Why Zeros in Partial Moments Do Not Bias the Measurement

Fred Viole

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Introduction

In the computation of canonical partial moments, such as lower partial moments (LPM) or upper partial moments (UPM), zeros arise when evaluating terms like $\max(0, t - x)^n$ for LPM, where t is the target value, x is an observation, and n is the degree. Some may worry that including these zeros—when $x \geq t$—"shrinks" or biases the moment's measurement. However, zeros are fundamental to the measure-theoretic framework, particularly in the context of the σ-algebra, and their inclusion is essential to preserve the integrity of the probability measure. Below, we explain why, emphasizing the σ-algebraic links.

Zeros and the σ-Algebraic Structure

The LPM of order n for a random variable X with respect to a target t is defined as the expectation:

$$\text{LPM}_n(t) = E[\max(0, t - X)^n] = \int_\Omega \max(0, t - x)^n \, dP(x),$$

where $(\Omega, \mathcal{F}, P)$ is a probability space, Ω is the sample space, $\mathcal{F}$ is a σ-algebra of measurable subsets, and P is the probability measure. The σ-algebra $\mathcal{F}$ defines the collection of measurable sets over which integration is performed, ensuring that events like $\{X < t\}$ and $\{X \geq t\}$ are measurable. The function $\max(0, t - x)^n$ is measurable with respect to $\mathcal{F}$, and the Lebesgue integral is taken over the entire space Ω.

The set Ω can be partitioned into measurable subsets defined by the σ-algebra, such as:

$$\Omega = \{x : x < t\} \cup \{x : x \geq t\},$$

where $\{x : x < t\}, \{x : x \geq t\} \in \mathcal{F}$. The LPM integral decomposes as:

$$\int_\Omega \max(0, t - x)^n dP(x) = \int_{\{x:x<t\}} (t - x)^n dP(x) + \int_{\{x:x\geq t\}} 0 dP(x).$$

The second integral evaluates to zero numerically but is defined over the measurable set $\{x : x \geq t\}$, which has probability $P(X \geq t)$. This set's contribution is essential to the expectation, as the σ-algebra ensures that all measurable subsets of Ω are accounted for in the probability measure.

Zeros in Empirical Computation

In practice, for a sample $\{x_1, x_2, \ldots, x_N\}$, the LPM is approximated as:

$$\text{LPM}_n(t) \approx \frac{1}{N} \sum_{i=1}^{N} \max(0, t - x_i)^n.$$

Here, the empirical measure assigns weight $1/N$ to each observation, reflecting the uniform weighting of the sample space. Zeros occur when $x_i \geq t$, corresponding to the measurable subset $\{x_i : x_i \geq t\}$. These zeros are valid evaluations of $\max(0, t - x_i)^n$ and represent the contribution of the corresponding measurable events in the empirical σ-algebra (the power set of the sample).

Zeros Do Not "Shrink" the Moment

The concern that zeros "shrink" the LPM arises because they reduce the numerical value of the average. However, this reduction is not a bias but a direct consequence of the probability measure over the σ-algebra. The LPM captures the expected shortfall across all of Ω, including the subset $\{x : x \geq t\}$ where no shortfall occurs. The zeros reflect the probability mass $P(X \geq t)$, ensuring the expectation respects the full structure of the σ-algebra.

For instance, if a dataset has many $x_i \geq t$, the LPM will be smaller, but this accurately represents the distribution's properties: fewer or smaller shortfalls. The σ-algebraic framework guarantees that all measurable subsets, including $\{x : x \geq t\}$, contribute to the expectation, making the inclusion of zeros a feature of the measure, not a distortion.

Excluding Zeros Disrupts the σ-Algebraic Framework

Excluding zeros from the computation (i.e., averaging only non-zero $\max(0, t - x_i)^n$) restricts the integral to the subset $\{x : x < t\}$, yielding a conditional moment:

$$E[(t - X)^n \mid X < t] \cdot P(X < t).$$

This is not the canonical LPM, as it ignores the measurable set $\{x : x \geq t\}$. The σ-algebra requires that the expectation be computed over all of Ω to maintain the additivity of the measure:

$$P(\Omega) = P(\{x : x < t\}) + P(\{x : x \geq t\}) = 1.$$

Excluding $\{x : x \geq t\}$ violates this property, altering the estimand and breaking the integrity of the probability space. In statistical comparisons, this leads to inconsistent moment definitions across groups, as the measurable sets being evaluated differ, undermining the validity of the analysis.

Zeros Are Valid Measurable Evaluations

Zeros from $\max(0, t - x)^n$, whether from $x = t$ or $x > t$, are valid evaluations of a measurable function over the σ-algebra. The set $\{x : x = t\}$ (if it has non-zero measure, e.g., in discrete distributions) and $\{x : x > t\}$ are both in $\mathcal{F}$, and their contributions are integral to the expectation. Excluding zeros arbitrarily discards measurable events, disrupting the mathematical structure of the LPM.

Conclusion

In canonical partial moments, zeros from $\max(0, t-x)^n$ do not bias or shrink the measurement—they are essential to the expectation defined over the entire σ-algebra $\mathcal{F}$. Including zeros ensures the LPM respects the probability measure, capturing the contributions of all measurable subsets, including $\{x : x \geq t\}$. Excluding zeros restricts the integral to a subset of the σ-algebra, altering the estimand and violating the measure-theoretic foundation of partial moments. This ensures that the LPM accurately reflects the distribution's properties without bias.