FX Quant Role: Barrier Options

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Context: Barrier options are path-dependent derivatives whose payoff depends on whether the underlying asset price reaches a predefined barrier level. In practice, double-barrier options, rebates, and first passage time probabilities play a crucial role in pricing and risk management.

Setup: Consider a foreign exchange rate S_t evolving under the risk-neutral measure as a Geometric Brownian Motion (GBM):

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t, \quad S_0 \in (L, H), \quad L < S_0 < H,$$

with maturity $T > 0$. A *double-barrier knock-out call* option is defined as follows:

- The option **knocks out** if S_t hits either the lower barrier L or the upper barrier H before time T.
- If no barrier is hit, the option pays $(S_T - K)^+$ at maturity.
- A rebate R is paid *instantly* if the upper barrier H is hit before expiry.

Problem Statement: Derive the pricing formula $V(S_0, 0)$ for this double-barrier knock-out call option using the following steps:

1. Apply the **log transformation** $X_t = \ln(S_t)$ to map the GBM dynamics to a Brownian motion with drift and reduce the option pricing PDE to a **heat equation** with *Dirichlet boundary conditions*.
2. Use the **reflection principle** and the **first passage time density** for Brownian motion with absorbing barriers to express the *probability of barrier breach*.
3. Express $V(S_0, 0)$ in closed-form when $L \to 0$, showing that the price admits an *infinite series representation* via the method of images.

Hint

For standard Brownian motion W_t, the first passage time density to a level a is given by:

$$f_{T_a}(t) = \frac{|a|}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right), \quad t > 0.$$

Challenge Extension: Analyze the *convergence rate* of the infinite series solution and discuss how *monitoring frequency* (discrete vs continuous) impacts the pricing accuracy of double-barrier options.