Derivatives and Fixed Income — Lecture Notes

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1 Motivation: Complex Ideas = Many Simple Ideas Combined

Concise version

Every "complicated" pricing result is built by stacking simple ideas:

1. Identify future cash flows.
2. Compute what they are worth today (present value).
3. If two strategies lead to the same cash flows, they must cost the same (no-arbitrage).

Narrative version

In finance, even the most intimidating formulas are built on simple building blocks. First, ask yourself: what money will change hands in the future? Then: what is that worth today? Finally: if there are two different ways to arrive at the same future payoff, their prices must line up, otherwise there is an arbitrage. Keeping this perspective prevents you from being overwhelmed by technical details.

2 Time Value of Money — Quick Refresher

Discrete compounding

Concise version

If you invest PV today at annual rate r, with m compounding periods per year for T years:

$$\mathrm{FV} = \mathrm{PV}\left(1 + \frac{r}{m}\right)^{mT}.$$

Example: $\mathrm{PV} = 1000$, $r = 5\%$, $m = 1$, $T = 3$:

$$\mathrm{FV} = 1000 \times (1 + 0.05)^3 = 1157.63.$$

Narrative version

Suppose you deposit €1000 at a 5% annual interest rate. If the interest is paid once per year, after one year you have €1050. After two years, interest also applies on the first year's interest, and so on. After three years, you end up with €1157.63. That extra €157.63 comes purely from letting your money grow over time.

Continuous compounding

Concise version

As $m \to \infty$:

$$\text{FV} = \text{PV}\, e^{rT}.$$

Narrative version

If instead of paying interest once a year, or once a month, the bank paid you in infinitely small steps, growth would be smooth and described by the exponential function e^{rT}. That's where the famous "e" comes from in finance: continuous compounding.

3 Forward Contracts: Definition and Intuition

Concise version

A forward is an agreement today to buy (long) or sell (short) at a fixed price in the future.

- Long gains if price rises.
- Short gains if price falls.

Narrative version

Imagine shaking hands today on a deal: in three months you will buy, or sell, something at a fixed price. You don't exchange anything today, just a promise. If the market price later moves in your favor, you gain; if it moves against you, you lose. One side's win is exactly the other's loss.

Worked Example (FX): GBP/USD

Contract: Buy £100,000 at $K = 1.36$ USD/GBP.

Case A: Market falls to $S_T = 1.34$

Concise version

$$\Delta = 1.34 - 1.36 = -0.02.$$

$$\text{P\&L} = 100{,}000 \times -0.02 = -2000 \text{ USD}.$$

Long loses $2000; short gains $2000.

Narrative version

You locked in a deal to buy pounds at 1.36. But when the contract matures, the market only requires 1.34 dollars per pound. That's two cents worse for you on every pound. With 100,000 pounds, that small difference multiplies into a $2000 loss. Meanwhile, your counterparty—the seller—makes exactly that $2000.

Case B: Market rises to $S_T = 1.38$

Concise version

$$\Delta = 1.38 - 1.36 = +0.02.$$

$$\text{P\&L} = 100{,}000 \times 0.02 = +2000 \text{ USD}.$$

Long gains $2000; short loses $2000.

Narrative version

Now imagine the market goes to 1.38. You still get to buy pounds at 1.36, which is cheaper than the market. For every pound, you gain two cents. Across 100,000 pounds, that's a tidy $2000 profit. Of course, the seller is losing the exact same amount.

General formula

Concise version

$$\text{P\&L} = \text{Quantity} \times (S_T - K) \times \text{Position},$$

with Position = +1 (long), -1 (short).

Narrative version

To avoid recalculating each case, we wrap everything into one formula. Just tell the formula whether you're long (+1) or short (-1), and it handles the signs automatically.

4 Futures vs Forwards

Concise version

- Forwards: private, risky (counterparty may default).
- Futures: exchange in the middle, collateral (margin), standardized.

Narrative version

Forwards are private promises: you and I agree, but it may be somehow easier for one of us to walking away. In other words, forward contracts entail more counterparty risk. Futures offer a solution to this: the exchange steps in as the trusted middleman. Both buyer and seller post collateral (called margin), like a deposit, and the exchange guarantees performance. Contracts are standardized and easy to trade.

5 Hedging with Futures: Cotton Example

Deal: 50,000 lbs of cotton at $0.66/lb.

Case A: Price falls to $0.64.

Concise version

$$\Delta = 0.64 - 0.66 = -0.02.$$

$$\text{Futures P\&L} = 50{,}000 \times -0.02 = -1000.$$

$$\text{Spot saving} = 50{,}000 \times (0.66 - 0.64) = +1000.$$

Net = 0 (hedged).

Narrative version

Suppose you're a T-shirt maker. You agree today to buy cotton at 66 cents in six months. If the market falls to 64 cents, your futures contract shows a \$1000 loss. But when you actually buy the cotton, it costs you \$1000 less than expected. One side cancels the other: your risk is neutralized.

Case B: Price rises to \$0.68.

Concise version

$$\Delta = 0.68 - 0.66 = +0.02.$$
$$\text{Futures P\&L} = 50{,}000 \times 0.02 = +1000.$$
$$\text{Extra cost} = 50{,}000 \times (0.68 - 0.66) = -1000.$$

Net = 0 (hedged).

Narrative version

Now imagine cotton rises to 68 cents. Your futures show a gain of \$1000. But in the real market, you pay \$1000 more for cotton. Again, the two effects offset. The futures protect you: your production costs are locked in.

6 Risk-Free Rate: Theory vs Reality

Concise version

In models: discount at risk-free rate r. In reality: nothing is perfectly risk-free. US Treasuries are just very low risk.

Narrative version

We often talk about the "risk-free" rate as if it were carved in stone. In practice, there is no such thing. Even governments can default, though some are much safer than others. Think of the risk-free rate as a practical shortcut: it's the lowest-risk rate available, not a truly riskless one.

7 Gold Forward Arbitrage

Given: Spot = \$3,683, Forward(1y) = \$3,850, $r = 3\%$.

Concise version

$$\text{FV of spot in 1y} \approx 3{,}683 \times 1.03 = 3{,}793.49 \;\Rightarrow\; \text{shown as } 3{,}793.$$

Arbitrage: Borrow 3,683, buy 1 oz gold, short forward at 3,850. One year later: Receive 3,850, repay 3,793. Profit = 3,850 − 3,793 = $\boxed{57}$.

Narrative version

Suppose gold costs \$3,683 today, but the one-year forward price is quoted at \$3,850. If you can borrow at 3%, you borrow \$3,683, buy an ounce of gold, and simultaneously agree to sell it forward at \$3,850. A year later, you deliver the gold, get \$3,850, and repay the loan of about \$3,793. The difference—\$57—is yours, essentially risk-free. That's arbitrage in action: aligning forwards and spots.

Key Insights

Concise version

- Pricing = present value of future cash flows.
- Long wins if price rises; short wins if price falls.
- P&L = Quantity × (S - K) × Position.
- Futures reduce counterparty risk with collateral.
- Hedging: buyers long futures, sellers short futures.

Narrative version

Every tool we saw boils down to the same idea: money in the future has a value today. Forwards and futures simply redistribute who gains when prices move. The formulas are compact, but the stories are intuitive: longs cheer when markets rise, shorts when they fall. Futures exchanges step in as referees, making sure promises are kept. And hedging? It's nothing more than using these contracts to sleep better at night.

Appendix A: Formula Sheet (Quick Reference)

Forwards/Futures P&L

$$\boxed{\text{P\&L} = \text{Quantity} \times (S_T - K) \times \text{Position}, \quad \text{Position} \in \{+1 \text{ (long)}, -1 \text{ (short)}\}}.$$

Spot–Forward (no-arbitrage), simple and continuous

If carry costs/dividends are ignored and r is the risk-free rate:

$$\boxed{F_0 \approx S_0(1+r)^T \quad \text{(discrete, per-year)} \qquad \text{or} \qquad F_0 = S_0\, e^{rT} \quad \text{(continuous)}}.$$

Time Value of Money

$$\text{FV}_{\text{disc}} = \text{PV}\left(1 + \frac{r}{m}\right)^{mT}, \qquad \text{FV}_{\text{cont}} = \text{PV}\, e^{rT}, \qquad \text{PV} = \frac{\text{FV}}{(1+r)^T} \ \text{ or } \ \text{FV}\, e^{-rT}.$$

Hedging Logic

- **Buyer of the good**: take **long futures** to protect against price *increases*.
- **Seller of the good**: take **short futures** to protect against price *decreases*.

Appendix B: Further Resources (Khan Academy)

Today's topics: forwards & futures

- Forward contract introduction
- Futures introduction
- Motivation for the futures exchange
- Futures margin mechanics

- Verifying hedge with futures margin mechanics
- Futures and forward curves

Review of first lecture (time value of money)

- The rule of 72 for compound interest
- e as a limit (interest and debt)
- Formula for continuously compounding interest
- Time value of money (present value)

For tomorrow's class (options & shorting)

- Call options intro (American finance & investing)
- Basic shorting
- American put options
- Call option as leverage
- Put vs. short and leverage
- Call payoff diagram
- Put payoff diagram
- Put as insurance