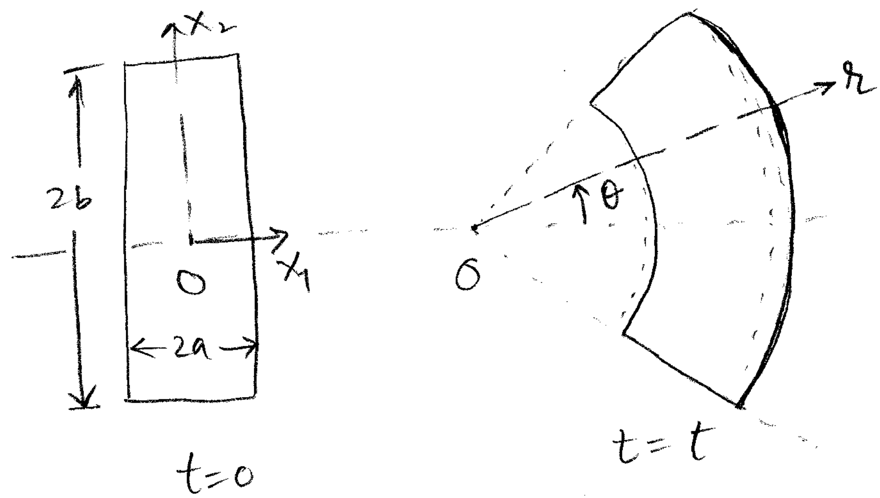
Incompressible Hyperelastic isotropic material:-

$$W = \frac{1}{2}\mu\left[\left(\frac{1}{2}+\beta\right)(I_1-3)+\left(\frac{1}{2}-\beta\right)(I_2-3)\right]$$

$$\left.\begin{array}{c}\mu > 0 \\ -\frac{1}{2} \le \beta \le \frac{1}{2}\end{array}\right] \rightarrow \text{material constants}$$

$$\Rightarrow \boxed{\underset{\sim}{T} = -p\,\underset{\sim}{I} + \mu\left(\frac{1}{2}+\beta\right)\underset{\sim}{B} - \mu\left(\frac{1}{2}-\beta\right)\underset{\sim}{B}^{-1}}$$

Bending of an incompressible isotropic rectangular bar



$$r_1 \le r \le r_2$$

$$-cb \le \theta \le cb$$

deformation of a rectangular bar into a curved bar can be described by:-

$$r = (2\alpha X_1 + \beta)^{1/2}, \quad \theta = cX_2, \quad z = X_3, \quad \alpha = 1/c$$

(X_1, X_2, X_3): Cartesian c.s. $\rightarrow$ reference configuration
(r, θ, z) : Cylindrical c.s. $\rightarrow$ current/spatial configuration

$X_1 = \pm a$ planes deform into surfaces $r = \sqrt{\pm 2\alpha a + \beta}$

$X_2 = \mp b$ " " " " $\theta = \mp cb$

Need to write $\underset{\sim}{B}$ in cylindrical coordinates

$\underset{\sim}{F}$ with respect to the current configuration $(\underline{e}_r, \underline{e}_\theta, \underline{e}_z)$ is:-

$$\underset{\sim}{F} = \begin{bmatrix} \dfrac{\partial r}{\partial X_1} & \dfrac{\partial r}{\partial X_2} & \dfrac{\partial r}{\partial X_3} \\[2ex] \dfrac{r\partial \theta}{\partial X_1} & \dfrac{r\partial \theta}{\partial X_2} & \dfrac{r\partial \theta}{\partial X_3} \\[2ex] \dfrac{\partial z}{\partial X_1} & \dfrac{\partial z}{\partial X_2} & \dfrac{\partial z}{\partial X_3} \end{bmatrix}$$

and,

$$\underset{\sim}{B} = \begin{bmatrix} B_{rr} & B_{r\theta} & B_{rz} \\ B_{\theta r} & B_{\theta\theta} & B_{\theta z} \\ B_{zr} & B_{z\theta} & B_{zz} \end{bmatrix}$$

$$B_{rr} = \underline{e}_r \cdot \underset{\sim}{F}\underset{\sim}{F}^T \underline{e}_r = \left(\frac{\partial r}{\partial X_1}\right)^2 + \left(\frac{\partial r}{\partial X_2}\right)^2 + \left(\frac{\partial r}{\partial X_3}\right)^2$$

$$B_{\theta\theta} = \underline{e}_\theta \cdot \underset{\sim}{F}\underset{\sim}{F}^T \underline{e}_\theta = \left(\frac{r\partial \theta}{\partial X_1}\right)^2 + \left(\frac{r\partial \theta}{\partial X_2}\right)^2 + \left(\frac{r\partial \theta}{\partial X_3}\right)^2$$

$$B_{zz} = \underline{e}_z \cdot \underset{\sim}{F}\underset{\sim}{F}^T \underline{e}_z = \left(\frac{\partial z}{\partial X_1}\right)^2 + \left(\frac{\partial z}{\partial X_2}\right)^2 + \left(\frac{\partial z}{\partial X_3}\right)^2$$

$$B_{r\theta} = \underline{e}_r \cdot \underset{\sim}{F}\underset{\sim}{F}^T \underline{e}_\theta = \left(\frac{\partial r}{\partial X_1}\right)\left(\frac{r\partial \theta}{\partial X_1}\right) + \left(\frac{\partial r}{\partial X_2}\right)\left(\frac{r\partial \theta}{\partial X_2}\right) + \left(\frac{\partial r}{\partial X_3}\right)\left(\frac{r\partial \theta}{\partial X_3}\right)$$

$$B_{rz} = \underset{\sim}{e}_r \cdot \underset{\sim}{F}\underset{\sim}{F}^T \underset{\sim}{e}_z = \left(\frac{\partial r}{\partial X_1}\right)\left(\frac{\partial z}{\partial X_1}\right) + \left(\frac{\partial r}{\partial X_2}\right)\left(\frac{\partial z}{\partial X_2}\right) + \left(\frac{\partial r}{\partial X_3}\right)\left(\frac{\partial z}{\partial X_3}\right)$$

$$B_{\theta z} = \underset{\sim}{e}_\theta \cdot \underset{\sim}{F}\underset{\sim}{F}^T \underset{\sim}{e}_z = \left(\frac{r\,\partial\theta}{\partial X_1}\right)\left(\frac{\partial z}{\partial X_1}\right) + \left(\frac{r\,\partial\theta}{\partial X_2}\right)\left(\frac{\partial z}{\partial X_2}\right) + \left(\frac{r\,\partial\theta}{\partial X_3}\right)\left(\frac{\partial z}{\partial X_3}\right)$$

Using these:-

$$\underset{\sim}{B} \equiv [B] = \begin{bmatrix} \alpha^2/r^2 & 0 & 0 \\ 0 & c^2 r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2/r^2 & 0 & 0 \\ 0 & r^2/\alpha^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{B}^{-1} \equiv \begin{bmatrix} r^2/\alpha^2 & 0 & 0 \\ 0 & 1/(c^2 r^2) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r^2/\alpha^2 & 0 & 0 \\ 0 & \alpha^2/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_1 = \frac{\alpha^2}{r^2} + \frac{r^2}{\alpha^2} + 1 = I_2$$

$$I_3 = \alpha^2 c^2 = 1$$

Using the generic constitutive equation for incompressible (hyperelastic) solid:-

$$\underset{\sim}{T} = -p\,\underset{\sim}{I} + 2W_1 \underset{\sim}{B} - 2W_2 \underset{\sim}{B}^{-1}$$

$$
\left.
\begin{aligned}
T_{rr} &= -p + 2W_1 \frac{\alpha^2}{r^2} - 2W_2 \frac{r^2}{\alpha^2} \\
T_{\theta\theta} &= -p + 2W_1 \frac{r^2}{\alpha^2} - 2W_2 \frac{\alpha^2}{r^2} \\
T_{zz} &= -p + 2W_1 - 2W_2 \\
T_{r\theta} &= T_{\theta z} = T_{rz} = 0 \\
W &= W(I_1, I_2)
\end{aligned}
\right] \quad \text{— (1)}
$$

Equilibrium equations in cylindrical coordinates:-

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{\partial T_{rz}}{\partial z} + \rho b_r = 0$$

$$\frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{T_{r\theta} + T_{\theta r}}{r} + \frac{\partial T_{\theta z}}{\partial z} + \rho b_\theta = 0$$

$$\frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r} + \rho b_z = 0$$

In the absence of body forces,
W as a function of 'r' only, (not θ, z),

$$
\left.
\begin{aligned}
&\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0 \\
&\frac{\partial T_{\theta\theta}}{\partial \theta} = 0 \\
&\frac{\partial T_{zz}}{\partial z} = 0
\end{aligned}
\right] \quad \text{— (2)}
$$

combining the previous two sets of equations (①, ②):-

$$\frac{\partial p}{\partial \theta} = 0 \quad , \quad \frac{\partial p}{\partial z} = 0$$

$$\Rightarrow \quad p = p(r)$$

$$\frac{dW}{dr} = \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial r} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial r} = \left(-\frac{2\alpha^2}{r^3} + \frac{2r}{\alpha^2}\right)(W_1 + W_2)$$

$$= -\frac{T_{rr} - T_{\theta\theta}}{r}$$

From the first eq^{b} eq^{n}:-

$$\Rightarrow \quad \frac{dT_{rr}}{dr} - \frac{dW}{dr} = 0$$

$$\Rightarrow \quad T_{rr} = W(r) + K$$

$$T_{\theta\theta} = r\frac{dT_{rr}}{dr} + T_{rr} = \frac{d(rT_{rr})}{dr} = \frac{d(rW)}{dr} + K$$

Boundary conditions are:-

$$T_{rr} = 0 \quad \text{at surfaces} \quad r = r_1, r_2$$

$$\Rightarrow \quad W(r_1) + K = 0 \quad , \quad W(r_2) + K = 0$$

$$\Rightarrow \quad W(r_1) = W(r_2)$$

Since, $W = W(I_1, I_2)$

and $I_1 = I_2 = \dfrac{\alpha^2}{r^2} + \dfrac{r^2}{\alpha^2} + 1$

$$\frac{\alpha^2}{r_1^2} + \frac{r_1^2}{\alpha^2} + 1 = \frac{\alpha^2}{r_2^2} + \frac{r_2^2}{\alpha^2} + 1$$

$$\Rightarrow \quad \alpha^2 = r_1 r_2$$

Since, $r_1 = \sqrt{-2\alpha a + \beta}$ & $r_2 = \sqrt{2\alpha a + \beta}$

$a = (r_2^2 - r_1^2)/4\alpha$ and $\beta = (r_1^2 + r_2^2)/2$

Normal force on the end planes $\theta = \pm Cb$ per unit length in z-direction :-

$$\int_{r_1}^{r_2} T_{\theta\theta}\, dr = \int_{r_1}^{r_2} \left(\frac{d(rW)}{dr} + K \right) dr = \Big[r\{W(r) + K\} \Big]_{r_1}^{r_2}$$

$$= 0$$

$\Rightarrow$ no resultant force on the end planes

Couple per unit width, M :-

$$M = \int_{r_1}^{r_2} r\, T_{\theta\theta}\, dr = \int_{r_1}^{r_2} \left(r \frac{d(rW)}{\partial r} + Kr \right) dr$$

$$M = \left[r^2 W(r)\right]_{r_1}^{r_2} - \int_{r_1}^{r_2} r\,W(r)\,dr + \left[\frac{kr^2}{2}\right]_{r_1}^{r_2}$$

$$= r_2^2 W(r_2) - r_1^2 W(r_1) - \int_{r_1}^{r_2} r\,W(r)\,dr + \frac{kr_2^2}{2} - \frac{kr_1^2}{2}$$

$$\Rightarrow \quad \boxed{M = \frac{k}{2}\left(r_1^2 - r_2^2\right) - \int_{r_1}^{r_2} r\,W(r)\,dr}$$

Torsion and Tension of an incompressible isotropic solid cylinder

The deformation is defined by the following configuration:

$$r = \lambda_1 R, \quad \theta = \Phi + KZ, \quad z = \lambda_3 Z$$

$(r, \theta, z) \longrightarrow$ spatial coordinates $t = t$

$(R, \Phi, Z) \longrightarrow$ material coordinates $t = 0$

$\Rightarrow$ Both in cylindrical coordinate systems.

λ_1: stretch in radial direction

λ_3: stretch in axial direction

Incompressibility $\Rightarrow I_3 = 1 \Rightarrow \lambda_1^2 \lambda_3 = 1$

Here, we need to derive $\underset{\sim}{B}$ for this combination of coordinate systems: cylindrical C.S. $\longrightarrow$ cylindrical C.S. for both reference and current configuration.

$$\underset{\sim}{F} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & \dfrac{\partial r}{R\partial \Phi} & \dfrac{\partial r}{\partial Z} \\[2ex] \dfrac{r\partial \theta}{\partial R} & \dfrac{r\partial \theta}{R\partial \Phi} & \dfrac{r\partial \theta}{\partial Z} \\[2ex] \dfrac{\partial z}{\partial R} & \dfrac{\partial z}{R\partial \Phi} & \dfrac{\partial z}{\partial Z} \end{bmatrix}$$

For $\underset{\sim}{B}$:-

$$B_{rr} = \left(\frac{\partial r}{\partial R}\right)^2 + \left(\frac{\partial r}{R\partial\phi}\right)^2 + \left(\frac{\partial r}{\partial Z}\right)^2$$

$$B_{r\theta} = \frac{r\partial\theta}{\partial R}\frac{\partial r}{\partial R} + \frac{r\partial\theta}{R\partial\phi}\frac{\partial r}{R\partial\phi} + \frac{r\partial\theta}{\partial Z}\frac{\partial r}{\partial Z}$$

$$B_{\theta\theta} = \left(\frac{r\partial\theta}{\partial R}\right)^2 + \left(\frac{r\partial\theta}{R\partial\phi}\right)^2 + \left(\frac{r\partial\theta}{\partial Z}\right)^2$$

$$B_{zz} = \left(\frac{\partial z}{\partial R}\right)^2 + \left(\frac{\partial z}{R\partial\phi}\right)^2 + \left(\frac{\partial z}{\partial Z}\right)^2$$

$$B_{rz} = \frac{\partial r}{\partial R}\frac{\partial z}{\partial R} + \frac{\partial r}{R\partial\phi}\frac{\partial z}{R\partial\phi} + \frac{\partial r}{\partial Z}\frac{\partial z}{\partial Z}$$

$$B_{z\theta} = \frac{\partial z}{\partial R}\frac{r\partial\theta}{\partial R} + \frac{\partial z}{R\partial\phi}\frac{r\partial\theta}{R\partial\phi} + \frac{\partial z}{\partial Z}\frac{r\partial\theta}{\partial Z}$$

$$\Rightarrow \quad \underset{\sim}{B} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_1^2 + r^2k^2 & rk\lambda_3 \\ 0 & rk\lambda_3 & \lambda_3^2 \end{bmatrix},$$

$$\underset{\sim}{B}^{-1} = \begin{bmatrix} 1/\lambda_1^2 & 0 & 0 \\ 0 & 1/\lambda_1^2 & -kr \\ 0 & -kr & \lambda_1^4 + \lambda_1^2 r^2 k^2 \end{bmatrix}$$

I_i of $\underset{\sim}{B}$:

$$I_1 = \frac{2}{\lambda_3} + r^2K^2 + \lambda_3^2$$

$$I_2 = 2\lambda_3 + \frac{1}{\lambda_3^2}(1 + \lambda_3 r^2 K^2)$$

$$I_3 = \lambda_1^4 \lambda_3^2 = 1$$

Note that I_i are function of 'r' only (not θ, z)

Thus, W and W_i will also depend on 'r' only.

Using the constitutive eq^n: $\underset{\sim}{T} = -p\underset{\sim}{I} + \varphi_1 \underset{\sim}{B} + \varphi_2 \underset{\sim}{B}^{-1}$

$\uparrow$ due to incompressibility

(Remember,
φ_i are scalar functions of I_i.
Thus, φ_1, φ_2 depend on 'r' only.)

$$T_{rr} = -p + \underbrace{\frac{\varphi_1}{\lambda_3} + \varphi_2 \lambda_3} = -p + \tau_{rr}$$

$$T_{\theta\theta} = -p + \underbrace{\varphi_1\left(\frac{1}{\lambda_3} + r^2K^2\right) + \varphi_2 \lambda_3} = -p + \tau_{\theta\theta}$$

$$T_{zz} = -p + \underbrace{\varphi_1 \lambda_3^2 + \frac{\varphi_2}{\lambda_3}\left(\frac{1}{\lambda_3} + r^2K^2\right)} = -p + \tau_{zz}$$

$$T_{\theta z} = K\lambda_3 r\left(\varphi_1 - \frac{\varphi_2}{\lambda_3}\right)$$

$$T_{rz} = T_{r\theta} = 0$$

Equations of equilibrium (same as in the last problem):

$b_i = 0$

$$\Rightarrow \begin{cases} \dfrac{\partial T_{rr}}{\partial r} + \dfrac{T_{rr} - T_{\theta\theta}}{r} = 0 \\ \dfrac{\partial T_{\theta\theta}}{\partial \theta} = 0 \\ \dfrac{\partial T_{zz}}{\partial z} = 0 \end{cases} \Rightarrow \frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial z} = 0$$

$$\Rightarrow \quad p = p(r) \qquad \text{(same as before)}$$

Total normal force on a cross-section plane:

$$N = \int_0^{r_0} T_{zz}\, 2\pi r\, dr$$

By rearranging T_{ij}:

$$2T_{zz} = (T_{rr} + T_{\theta\theta}) - \tau_{rr} - \tau_{\theta\theta} + 2\tau_{zz}$$

and from the first eqlb eqn (of 'r'):

$$T_{rr} + T_{\theta\theta} = 2T_{rr} + r\frac{\partial T_{rr}}{\partial r} = \frac{1}{r}\frac{d(r^2 T_{rr})}{dr}$$

$$\Rightarrow \quad 2T_{zz} = \frac{1}{r}\frac{d(r^2 T_{rr})}{dr} - \tau_{rr} - \tau_{\theta\theta} + 2\tau_{zz}$$

$$\Rightarrow \quad N = \underbrace{\int_0^{r_o} \frac{d}{dr}(r^2 T_{rr})\,dr}_{=0} + \pi \int_0^{r_o} (2\tau_{zz} - \tau_{rr} - \tau_{\theta\theta})\, r\, dr$$

$T_{rr}(r_o) = 0$

$$N = \pi \int_0^{r_o} (2\tau_{zz} - \tau_{rr} - \tau_{\theta\theta})\, r\, dr$$

Plugging in τ_{ii},

and, $\quad r\,dr = \lambda_1^2 R\,dR = R\,dR/\lambda_3 \quad , \quad R_o = r_o/\lambda_1$

$r = \lambda_1 R \qquad \lambda_1^2 \lambda_3 = 1$

$$\boxed{\Rightarrow \quad N = 2\pi\left(\lambda_3 - \frac{1}{\lambda_3^2}\right)\int_0^{R_o}\left(\varphi_1 - \frac{\varphi_2}{\lambda_3}\right) R\,dR - \frac{\pi k^2}{\lambda_3^2}\int_0^{R_o}\left(\varphi_1 - \frac{2\varphi_2}{\lambda_3}\right) R^3\,dR}$$

Similarly, the twisting moment 'M' can be calculated:

$$M = \int_0^{r_o} r\, T_{\theta z}\, 2\pi r\, dr = \frac{2\pi K}{\lambda_3}\int_0^{R_o}\left(\varphi_1 - \frac{\varphi_2}{\lambda_3}\right) R^3\,dR$$

if K (angle of twist) is very small, ($K \to 0$)

$$I_1 \simeq \frac{2}{\lambda_3} + \lambda_3^2 \,, \qquad I_2 \simeq 2\lambda_3 + \frac{1}{\lambda_3^2}$$

↳ independent of R'

$\Rightarrow$ φ_1, φ_2 are also independent of R'.

$$\Rightarrow \quad \boxed{N = \pi R_0^2 \left(\lambda_3 - \frac{1}{\lambda_3^2}\right)\left(\varphi_1 - \frac{\varphi_2}{\lambda_3}\right) + O(K^2)}$$

and,

$$\boxed{M = \frac{K \pi R_0^4}{2\lambda_3}\left(\varphi_1 - \frac{\varphi_2}{\lambda_3}\right)}.$$

for $K \to 0$,

$$\boxed{\frac{M}{K} = \frac{R_0^2}{2} \frac{N}{\left(\lambda_3^2 - \frac{1}{\lambda_3}\right)}}$$

↓

Rivlin's universal relation that gives:

* torsional stiffness as a function of stretch in the axial direction.

$\Rightarrow$ simple extension experiment yields torsional stiffness given small angle of twist.