Homework - 5

(1) $\underline{n}^{(BC)} = [0 \quad 1 \quad 0]$

$\underline{t}^{(BC)} = [0 \quad 0 \quad 0]$

$$t_i^{(BC)} = T_{ij} n_j^{(BC)} \Rightarrow \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} T_{12} \\ T_{23} \\ T_{32} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$\underline{n}^{(OB)} = [-1 \quad 0 \quad 0]$

$t_i^{(OB)} = [\rho_a g (h - x_2) \quad 0 \quad 0]$

$$\Rightarrow \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{Bmatrix} -1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -T_{11} \\ -T_{21} \\ -T_{31} \end{Bmatrix} = \begin{Bmatrix} \rho_a g (h - x_2) \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad T_{11} = -\rho_a g (h - x_2)$$

$$T_{21} = 0 \; ; \; T_{31} = 0$$

On face AC, $\underline{n}^{(AC)} = [1 \quad 0 \quad 0]$

$$t_i^{(AC)} = [0 \quad 0 \quad 0]$$

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} T_{11} \\ T_{21} \\ T_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\frac{D}{Dt}\int_V \Phi(\underline{x},t)\, dV = \int_V \left[\frac{D}{Dt}\Phi(\underline{x},t) + \Phi(\underline{x},t)\frac{\partial v_p}{\partial x_p}\right] dV$$

$$\Phi = \rho P_{ij\ldots}$$

$$\Rightarrow \frac{D}{Dt}\int_V \rho P_{ij\ldots}\, dV = \int_V \left[\frac{D}{Dt}(\rho P_{ij\ldots}) + \rho P_{ij\ldots}\frac{\partial v_p}{\partial x_p}\right] dV$$

$$= \int_V \left[\rho \frac{D}{Dt} P_{ij\ldots} + P_{ij\ldots}\frac{D\rho}{Dt} + \rho P_{ij\ldots}\frac{\partial v_k}{\partial x_k}\right] dV$$

$$= \int_V \left[\rho \frac{D}{Dt} P_{ij\ldots} + P_{ij\ldots}\underbrace{\left(\frac{D\rho}{Dt} + \rho\frac{D v_k}{D x_k}\right)}_{\to\, 0 \text{ (Equation of mass continuity)}}\right] dV$$

$$\Rightarrow \frac{D}{Dt}\int_V \rho P_{ij\ldots}\, dV = \int_V \left[\rho\frac{DP_{ij\ldots}}{Dt}\right] dV$$

(3)

Using the principle of conservation of linear momentum and equation of motion:-

$$\underline{\nabla} \cdot \underset{\sim}{T} + \rho \underline{b} = \rho \dot{\underline{v}} = \rho \underline{a}$$

$$\rho \underline{b} = \rho \underline{a} - \underline{\nabla} \cdot \underset{\sim}{T}$$

$$\underline{a} = \frac{\partial \underline{v}(\underline{x}, t)}{\partial t} + \frac{\partial \underline{v}(\underline{x}, t)}{\partial \underline{x}} \cdot \frac{\partial \underline{x}}{\partial t}$$

$$\Rightarrow \quad a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}$$

where,

$$\frac{\partial v_i}{\partial t} = \begin{Bmatrix} 0 \\ x_2^2 \\ x_2 x_3 \end{Bmatrix} ; \quad \frac{\partial v_i}{\partial x_j} = \begin{bmatrix} x_3 & 0 & x_1 \\ 0 & 2x_2 t & 0 \\ 0 & x_3 t & x_2 t \end{bmatrix}$$

Thus,

$$a_i = \begin{Bmatrix} 0 \\ x_2^2 \\ x_2 x_3 \end{Bmatrix} + \begin{bmatrix} x_3 & 0 & x_1 \\ 0 & 2x_2 t & 0 \\ 0 & x_3 t & x_2 t \end{bmatrix} \begin{Bmatrix} x_1 x_3 \\ x_2^2 t \\ x_2 x_3 t \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ x_2^2 \\ x_3 x_2 \end{Bmatrix} + \begin{Bmatrix} x_1 x_3^2 + x_1 x_2 x_3 t \\ 2 x_2^3 t \\ x_3 x_2^2 t^2 + x_2^2 x_3 t^2 \end{Bmatrix}$$

$$\Rightarrow \quad a_i = \begin{Bmatrix} x_1 x_3^2 + x_1 x_2 x_3 t \\ x_2^2 + 2x_2^3 t \\ x_2 x_3 + x_3 x_2^2 t^2 + x_2^2 x_3 t^2 \end{Bmatrix}$$

$$\nabla \cdot \underset{\sim}{T} \Rightarrow$$

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = \alpha(x_2 - x_3)$$

$$\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} = \alpha(2x_2)$$

$$\frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} = \alpha(2x_3 - 1)$$

Now, use $\rho \underline{b} = \rho \underline{a} - \nabla \cdot \underset{\sim}{T}$

$$\rho b_i = \rho a_i - \frac{\partial T_{ij}}{\partial x_j}$$

$$\Rightarrow \rho b_i = \rho \begin{Bmatrix} x_1 x_3^2 + x_1 x_2 x_3 t \\ x_2^2 + 2x_2^3 t \\ x_2 x_3 + x_3 x_2^2 t^2 + x_2^2 x_3 t^2 \end{Bmatrix} - \alpha \begin{Bmatrix} x_2 - x_3 \\ 2x_2 \\ 2x_3 - 1 \end{Bmatrix}$$