Homework #2 Answers

1. $$\underset{\sim}{A} = \underline{a} \otimes \underline{a} + \underline{e}_3 \otimes \underline{e}_3$$

$$\underline{a} = \underline{e}_1 - 2\underline{e}_2$$

(i) $$\underset{\sim}{A} = (\underline{e}_1 - 2\underline{e}_2) \otimes (\underline{e}_1 - 2\underline{e}_2) + \underline{e}_3 \otimes \underline{e}_3$$

$$\underset{\sim}{A} = \underline{e}_1 \otimes \underline{e}_1 - 2\underline{e}_1 \otimes \underline{e}_2 - 2\underline{e}_2 \otimes \underline{e}_1 + 4\underline{e}_2 \otimes \underline{e}_2 + \underline{e}_3 \otimes \underline{e}_3$$

$$\Rightarrow \quad \underset{\sim}{A} = [A] = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det[A] = 1(4-0) - (-2)(-2) + (0)(0)$$

$$= 4 - 4 = 0$$

Since $\det[A] = 0$, $\underset{\sim}{A}^{-1}$ does not exist.

(ii) $$\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = 0$$

$$\Rightarrow \quad \begin{bmatrix} 1-\lambda & -2 & 0 \\ -2 & 4-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)\begin{vmatrix} 4-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - (-2)\begin{vmatrix} -2 & 0 \\ 0 & 1-\lambda \end{vmatrix} + 0 = 0$$

$$(1-\lambda)(4-4\lambda-\lambda+\lambda^2-4) = 0$$

$$\lambda(1-\lambda)(\lambda-5) = 0$$

$$\Rightarrow \quad \lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 5$$

2 (i) Bases are orthonormal if (mutually perpendicular)

$$\underline{e}_i' \cdot \underline{e}_j' = 0$$

check: $\underline{e}_1' \cdot \underline{e}_2' = \left(\frac{1}{\sqrt{5}}\right)\left(-\frac{2}{\sqrt{6}}\right) + \left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{6}}\right) + 0\left(-\frac{1}{\sqrt{6}}\right)$

$$= 0 \quad \checkmark$$

$$\underline{e}_1' \cdot \underline{e}_3' = \left(\frac{1}{\sqrt{5}}\right)\left(\frac{-2}{\sqrt{30}}\right) + \left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{30}}\right) + 0\left(\frac{5}{\sqrt{30}}\right) = 0$$

$$\underline{e}_2' \cdot \underline{e}_3' = \left(\frac{-2}{\sqrt{6}}\right)\left(\frac{-2}{\sqrt{30}}\right) + \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{30}}\right) + \left(-\frac{1}{\sqrt{6}}\right)\left(\frac{5}{\sqrt{30}}\right) = 0$$

$\Rightarrow \quad \underline{e}_i' \cdot \underline{e}_j' = 0 \quad \Rightarrow \quad \underline{e}_i'$ are orthonormal.

2.(ii)

$$\underset{\sim}{e}'_i = M_{ij}\, \underset{\sim}{e}_j$$

$$\underset{\sim}{e}'_1 = M_{1j}\, \underset{\sim}{e}_j = M_{11}\, \underset{\sim}{e}_1 + M_{12}\, \underset{\sim}{e}_2 + M_{13}\, \underset{\sim}{e}_3$$

$$\underset{\sim}{e}'_2 = M_{2j}\, \underset{\sim}{e}_j = M_{21}\, \underset{\sim}{e}_1 + M_{22}\, \underset{\sim}{e}_2 + M_{23}\, \underset{\sim}{e}_3$$

$$\underset{\sim}{e}'_3 = M_{3j}\, \underset{\sim}{e}_j = M_{31}\, \underset{\sim}{e}_1 + M_{32}\, \underset{\sim}{e}_2 + M_{33}\, \underset{\sim}{e}_3$$

$$\Rightarrow \quad [M] = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ -2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \\ -2/\sqrt{30} & 1/\sqrt{30} & 5/\sqrt{30} \end{bmatrix}$$

3

(i) $$\underset{\sim}{D} = \underline{a} \otimes \underline{b} + \underline{b} \otimes \underline{a} + \underline{c} \otimes \underline{c}$$

$$\underset{\sim}{D} = a_i b_j\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j + b_i a_j\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j + c_i c_j\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

$$\underset{\sim}{D} = (a_i b_j + b_i a_j + c_i c_j)\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

$$\Rightarrow \quad D_{ij} = a_i b_j + b_i a_j + c_i c_j$$

$$\Rightarrow \quad \underset{\sim}{D} = \begin{bmatrix} 12 & -5 & 0 \\ -5 & -12 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

3.(i) $\underset{\sim}{F} = \underline{a} \otimes \underline{b} \otimes \underline{c}$

$= a_i b_j c_k (\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k) = F_{ijk} (\underline{e}_i \otimes \underline{e}_j \otimes \underline{e}_k)$

For $k = 1, 2$, $F_{ijk} = 0$.

Also, for $i = j = 3$, $F_{ijk} = 0$

Only non-zero terms are:

$\underset{\sim}{F} = a_1 b_1 c_3\, \underline{e}_1 \otimes \underline{e}_1 \otimes \underline{e}_3 + a_2 b_1 c_3\, \underline{e}_2 \otimes \underline{e}_1 \otimes \underline{e}_3$
$+ a_1 b_2 c_3\, \underline{e}_1 \otimes \underline{e}_2 \otimes \underline{e}_3 + a_2 b_2 c_3\, \underline{e}_2 \otimes \underline{e}_2 \otimes \underline{e}_3$

$\Rightarrow$ $\underset{\sim}{F} = (-12)\, \underline{e}_1 \otimes \underline{e}_1 \otimes \underline{e}_3 + (-8)\, \underline{e}_2 \otimes \underline{e}_1 \otimes \underline{e}_3$
$+ 18\, \underline{e}_1 \otimes \underline{e}_2 \otimes \underline{e}_3 + 12\, \underline{e}_2 \otimes \underline{e}_2 \otimes \underline{e}_3$

(3. ii) $D_{ii} = a_i b_i + a_i b_i + c_i c_i$

$= 12 - 12 + 4 = 4$

$F_{ii3} = a_i b_i c_3 = 3 \times 2 \times (-2) + 2 \times (-3) \times (-2)$
$= 0$

4 (i) Solve equations $x_i = x_i(X_R, t)$

Sum $x_2 + x_3 = e^{t/\tau}(X_2 + X_3)$

$$\Rightarrow \quad X_2 + X_3 = e^{-t/\tau}(x_2 + x_3) \quad \text{—} 1$$

Similarly $(X_2 - X_3) = e^{t/\tau}(x_2 - x_3) \quad \text{—} 2$

sum 1 + 2

$$\Rightarrow \quad X_2 = \frac{1}{2} x_2 \left(e^{-t/\tau} + e^{t/\tau}\right) + \frac{1}{2} x_3 \left(e^{-t/\tau} - e^{t/\tau}\right)$$

Plug into $x_3 = x_3(X_R, t)$

$$\Rightarrow \quad X_3 = \frac{1}{2} x_2 \left(e^{-t/\tau} - e^{t/\tau}\right) + \frac{1}{2} x_3 \left(e^{-t/\tau} + e^{t/\tau}\right)$$

and $X_1 = x_1$

$\rightarrow X_R(x_i, t)$.

(4.ii) Velocity in material description:

$$v_r(X_r, t) = \frac{\partial x_i(X_r, t)}{\partial t}$$

$$V_1 = 0$$

$$V_2 = \frac{e^{t/\tau}}{2\tau}(X_2 + X_3) - \frac{e^{-t/\tau}}{2\tau}(X_2 - X_3)$$

$$V_3 = \frac{1}{2\tau}e^{t/\tau}(X_2 + X_3) + \frac{1}{2\tau}e^{-t/\tau}(X_2 - X_3)$$

Velocity in spatial description:

Substitute X_1, X_2, X_3 with x_1, x_2, x_3 using the relations derived in the $X_r = X_r(x_i, t)$ relations derived in the previous part.

$$\Rightarrow \quad V_1 = 0$$

$$V_2 = \frac{1}{2\tau}(x_2 + x_3) - \frac{1}{2\tau}(x_2 - x_3) = \frac{x_3}{\tau}$$

$$V_3 = \frac{1}{2\tau}(x_2 + x_3) + \frac{1}{2\tau}(x_2 - x_3) = \frac{x_2}{\tau}.$$

5

$$f_i = \frac{Dv_i}{Dt} = \frac{\partial v_i(x_j,t)}{\partial t} + v_k \frac{\partial v_i(x_j,t)}{\partial x_k}$$

$$f_1 = \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3}$$

$$f_1 = 0 + 0 + (-Va^2 x_1 x_3)(Va^2 x_3) + (Vax_3)(-Va^2 x_1)$$

$$f_1 = -V^2 a^4 x_1 x_3^2 + V^2 a^3 x_2 x_3$$

$$f_2 = \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3}$$

$$f_2 = Va^2 x_2 x_3(-Va^2 x_3) + Vax_3(-Va^2 x_1)$$

$$f_2 = -V^2 a^4 x_2 x_3^2 - V^2 a^3 x_1 x_3$$

$$f_3 = \frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x_1} + v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3}$$

$$f_3 = (Vax_3)(Va) = V^2 a^2 x_3$$