Homework #6 - Answers

(1)

$$\underset{\sim}{E}(r,\theta,z) = \begin{bmatrix} a\sin\theta & \frac{a\cos\theta}{2} & 0 \\ \frac{a\cos\theta}{2} & a\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{T} = \lambda\, tr(\underset{\sim}{E})\underset{\sim}{I} + 2\mu \underset{\sim}{E}$$

$$tr(\underset{\sim}{E}) = 2a\sin\theta$$

$$\Rightarrow \quad \underset{\sim}{T} = \lambda a\sin\theta \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} a\sin\theta & \frac{a\cos\theta}{2} & 0 \\ \frac{a\cos\theta}{2} & a\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{T} = \begin{bmatrix} \lambda 2a\sin\theta + 2\mu a\sin\theta & \mu a\cos\theta & 0 \\ \mu a\cos\theta & 2\lambda a\sin\theta + 2\mu a\sin\theta & 0 \\ 0 & 0 & 2\lambda a\sin\theta \end{bmatrix}$$

$$\underline{t}^{(n)} = \underset{\sim}{T}\cdot\underline{n} \qquad \text{with} \quad \underline{n} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\underline{t}^{(n)} = \begin{Bmatrix} 2\lambda a\sin\theta + 2\mu\sin\theta \\ \mu a\cos\theta \\ 0 \end{Bmatrix}$$

(2)

2.a

$$u_1 = a(x_1^2 - 5x_2^2)$$

$$u_2 = 2a x_1 x_2$$

$$u_3 = 0$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \begin{bmatrix} 2x_1 a & -4a x_2 & 0 \\ -4a x_2 & 2a x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.b

Principal strains:

$$\det \begin{pmatrix} 2x_1 a - \lambda & -4a x_2 \\ -4a x_2 & 2a x_1 - \lambda \end{pmatrix} = 0$$

$$\Rightarrow (2a x_1 - \lambda)^2 - (4a x_2)^2 = 0$$

$$\Rightarrow (2a x_1 - \lambda)^2 = (4a x_2)^2$$

$$\Rightarrow 2a x_1 - \lambda = \pm 4a x_2$$

$$\Rightarrow \left.\begin{aligned} \lambda_1 &= 2a x_1 - 4a x_2 \\ \lambda_2 &= 2a x_1 + 4a x_2 \end{aligned}\right\} \rightarrow \text{Principal strains.}$$

2.c

$$\underline{\nabla} \cdot \underset{\sim}{T} + \rho \underset{\downarrow \atop 0}{\underline{b}} = \underline{0}$$

$$\frac{\partial T_{ij}}{\partial x_j} = 0$$

$$T_{ij} = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij}$$ ← Constitutive equation.

$$\Rightarrow T_{ij} = \lambda (4ax_1) \delta_{ij} + 2\mu E_{ij}$$

$$\Rightarrow \quad T_{11} = \lambda 4ax_1 \delta_{11} + 2\mu E_{11} = \lambda 4ax_1 + 2\mu(2x_1 a)$$

$$T_{11} = 4ax_1(\lambda + \mu)$$

$$T_{12} = \lambda 4ax_1 \delta_{12} + 2\mu E_{12} = 2\mu(-4ax_2) = -8\mu a x_2$$

$$T_{13} = 0$$

$$\frac{\partial T_{1j}}{\partial x_j} = 0 \Rightarrow \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = 0$$

$$\Rightarrow 4a(\lambda + \mu) - 8\mu a = 0$$

$$\lambda + \mu = 2\mu$$

$$\Rightarrow \boxed{\lambda = \mu = G}$$

Also, note that $E = \dfrac{G(3\lambda + 2G)}{\lambda + G}$

by using,

$\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$ and $\mu = G = \dfrac{E}{2(1+\nu)}$

$$\Rightarrow \quad E = \frac{G(3\lambda + 2G)}{\lambda + G} = \frac{G(3G + 2G)}{G + G} = 2.5\,G$$

(3)

3.a

$$E_{ij} = \begin{bmatrix} 0 & -\sqrt{3}\times 10^{-3} & 0 \\ -\sqrt{3}\times 10^{-3} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Volumetric deformation, $\Delta = E_{11} + E_{22} + E_{33}$

$$= -2\times 10^{-3}$$

$$\Rightarrow E_{22} = -2\times 10^{-3}$$

$$\Rightarrow \quad E_{ij} = \begin{bmatrix} 0 & -\sqrt{3} & 0 \\ -\sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3}$$

This indicates a plane strain situation.

Principal strains:

$$\det\begin{pmatrix} 0-\lambda & -\sqrt{3} \\ -\sqrt{3} & -2-\lambda \end{pmatrix} = 0 \quad \Rightarrow \lambda^2 + 2\lambda - 3 = 0$$

$$\Rightarrow \lambda_1 = 1$$

$$\lambda_2 = -3$$

$$\Rightarrow \quad E_1 = 1\times 10^{-3}$$

$$E_2 = -3\times 10^{-3} \quad \Rightarrow \quad E'_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \times 10^{-3}$$

3.2

To apply, $T_{ij} = \lambda \, tr(\underset{\sim}{E}) \delta_{ij} + 2\mu E_{ij}$

with $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)} = 20 \text{ MPa}$

$\mu = \dfrac{E}{2(1+\nu)} = 20 \text{ MPa}$, $tr(\underset{\sim}{E}) = -2\times10^{-3}$

Thus,

$$T_{ij} = \lambda E_{kk} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} 0 & -\sqrt{3} & 0 \\ -\sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \times 10^{-3}$$

$$= \left(\begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix} + 40 \begin{bmatrix} 0 & -\sqrt{3} & 0 \\ -\sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \underbrace{\times 10^{-3} \text{ MPa}}_{\text{KPa}}$$

$$T_{ij} = \begin{bmatrix} -40 & -40\sqrt{3} & 0 \\ -40\sqrt{3} & -120 & 0 \\ 0 & 0 & -40 \end{bmatrix} \text{kPa}$$

3.3 strain energy density:

$$W = \frac{1}{2} T_{ij} E_{ij} = \frac{1}{2}(T_1E_1 + T_2E_2 + T_3E_3)$$

$$W = \frac{1}{2}\left[(0)(1) + (-160\times10^{3})(-3\times10^{-3}) + (-40\times10^{3})(0)\right] = 240 \text{ Pa} \frac{m}{m} = 240 \frac{Nm}{m^2 m} = 240 \frac{J}{m^3}$$