Yield Curve Construction: A Mathematical Guide

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Notation

- $P(t,T)$: price at time t of a ZCB paying 1 at T; $D(t,T) \equiv P(t,T)$ ("discount factor").
- $r(t)$: risk–free short rate; $y(t,T)$: (continuously compounded) spot yield for maturity T.
- $f(t,T)$: instantaneous forward rate; $f(t,T) = -\partial_T \ln P(t,T)$ and $P(t,T) = e^{-\int_t^T f(t,u)\,du}$.
- α_i: accrual fraction of period $[T_{i-1}, T_i]$; c: coupon rate; S: par swap rate.

1 Bootstrapping Methods

1.1 Par Curve Bootstrapping (Coupons/Bonds)

Given a set of coupon bonds with maturities $T_1 < \cdots < T_n$ and price B_k for maturity T_k,

$$B_k \;=\; \sum_{i=1}^{k-1} c\,\alpha_i\, D(t,T_i) \;+\; \big(1 + c\,\alpha_k\big)\, D(t,T_k). \tag{1}$$

Solve recursively for $D(t,T_k)$:

$$D(t,T_k) \;=\; \frac{B_k - \sum_{i=1}^{k-1} c\,\alpha_i\, D(t,T_i)}{1 + c\,\alpha_k}. \tag{2}$$

When issuing prices are quoted in YTM, convert to clean prices or equivalently to par coupon quotes.

1.2 On-the-Run Treasuries Bootstrapping

Select most liquid benchmarks; apply the same recursion. If only a sparse set exists, combine with bills (short end) and extrapolate smoothly between benchmarks (see §2).

1.3 Swap Curve Bootstrapping (Deposits, Futures/FRAs, Swaps)

Step 1: Short end (Deposits). For an overnight/term deposit over $[t, T_1]$ with simple rate L,

$$D(t,T_1) \;=\; \frac{1}{1 + L\,\alpha_1}. \tag{3}$$

Step 2: Forwards via FRAs/futures. If forward simple rates $L(T_{i-1}, T_i)$ (from FRAs or futures, with convexity adjustment if needed) are available, then

$$\frac{D(t,T_i)}{D(t,T_{i-1})} \;=\; \frac{1}{1 + L(T_{i-1},T_i)\,\alpha_i}, \qquad i = 2, \ldots, m. \tag{4}$$

Step 3: Swaps (par conditions). For a fixed–for–floating swap maturing at T_n (floating leg on tenor α_i), the par swap rate S_n satisfies

$$S_n \sum_{i=1}^{n} \alpha_i D(t,T_i) \;=\; 1 - D(t,T_n). \tag{5}$$

Given $D(t,T_1),\dots,D(t,T_{n-1})$, solve (5) for $D(t,T_n)$ and iterate along maturities. In the modern multi-curve setting, discounting is done on an OIS curve D_d while forwards come from a tenor–specific forwarding curve D_f; then the par condition becomes

$$S_n \sum_{i=1}^{n} \alpha_i D_d(t,T_i) \;=\; 1 - D_d(t,T_n), \tag{6}$$

with forward cashflows projected from *forwarding* rates consistent with D_f.

2 Interpolation and Smoothing

Let $\{T_k\}$ be bootstrapped nodes with discount factors $D_k = D(t,T_k)$. Interpolate either D, $\ln D$, y, or f.

2.1 Linear Interpolation

Piecewise linear on D. For $T \in [T_k, T_{k+1}]$,

$$D(T) \;=\; D_k + \frac{D_{k+1} - D_k}{T_{k+1} - T_k}\,(T - T_k). \tag{7}$$

Piecewise constant/linear on forwards f is also common: choose f constant on each bucket, then D integrates exactly.

2.2 Cubic Splines (Smoothing Splines)

Estimate a smooth function $g(T)$ (yield or log-discount) minimizing

$$\sum_j w_j \big(g(T_j) - \hat{g}_j\big)^2 \;+\; \lambda \int \big(g''(T)\big)^2 dT, \tag{8}$$

with a possibly time-varying roughness penalty $\lambda = \lambda(T)$ (VRP). Afterwards derive D or f from g.

2.3 B-Splines / Penalized Splines

Represent $g(T) = \sum_\ell \beta_\ell B_\ell(T)$ and penalize differences of coefficients (or integrated curvature). Ensures local control and stable fitting.

2.4 Monotone Convex (Hagan–West)

Interpolate the *forward* curve with a piecewise cubic that enforces monotonicity and shape-preserving convexity between nodes, avoiding spurious oscillations. Given node times T_k and forwards f_k, construct cubic segments with slope choices guaranteeing

$$f'(T_k) \text{ consistent with data}, \qquad D(T) = \exp\Big(-\int_t^T f(u)\,du\Big).$$

This yields arbitrage-free (non-negative) instantaneous forward rates under mild conditions and excellent locality.

2.5 Hermite (PCHIP)

Use piecewise cubic Hermite polynomials with data-dependent slopes to preserve monotonicity. Applied to D, $\ln D$, or f, PCHIP avoids overshoot while ensuring C^1 continuity.

3 Parametric Curve Fitting

3.1 Nelson–Siegel (NS)

Forward rate (time to maturity $m = T - t$):

$$f(m) \;=\; \beta_0 + \beta_1 \frac{1 - e^{-m/\tau_1}}{m/\tau_1} + \beta_2 \left(\frac{1 - e^{-m/\tau_1}}{m/\tau_1} - e^{-m/\tau_1} \right). \tag{9}$$

Spot yield follows by integrating f:

$$y(m) \;=\; \frac{1}{m} \int_0^m f(u)\,du = \beta_0 + \beta_1 \Big(\frac{1 - e^{-m/\tau_1}}{m/\tau_1} \Big) + \beta_2 \Big(\frac{1 - e^{-m/\tau_1}}{m/\tau_1} - e^{-m/\tau_1} \Big). \tag{10}$$

$\beta_0, \beta_1, \beta_2$ map to level, slope, curvature; τ_1 controls decay.

3.2 Svensson (Extended NS)

Add a second curvature term:

$$f(m) \;=\; \beta_0 + \beta_1 \frac{1 - e^{-m/\tau_1}}{m/\tau_1} + \beta_2 \left(\frac{1 - e^{-m/\tau_1}}{m/\tau_1} - e^{-m/\tau_1} \right) + \beta_3 \left(\frac{1 - e^{-m/\tau_2}}{m/\tau_2} - e^{-m/\tau_2} \right). \tag{11}$$

Gives more flexibility for medium/long maturities.

3.3 Smith–Wilson (Solvency II Extrapolation)

Given market instruments (cashflows $\boldsymbol{c}$ at maturities $\boldsymbol{u}$) and an ultimate forward rate (UFR) with long-run convergence parameter α, the discount function is

$$P(m) \;=\; e^{-\mathrm{UFR}\cdot m} \;+\; \boldsymbol{W}(m)^\top \boldsymbol{\zeta}, \qquad \boldsymbol{\zeta} = (\boldsymbol{W}\boldsymbol{C})^{-1} (\boldsymbol{p} - \boldsymbol{e}), \tag{12}$$

where $m = T - t$, $W(m,u)$ is the Wilson kernel

$$W(m,u) \;=\; e^{-\mathrm{UFR}(m+u)} \Big(\alpha \min(m,u) - e^{-\alpha \max(m,u)} \sinh(\alpha \min(m,u)) \Big), \tag{13}$$

$\boldsymbol{C}$ maps instrument prices to cashflows, $\boldsymbol{p}$ are observed prices, and $\boldsymbol{e}$ are the discounted cashflows under UFR alone. This enforces smooth convergence of forward rates to UFR beyond the last liquid point (LLP).

3.4 Exponential Polynomials

Fit $y(m) = \sum_{k=0}^{K} a_k m^k + \sum_{j=1}^{J} b_j e^{-\lambda_j m}$ (or analogous forward specification). Choose (K, J) to balance fit and parsimony; impose positivity/arbitrage constraints if working on f.

4 Stochastic / Model-Based Approaches

4.1 Short-Rate Models

Vasicek. $dr_t = \kappa(\theta - r_t)dt + \sigma\,dW_t$. Zero-coupon bond:

$$P(t,T) \;=\; \exp\Big(A(T-t) - B(T-t)\,r_t \Big), \quad B(\tau) = \frac{1 - e^{-\kappa\tau}}{\kappa}, \; A(\tau) = \Big(\theta - \frac{\sigma^2}{2\kappa^2} \Big) (B(\tau) - \tau) - \frac{\sigma^2}{4\kappa} B(\tau)^2. \tag{14}$$

Hull–White (extended Vasicek). $dr_t = \big(\theta(t) - \kappa r_t\big)dt + \sigma\, dW_t$; retains exponential-affine $P(t,T) = \exp\{A(t,T) - B(t,T)r_t\}$ with $B(t,T) = (1 - e^{-\kappa(T-t)})/\kappa$ and time-dependent A calibrated to today's curve.

CIR / Black–Karasinski. Mean-reverting specifications with positivity (CIR) or lognormal rates (BK) implying closed/semi-closed bond prices; parameters are calibrated to the initial $P(0,T)$.

4.2 HJM (Forward-Rate) Framework

Model $f(t,T)$ directly under risk-neutral measure:

$$df(t,T) = \mu(t,T)\,dt + \sum_{k=1}^{d} \sigma_k(t,T)\,dW_k(t), \qquad \mu(t,T) = \sum_{k=1}^{d} \sigma_k(t,T) \int_t^T \sigma_k(t,u)\,du \tag{15}$$

(no-arbitrage drift restriction). Choosing σ_k specifies the model; the initial curve $P(0,T)$ is matched exactly.

4.3 LIBOR Market Model (BGM/LMM)

For tenor dates $\{T_j\}$, forward LIBOR $L_j(t)$ for $[T_j, T_{j+1}]$ under $Q_{T_{j+1}}$:

$$dL_j(t) \;=\; \sigma_j(t)\, L_j(t)\, dW^{Q_{T_{j+1}}}(t), \tag{16}$$

(lognormal under its own forward measure). Under a common numéraire Q_{T_p}, drifts adjust via Girsanov:

$$dL_j(t) \;=\; L_j(t)\sigma_j(t)\, dW^{Q_{T_p}}(t) \pm L_j(t) \sum_k \frac{\delta L_k(t)}{1 + \delta L_k(t)} \sigma_j(t)\sigma_k(t)\rho_{jk}\, dt, \tag{17}$$

linking the entire family of forwards; caplets/swaptions recover Black-style prices.

5 Market-Specific Curves

5.1 Government Bond Curves

Estimate ZC curves from sovereign bonds/bills; common choices are Svensson/NS fits or smoothing splines with VRP. Illiquid points are filtered; callable/short residual maturities often excluded.

5.2 Swap Curves: OIS Discounting vs LIBOR Forwarding

Post–2007, collateralized trades are discounted on the overnight indexed swap (OIS) curve. For pricing a fixed–float IRS:

$$\text{NPV} \;=\; S \sum_i \alpha_i D_{\text{OIS}}(t, T_i) \;-\; \sum_i \alpha_i D_{\text{OIS}}(t, T_i)\, \mathbb{E}^{Q_{T_i}}\big[L^{(\text{tenor})}(T_{i-1}, T_i)\big], \tag{18}$$

with forwards generated from a tenor-specific forwarding curve (3M, 6M, ...). Multi-curve bootstrapping builds D_{OIS} and forwarding curves jointly using deposits, FRAs/futures, basis swaps, and IRS quotes.

5.3 Corporate Bond Curves

From government/swap curve plus spread: price $B = \sum c_i \alpha_i D(t, T_i) e^{-s(T_i)T_i}$. If a constant Z-spread is assumed, discount by $D(t, T_i)e^{-ZT_i}$; otherwise fit term-dependent $s(T)$ (e.g., spline on OAS).

5.4 Inflation-Linked Curves

Construct nominal P^N and real P^R curves from nominals and linkers (TIPS/ILBs). The breakeven inflation curve follows from

$$1 + \pi_{BE}(t, T) \approx \frac{(P^N(t,T))^{-1/T}}{(P^R(t,T))^{-1/T}}. \tag{19}$$

Interpolate P^R (or real forwards) with methods as above; price cashflows using real discounting and indexation mechanics.

Implementation Notes (Practical)

- **Choice of state space:** Interpolate $\ln D$ or f to preserve positivity of D and avoid arbitrage.
- **Locality vs smoothness:** Hagan–West/PCHIP give strong locality with shape control; smoothing splines give global smoothness with penalty tuning.
- **Multi-curve:** Calibrate OIS discounting first; then each forwarding curve using matching-tenor instruments (FRAs/futures, IRS, basis swaps).
- **Extrapolation:** Use Smith–Wilson (with LLP, UFR, α) when regulatory convergence to a long-run rate is required.