Derivatives and Fixed Income — Lecture Notes

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Lecturer: Alexandre Landi

Black–Scholes vs Monte Carlo

1 Recap of Option Pricing with Black–Scholes

The Black–Scholes model is considered a cornerstone of modern option pricing. It gives us a way to take market observables (spot price, strike, volatility, interest rate, time) and translate them into "fair" prices for European options in closed-form solutions.

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2),$$
$$P = Ke^{-rT} N(-d_2) - S_0 N(-d_1),$$

with

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T}.$$

Each input plays a clear role in the option's value:

- $T \uparrow \Rightarrow$ option value increases.
- $\sigma \uparrow \Rightarrow$ option value increases.
- $S_0 \uparrow \Rightarrow$ calls more valuable, puts less valuable.
- $r \uparrow \Rightarrow$ calls more valuable, puts less valuable.

In words: the more time you give an option, the more chances it has to end up profitable. The more volatile the market, the more valuable optionality becomes. And naturally, if the underlying asset goes up, calls benefit while puts lose. Interest rates tilt the value slightly in favor of calls.

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2 Implied vs. Historical Volatility

Volatility is the heartbeat of option pricing. But there are two different notions:

- **Historical volatility:** the measured standard deviation of past returns. It tells us how bumpy the road has been.
- **Implied volatility:** the forward-looking number hidden inside market option prices. Traders use the Black–Scholes formula in reverse: they ask, "What volatility must I plug in so the model price matches today's market price?"

Think of historical volatility as the rear-view mirror, and implied volatility as the windshield. One shows where we've been, the other hints at where the market expects us to go.

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3 Black–Scholes Assumptions

The elegance of the Black–Scholes formula comes at a price: a set of restrictive assumptions.

- Continuous trading (markets never close).
- Assets are perfectly divisible.
- Volatility σ is constant and strictly positive.
- Options are European (exercise only at maturity).
- No dividends, or constant continuous yield q.
- Returns $\sim \mathcal{N}(\mu, \sigma^2)$; prices are lognormal.
- No transaction costs.
- Constant risk-free rate r.
- The underlying follows a geometric Brownian motion.

Each assumption is mathematically convenient but unrealistic in practice. Markets close every evening, volatility clusters and shifts, investors face transaction costs, and many traded options are American style. This gap between theory and reality must be taken into account by practitioners.

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4 Geometric Brownian Motion (GBM)

At the heart of the model lies the idea that stock prices follow a geometric Brownian motion:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t,$$

where:

- dS_t: change in stock price,
- μ: drift (expected return),
- σ: volatility,
- dW_t: increment of Brownian motion ($\mathbb{E}[dW_t] = 0$, $\mathrm{Var}[dW_t] = dt$).

The intuition is straightforward. The drift term $\mu S_t dt$ reflects the "trend" — on average the asset is expected to grow at some rate. The stochastic term $\sigma S_t dW_t$ captures randomness: prices jitter around the trend because markets are noisy.

Normal vs. Lognormal

The model assumes returns are normal, but prices themselves are lognormal:

$$\begin{aligned}
\ln\left(\frac{S_t}{S_{t-1}}\right) &\sim \mathcal{N}(\mu, \sigma^2) && \text{(log-returns are normal)} \\
S_t &\sim \text{Lognormal}(\mu, \sigma^2) && \text{(prices are lognormal)}
\end{aligned}$$

This distinction is crucial: it prevents negative prices, something a normal distribution would allow.

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5 Analytical vs. Numerical Pricing Methods

Closed-form models like Black–Scholes are beautiful: they give an immediate answer. But the real world is messier, and sometimes we need brute-force numerical methods. Let's compare.

$$\begin{aligned}
\text{Analytical (Closed-Form):}\ & \text{e.g., Black–Scholes} \\
& \rightarrow \text{Simple plug-in formula} \\
& \rightarrow \text{Requires restrictive assumptions} \\
\text{Numerical (Simulation):}\ & \text{e.g., Monte Carlo} \\
& \rightarrow \text{Simulate many paths for } S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z},\ Z \sim N(0,1) \\
& \rightarrow \text{Compute payoff for each path} \\
& \rightarrow \text{Discount and average: } \ \text{Option Price} = e^{-rT}\frac{1}{N}\sum_{i=1}^{N}\text{Payoff}_i
\end{aligned}$$

In essence: analytical models are like calculators, fast and elegant but inflexible. Monte Carlo is like a simulation lab, slower but capable of handling almost any payoff or market assumption.

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6 Worked Example: European Call and Put (October 2, 2025)

Market Data (Yahoo Finance):

- $S_0 = 227.35$ USD (Apple stock)
- $K = 230$ USD
- $T = 0.25$ years
- $r = 5.25\%$
- $\sigma = 22\%$

Step 1: Compute d_1, d_2

General formulas:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

Substitution:

$$\begin{aligned} d_1 &= \frac{\ln(227.35/230) + (0.0525 + 0.5 \times 0.22^2) \times 0.25}{0.22\sqrt{0.25}} \\ &\approx -0.02 \\ d_2 &= d_1 - \sigma\sqrt{T} \\ &= -0.02 - 0.22 \times 0.5 \\ &\approx -0.13 \end{aligned}$$

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Step 2: Call Price

General formula:

$$C = S_0 N(d_1) - Ke^{-rT}N(d_2)$$

Substitution:

$$\begin{aligned} C &= 227.35\, N(-0.02) - 230e^{-0.0525\times 0.25}\, N(-0.13) \\ &\approx 10.28\,\text{USD} \end{aligned}$$

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Step 3: Put Price

General formula:

$$P = Ke^{-rT}N(-d_2) - S_0 N(-d_1)$$

Substitution:

$$\begin{aligned} P &= 229.70 \times 0.5517 - 227.35 \times 0.5080 \\ &\approx 11.25\,\text{USD} \end{aligned}$$

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$$\boxed{C \approx 10.28\,\text{USD}, \quad P \approx 11.25\,\text{USD}}$$

This tells us that with Apple stock trading at $227.35, the right to buy at 230 in three months costs about $10.28, while the right to sell at 230 costs about $11.25. The higher put price reflects the fact that this option is slightly in-the-money.

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7 Monte Carlo Simulation Example

Now let's check this with a numerical method. Monte Carlo simulates thousands of random future price paths and averages their payoffs.

Listing 1: Monte Carlo simulation for a European call option

```python
import numpy as np

def monte_carlo_call(S0, K, T, r, sigma, n_paths=100000):
    Z = np.random.standard_normal(n_paths)
    ST = S0 * np.exp((r - 0.5*sigma**2)*T + sigma*np.sqrt(T)*Z)
    payoffs = np.maximum(ST - K, 0)
    return np.exp(-r*T) * np.mean(payoffs)

price = monte_carlo_call(227.35, 230, 0.25, 0.0525, 0.22)
print(f"Monte Carlo Call Price: {price:.2f} USD")
```

If you run this, you'll see the simulated call price hovers close to the theoretical $10.28. The more paths you simulate, the closer it gets. This is a powerful way to handle exotic options or situations where closed-form formulas do not exist.

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Appendix A: Formula Sheet

- Call payoff: $\max(S_T - K, 0)$
- Put payoff: $\max(K - S_T, 0)$
- Black–Scholes:

$$C = S_0 N(d_1) - K e^{-rT} N(d_2), \quad P = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

- $d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$
- Monte Carlo pricing:

$$\text{Option Price} = e^{-rT} \frac{1}{N} \sum_{i=1}^{N} \text{Payoff}_i$$