Derivatives and Fixed Income — Lecture Notes

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Lecturer: Alexandre Landi

Options and Black–Scholes Model

1 Vanilla Options: Calls and Puts

- Call = right to **buy**. Put = right to **sell**.
- Buyer: has rights. Seller: has obligations.
- Payoff at maturity:

$$\text{Call: } \max(S_T - K, 0), \quad \text{Put: } \max(K - S_T, 0)$$

- Profit = payoff − premium.
- Potential outcomes:

	Max Profit	Max Loss
Call Buyer	∞	Premium
Call Seller	Premium	∞
Put Buyer	$K - S_T -$ Premium	Premium
Put Seller	Premium	K

- Break-even:

$$\text{Call: } K + \text{Premium}, \quad \text{Put: } K - \text{Premium}$$

Options give the buyer a **right but not an obligation**. A call option allows the buyer to purchase an underlying asset at a fixed strike price, while a put allows selling at that strike. The seller, on the other hand, must fulfill the opposite obligation if the buyer exercises.

At maturity, the call is valuable if the underlying is above the strike, while the put is valuable if the underlying is below the strike. The buyer pays a premium upfront; this is the maximum they can lose. The seller receives the premium but faces potentially unlimited losses (for calls).

Think of a call buyer as having lottery-ticket–like upside, while the call seller is “short the lottery.”

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2 Determinants of Option Value

- Option value before maturity > payoff at maturity.
- Drivers of option price:
  1. S_0 = underlying spot price.
  2. K = strike.
  3. T = time to maturity.
  4. σ = volatility of returns.
  5. r = risk-free rate.
- More time $\Rightarrow$ higher value.
- Higher volatility $\Rightarrow$ higher value.
- Impact of r: raises call values, lowers put values.

Before expiration, options carry a "hope value" — the possibility that market moves may turn them profitable.

The main determinants of option price are:

- the stock price,
- the strike price,
- the time remaining,
- the volatility (uncertainty of future moves),
- and the interest rate.

In a nutshell:

- Time increases value because it gives more time for the option to have the opportunity to end up in the money.
- Volatility also boosts value, thanks to the asymmetry: downside is capped at zero, upside is unlimited.
- Finally, higher interest rates generally make calls more attractive and puts less so, due to discounting effects.

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3 Black–Scholes Model

European call option:

$$C = S_0 N(d_1) - K e^{-rT} N(d_2)$$

with

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

European put option (put–call parity):

$$P = K e^{-rT} N(-d_2) - S_0 N(-d_1).$$

Where:

- S_0 = spot price,
- K = strike,
- T = time to maturity (years),
- r = risk-free rate,
- σ = volatility,
- $N(\cdot)$ = CDF of standard normal.

$$\boxed{\text{Value rises with } T \text{ and } \sigma. \text{ Calls rise with } r\text{, puts fall with } r.}$$

The Black–Scholes model, developed by Fischer Black, Myron Scholes, and later extended by Robert Merton, gave finance its first analytical framework for valuing options.

The formula expresses an option's value as the difference between:

- the expected value of receiving the stock, adjusted by the probability of being in-the-money,
- and the discounted cost of paying the strike.

The parameters each have intuitive roles:

- S_0: the current asset price (higher makes calls worth more, puts less),
- K: the strike (the hurdle),
- T: more time means more optionality,
- σ: the volatility — the key driver, since more uncertainty increases potential upside,
- r: higher interest rates raise calls (discounting the strike more), and depress puts.

Black–Scholes compresses all market uncertainty into a single number: volatility. Traders invert the formula to extract the market's "implied volatility," which reveals what the market expects about future uncertainty.

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4 Worked Example (as of October 2, 2025)

Market data (Yahoo Finance, Oct 2, 2025):

- Underlying: Apple stock ($S_0 = 227.35$ USD)
- Strike: $K = 230$ USD
- Time to maturity: $T = 0.25$ years (3 months)
- Risk-free rate: $r = 5.25\%$
- Volatility: $\sigma = 22\%$

Step 1: Compute d_1 and d_2

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

$$\sigma\sqrt{T} = 0.22 \times \sqrt{0.25} = 0.11, \quad \ln\left(\tfrac{227.35}{230}\right) = -0.011589, \quad (r + \tfrac{1}{2}\sigma^2)T = 0.019175.$$

$$d_1 = \frac{-0.011589 + 0.019175}{0.11} = 0.069, \qquad d_2 = 0.069 - 0.11 = -0.041.$$

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Step 2: Price of the Call Option

$$C = S_0 N(d_1) - Ke^{-rT}N(d_2)$$

$$N(d_1) = 0.5275, \quad N(d_2) = 0.4836, \quad e^{-rT} = e^{-0.0525\times0.25} = 0.9870.$$

$$C = 227.35 \times 0.5275 - 230 \times 0.9870 \times 0.4836 = \boxed{10.14\,\text{USD}}.$$

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Step 3: Price of the Put Option

$$P = Ke^{-rT}N(-d_2) - S_0 N(-d_1)$$

$$N(-d_2) = 0.5164, \quad N(-d_1) = 0.4725.$$

$$P = 230 \times 0.9870 \times 0.5164 - 227.35 \times 0.4725 = \boxed{9.79\,\text{USD}}.$$

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$$\boxed{C \approx 10.14\,\text{USD}, \quad P \approx 9.79\,\text{USD}}$$

The call is slightly more expensive than the put, consistent with **put–call parity**:

$$C - P = S_0 - Ke^{-rT} = 227.35 - 230 \times 0.9870 = 0.35.$$

Using today's Apple stock as the underlying, a 3-month European call with strike 230 costs about \$10.14, while the corresponding put costs about \$9.79.

These prices are consistent with the Black–Scholes framework and current interest rates. Because the strike (230) is just above the current spot (227.35), the call is slightly out-of-the-money and has most of its value from time and volatility rather than intrinsic value.

At the same time, the relatively high short-term interest rate (5.25%) slightly favors the call, since the strike payment is discounted more heavily. Hence, both options trade at nearly the same price, with the call marginally more expensive.

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Appendix A: Formula Sheet

- Call payoff: $\max(S_T - K, 0)$
- Put payoff: $\max(K - S_T, 0)$
- Break-even: Call $=$ $K +$ Premium, Put $=$ $K -$ Premium
- Black–Scholes:

$$C = S_0 N(d_1) - K e^{-rT} N(d_2), \quad P = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

- $d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$