Partial Moments and Sigma Algebras: A Measure-Theoretic Reconciliation of Zeros and Measurability

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Abstract

Partial moments are fundamental in probability and finance, yet practical concerns persist about the role of zeros in computations. This note provides a measure-theoretic reconciliation: the σ-algebra structure ensures zeros are mathematically necessary, preserving expectation over the entire probability space. Empirical implementations should retain zeros to maintain probabilistic integrity.

1 Context and Motivation

Partial moments have been fundamental in decision theory since Bawa (1975) and Fishburn (1977), with widespread adoption in finance and risk management. However, applied discussions frequently question whether zeros in $\max(0, t - X)^n$ computations "dilute" or bias the moment estimation. This note provides a definitive measure-theoretic clarification.

Remark 1 (Practical Concern)**.** *Empirical researchers often observe that* $\max(0, t - x_i)^n$ *yields many zeros and question whether this artificially reduces the moment. We demonstrate this is a misunderstanding: zeros are mathematically necessary features, not computational artifacts.*

2 Probability Spaces and σ-Algebras

Definition 1 (Probability space)**.** *A probability space is a triple* $(\Omega, \mathcal{F}, P)$ *where* Ω *is a sample space,* $\mathcal{F}$ *is a* σ*-algebra on* Ω*, and* $P : \mathcal{F} \to [0, 1]$ *is a probability measure.*

A random variable $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ is measurable if $X^{-1}(B) \in \mathcal{F}$ for all Borel sets $B \in \mathcal{B}$. For any threshold q, the events $\{X > q\}$ and $\{X \leq q\}$ are measurable, guaranteeing well-defined Lebesgue integrals.

3 Partial Moments as Lebesgue Integrals

For $k \geq 0$ and $q \in \mathbb{R}$, define:

$$\mathrm{UPM}_k(q) = \int_{\{X>q\}} (X-q)^k \, dP = \mathbb{E}\left[(X-q)^k \mathbf{1}_{\{X>q\}}\right], \tag{1}$$

$$\mathrm{LPM}_k(q) = \int_{\{X\leq q\}} (q-X)^k \, dP = \mathbb{E}\left[(q-X)^k \mathbf{1}_{\{X\leq q\}}\right]. \tag{2}$$

These are expectations over all of Ω restricted by measurable indicators, inheriting measure-theoretic guarantees from $(\Omega, \mathcal{F}, P)$.

4 Zeros and Canonical Partial Moments

For $t \in \mathbb{R}$ and integer $n \geq 0$, the canonical LPM is:

$$\mathrm{LPM}_n(t) = \mathbb{E}\left[\max(0,\, t-X)^n\right] = \int_{\Omega} \max(0,\, t-x)^n \, dP(x). \tag{3}$$

Partitioning Ω yields:

$$\mathrm{LPM}_n(t) = \int_{\{x\leq t\}} (t-x)^n \, dP(x) \; + \underbrace{\int_{\{x>t\}} 0 \, dP(x)}_{=0}. \tag{4}$$

Zeros for $x > t$ reflect the probability mass of $\{X > t\}$ and ensure expectation spans the entire probability space.

Remark 2 (Excluding Zeros Changes the Estimand)**.** *Removing zero evaluations computes a conditional moment on $\{X \leq t\}$ multiplied by $P(X \leq t)$, not the canonical LPM. This breaks additivity and yields an estimand that is not the expectation over Ω.*

5 Empirical Estimation

Given a sample $x_1, \ldots, x_N$, the empirical LPM is:

$$\widehat{\mathrm{LPM}}_n(t) = \frac{1}{N} \sum_{i=1}^{N} \max(0,\, t-x_i)^n. \tag{5}$$

Zeros when $x_i > t$ are legitimate evaluations representing $\{X > t\}$ under the empirical measure.

Reproducible R Implementation

Using the `NNS` package (Viole, 2023):

```
set.seed(123)
x  <- rnorm(100, mean = 0, sd = 1)
t  <- mean(x)

# Sample Variance (base R):
var(x)
[1] 0.8332328


est_lpm <- NNS::LPM(degree = 2, target = t, variable = x)
est_upm <- NNS::UPM(degree = 2, target = t, variable = x)

# Sample Variance:
(est_lpm + est_upm) * (length(x) / (length(x) - 1))
[1] 0.8332328

# Population Adjustment of Sample Variance (base R):
var(x) * ((length(x) - 1) / length(x))
[1] 0.8249005

# Population Variance:
est_lpm + est_upm
[1] 0.8249005
```

The output demonstrates the mathematical relationship: the sum of LPM and UPM equals the population variance, while the bias-adjusted sum equals the sample variance. This validates that partial moments provide a complete decomposition of distributional moments when zeros are properly included.

6 Conclusion

The measure-theoretic perspective definitively resolves practical concerns: zeros in partial moment computations are not biases but necessary features that preserve expectation over Ω. The σ-algebra structure ensures mathematical integrity, while empirical implementations must retain zeros to maintain probabilistic correctness. This reconciliation validates standard computational practice and clarifies a frequent point of confusion in applied work.

References

- Bawa, V. S. (1975). Optimal rules for ordering uncertain prospects. *Journal of Financial Economics*, 2(1), 95–121.
- Fishburn, P. C. (1977). Mean-risk analysis with risk associated with below-target returns. *The American Economic Review*, 67(2), 116–126.
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