_y(p,q) \\ &= \operatorname{dist}(q,p) + D_x(p,q) \\ &= \operatorname{dist}(q,p) + \operatorname{dist}(p,x) - \operatorname{dist}(q,x) \\ &= 0,\end{aligned}$$

and hence p is on the minimizing geodesic from q to y.[2]

Let α denote a minimizing geodesic segment from p to y with length $\ell_y - s$. Then the union $\gamma_{q,\eta}([0, s]) \cup \alpha$ is a distance minimizing curve from q to y and is thus a geodesic. Hence α is the continuation of $\gamma_{q,\eta}([0, s])$, and so $y = \gamma_{q,\eta}(\ell_y)$. Since $\gamma_{q,\eta}([0, \ell_x])$ and $\gamma_{q,\eta}([0, \ell_y])$ are distance minimizing geodesics from q to x, and from q to y, respectively, and $x \neq y$, we conclude that $\ell_x \neq \ell_y$ and hence that $\ell_x < \ell_y$.

Let $q' \in \operatorname{int}(\Omega)$ be such that $q' \notin \gamma_{q,\eta}(\mathbb{R})$ (which clearly exists as $n \geq 2$), and let $\ell'_x = \operatorname{dist}(q', x)$ and $\ell'_y = \operatorname{dist}(q', y)$. Then as before, there exists $\eta' \in S_{q'} M$ such that $\gamma_{q',\eta'}([0, \ell'_x])$ and $\gamma_{q',\eta'}([0, \ell'_y])$ are distance minimizing geodesics from q' to x and from q' to y, respectively, again since $x \neq y$, we have that $\ell'_x \neq \ell'_y$.

Let β denote the minimizing geodesic segment with length $|\ell_y - \ell_x|$ (accounting for either possibility of $\ell'_x < \ell'_y$ or $\ell'_y < \ell'_x$). Then the union $\gamma_{q,\eta}([0, \ell_x]) \cup \beta$ is a distance minimizing geodesic from q to y, and is thus a geodesic. In particular, this implies that

$$\gamma'_{q,\eta}(\ell_x) = \pm \gamma'_{q',\eta'}(\ell'_x),$$

and hence that $q' \in \gamma_{q,\eta}(\mathbb{R})$, a contradiction. Thus $x = y$.

Finally, since M is compact and $(C(\Omega \times \Omega), \|\cdot\|_{L^\infty})$ is Hausdorff, we conclude via a basic topological result (cf. the Closed Map Lemma in [12]) that $\mathcal{D} : M \to \mathcal{D}(M)$ is a homeomorphism, and hence that $\mathcal{D} : N \to \mathcal{D}(N) \subset \mathcal{D}(M)$ is a homeomorphism.

$\square$

[1] The idea follows from the standard representation: $r_1(p) = \operatorname{dist}(x_1, p)$ and $r_2(p) = \operatorname{dist}(x_2, p)$. If $r_1 = r_2$ on open set, then outside $\operatorname{Cut}(x_1) \cup \operatorname{Cut}(x_2)$ $\operatorname{grad}(r_1)_p = \operatorname{grad}(r_2)_p$, and hence $x_1 = \gamma_{p,-\xi}(r_1(p)) = \gamma_{p,-\xi}(r_2(p)) = x_2$. However, the lengths aren't the same in the above proof, and must be circumvented.

[2] See *Riemannian Geodesics* notes, "Cut Points" section.

We note that if $X \subseteq M$ is a dense subset, then since $\mathcal{D}$ is a homeomorphism, that

$$\mathcal{D}(M) = \overline{\mathcal{D}(X)},$$

where the closure is taken with respect to the topology of $C(\Omega, \Omega)$. That is, the distance difference functions corresponding to x in a dense subset X determine the distance difference functions on the whole space M.

3.2 M and $\mathcal{D}(M)$ are Diffeomorphic

We first need a linear algebra lemma.

thm:linAlgLemma

Lemma 3.2. *Let $(V, \langle \cdot, \cdot \rangle)$ be an n-dimensional, inner product space, with a fixed $v \in V \setminus \{0\}$. Then there exists a a basis $v_1, .., v_n \in V$ such that $\|v_j\| = 1$ for $1 \leq j \leq n$, and*

$$v = a^1 v_1 + a^2 v_2,$$

with

$$c^i \neq \frac{\|v\|^2}{\langle v_i, v \rangle}, \qquad ci \neq 0,$$

for $i = 1, 2$.

Moreover, for a basis, there exists $\epsilon > 0$ such that the vectors

$$\left\{ \frac{v + tv_1}{\|v + tv_1\|} - \frac{v}{\|v\|}, ..., \frac{v + tv_n}{\|v + tv_n\|} - \frac{v}{\|v\|} \right\}$$

are linearly independent for any $t \in (0, \epsilon)$.

Proof: Let $v^\perp \in V$ be such that $\langle v, v^\perp \rangle = 0$, $\|v^\perp\| = \|v\|$ (in fact, nonzero is all that's needed). Then for $i = 1, 2$, let

$$v_i = \frac{v + (-1)^i v^\perp}{\sqrt{2}\,\|v\|},$$

and

$$c^i = \frac{\|v\|}{\sqrt{2}}.$$

Complete $\{v_1, v_2\}$ to a basis $\{v_1, ..., v_n\}$ and we've satisfied the first claim.

Let $v = c^j v_j$, where c^1, c^2 are as above and $c^j = 0$ for $3 \leq j \leq n$. Then for each $1 \leq j \leq n$, let

$$f_j(t) = t\,\|v\| + c^j(\|v\| - \|tv_j + v\|).$$

Then $f_j(0) = 0$, and

$$\frac{df_j}{dt}(0) = \|v\| - c^j \frac{\langle v_j, v \rangle}{\|v\|},$$

which is nonzero by our choices of c^j. That is, there exists $\epsilon > 0$ such that $f_j(t) \neq 0$ for all $t \in (0, \epsilon)$ for each $1 \leq j \leq n$. Let $\{a_1, ..., a_n\} \subset \mathbb{R}$ be such that

$$0 = \sum_{j=1}^{n} a_j \left(\frac{v + tv_j}{\|v + tv_j\|} - \frac{v}{\|v\|} \right) = \sum_{j=1}^{n} a_j \left(\frac{t + c^j}{\|v + tv_j\|} - \frac{c^j}{\|v\|} \right) v_j.$$

Since $\{v_j\}$ forms a basis, we see that

$$a_j \left(\frac{t + c^j}{\|v + tv_j\|} - \frac{c^j}{\|v\|} \right) = 0,$$

for each $1 \leq j \leq n$. Since the expression in the parentheses vanishes exactly when $f_j(t) = 0$, we see that it's nonzero for our chosen $t \in (0, \epsilon)$, and hence

$$a_1 = \cdots = a_n = 0,$$

as desired. $\square$

Proposition 3.3. *Fix $(x, \xi) \in SM$ and let $\gamma_{x,\xi} : [0, b] \to M$ be a distance minimizing geodesic segment. For any $a \in (0, b)$, let $z = \gamma_{x,\xi}(a)$ and $\zeta = \gamma'_{x,\xi}(a) \in S_z M$. Then there exists a basis $\{\eta_j \in T_z M : 1 \leq j \leq n\}$ of $T_z M$ and $\epsilon > 0$ such that for all $s \in (0, \epsilon)$, there is a neighborhood $U \subseteq M$ of x such that the function $F : U \to \mathbb{R}^n$,*

$$F(y) = (D_y(z, z_j) : 1 \leq j \leq n), \qquad z_j = \gamma_{z,\eta_j}(s),$$

is a smooth coordinate map.

Proof: Since $\gamma_{x,\xi}|_{[0,b]}$ is minimizing, $\gamma_{x,\xi}|_{[0,a]}$ has no cut points from x to z. Then there exists neighborhoods $U_x \subset M$ of x and $U_z \subset M$ of z such that $(p, q) \mapsto \mathrm{dist}(p, q)$ is smooth on U_x and U_z. Letting $v := a\xi$, we have that $d(\exp_x)_v$ is nonsingular. Choose vectors $v_1, v_2 \in T_x M$ as in Lemma 3.2 and complete to a basis $\{v_j\}$ of $T_x M$. Then again by Lemma 3.2, there exists $\delta > 0$ such that for any $t \in (0, \delta)$ that

$$\left\{ \frac{v + tv_1}{\|v + tv_1\|} - \frac{v}{\|v\|}, ..., \frac{v + tv_n}{\|v + tv_n\|} - \frac{v}{\|v\|} \right\}$$

forms a basis for $T_x M$.

Let $\eta_j = d(\exp_x)_v(v_j) \in T_z M$, and note that $\{\eta_j\}$ then forms a basis for $T_z M$ by the nondegeneracy of $d(\exp_x)_v$. Let

$$c_j(t) = \exp_x^{-1}(\gamma_{z,\eta_j}(t))$$

denote a curve in T_xM by possibly restricting U_z so that $\exp_x^{-1}|_{U_z}$ is a diffeomorphism. Then

$$\begin{aligned} c_j(0) &= \exp_x^{-1}(\gamma_{z,\eta_j}(0)) \\ &= \exp_x^{-1}(z) \\ &= a\xi \\ &= v, \end{aligned}$$

and

$$\begin{aligned} c_j'(0) &= \left.\frac{d}{dt}\right|_{t=0} (\exp_x^{-1}(\gamma_{z,\eta_j}(t))) \\ &= d(\exp_x^{-1})_z(\eta_j) \\ &= (d(\exp_x^{-1})_v)^{-1}(\eta_j) \\ &= v_j. \end{aligned}$$

Hence for fixed $1 \leq j \leq n$, the two curves in T_xM given by

$$t \mapsto \left(\frac{c_j(t)}{|c_j(t)|_g} - \frac{v}{|v|_g} \right), \qquad t \mapsto \left(\frac{v + tv_j}{|v + tv_j|_g} - \frac{v}{|v|_g} \right),$$

satisfy the same initial point

$$0 \mapsto 0$$

and the same initial velocity

$$0 \mapsto \frac{v_j |v|_g^2 - vg(v, v_j)}{|v|_g^3}.$$

Thus by considering the determinants obtained by the coefficients of the above vectors in the basis $\{v_j\}$, we see there exists $\epsilon > 0$ such that any $t \in (0, \epsilon)$, the vectors

$$\left\{ \frac{c_1(t)}{|c_1(t)|_g} - \frac{v}{|v|_g}, \ldots, \frac{c_n(t)}{|c_n(t)|_g} - \frac{v}{|v|_g} \right\}$$

are linearly independent.

I don't see why this is true. Linear algebra fact?

Letting

$$z_j = \gamma_{z,\eta_j}(s) = \exp_x(c_j(s))$$

as in the statement of the proof, we then see that

$$z_j = \gamma_{x,c_j(s)}(1) = \gamma_{x,\frac{c_j(s)}{|c_j(s)|_g}}(|c_j(s)|_g) = \gamma_{xz_j}(|c_j(s)|_g),$$

and hence

$$\gamma'_{x,\frac{c_j(s)}{|c_j(s)|_g}}(0) = \frac{c_j(s)}{|c_j(s)|_g}.$$

Thus

$$\begin{aligned}
\operatorname{grad}\left(D_{(\cdot)}(z,z_j)\right)_x &= \operatorname{grad}\left(\operatorname{dist}(z,\cdot) - \operatorname{dist}(z_j,\cdot)\right)_x \\
&= \gamma'_{zx}(\operatorname{dist}(z,x)) - \gamma'_{z_jx}(\operatorname{dist}(z_j,x)) \\
&= -\gamma'_{xz}(0) + \gamma'_{xz_j}(0) \\
&= -\frac{v}{|v|_g} + \frac{c_j(s)}{|c_j(s)|_g}.
\end{aligned}$$

As this is true for each $j \in \{1,...,n\}$, we conclude that F is regular at x, and hence by the Inverse Function Theorem, there exists an open neighborhood $U \subseteq M$ of x such that $F : U \to \mathbb{R}^n$ is a coordinate map. $\square$

These charts $\{(U,F)\}$ on M thus must form a compatible smooth structure on M.

Theorem 3.4. *The map $\mathcal{D} : M \to \mathcal{D}(M)$ is a diffeomorphism.*

Proof: Let $\{U_\alpha, F_\alpha\}$ denote the coordinate charts on M as defined above. Since $\mathcal{D}$ is a homeomorphism, $\{\mathcal{D}(U_\alpha)\}$ is an open cover of $\mathcal{D}(M)$ and consider the maps $F_\alpha \circ \mathcal{D}^{-1} : \mathcal{D}(U_\alpha) \to \mathbb{R}^n$. Then for any $U_\alpha \cap U_\beta \neq \emptyset$, we see that on this intersection that the maps between

$$(F_\alpha \circ \mathcal{D}^{-1}) \circ (F_\beta \circ \mathcal{D}^{-1})^{-1} = F_\alpha \circ F_\beta^{-1}$$

which are smooth, and thus determine a charts on $\mathcal{D}(M)$ turning $\mathcal{D}(M)$ into a smooth n-dimensional manifold.

Finally, fix some chart (U,F) on M and $(\mathcal{D}(U), F \circ \mathcal{D}^{-1})$ on $\mathcal{D}(M)$. Then on U, we have that

$$\mathcal{D} = (\mathcal{D} \circ F)^{-1} \circ F,$$

which is smooth, and hence D is smooth. Similarly, on $\mathcal{D}(U)$, we have that

$$\mathcal{D}^{-1} = F^{-1} \circ (F \circ \mathcal{D}^{-1}),$$

which is also smooth. Since $\mathcal{D}$ is smooth homeomorphism with smooth inverse, we conclude that $\mathcal{D}$ is a diffeomorphism. $\square$

For pregeodesics and geodesic equivalence and projective equivalence and such, cf. [4], [15], [2], [1], [16], [13], [6], [14], [3], [5].

4 Constructing a Connection from Pregeodesics

Let M be a connected smooth manifold of dimension $n \geq 2$. Suppose we are given a collection of curves $\{t \mapsto \gamma(t;\alpha) : \alpha \in A\}$ on M.

Goal: Does there exist a symmetric affine connection ∇ on M such that for each $\alpha \in A$, the curve $t \mapsto \gamma(t;\alpha)$ is a pregeodesic with respect to ∇?

We first note that if ∇ is a connection on M, then in coordinates $(U,(x^j))$, we have the Christoffel symbols $\Gamma^k_{ij} : U \to \mathbb{R}$ given by

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k.$$

Thus if we know Γ^k_{ij} with respect to $(U,(x^j))$, we know the connection $\nabla|_U$. Hence this is a local problem, and so we may assume $U \subset \mathbb{R}^n$ is small with our usual coordinates (x^j), and we've reduced the problem to the following.

New Goal: Does there exist a symmetric affine connection (now denoted) Γ^k_{ij} on U such that for each $\alpha \in A$, the curve $t \mapsto \gamma(t;\alpha)$ is a pregeodesic with respect to Γ^k_{ij}? Note that now the collection of curves lives in U.

We first specify that our curves in the collection each come with a domain, i.e., γ is in our collection if $\gamma = \gamma(t;\alpha)$ and $\gamma : (a,b) \to M$ for some $a \leq -\infty < b \leq \infty$. We let $I(\gamma)$ or $I(\alpha)$ denote the domain of γ.

We need our collection of curves to be "large". That is, for any $x \in U$, define the set of vectors

$$\Omega_x := \{\xi \in T_xU : \text{ there exists } \alpha \in A,\, t_0 \in I(\alpha) \text{ such that } \gamma'(t_0;\alpha) \propto \xi\}.$$

Our collection of curves will be large enough if each $x \in U$ the set Ω_x contains an open subset of T_xU. Set $\Omega = \bigcup_{x\in U}\Omega_x$. We call a pair $(t_0;\alpha)$, x_0-admissible if $\gamma'(t_0;\alpha) \in \Omega_{x_0}$.

4.1 Pregeodesics

Let M be a smooth manifold with symmetric connection ∇. Recall that a curve $\gamma : J \to M$ is a *geodesic* if γ satisfies the *geodesic equation*, $D_t\gamma' = 0$ which in local coordinates with Christoffel symbols Γ^k_{ij} is written

$$\frac{d^2\gamma^k}{dt^2} + \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma^k_{ij} = 0. \tag{4.1}$$ {eq:geoEqun}

A curve $\hat{\gamma} : I \to M$ is said to be a *pregeodesic* if it satisfies the *pregeodesic equation*

$$D_t\hat{\gamma}' = f(t)\hat{\gamma}',$$

for some continuous $f : I \to \mathbb{R}$. In coordinates, this reads

$$\frac{d^2\hat{\gamma}^k}{dt^2} + \frac{d\hat{\gamma}^i}{dt}\frac{d\hat{\gamma}^j}{dt}\Gamma_{ij}^k = f(t)\frac{d\hat{\gamma}^k}{dt}. \tag{4.2}$$

{eq:pregeoEqunGen}

We shall let s denote our affine parameters for geodesics and t our arbitrary parameters.

Proposition 4.1. *A curve $\hat{\gamma} : I \to M$ is a pregeodesic if and only if there exists a diffeomorphism $\phi : J \to I$ such that the curve $\gamma := \hat{\gamma} \circ \phi$ is a geodesic.*

Cf. Lemma 6.1.66 in
Also in the same source Theorem 8.4.16.

Proof: Suppose there exists a diffeomorphism $\phi : J \to I$ with $t = \phi(s)$ such that $\gamma = \hat{\gamma} \circ \phi$ is a geodesic. Then

$$\frac{d\gamma^k}{ds} = \frac{d\hat{\gamma}^k}{dt}\frac{d\phi}{ds},$$

and

$$\frac{d^2\gamma^k}{ds^2}(s) = \frac{d^2\hat{\gamma}^k}{dt^2}(\phi(s))(\phi'(s))^2 + \frac{d\hat{\gamma}^k}{dt}(\phi(s))\phi''(s).$$

Moreover, for $\psi : I \to J$, $\psi = \phi^{-1}$, we have that

$$\psi'(t) = \frac{1}{\phi'(\psi(t))},$$

and

$$\phi''(\psi(t)) = -(\phi'(\psi(t)))^3\psi''(t).$$

Now, since γ is a geodesic, we have that

$$\begin{aligned} 0 &= \frac{d^2\gamma^k}{ds^2} + \frac{d\gamma^i}{ds}\frac{d\gamma^j}{ds}\Gamma_{ij}^k \\ &= \left.\frac{d^2\hat{\gamma}^k}{dt^2}\right|_{\phi(s)}(\phi'(s))^2 + \left.\frac{d\hat{\gamma}^k}{dt}\right|_{\phi(s)}\phi''(s) + \left.\frac{d\hat{\gamma}^i}{dt}\right|_{\phi(s)}\left.\frac{d\hat{\gamma}^j}{dt}\right|_{\phi(s)}(\phi'(s))^2\Gamma_{ij}^k, \end{aligned}$$

or rather

$$0 = (\phi'(\psi(t)))^2 D_t\hat{\gamma}' + \phi''(\psi(t))\hat{\gamma}',$$

and simplifying

$$\begin{aligned} D_t\hat{\gamma}' &= -\frac{\phi''(\psi(t))}{(\phi'(\psi(t)))^2}\hat{\gamma}' \\ &= \phi'(\psi(t))\psi''(t)\hat{\gamma}', \end{aligned}$$

thus showing $\hat{\gamma}$ is a pregeodesic.

Conversely, suppose $D_t\hat{\gamma}' = f(t)\hat{\gamma}'$ for some continupus $f : I \to \mathbb{R}$. Then fix any $a \in I$ and define a function $g : I \to \mathbb{R}$ by

$$g(t) = \exp\left\{\int_a^t f(\lambda)d\lambda\right\}.$$

Then $g'(t) = f(t)g(t)$ which is continuous, so $g \in C^1(I)$. Moreover, fix any $b \in I$ and define $\psi : I \to \mathbb{R}$ by

$$\psi(t) = \int_b^t g(\tau)d\tau.$$

Then $\psi \in C^2(I)$ by construction, and $\psi'(t) = g(t) \neq 0$ for all $t \in I$, and so $\psi : I \to J$ is a diffeomorphism for some $J \subseteq \mathbb{R}$. Let $\phi := \psi^{-1} : J \to I$ with $t = \phi(s)$ and let $\gamma = \hat{\gamma} \circ \phi$. Then from our previous calculation

$$\begin{aligned} D_s\gamma' &= (\phi'(\psi(t)))^2 D_t\hat{\gamma}' + \phi''(\psi(t))\hat{\gamma}' \\ &= \hat{\gamma}'(f(t)\phi'(\psi(t))^2 + \phi''(\psi(t))) \\ &= \hat{\gamma}'(f(t)\phi'(\psi(t))^2 - (\phi'(\psi(t)))^3\psi''(t)) \\ &= \hat{\gamma}'(f(t)\phi'(\psi(t))^2 - (\phi'(\psi(t)))^3 f(t)\psi'(t)) \\ &= f(t)\hat{\gamma}'((\phi'(\psi(t)))^2 - (\phi'(\psi(t)))^2) \\ &= 0, \end{aligned}$$

thus showing that γ is a geodesic. $\square$

From the above proof, we note that for an affine change of parameter $s \mapsto as + b$ geodesics go to geodesics, that is, given a geodesic $\gamma(s)$, the curve $\gamma(as + b)$ is also a geodesic. However, $\gamma(t)$ is a pregeodesic, then from the above computation, if $\tilde{\gamma}(t) = \gamma(at + b)$, we get that

$$D_t\tilde{\gamma}' = a^2 D_t\gamma = a^2 f(t)\gamma' = af(t)\tilde{\gamma}'.$$

So $\tilde{\gamma}$ is still a pregeodesic, but satisfies a different pregeodesic equation with af instead of f.

We with impose further restrictions on f so that pregeodesics satisfy the same pregeodesic equation under affine changes in parameter. To this end, let $F : TM \to \mathbb{R}$ be a continuous function on the tangent bundle, that's homogeneous of degree 1, i.e.,

$$F(a\xi) = aF(\xi)$$

for any $a \in \mathbb{R}$, $\xi \in TU$, and consider the equation

$$D_t\gamma' = F(\gamma')\gamma'. \tag{4.3}$$ {eq:pregeoEqun}

If $\gamma : I \to \mathbb{R}$, then letting $f(t) = F(\gamma'(t))$, we see that $f : I \to \mathbb{R}$ is continuous and hence that γ is a pregeodesic. Moreover, by previous remarks, this equation is invariant under affine changes of parameter.

In particular, if $\xi \in T_xM$ is such that $\gamma'(t_0) = \xi$ for some pregeodesic γ which satisfies

$$D_t\gamma' = F(\gamma')\gamma',$$

for some F with the above properties. Then for any $a \in \mathbb{R} \setminus \{0\}$, the pregeodesic $\tilde{\gamma}(t) = \gamma(at)$ solves the same pregeodesic equation and satisfies

$$\tilde{\gamma}'(a^{-1}t_0) = a\gamma(t_0) = a\xi.$$

The relates our interlude of pregeodesics to our topic at hand. Namely, if $\gamma(t, \alpha)$ is in our collection of curves, we assume without loss of generality that all affine reparametrization of $\gamma(t; \alpha)$ are in the collection as well, since they solve the exact same pregeodesic equation, Equation (4.3).

Moreover, by this assumption on our collection of curves, if $\xi \in \Omega_x$, then $a\xi \in \Omega_x$ for all $a \neq 0$. Moreover, since we assume Ω_x contains an open subset of T_xU, we conclude that Ω_x contains a nonempty open double C_x.

Given a pregeodesic γ, can we always find such an F? Seems to be true, but I can't find a source for this. Check Levi-Civita's paper. Yes! If $\hat{\gamma}$ is a pregeodesic starting at $(x, v) \in TM$, and let $\gamma = \hat{\gamma} \circ \phi$ denote the geodesic starting (x, v) after reparametrization by ϕ. Then

$$f(\phi(s)) = -\frac{\phi''(s)}{(\phi'(s))^2},$$

and since $\phi(0) = 0$, define

$$F(v) = f(0).$$

4.2 Uniqueness of Connection up to Gauge Freedom

We first describe a gauge freedom on the problem. Suppose we have a symmetric affine connection Γ_{ij}^k on U, and let $\alpha \in \Omega^1(U)$ be any differential 1-form. Then define a new symmetric affine connection by

$$\overline{\Gamma}_{ij}^k = \Gamma_{ij}^k + \delta_i^k\alpha_j + \delta_j^k\alpha_i.$$

Suppose $\gamma : I \to U$ is a curve that satisfies the pregeodesic equation

$$\ddot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \Gamma^k_{ij} = f(\gamma')\dot{\gamma}^k,$$

for some prescribed continuous, homogeneous of degree 1 map $f : TU \to \mathbb{R}$. Then

$$\begin{aligned} f(\gamma')\dot{\gamma}^k &= \ddot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \Gamma^k_{ij} \\ &= \ddot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \overline{\Gamma}^k_{ij} - \dot{\gamma}^i \dot{\gamma}^j \delta^k_i \alpha_j - \dot{\gamma}^i \dot{\gamma}^j \delta^k_j \alpha_i \\ &= \ddot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \overline{\Gamma}^k_{ij} - 2\alpha(\gamma')\dot{\gamma}^k, \end{aligned}$$

and so γ solves the pregeodesic equation with respect to $(\overline{\Gamma}, \overline{f})$, where

$$\overline{f} = f + 2\alpha.$$

Thus any pregeodesic γ with respect to (Γ, f) is also a pregeodesic with respect to $(\overline{\Gamma}, \overline{f})$ for any 1-form α. This means our data cannot distinguish between connections up to gauge freedom.

We now show that such a gauge freedom is our only obstruction to uniqueness when such a connection exists. To this end, let change our thinking of the pregeodesic equation, and firstly, we shall only work at a fixed point $x_0 \in U$. Let $(t_0; \alpha)$ be an x_0-admissible curve, and consider the system of equations

$$\ddot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \Gamma^k_{ij}(x_0) = f(\gamma')\dot{\gamma}^k. \qquad (*)$$

As γ is part of our data, we wish to find Γ^k_{ij} and $f|_{\Omega_{x_0}}$ for which $(*)$ is satisfied. We note that if γ solves $(*)$ then infinitely many do as well, by the homogeneity of f and using infinitely many affine changes of parameter of γ.

Proposition 4.2. *For all $x_0 \in U$, suppose (Γ, f) and $(\overline{\Gamma}, \overline{f})$ solve $(*)$ for all x_0-admissible $(t_0; \alpha)$'s. Then there exists $\alpha \in \Omega^1(U)$ such that*

$$\overline{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta^k_i \alpha_j + \delta^k_j \alpha_i$$

and

$$\overline{f} = f + 2\alpha.$$

Proof: Fix $x_0 \in U$, and since (Γ, f) and $(\overline{\Gamma}, \overline{f})$ both solve $(*)$, we subtract the expressions to obtain

$$\hat{\Gamma}^k_{ij}(x_0)\dot{\gamma}^i \dot{\gamma}^j = \hat{f}(\gamma')\dot{\gamma}^k$$

for all x_0-admissible $(t_0; \alpha)$'s, where

$$\hat{\Gamma}^k_{ij} = \Gamma^k_{ij} - \overline{\Gamma}^k_{ij}, \qquad \hat{f} = f - \overline{f}.$$

In particular, we have that

$$\hat{\Gamma}^k_{ij} v^i v^j = \hat{f}(v) v^k$$

for all $v \in \Omega_{x_0}$.

For $k \in \{1, ..., n\}$, let $\sigma^k : T_{x_0}U \times T_{x_0}U \to \mathbb{R}$ denote the symmetric, bilinear form defined by

$$\sigma^k(u, v) = \hat{\Gamma}^k_{ij} u^i v^j.$$

In particular, σ^k satisfies the parallelogram law, i.e.,

$$0 = \sigma^k(u + v, u + v) + \sigma^k(u - v, u - v) - 2\sigma^k(u, u) - 2\sigma^k(v, v).$$

Hence, for $u, v \in \Omega_{x_0}$ with $u + v, u - v \in \Omega_{x_0}$ as well (which clearly exists by continuity and that Ω_{x_0} contains an open subset of $T_{x_0}U$), we have that

$$\begin{aligned} 0 &= \hat{f}(u+v)(u^k + v^k) + \hat{f}(u-v)(u^k - v^k) - 2\hat{f}(u)u^k - 2\hat{f}(v)v^k \\ &= \left(\hat{f}(u+v) + \hat{f}(u-v) - 2\hat{f}(u)\right) u^k + \left(\hat{f}(u+v) - \hat{f}(u-v) - 2\hat{f}(v)\right) v^k, \end{aligned}$$

for all $k = 1, ..., n$, and in particular, as a vectorial equation

$$0 = \left(\hat{f}(u+v) + \hat{f}(u-v) - 2\hat{f}(u)\right) u + \left(\hat{f}(u+v) - \hat{f}(u-v) - 2\hat{f}(v)\right) v.$$

Since Ω_{x_0} contains a nonempty open subset of $T_{x_0}U$, we may choose $u, v \in \Omega_{x_0}$ which are linearly independent. Hence we get the two equations

$$\begin{aligned} 0 &= \hat{f}(u+v) + \hat{f}(u-v) - 2\hat{f}(u), \\ 0 &= \hat{f}(u+v) - \hat{f}(u-v) - 2\hat{f}(v), \end{aligned}$$

which then yields

$$\hat{f}(u+v) = \hat{f}(u) + \hat{f}(v).$$

Let $V \subset \Omega_{x_0}$ denote this open subset for which on, the above calculations are performed. Then since $\hat{f}$ is homogeneous of degree 1, we conclude that $\hat{f}\Big|_V$ is linear, and hence there exists $\alpha \in T^*_{x_0}U$ such that $\hat{f}\Big|_V = 2\,\alpha|_V$. Since V is open, by linearly, we conclude that $\hat{f} = 2\alpha$ on $T_{x_0}U$, and hence that

$$f = \overline{f} + 2\alpha.$$

Now, define a new connection

$$\tilde{\Gamma}^k_{ij} = \overline{\Gamma}^k_{ij} + \delta^k_i \alpha_j + \delta^k_j \alpha_i.$$

Then for all x_0-admissible $(t_0; \alpha)$'s, we have that

$$\begin{aligned}
\ddot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \tilde{\Gamma}^k_{ij} &= \ddot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \overline{\Gamma}^k_{ij} + 2\alpha(\gamma')\dot{\gamma}^k \\
&= \ddot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \overline{\Gamma}^k_{ij} + \hat{f}(\gamma')\dot{\gamma}^k \\
&= \ddot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \overline{\Gamma}^k_{ij} + f(\gamma')\dot{\gamma}^k - \overline{f}(\gamma')\dot{\gamma}^k \\
&= f(\gamma')\dot{\gamma}^k \\
&= \ddot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \Gamma^k_{ij},
\end{aligned}$$

that is,

$$\tilde{\Gamma}^k_{ij} v^i v^j = \Gamma^k_{ij} v^i v^j$$

for all $v \in C_{x_0}$. Since this set is open, we have that

$$\tilde{\Gamma}^k_{ab} = \partial_{v^a} \partial_{v^b} (\tilde{\Gamma}^k_{ij} v^i v^j) = \partial_{v^a} \partial_{v^b} (\Gamma^k_{ij} v^i v^j) = \Gamma^k_{ab},$$

and hence that

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij}.$$

□

> This is missing a lot of details, and potentially needs more work. I think I'm missing why Teemu needed to add some of the details he did.

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