# MEMS 5703

# ANALYSIS OF ROTARY-WING SYSTEMS

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## Comment on *The Iliad*

by Robert Fagles, translator

Troy will fall to the Achaeans, to become the pattern for all time of the death of a city. The images of that night assault will be stamped indelibly on the consciousness of the Greeks throughout their history, immortalized in lyric poetry, in tragedy, on temple pediments, and painted vases—to reinforce the stern lesson of Homer's presentation of the war that:

**No civilization, no matter how rich, no matter how refined, can long survive once it loses the power to meet force with equal or superior force.**

# Part I

# Rotor Basics

pp. 1-66

## OUTLINE

I. Introduction

- A. How a Helicopter Works
- B. Modelling Issues

II. Static Momentum Theory/Blade-Element

III. Rigid Blade Flapping

- A. Equations Hover and Forward Flight
- B. Forced Response
- C. Transients

IV. Advanced Methods

- A. Genralized Harmonic Balance
- B. Floquet Theory
- C. Extended Dynamic Inflow

V. Flap-Lag Dynamics With Body DOF

- A. Flap-Lag Stability
- B. Multi-Blade Coordinates and Rotor/Body Coupling
- C. Ground Resonance and Whirl

VI. Elastic Blade Equations

- A. Flap-Lag-Torsion
- B. Energy Methods

VII. Advanced Aerodynamic Models for Aeroelasticity

- A. Dynamic Stall
- B. Linear, Unsteady Aero.

## REFERENCES

| **Topic** | **Gessow & Meyers 1952** | **Bramwell 1976** | **W. Johnson 1980** | **W. Johnson 2013** |
|---|---|---|---|---|
| Introduction | 1, 2 | 1, 2 | 1 | 1, 2 |
| Momentum Theory | 3 | 3.1 | 2.1 – 2.3 | 3.1 – 3.4, 4.1 – 4.5 |
| Blade-Element Theory | 3, 4 | 3.4 | 2.4 – 2.5 | 3.5 |
| Tip Loss | pp. 72-75 | pp. 110-116 | pp. 58-61, 81-88 133-134 | 3.6, 4.8 |
| Lifting Line | ----- | ---- | 10.1-10.5 | 3.7 |
| Efficiency | 5 | 6 | 2.6.5 – 2.6.10 | 3.8 – 3.9 |
| Rigid-Blade Flapping | 8 | 2, 5 | 5 | 6.1 – 6.17 |
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| Flap-Lag | ---- | 2.8 – 2.9, 11.2 | 9.3, 11.3, 12.3 | 6.20, 20.2 |
| Floquet Theory | ---- | 2.6 – 2.7 | 8, 12.1 | 15.6 – 15.7 |
| Elastic Blade | ---- | 9 | 9.8 – 9.9 | 17 |
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| Dynamic Inflow | ---- | ---- | 10.6 | 11 |
| Stability and Control | ---- | 7 (pp. 212-229) | 13 (pp. 778-821) | 21 |
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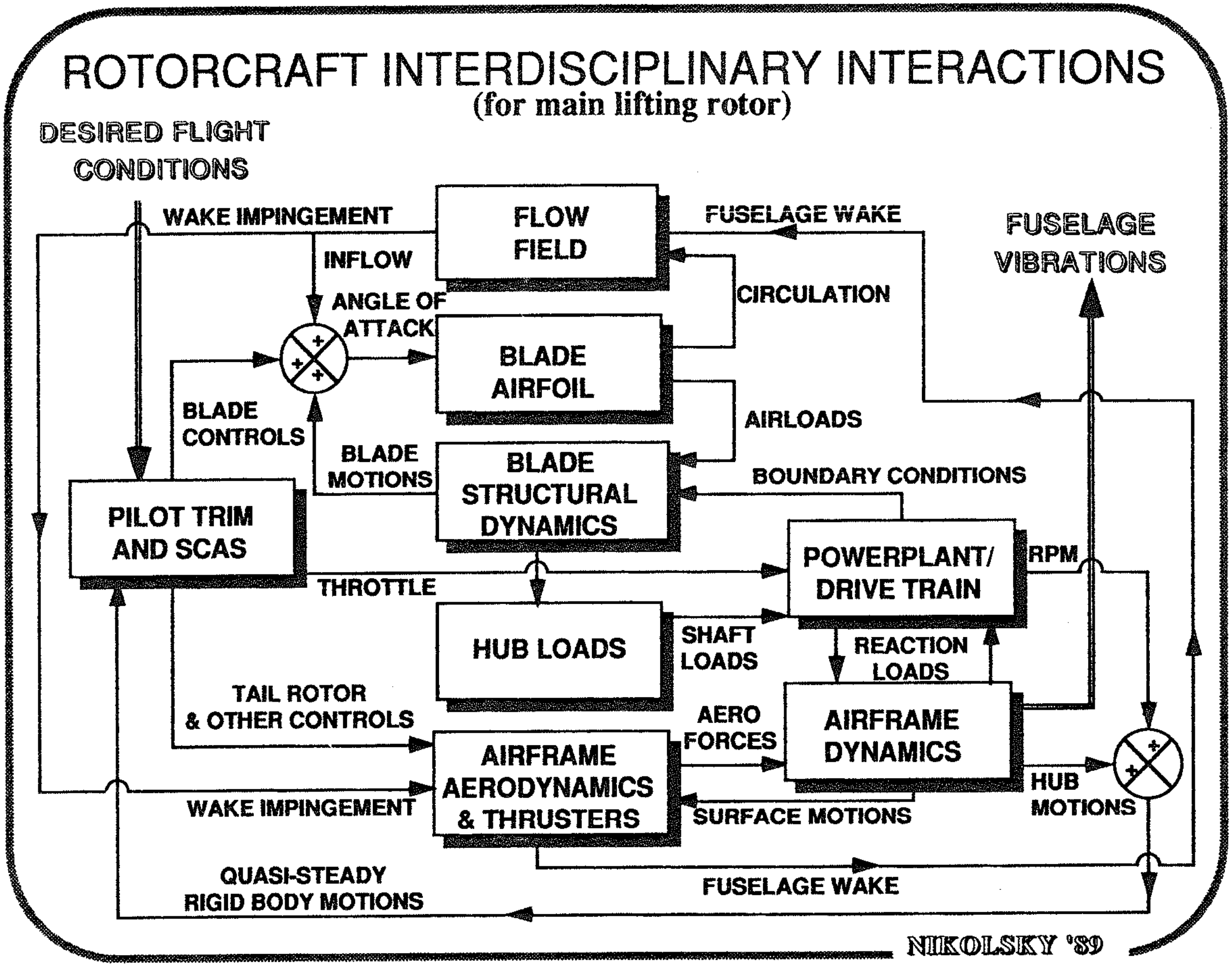
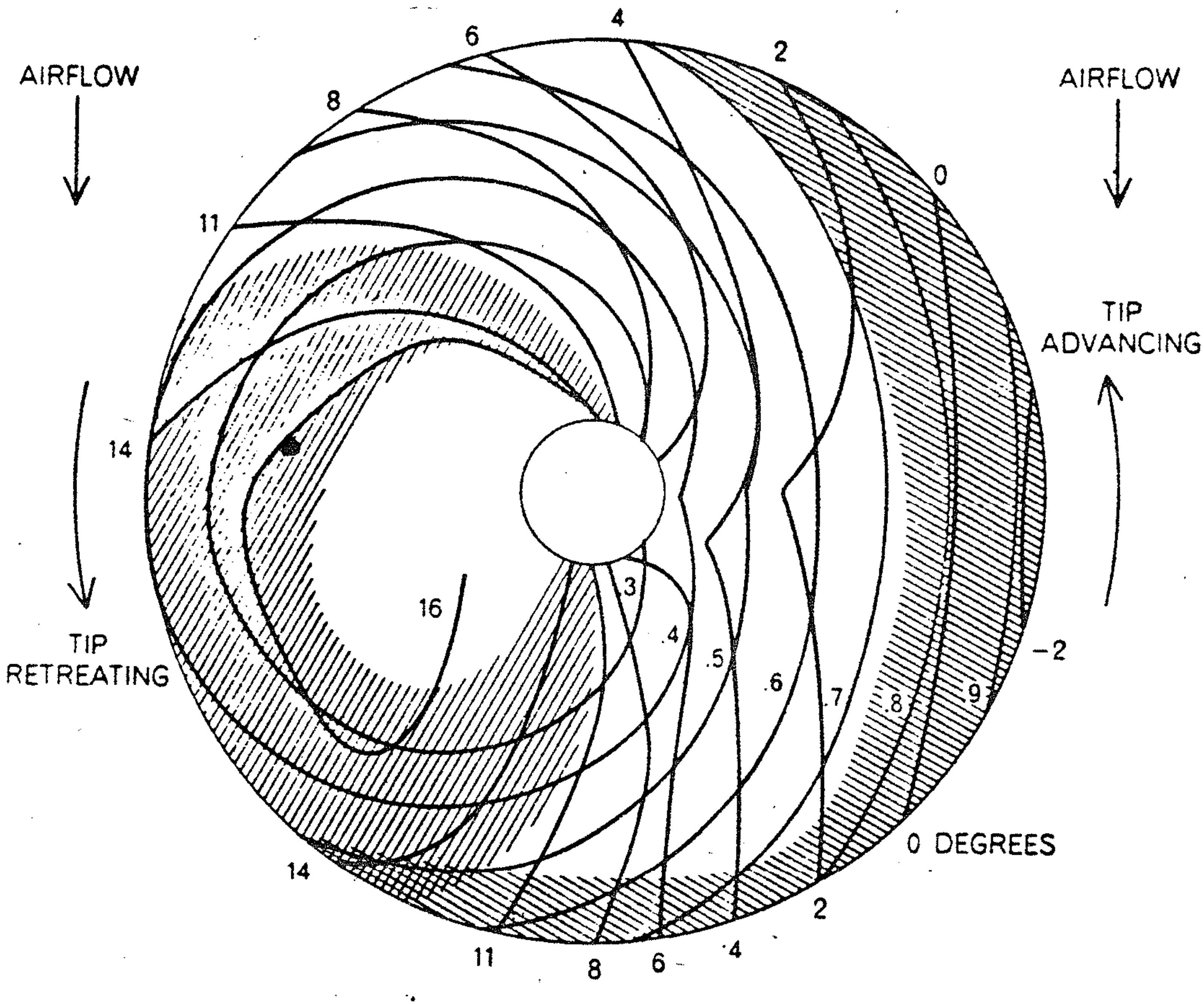
| Dynamic Stall | ---- | ---- | 16 (pp. 879-899) | 12 |

# I. INTRODUCTION

A. How a Helicopter Works
B. Modeling Issues



ROTORCRAFT INTERDISCIPLINARY INTERACTIONS
(for main lifting rotor)
DESIRED FLIGHT CONDITIONS
WAKE IMPINGEMENT
INFLOW
ANGLE OF ATTACK
FLOW FIELD
FUSELAGE WAKE
FUSELAGE VIBRATIONS
CIRCULATION
BLADE AIRFOIL
BLADE CONTROLS
AIRLOADS
BLADE MOTIONS
BLADE STRUCTURAL DYNAMICS
BOUNDARY CONDITIONS
PILOT TRIM AND SCAS
THROTTLE
POWERPLANT/ DRIVE TRAIN
RPM
HUB LOADS
SHAFT LOADS
REACTION LOADS
TAIL ROTOR & OTHER CONTROLS
AERO FORCES
AIRFRAME DYNAMICS
AIRFRAME AERODYNAMICS & THRUSTERS
WAKE IMPINGEMENT
SURFACE MOTIONS
HUB MOTIONS
QUASI-STEADY RIGID BODY MOTIONS
FUSELAGE WAKE
NIKOLSKY '89


AIRFLOW
AIRFLOW
TIP
ADVANCING
TIP
RETREATING
0 DEGREES
-2
0
2
4
6
8
11
14
16
14
11
8
6
4
2
.3
.4
.5
.6
.7
.8
.9

5

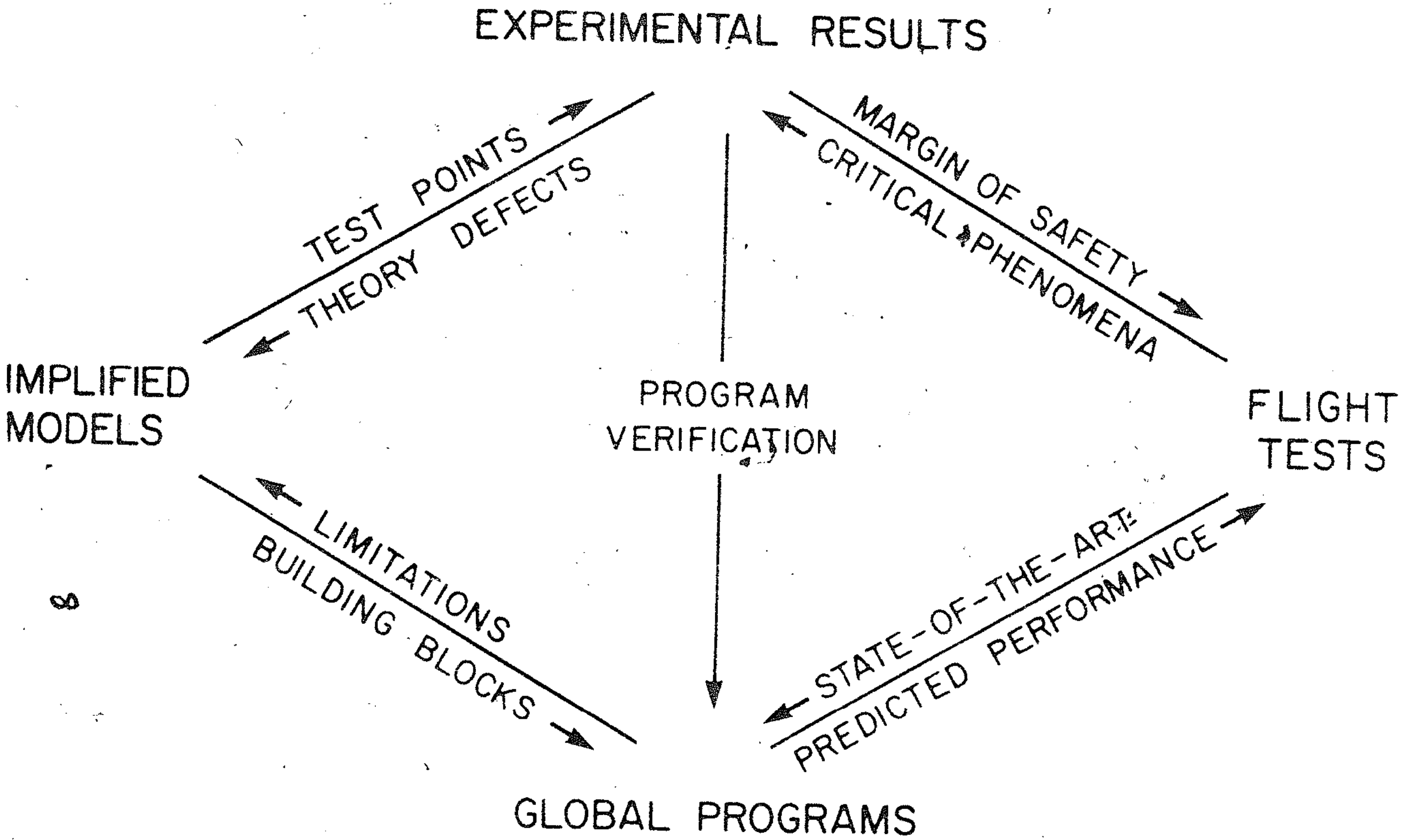
# METHODS OF INVESTIGATION

- SIMPLIFIED MATHEMATICAL MODELS
- GLOBAL MATHEMATICAL MODELS
- EXPERIMENTAL INVESTIGATIONS

# GENERAL PROBLEMS IN ROTOR DYNAMICS

- AEROELASTIC AND
  AEROMECHANICAL STABILITY
- VIBRATIONS
- CONTROL AND HANDLING QUALITIES

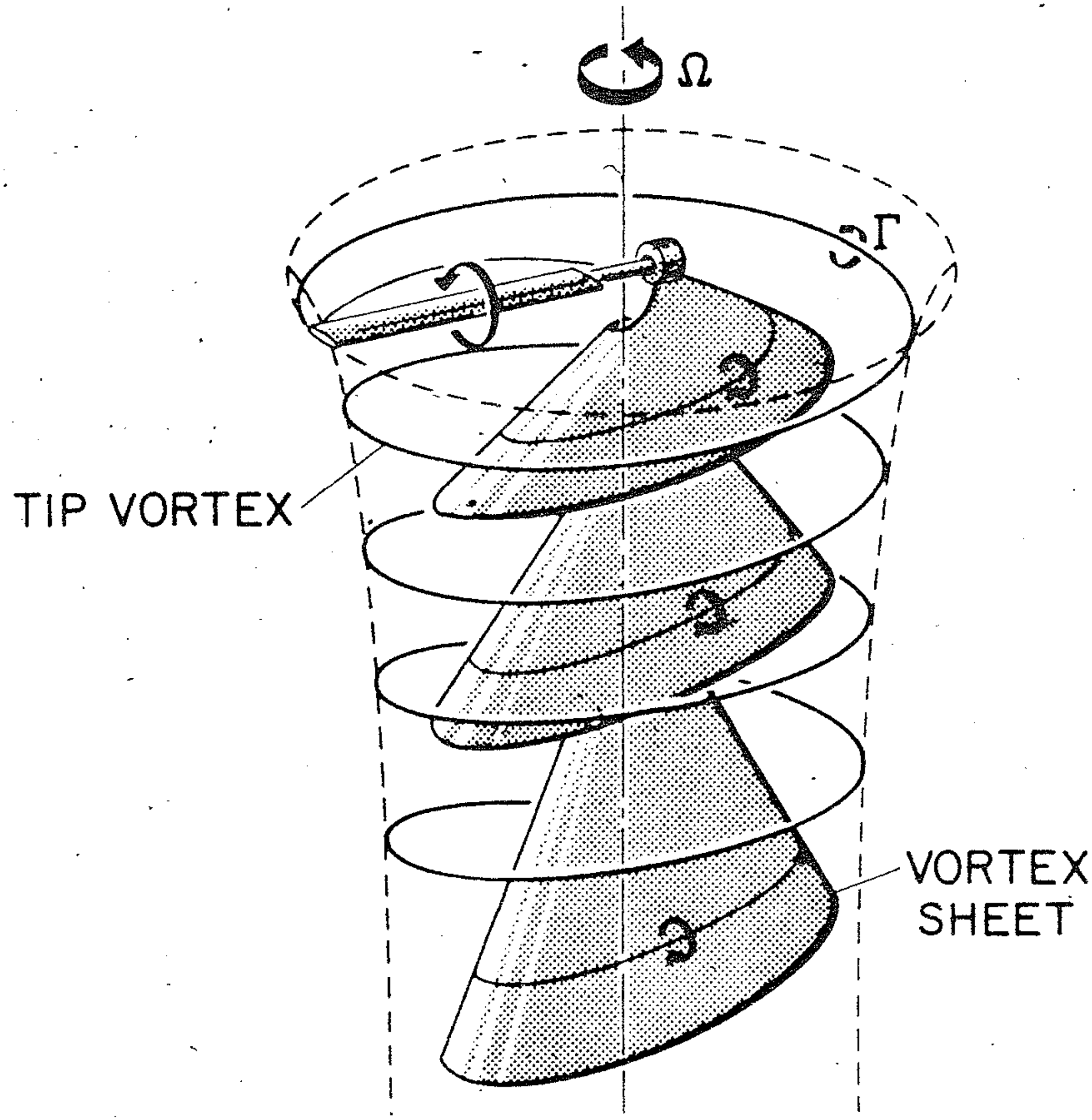
# RESEARCH PHILOSOPHY



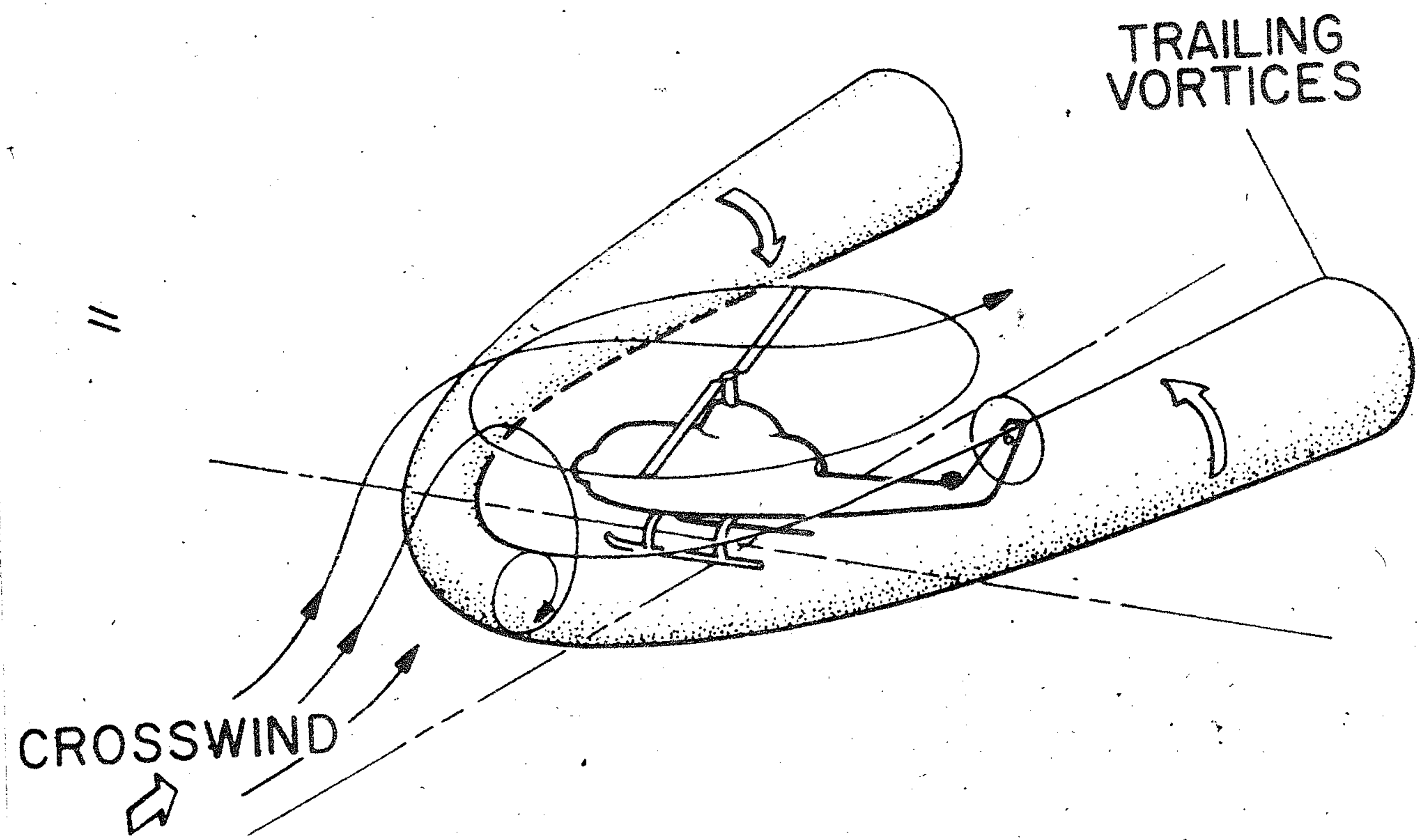
# AERODYNAMIC ENVIRONMENT

- LOW FREQUENCY WAKE
  - WAKE GEOMETRY
  - WAKE/ROTOR COUPLING
- HIGH FREQUENCY EFFECTS
  - PERIODICITY
  - DYNAMIC STALL

# HOVERING ROTOR WAKE STRUCTURE

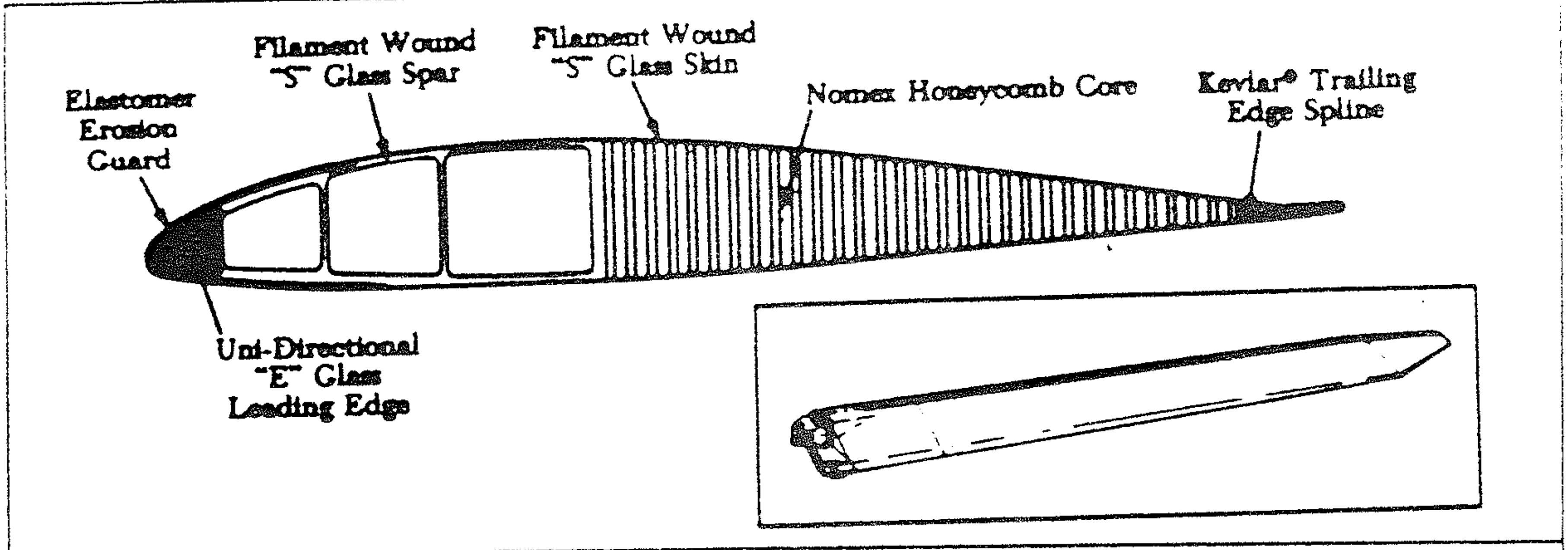


# GROUND VORTEX PHENOMENA

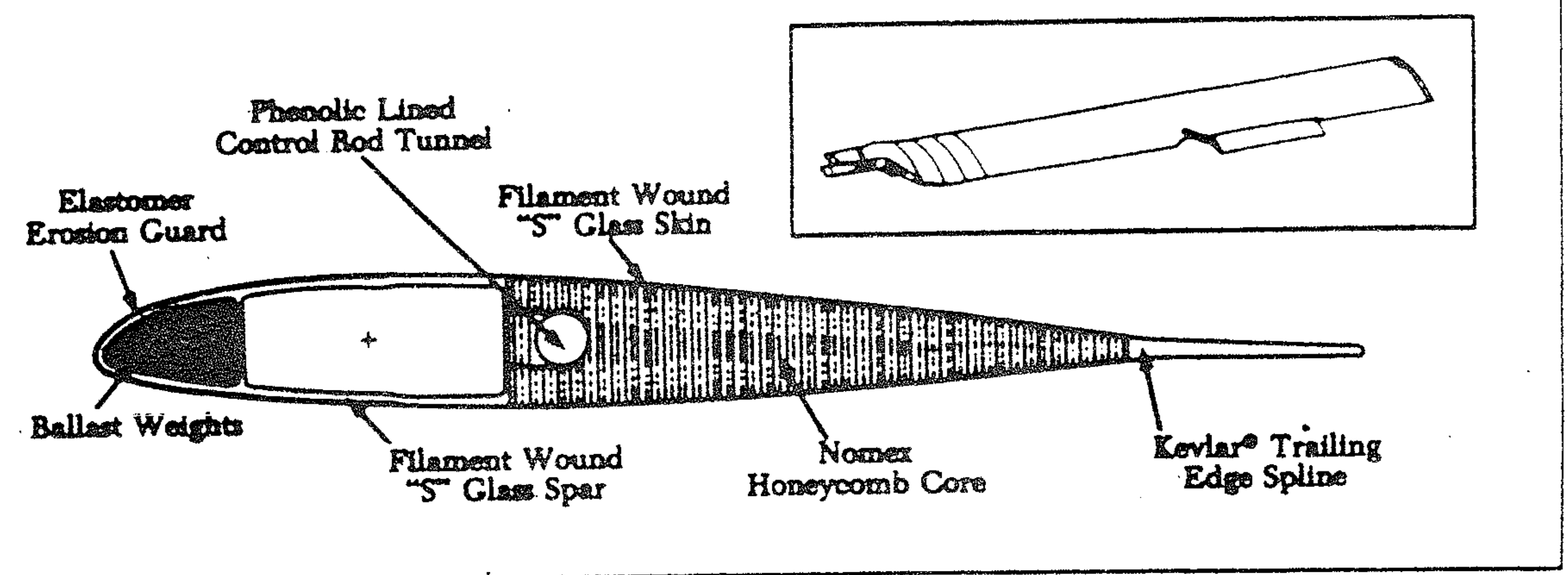


# THE ROTOR BLADE

- ELASTICITY
  - LARGE DEFORMATIONS
  - NONHOMOGENEOUS BLADES
- KINEMATICS
  - CONTROL LINKAGES AND GEOMETRY
  - EFFECTS OF BLADE ROTATION


Elastomer Erosion Guard
Filament Wound "S" Glass Spar
Filament Wound "S" Glass Skin
Nomex Honeycomb Core
Kevlar® Trailing Edge Spline
Uni-Directional "E" Glass Leading Edge

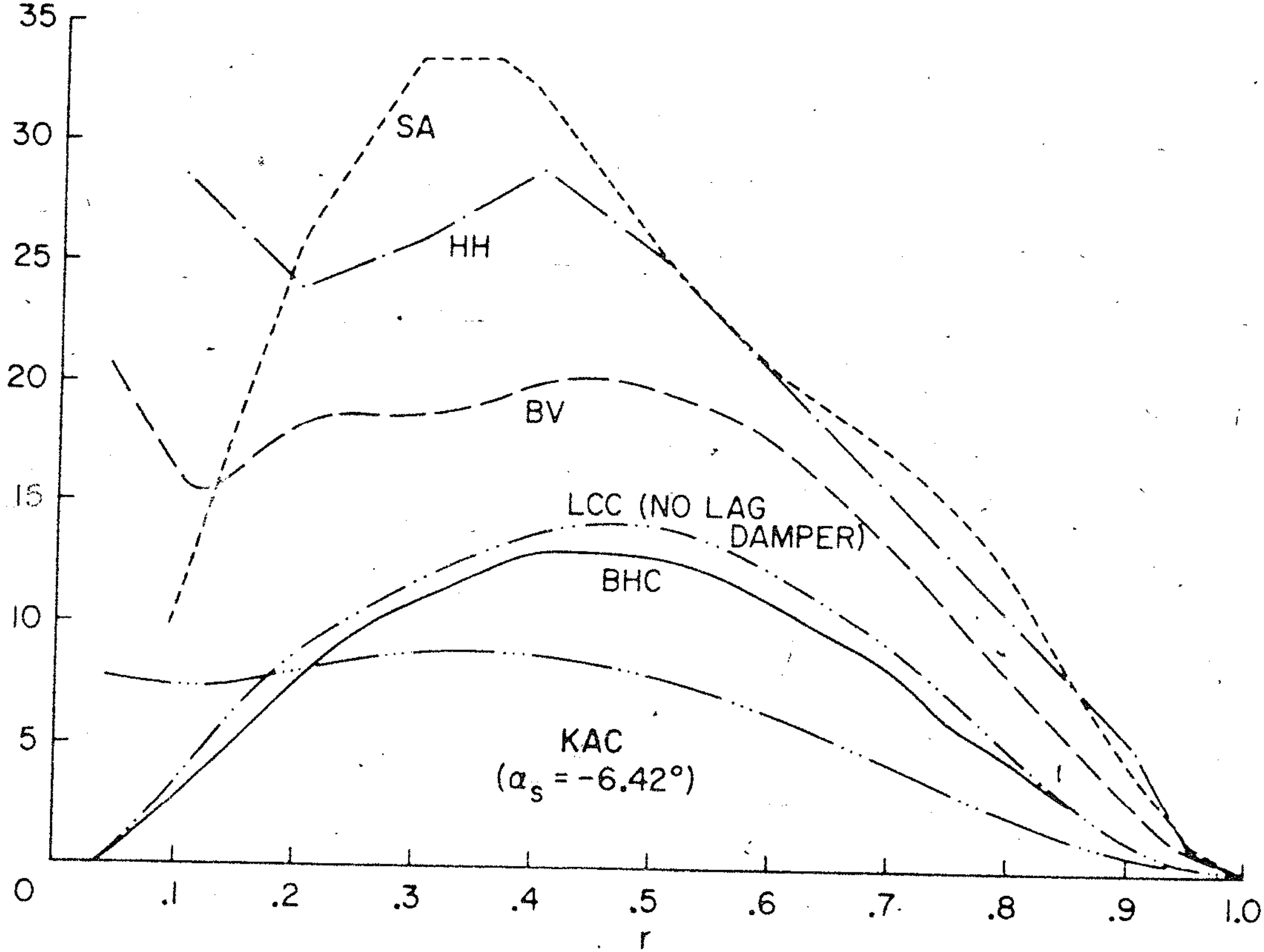
13

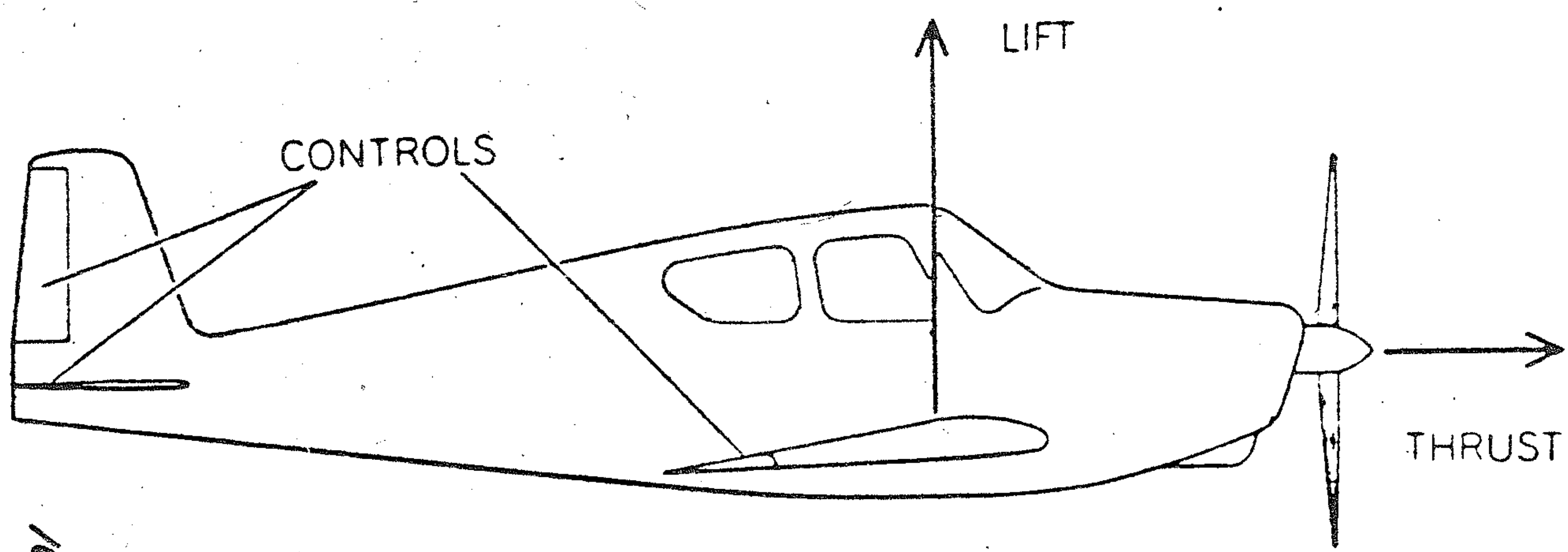

Phenolic Lined Control Rod Tunnel
Elastomer Erosion Guard
Filament Wound "S" Glass Skin
Ballast Weights
Filament Wound "S" Glass Spar
Nomex Honeycomb Core
Kevlar® Trailing Edge Spline

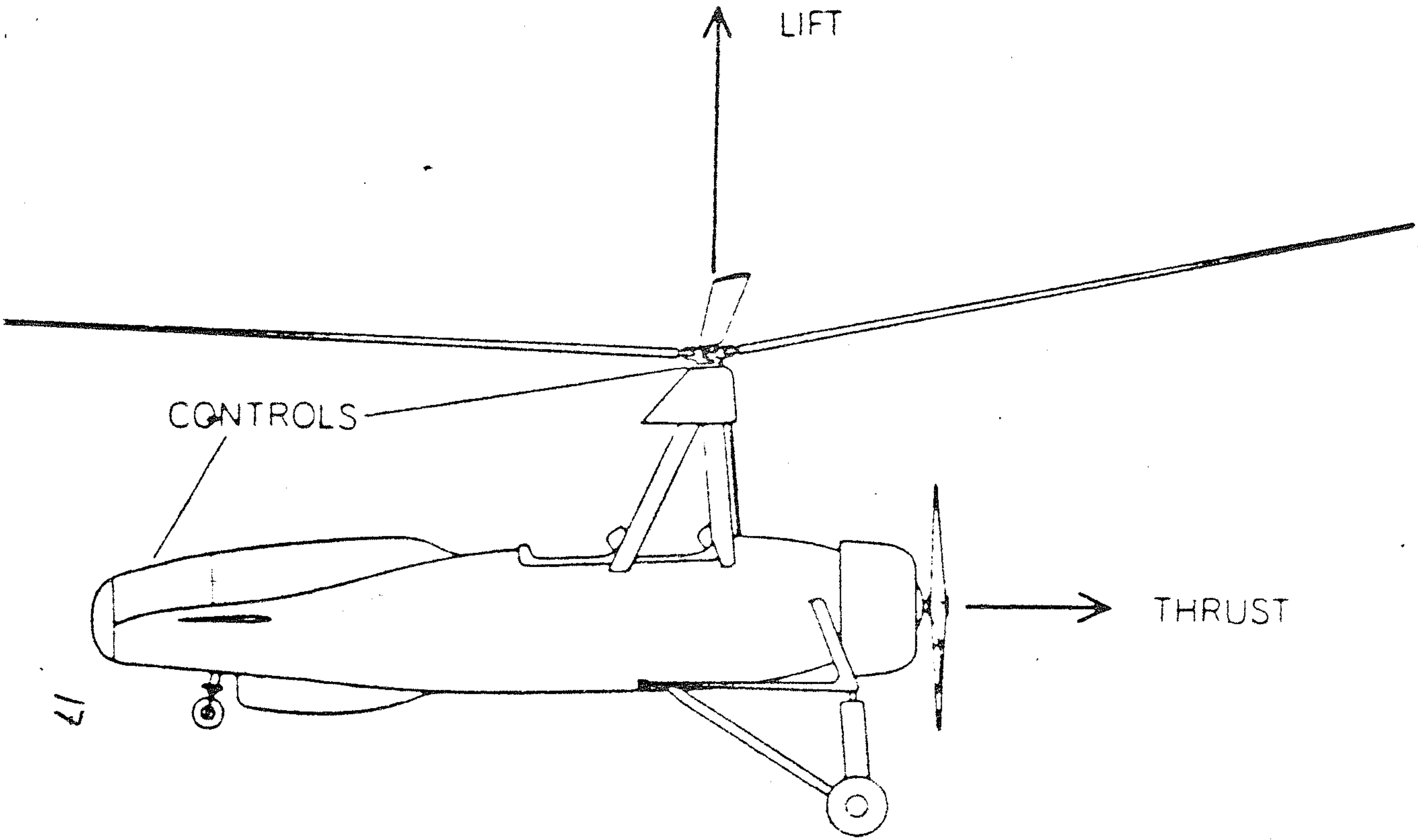
# ROTOR VEHICLE INTERACTIONS

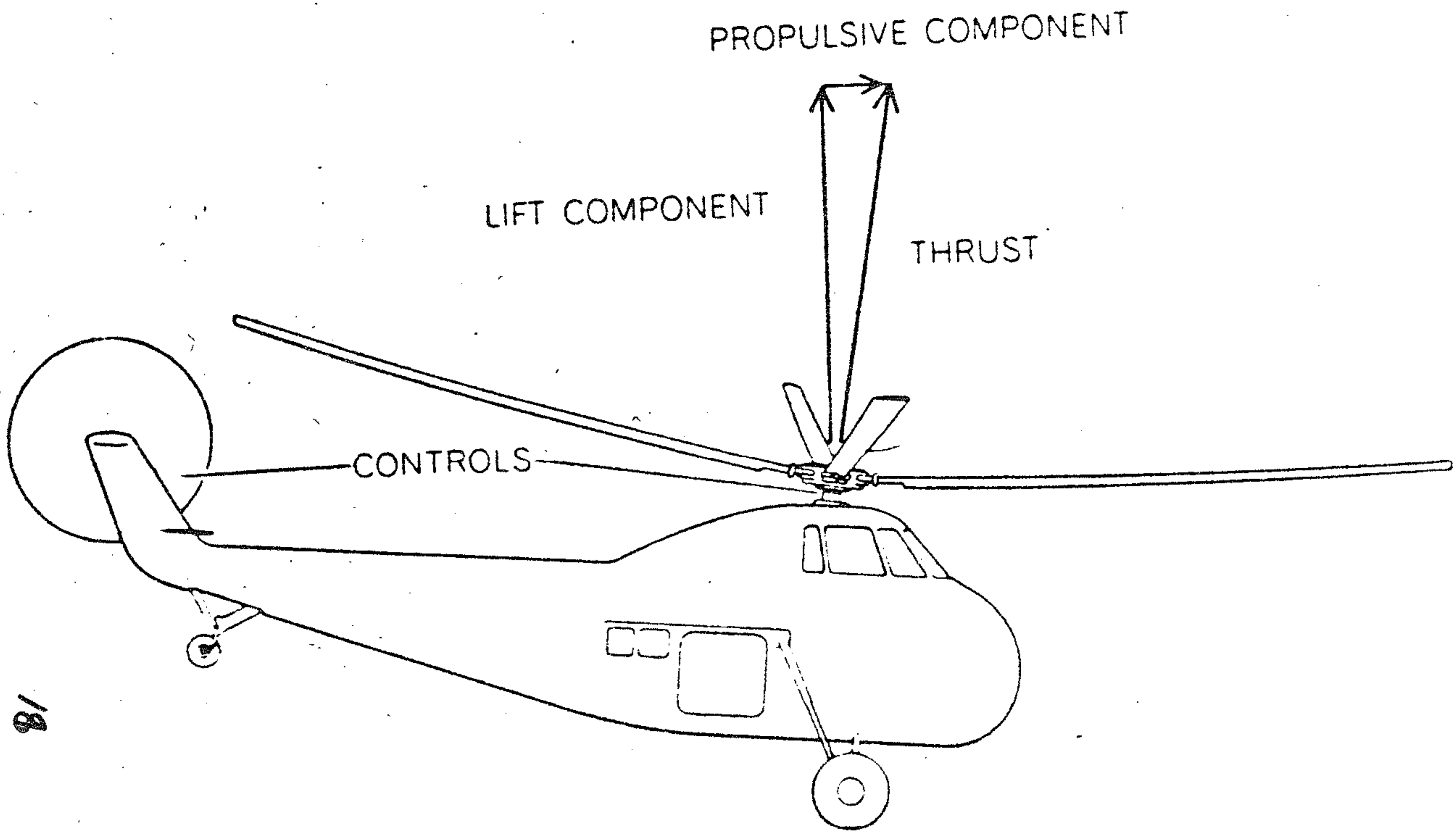
- FUSELAGE DYNAMICS
  - AIR AND GROUND RESONANCE
  - VIBRATIONS AND AIR LOADS
- FEEDBACK SYSTEMS
  - GUST ALLEVIATION
  - CARGO HANDLING

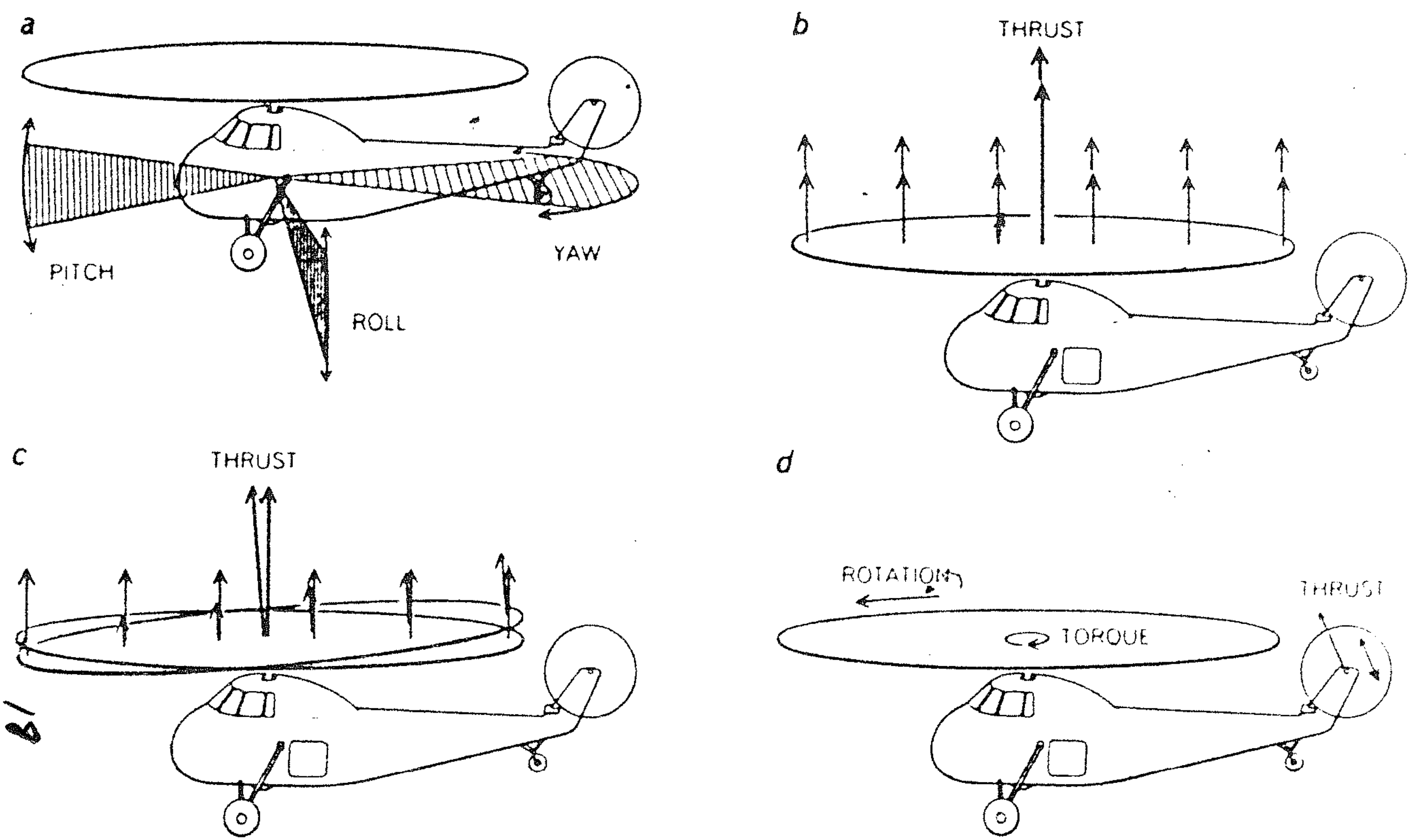
CHORDWISE MOMENTS, 1/2 PTP, $10^3$ in.-lb




LIFT
CONTROLS
THRUST


LIFT
CONTROLS
THRUST


PROPULSIVE COMPONENT
LIFT COMPONENT
THRUST
CONTROLS

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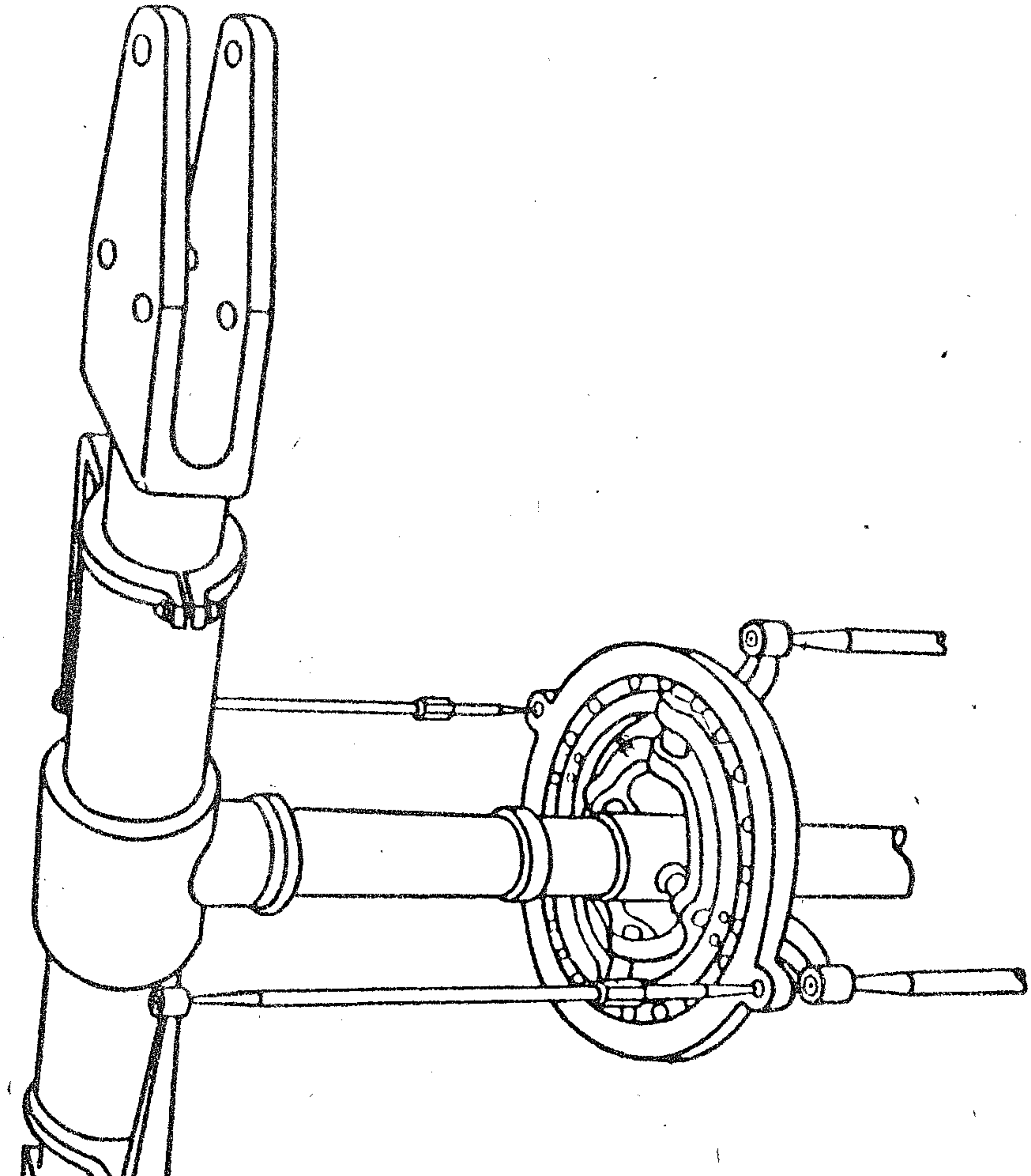
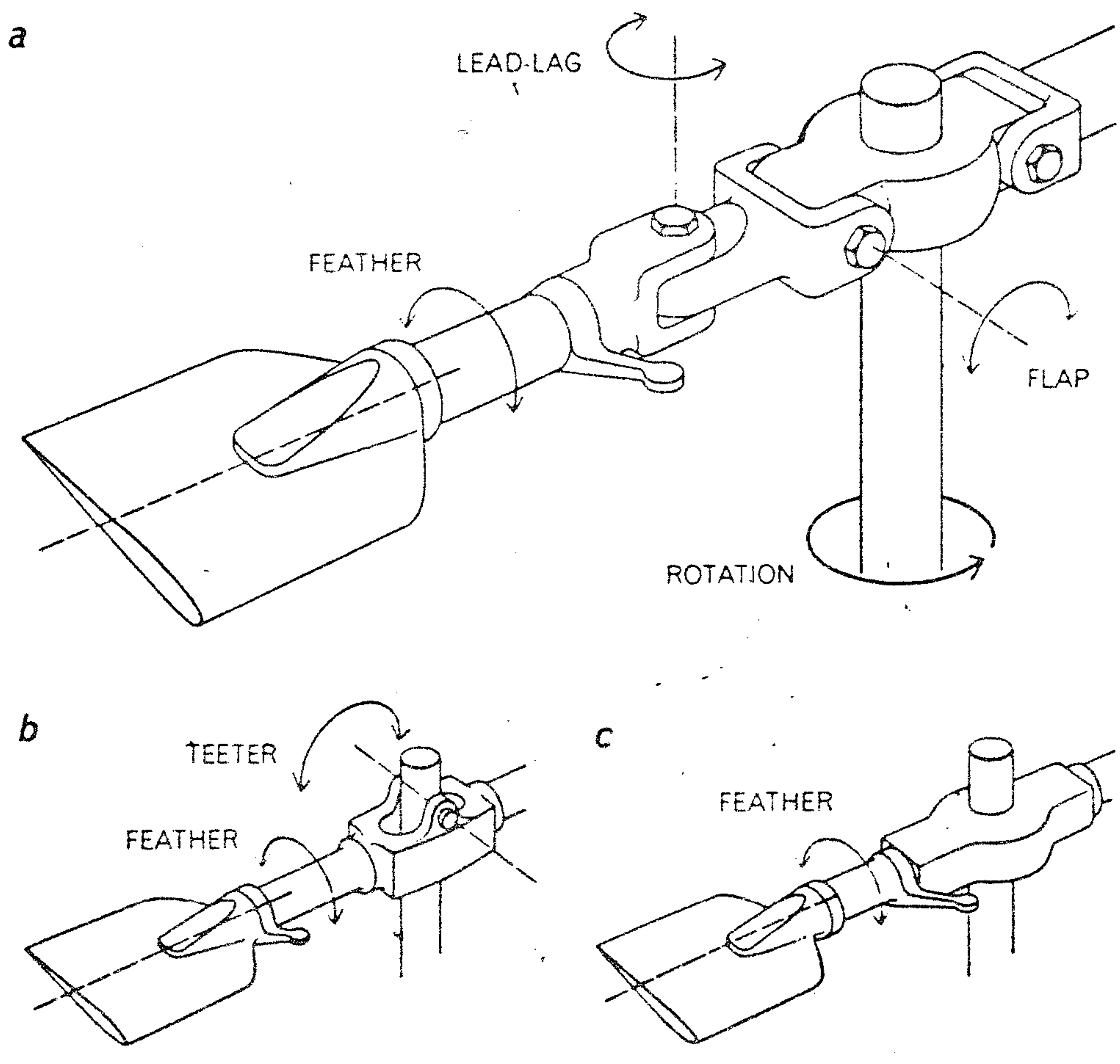
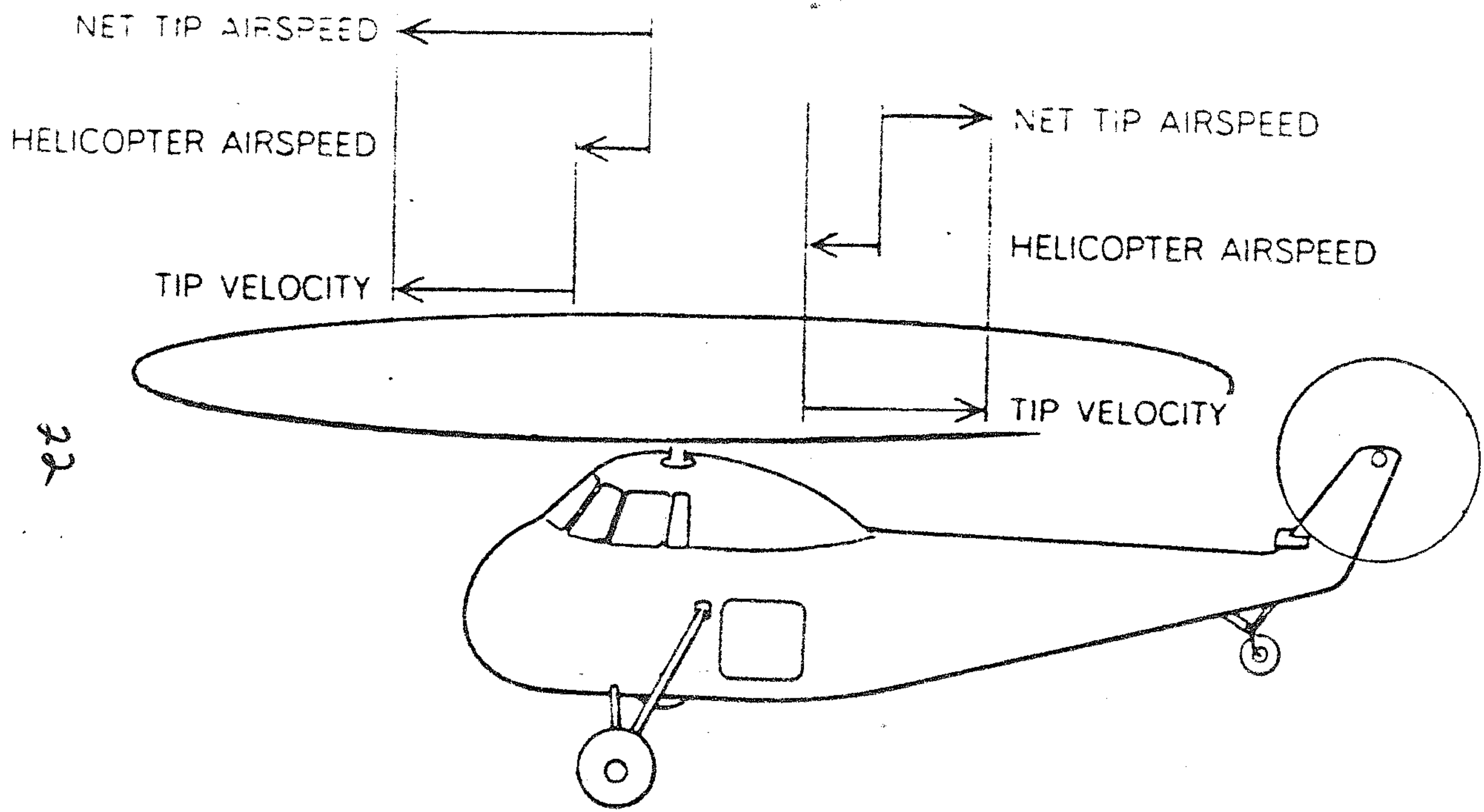
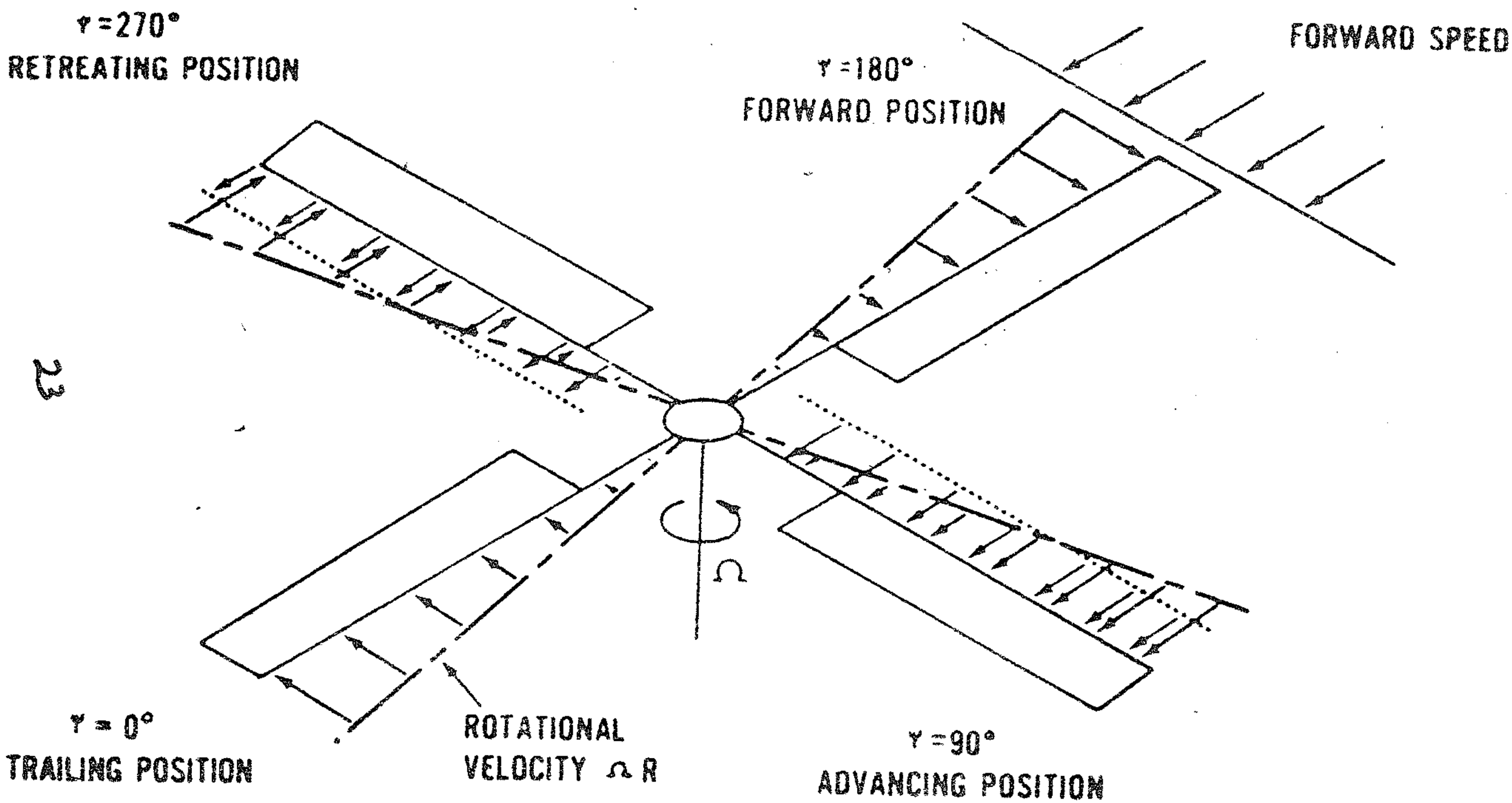
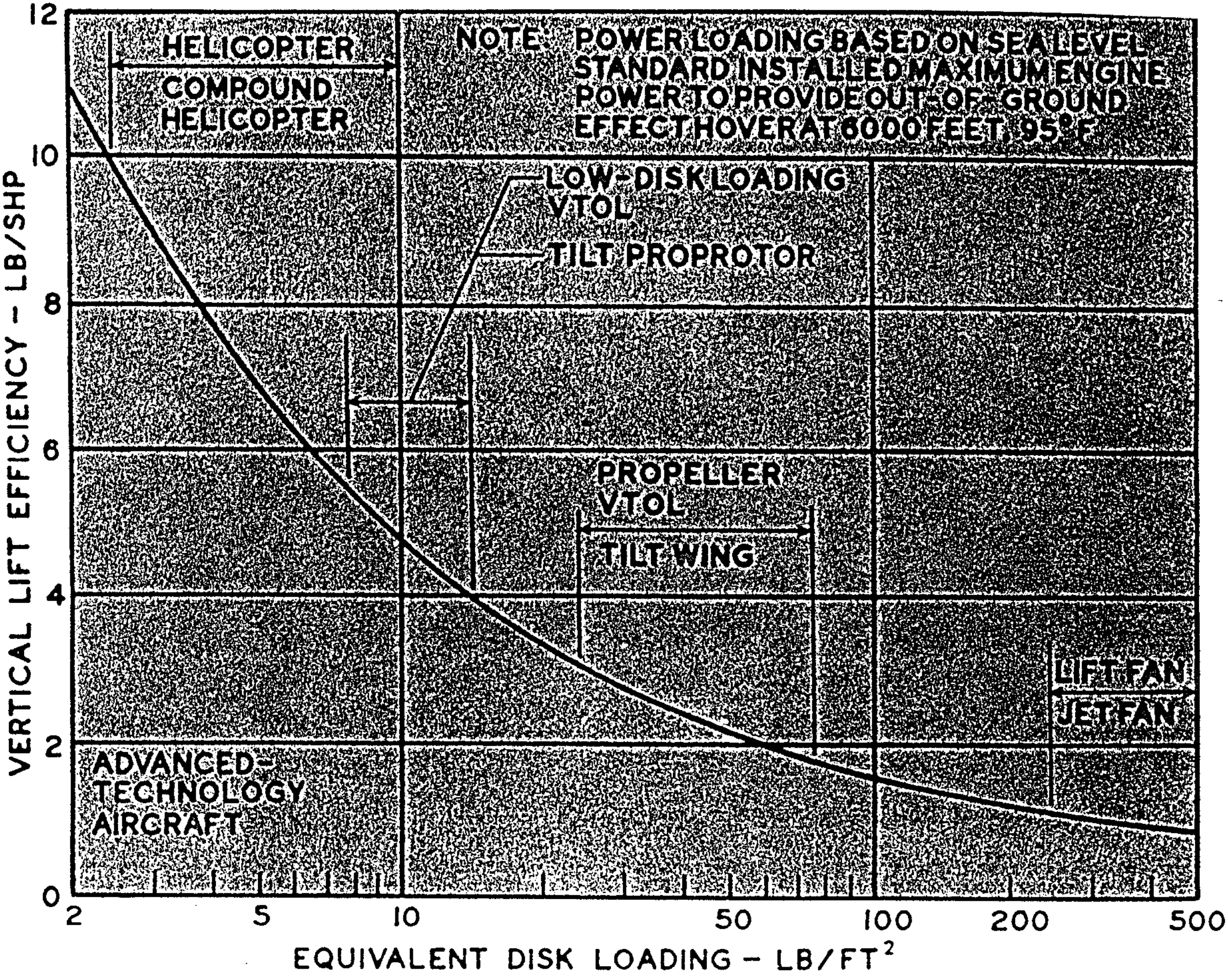



a
LEAD-LAG
FEATHER
FLAP
ROTATION
b
TEETER
FEATHER
c
FEATHER

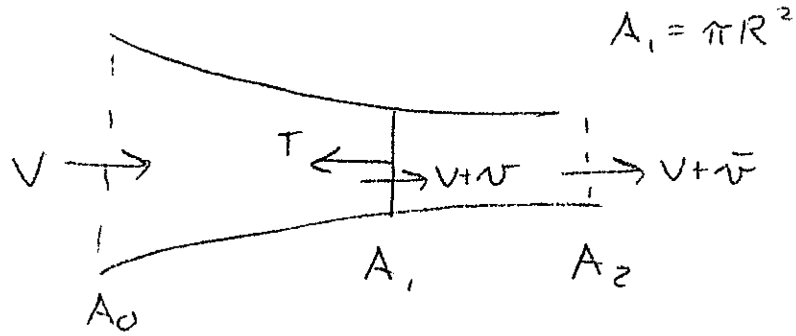

NET TIP AIRSPEED
HELICOPTER AIRSPEED
TIP VELOCITY
NET TIP AIRSPEED
HELICOPTER AIRSPEED
TIP VELOCITY


$\Psi = 270°$
RETREATING POSITION
$\Psi = 180°$
FORWARD POSITION
FORWARD SPEED
$\Omega$
$\Psi = 0°$
TRAILING POSITION
ROTATIONAL
VELOCITY $\Omega R$
$\Psi = 90°$
ADVANCING POSITION


HELICOPTER
COMPOUND HELICOPTER
NOTE: POWER LOADING BASED ON SEA LEVEL STANDARD INSTALLED MAXIMUM ENGINE POWER TO PROVIDE OUT-OF-GROUND EFFECT HOVER AT 6000 FEET, 95°F
LOW-DISK LOADING VTOL
TILT PROPROTOR
PROPELLER VTOL
TILT WING
LIFT FAN
JET FAN
ADVANCED-TECHNOLOGY AIRCRAFT
VERTICAL LIFT EFFICIENCY - LB/SHP
EQUIVALENT DISK LOADING - LB/FT²
0
2
4
6
8
10
12
2
5
10
50
100
200
500

# II. MOMENTUM / BLADE-ELEMENT THEORY

| | |
|---|---|
| Gessow & Meyers | 3 |
| Bramwell | 3.1 |
| Johnson | 2.1 - 2.3 |




$$Continuity : A_o V = A_1(V+v) = A_2(V+\overline{v})$$

(1) $$Momentum : T = \dot{m}\Delta v$$

$$\dot{m} = \rho A_1 \left|V+v\right| \qquad \Delta v = \overline{v}$$

$$T = \rho \pi R^2 \left|V+v\right| \overline{v}$$

(2) $$Power : T(V+v) = \tfrac{1}{2}\dot{m}(V+\overline{v})^2 - \tfrac{1}{2}\dot{m}V^2$$

From (1) and (2) combined

$$\dot{m}\overline{v}(V+v) = \tfrac{1}{2}\dot{m}\left[V^2 + 2V\overline{v} + \overline{v}^2 - V^2\right]$$

$$2\overline{v}V + 2\overline{v}v = 2V\overline{v} + \overline{v}^2$$

$$2\overline{v}v = \overline{v}^2$$

$$\boxed{\overline{v} = 2v}$$

$$F = \rho\pi R^2 \upsilon$$ Induced Flow

$$\dot{m} = \rho\pi R^2 \left|V + \upsilon\right|$$ Mass Flow

$$T = 2\rho\pi R^2 \left|V + \upsilon\right| \upsilon$$ Thrust

$$P = 2\rho\pi R^2 \left|V + \upsilon\right| (V + \upsilon)\upsilon$$ Power (Ideal)

Nondimensional Coefficients

$$C_{\dot{m}} = \frac{\dot{m}}{\rho\pi R^2(\Omega R)} = \left|\frac{V}{\Omega R} + \frac{\upsilon}{\Omega R}\right| + \left|\eta + \nu\right|$$

$$C_F = \frac{F}{\rho\pi R^2(\Omega R)} = \frac{\upsilon}{\Omega R} = \nu$$

$$C_T = \frac{T}{\rho\pi R^2(\Omega^2 R^2)} = 2\left|\eta + \nu\right| \nu$$

$$C_{P_I} = \frac{P}{\rho\pi R^2(\Omega^3 R^3)} = 2\left|\eta + \nu\right| (\eta + \nu)\nu$$

$$C_{P_I}/C_T = \eta + \nu = \lambda$$

Normalized Values

$$\overline{\nu} = \frac{\nu}{\sqrt{C_T/2}}, \; \overline{\eta} = \frac{\eta}{\sqrt{C_T/2}}, \; \overline{\lambda} = \overline{\eta} + \overline{\nu}$$

$$1 = \overline{\nu}\left|\overline{\nu} + \overline{\eta}\right| = \overline{\nu}\left|\overline{\lambda}\right|$$

## APPLICATION OF MOMENTUM THEORY TO WIND TURBINES

$W = -V$ (velocity negative, in direction of thrust)

$Q = -P$ (power extracted from wind)

$Q = 2\pi\rho R^2(W-v)^2 v$ (power as function of v)

$Q_T = (1/2)\, \pi\rho R^2 W^3$ (total power through disk)

$EF = Q/Q_T = 4(1-z)^2 z$ (power efficiency)

where $z = V/W$ (percent of wind stopped)

# OPTIMUM POWER POINT

$d(EF)/dz = 4(1 - 4z + 3z^2) = 4(1-z)(1-3z) = 0$

Root $z=1$ is a minimum power $=0$ (wind stopped)

Root $z=1/3$ is maximum power $EF = 16/27 = 0.59$

This is 1/3 of wind stopped at rotor
2/3 of wind stopped downstream

In terms of normalized variables, optimum point is:

$\nu = -\eta/3$; $\nu = \mathrm{sqrt}(C_T)$; $\eta = -3\mathrm{sqrt}(C_T)$

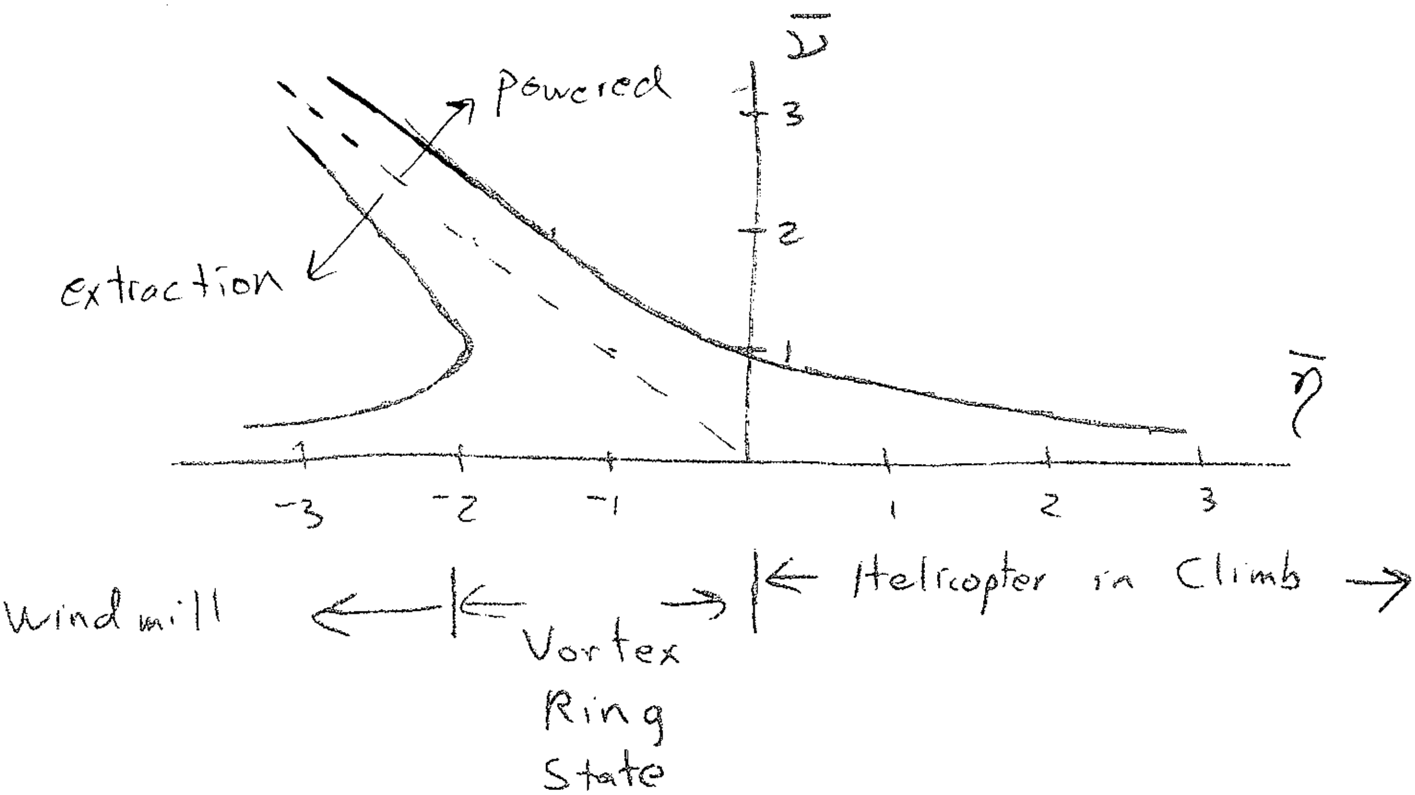
$$\overline{C}_{P_I} = \frac{C_{P_I}/2}{(C_T/2)^{3/2}} = \frac{\sqrt{2}C_{P_I}}{C_T^{3/2}} = |\overline{\nu}+\overline{\eta}|\,(\overline{\nu}+\overline{\eta})\overline{\nu}$$

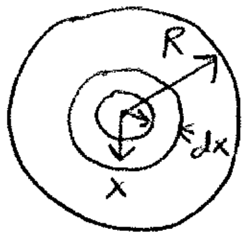
$$\overline{C}_{P_I} = \overline{\nu}+\overline{\eta} = \overline{\lambda}$$

$$C_P\,(\text{actual}) \geq C_{P_I} \qquad \text{[Hover Only]}$$

$$\text{Figure of Merit} \equiv \frac{C_T^{3/2}}{C_P\sqrt{2}} = \frac{1}{\overline{C}_P} \leq 1$$

$$P = \rho^{-\frac{1}{2}}\pi^{-\frac{1}{2}}\,T^{3/2}R^{-1}\frac{1}{\sqrt{2}\,FM}$$






$\frac{x}{R} = r$

# Radial Variations in $\upsilon$

For an anular ring :

$$dC_F = 2\nu r dr \qquad dC_{\dot{m}} = 2\left|\eta + \nu\right| r dr$$
$$dC_T = 4\left|\eta + \nu\right| \nu r dr \qquad dC_{PI} = 4\left|\eta + \nu\right|(\eta + \nu)\nu r dr$$

Minimize $C_{P_I}$ given $C_T$ for $\eta = 0 \Rightarrow \nu =$ constant

Dimensional Forms:

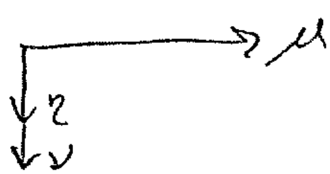
$$d\dot{m} = \rho 2\pi x dx\left|V + \upsilon\right|$$

$$dT = \rho 4\pi x dx\left|V + \upsilon\right|\upsilon$$

$$dP_I = \rho 4\pi x dx\left|V + \upsilon\right|(V + \upsilon)\upsilon$$

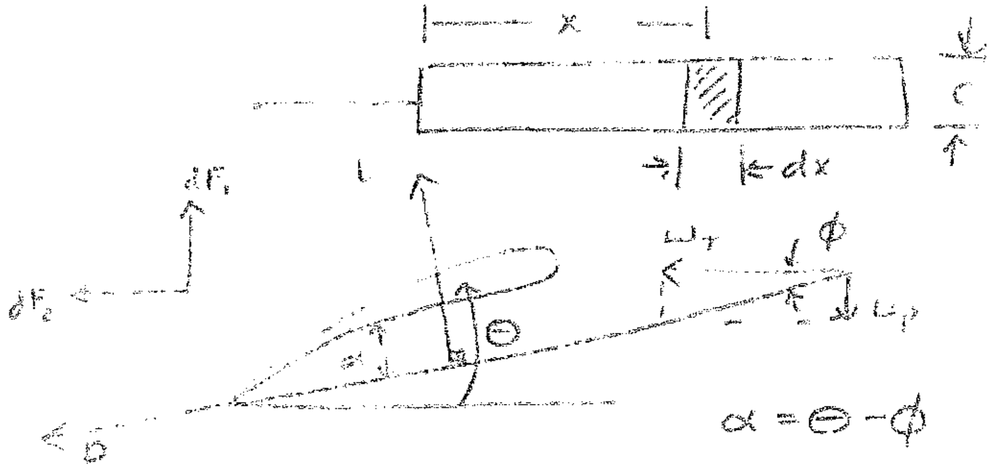
Forward Flight

$$C_T = 2\nu\sqrt{\mu^2 + (\eta + \nu)^2}$$



# BLADE - ELEMENT THEORY

| | |
|---|---|
| Gessow & Meyers | 3 & 4 |
| Bramwell | 3.4 |
| Johnson | 2.4, 2.5 |




$$\mathbb{L} = \frac{1}{2}\rho A\mathbb{V}^2 C_L \quad D = \frac{1}{2}\rho A\mathbb{V}^2 C_D$$

$$\mathrm{d}A = c\mathrm{d}x \;\; , \;\; \mathbb{V}^2 = \sqcup_T^2 + \sqcup_P^2$$

$$C_L = a\sin\alpha = a\sin(\theta - \phi)$$

$$a = \text{slope of lift curve}$$

$$F_1 = \mathbb{L}\cos\phi - D\sin\phi$$

$$F_2 = D\cos\phi + \mathbb{L}\sin\phi$$

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

$$\cos\phi = \frac{\sqcup_T}{\sqrt{\sqcup_P^2 + \sqcup_T^2}} \quad \sin\phi = \frac{\sqcup_P}{\sqrt{\sqcup_P^2 + \sqcup_T^2}}$$

$$\mathrm{d}F_1 = \frac{a}{2}\rho c \mathrm{d}x\left[\sin\theta\, \sqcup_T^2 - \cos\theta\, \sqcup_P \sqcup_T\right] - \frac{1}{2}\rho c \mathrm{d}x\left[\sqcup_P\sqrt{\sqcup_T^2 + \sqcup_P^2}\right] C_D$$

$$\approx \frac{1}{2}a\rho c\mathrm{d}x\left[\sin\theta\, \sqcup_T^2 - \sqcup_P \sqcup_T\left(\cos\theta + \frac{C_D}{a}\right)\right]$$

$$\mathrm{d}F_2 = \frac{1}{2}\rho c\mathrm{d}x\left[\sqcup_T\sqrt{\sqcup_T^2 + \sqcup_P^2}\right] C_D + \frac{1}{2}\rho c\mathrm{d}x\left[\sin\theta\, \sqcup_P \sqcup_T - \cos\theta\, \sqcup_P^2\right] a$$

$$= \frac{1}{2}a\rho c\mathrm{d}x\left[\sqcup_T^2\frac{C_D}{a} + \sin\theta\, \sqcup_P \sqcup_T - \sqcup_P^2\left(\cos\theta - \frac{1}{2}\frac{C_D}{a}\right)\right]$$

$$\text{Approximations : } \sqrt{1 + \frac{\sqcup_P^2}{\sqcup_T^2}} = 1 + \frac{1}{2}\frac{\sqcup_P^2}{\sqcup_T^2} - \frac{1}{8}\frac{\sqcup_P^4}{\sqcup_T^4}\cdots$$

$$\sin\theta \approx \theta \quad , \quad \cos\theta \approx 1$$

$$C_D \ll a \qquad (\text{e.g.}\,,\ C_D = .01, a = 6.0)$$

$$\mathrm{d}F_1 = \frac{a}{2}\rho c\mathrm{d}x\left[\sqcup_T^2\theta - \sqcup_P\sqcup_T\right]$$

$$\mathrm{d}F_2 = \frac{a}{2}\rho c\mathrm{d}x\left[\sqcup_T^2\frac{C_D}{a} + \sqcup_P \sqcup_T\, \theta - \sqcup_P^2\right]$$

$$T = b\int_0^R \mathrm{d}F_1 \qquad (b = \text{number of blades})$$

$$P = b\int_0^R \Omega\mathrm{d}F_2 x \qquad \text{Axial Flow } \Rightarrow \sqcup_T = \Omega x\,, \sqcup_P = V + \upsilon$$

(3) $$T = \int_0^R \frac{1}{2}\rho abc\left[\Omega^2x^2\theta - \Omega x(V + \upsilon)\right]\mathrm{d}x$$

(4) $$P = \int_0^R \frac{1}{2}\rho abc\,\underbrace{\left[\Omega^3x^3\theta\frac{C_D}{a}\right.}_{\text{Profile Power}} + \underbrace{\left.\Omega^2x^2(V+\upsilon)\theta - \Omega x\,(V+\upsilon)^2\right]}_{\text{Induced Power}}\mathrm{d}x$$

Notice that $induced\,power = (V + v)$ times $thrust$.
This is identical results to momentum-theory.
Thus momentum-theory power equation now superseded by (4).

Nondimensional Versions: $V + v \equiv \Omega x \phi$

$$\frac{C_T}{\sigma a} = \int_0^1 \tfrac{1}{2}\left[r^2\theta - r^2\phi\right]\mathrm{d}r = \int_0^1 \tfrac{1}{2}r^2(\theta - \phi)\mathrm{d}r$$

$$\sigma \equiv \frac{bc}{\pi R}$$

$$\frac{C_P}{\sigma a} = \int_0^1 \tfrac{1}{2}\left[r^3\tfrac{C_D}{a} - r^3(\theta - \phi)\phi\right]\mathrm{d}r$$

For constant $C_D, \theta, \phi$

$$\frac{C_T}{\sigma a} = \tfrac{1}{6}(\theta - \phi)$$

$$\frac{C_P}{\sigma a} = \tfrac{1}{8}\left[\tfrac{C_D}{a} + \phi(\theta - \phi)\right]$$

$$\frac{\mathrm{d}C_T}{\mathrm{d}r} = \tfrac{\sigma a}{2}\left[r^2\theta - r(\eta + \nu)\right]$$

$$\frac{\mathrm{d}C_P}{\mathrm{d}r} = \tfrac{\sigma a}{2}\left[r^3\tfrac{C_D}{a} + r^2\theta(\eta + \nu) - r(\eta + \nu)(\eta + \nu)\right]$$

$$\lambda = \eta + \nu = \phi r$$

Formulas for $C_T + C_P$ with constant $\lambda = \eta + \nu$

$$C_T = \frac{\sigma a}{6}\theta - \frac{\sigma a}{4}\lambda$$

$$C_P = \frac{\sigma a}{8}\frac{C_D}{a} + \frac{\sigma a}{6}\theta\lambda - \frac{\sigma a}{4}\lambda^2$$

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Note :

For $C_T$, $\quad \phi_{eq} = \frac{3}{2}\lambda_{eq}$

For $C_P$, $\quad \phi_{eq} = \frac{4}{3}\lambda_{eq}$ or $\phi_{eq} = \sqrt{2}\lambda_{eq}$

$$\phi = \frac{\lambda}{r} \quad \Rightarrow \quad r_{eq} = \begin{array}{l} .667 \\ .707 \\ .750 \end{array}$$

## Combined Blade – Element Momentum Theory

(Hover)

Momentum $\quad dT = 4\pi x dx \rho v^2$

Blade Element $\quad dT = \frac{1}{2}\rho abc\left(\Omega^2 x^2 \theta - \Omega x v\right) dx$

$$4\pi x dx \rho v^2 = \frac{1}{2}\rho abc\left(\Omega^2 x^2 \theta - \Omega x v\right) dx$$

$$8\pi x v^2 = abc\left(\Omega^2 x^2 \theta - \Omega x v\right) \qquad \nu = \frac{v}{\Omega R}$$

$$\nu^2 + \frac{\sigma a}{16}\nu - \frac{\sigma a}{8} r\theta = 0$$

$$\nu = -\frac{\sigma a}{16} \pm \sqrt{\left(\frac{\sigma a}{16}\right)^2 + \frac{\sigma a}{8} r\theta}$$

$$\boxed{\nu = \frac{\sigma a}{16}\left[-1 + \sqrt{1 + \frac{32 r\theta}{\sigma a}}\right]}$$

## Tip-Loss Factor

Gessow & Meyers pp. 72-75

Bramwell pp. 110 - 116

Johnson pp. 58-61, 81-88, 133-134

$$B = 1 - 2.50\frac{s\ln(2)}{R\pi}\left[\frac{1 + .1564\left(\frac{s}{c}\right)}{1 + 1.080\left(\frac{s}{c}\right) + \left(.2593\left(\frac{s}{c}\right)\right)^2}\right]$$

$$B = 1 - 1.283\frac{c}{R}\left[\frac{1 + 6.394\left(\frac{c}{s}\right)}{1 + 16.06\left(\frac{c}{s}\right) + \left(3.856\left(\frac{c}{s}\right)\right)^2}\right]$$

An altered momentum theory

$$dT = 4\pi x dx \rho v^2 k$$

$$k = \frac{2}{\pi}\cos^{-1}\left(e^{-f}\right)$$

$$s = \frac{2\pi\lambda}{b\sqrt{1+\lambda^2}}$$

$$f = \frac{1}{2}\frac{b(1-r)}{\sin\phi_A} = +\frac{\pi x}{s}$$

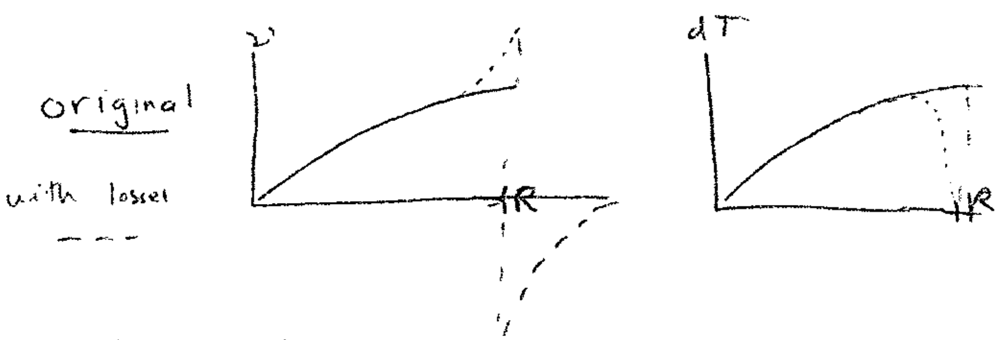
$$\Rightarrow \nu = \frac{\sigma a}{16k}\left[-1+\sqrt{1+\frac{32r\theta k}{\sigma a}}\right]$$

$$\boxed{\phi_A \equiv \frac{\nu}{r} \text{ when } k \equiv 1}$$ at the tip (no tiploss)

with tiploss:

at tip $r \to 1$ $f \to 0$ $e^{-f} \to 1$ $k \to 0$ (true $k$)

$v = r\theta$ (just cancels lift)




Approximation, cut off Thrust at $x = BR$

$$B = 1 - \frac{2\ln 2 \sin\phi}{b}$$

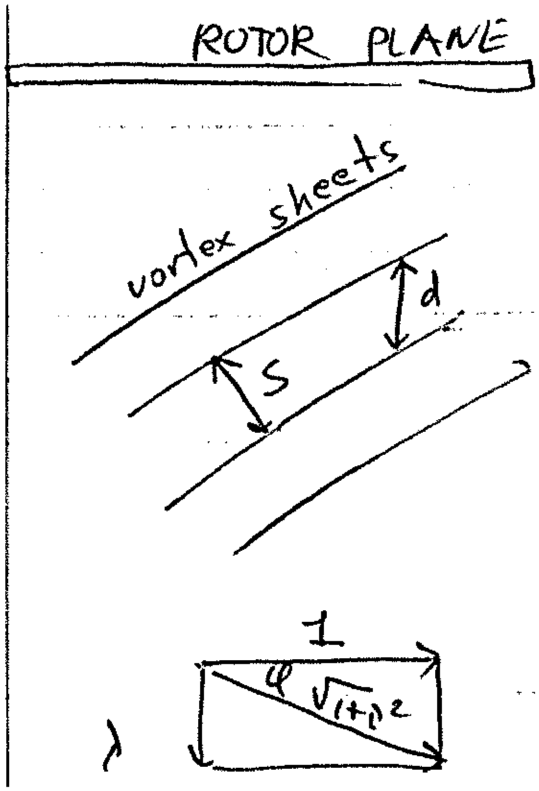
$$2\ln 2 \approx \sqrt{2}$$

$$\sin\phi \approx \lambda = \sqrt{C_T}$$

$$B \approx 1 - \frac{\sqrt{2C_T}}{b}$$

$$B = 1 - \frac{1}{2}\frac{c}{R} \text{ (alternate)}$$

# Wake Geometry at tip



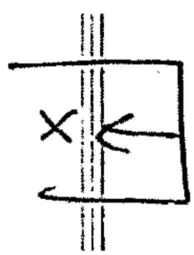

period $= \frac{2\pi}{\Omega} = T$
distance between layers vertically $=$

$$d = \frac{(V+v)T}{b} = \frac{2\pi(V+v)}{\Omega b}$$

$$d = \frac{2\pi\lambda}{b} R$$

$$\sin\phi = \frac{\lambda}{\sqrt{1+\lambda^2}}$$

$$\cos\phi = \frac{1}{\sqrt{1+\lambda^2}}, \quad s = d\cos\phi = \frac{2\pi\lambda}{b\sqrt{1+\lambda^2}} R$$




$$s = [2\pi\sin\phi/b]\, R$$

$$f = \frac{\pi x}{s} = \frac{\pi(1-r)Rb}{2\pi\sin\phi R}$$

$$f = \frac{1}{2}\frac{b(1-r)}{\sin\phi}$$

$$\mathrm{d}C_T = 4\nu(\eta+\nu)kr\mathrm{d}r$$
$$\mathrm{d}C_T = \frac{\sigma a}{2}\left[r^2\sin\theta - r(\eta+\nu)\cos\theta\right]\mathrm{d}r$$

Let $\frac{\sigma a}{8k}\cos\theta \equiv Q$

$$\nu^2 + \nu\eta = Q\left[r\tan\theta - (\eta+\nu)\right]$$
$$\nu^2 + \nu(\eta+Q) - Q(r\tan\theta - \eta) = 0$$
$$\nu = -\left(\frac{\eta+Q}{2}\right) \pm \sqrt{\left(\frac{\eta+Q}{2}\right)^2 + Q(r\tan\theta-\eta)}$$

$$\nu = -\left[\frac{\eta}{2} + \frac{\sigma a}{16k}\cos\theta\right] + \sqrt{\left(\frac{\eta}{2} + \frac{\sigma a\cos\theta}{16k}\right)^2 + \frac{\sigma a\cos\theta}{8k}(r\tan\theta-\eta)}$$

Tip Loss, No Small Angles, Climb

# Summary Thus Far

## I. Momentum Theory (axial flow)

$$\frac{\mathrm{d}C_T}{\mathrm{d}r} = 4(\eta+\nu)\nu r = 4\lambda\nu r = 4\phi\nu r^2$$

$$\lambda = \eta+\nu,\ \phi = \tfrac{\eta+\nu}{r}$$

## II. Blade element theory (axial flow)

$$\frac{\mathrm{d}C_T}{\mathrm{d}r} = \tfrac{\sigma a}{2}\left(r^2\theta - r\lambda\right) = \tfrac{\sigma a}{2}r^2(\theta-\phi)$$

$$\frac{\mathrm{d}C_P}{\mathrm{d}r} = \tfrac{\sigma a}{2}\left[r^3\tfrac{C_d}{a} + r^2\theta\lambda - r\lambda^2\right]$$

$$= \tfrac{\sigma a}{2}r^3\left[\tfrac{C_d}{a} + (\theta-\phi)\phi\right]$$

## III. Tip-loss correction

$$\frac{\mathrm{d}C_T}{\mathrm{d}r} = 4(\eta+\nu)\nu r k$$

$$k = \tfrac{2}{\pi}\cos^{-1}\left(e^{-f}\right) \quad f = \tfrac{1}{2}\tfrac{b(1-r)}{\sin\phi_A}$$

$$\phi_A = \phi(r=1)\ \text{ w/o tip-loss}$$

Poor-man approximation:

$$B = 1 - \tfrac{2\ln(2)\sin\phi_A}{b}$$

## Ground Effect

$h =$ distance from ground$/R$

Hayden

$$k_G = .9926 + .03794(2/h)^2$$

Newman

$$k_G = [1 - \exp(-.275h)]^{-3/2}$$

Cheeseman

$$k_G = [1 - 1/(4h)]^{-3/2}$$

Replace $k$ in momentum theory and blade-element momentum theory with $k \cdot k_G$.

Lift – Coefficient Versions Dimensional

$$\frac{\mathrm{d}T}{\mathrm{d}x} = 4\pi\rho(V+v)vxk$$
$$= 4\pi\rho(W-v)vxk$$

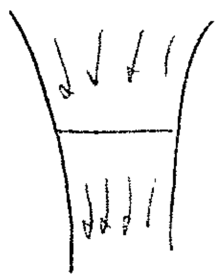
$$\frac{\mathrm{d}T}{\mathrm{d}x} = \tfrac{1}{2}\rho bc\Omega^2 x^2 C_l$$

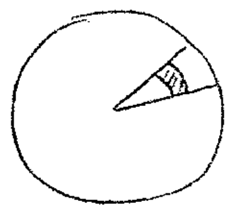
$$\frac{\mathrm{d}P}{\mathrm{d}x} = \tfrac{1}{2}\rho bc\Omega^2 x^2 \left[C_d\Omega x + (V+v)C_l\right]$$
$$= \tfrac{1}{2}\rho bc\Omega^2 x^2 \left[C_d\Omega x - (W-v)C_l\right]$$

$$k = \tfrac{2}{\pi}\cos^{-1}\left(e^{-f}\right)$$
$$f = \tfrac{1}{2}\tfrac{b(R-x)\Omega}{V+v} = \tfrac{1}{2}\tfrac{b(R-x)\Omega}{W-v}$$
$$C_l = a\left[\theta - \tfrac{(V+v)}{\Omega x}\right] = a\left[\theta + \tfrac{(W-v)}{\Omega x}\right]$$
$$\alpha = \theta - \tan^{-1}\left(\tfrac{V+v}{\Omega x}\right) = \theta + \tan^{-1}\left(\tfrac{W-v}{\Omega x}\right)$$

# Dynamic Inflow
# (Hover)

## I. Quasi-steady





$$\mathrm{d}A = x\mathrm{d}x\mathrm{d}\psi$$

$$\mathrm{d}\dot{m} = \rho\left(V + v_i\right)\mathrm{d}A, \quad \mathrm{d}T = \mathrm{d}\dot{m}\Delta v$$

$$v_i = \overline{v_i} + \widetilde{v_i}, \quad \widetilde{v_i} = v_o + \tfrac{x}{R}v_s\sin\psi + \tfrac{x}{R}v_c\cos\psi$$

$$\widetilde{v_i} \ll \overline{v_i}$$

$$\mathrm{d}T = 2\rho\left(V + \overline{v_i} + \widetilde{v_i}\right)\left(\overline{v_i} + \widetilde{v_i}\right)x\mathrm{d}x\mathrm{d}\psi$$

$$T = \iint_A \mathrm{d}T = \int_0^{2\pi}\int_0^R 2\rho\left(V + \overline{v_i} + v_o + \tfrac{x}{R}v_s\sin\psi + \tfrac{x}{R}v_c\cos\psi\right)\cdot \left(\overline{v_i} + v_o + \tfrac{x}{R}v_s\sin\psi + \tfrac{x}{R}v_c\cos\psi\right)x\mathrm{d}x\mathrm{d}\psi$$

Neglect $v_o^2,\ v_s^2,\ v_c^2,\ v_ov_s,\ v_ov_c,\ v_sv_c$

$$T = 4\pi\rho\int_0^R\left[\left(V + \overline{v_i}\right)\overline{v_i} + \left(V + \overline{v_i}\right)v_o + \overline{v_i}v_o\right]x\mathrm{d}x$$

$$T = 2\pi R^2\rho\left[\left(V + \overline{v_i}\right)\overline{v_i} + \left(V + 2\overline{v_i}\right)v_o\right]$$

$$T = \overline{T} + \widetilde{T} \qquad \begin{aligned}\overline{T} &= 2\pi R^2\rho\left(V + \overline{v_i}\right)\overline{v_i}\\ \widehat{T} &= 2\pi R^2\rho\left(V + 2\overline{v_i}\right)v_o\end{aligned}$$

$$\text{Note :}\quad \widehat{T} = \frac{\partial\overline{T}}{\partial\overline{v_i}}v_o$$

<u>Nondimensional</u> $\quad \overline{\eta} = \frac{V}{\Omega R} \quad \overline{\lambda} = \frac{V+\overline{\upsilon_i}}{\Omega R}$

$$C_T = \overline{C_T} + \widetilde{C_T} \qquad \frac{\upsilon_o}{\Omega R} = \nu_o, \qquad \frac{\overline{\upsilon_i}}{\Omega R} = \overline{\nu}$$

$$\overline{C_T} = 2\overline{\nu}\left(\lambda\right) = 2\overline{\nu}\left(\overline{\eta} + \overline{\nu}\right)$$

$$\widetilde{C_T} = 2\left(\overline{\eta} + 2\overline{\nu}\right)\nu_o$$

Roll Moment, $L = \iint_A \left(-x \sin\psi\right) \mathrm{d}T$

Pitch Moment, $M = \iint_A \left(-x \cos\psi\right) \mathrm{d}T$

$$L = \int_0^{2\pi} \int_0^R (-2\rho)[(V + \overline{\upsilon_i})\,\overline{\upsilon_i} \sin\psi + (V + \overline{\upsilon_i})\left(\upsilon_o + \tfrac{x}{R}\upsilon_s \sin\psi + \tfrac{x}{R}\upsilon_c \cos\psi\right)\sin\psi$$
$$+ \overline{\upsilon_i}\left(\upsilon_o + \tfrac{x}{R}\upsilon_s \sin\psi + \tfrac{x}{R}\upsilon_c \cos\psi\right)\sin\psi]x^2 \mathrm{d}x\mathrm{d}\psi$$

$$L = \int_0^R \tfrac{2\pi(-\rho)}{R}\left[\left(V + 2\overline{\upsilon_i}\right)\upsilon_s\right] x^3 \mathrm{d}x = \tfrac{-\pi R^4 \rho}{2R}\left(V + 2\overline{\upsilon_i}\right)\upsilon_s$$

$$C_L = -\tfrac{1}{2}\left(\overline{\eta} + 2\overline{\nu}\right)\nu_s = -\tfrac{1}{2}\left(\overline{\lambda} + \overline{\nu}\right)\nu_s$$
$$C_M = -\tfrac{1}{2}\left(\overline{\eta} + 2\overline{\nu}\right)\nu_c = -\tfrac{1}{2}\left(\overline{\lambda} + \overline{\nu}\right)\nu_c$$

$$C_L = -\frac{V}{2}\nu_s, \; C_M = -\frac{V}{2}\nu_c$$

$$V \equiv \left|\overline{\eta} + 2\overline{\nu}\right|$$

## II. Equivalent Lock Number (cycle flapping)

$$\overset{**}{\beta} + \tfrac{\gamma}{8}\overset{*}{\beta} + p^2\beta = \tfrac{\gamma}{8}\left(\theta - \phi\right)$$

$$\overset{**}{\beta} + p^2\beta = \tfrac{\gamma}{8}\left(\theta_s \sin\psi + \theta_c \cos\psi - \nu_s \sin\psi - \nu_c \cos\psi - \overset{*}{\beta}\right)$$

$$\beta = \beta_s \sin\psi + \beta_c \cos\psi$$

$$\left(p^2 - 1\right)\beta_s = \tfrac{\gamma}{8}\left(\theta_s - \nu_s + \beta_c\right)$$

$$\left(p^2 - 1\right)\beta_c = \tfrac{\gamma}{8}\left(\theta_c - \nu_c - \beta_s\right)$$

$$\frac{-\beta_s\left(p^2-1\right)}{2}\frac{\sigma a}{\gamma} = C_L = -\frac{V}{2}\nu_s \Rightarrow \nu_s = \frac{\sigma a}{V}\frac{\left(p^2-1\right)\beta_s}{\gamma}$$

$$\frac{-\beta_c\left(p^2-1\right)}{2}\frac{\sigma a}{\gamma} = C_M = -\frac{V}{2}\nu_c \Rightarrow \nu_c = \frac{\sigma a}{V}\frac{\left(p^2-1\right)\beta_c}{\gamma}$$

$$\left(p^2 - 1\right)\beta_s\left[1 + \frac{\sigma a}{8V}\right] = \frac{\gamma}{8}\left(\theta_s + \beta_c\right)$$

$$\left(p^2 - 1\right)\beta_c\left[1 + \frac{\sigma a}{8V}\right] = \frac{\gamma}{8}\left(\theta_c - \beta_s\right)$$

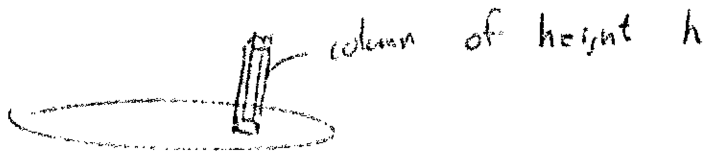
$$\left(p^2 - 1\right)\beta_s = \frac{\gamma^*}{8}\left(\theta_s + \beta_c\right)$$

$$\left(p^2 - 1\right)\beta_c = \frac{\gamma^*}{8}\left(\theta_c - \beta_s\right)$$

$$\gamma^* = \frac{\gamma}{1 + \frac{\sigma a}{8V}} = \gamma\left[1 - \frac{1}{1 + \frac{8V}{\sigma a}}\right]$$

# III. Unsteady Perturbations

$$\mathrm{d}T = \mathrm{d}\dot{m}\Delta v + \mathrm{d}m\Delta\dot{v}$$




$$\mathrm{d}m = \rho h \mathrm{d}A$$

Impermeable Disc $m = \frac{8}{3}\rho R^3, \quad I_A = \frac{16}{45}\rho R^5$

Nondimensional $\quad K_m = \frac{8}{3\pi}, \quad K_I = \frac{16}{45\pi}$

$$K_m \overset{*}{\nu}_o + 2\nu_o V = \widetilde{C_T} = \frac{\sigma a}{6}\left(\widetilde{\theta}_o - \frac{3}{2}\nu_o - \overset{*}{\beta}_o\right)$$

$$K_I \overset{*}{\nu}_s + \frac{V}{2}\nu_s = -\widetilde{C_L} = \frac{\sigma a}{16}\left(\widetilde{\theta}_s - \nu_s + \beta_c - \overset{*}{\beta}_s\right)$$

$$K_I \overset{*}{\nu}_c + \frac{V}{2}\nu_c = -\widetilde{C_M} = \frac{\sigma a}{16}\left(\widetilde{\theta}_c - \nu_c - \beta_s - \overset{*}{\beta}_c\right)$$

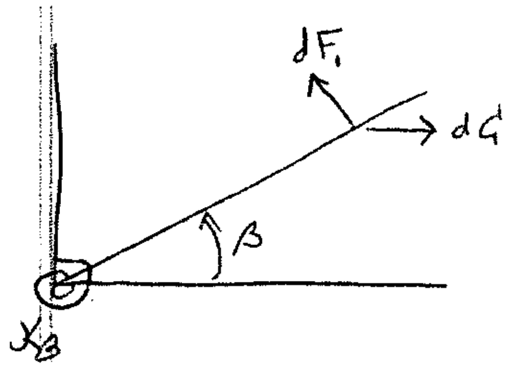
$$V = \eta + 2\overline{\nu} \qquad \overline{C_T} = 2\overline{\nu}\left(\eta + \overline{\nu}\right)$$

---

Frequency Response, cyclic, $V \to (V + 2K_I i\omega)$

$$\gamma^* = \gamma\left[1 - \frac{1}{1 + \frac{8V}{\sigma a} + \frac{16K_I}{\sigma a} i\omega}\right]$$

# Derivation of Equations




$$I_\beta = \int_0^R m x^2 \mathrm{d}x$$

$$I_\beta \ddot{\beta} = \int_0^R \left(\mathrm{d}F_1 x - \mathrm{d}C x \sin\beta\right) - K_\beta \beta$$

$$\mathrm{d}F_1 = \tfrac{1}{2}\rho a c \mathrm{d}x \left(\sqcup_T^2 \theta - \sqcup_P \sqcup_T\right)$$

$$\mathrm{d}C = \mathrm{d}m \Omega^2 x \cos\beta \approx \mathrm{d}x \Omega^2 x m$$

$$\sqcup_T = \Omega x, \quad \sqcup_P = \dot{\beta} x + V + v$$

$$I_\beta \ddot{\beta} = \int_0^R \left[\tfrac{1}{2}\rho a c \Omega^2 x^2 \theta - \tfrac{1}{2}\rho a c \Omega x \left(\dot{\beta} x + V + v\right)\right] x \mathrm{d}x - \int_0^R \beta m \Omega^2 x^2 \mathrm{d}x - K_\beta \beta$$

$$I_\beta \ddot{\beta} + \Omega^2 I_\beta \beta = \tfrac{1}{8}\rho a c \left[\Omega^2 R^4 \theta - R^4 \dot{\beta} \Omega - R^4 \phi \Omega^2\right] - K_\beta \beta$$

$$\overset{*}{()} = \frac{\mathrm{d}}{\mathrm{d}(\Omega t)}, \quad \gamma = \frac{\rho a c R^4}{I_\beta}$$

$$\overset{**}{\beta} + \tfrac{\gamma}{8}\overset{*}{\beta} + p^2 \beta = \tfrac{\gamma}{8}(\theta - \phi)$$

$$p^2 = 1 + \frac{K_\beta}{\Omega^2 I_\beta}$$

# III. Rigid-Blade Flapping

| | |
|---|---|
| Gessow & Meyers | 8 |
| Johnson | 5 |
| Bramwell | 2,5 |

$$\overset{**}{\beta} + \tfrac{\gamma}{8}\overset{*}{\beta} + p^2\beta = \tfrac{\gamma}{8}(\theta - \phi)$$

$$\overset{*}{()} = \frac{\mathrm{d}}{\mathrm{d}\psi}$$

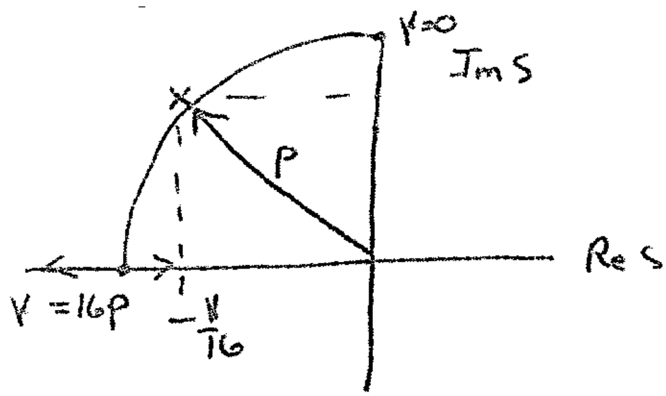
$$\gamma = \frac{\rho a c R^4}{I_\beta}$$

$$p^2 = 1 + \frac{K_\beta}{\Omega^2 I_\beta}$$

Transients:

$$s^2 + \tfrac{\gamma}{8}s + p^2 = 0$$

$$s = -\tfrac{\gamma}{16} \pm \sqrt{\left(\tfrac{\gamma}{16}\right)^2 - p^2} = -\tfrac{\gamma}{16} \pm i\sqrt{p^2 - \left(\tfrac{\gamma}{16}\right)^2}$$




$$\beta = e^{-\frac{\gamma}{16}\psi}\left[A\cos\sqrt{p^2 - \left(\tfrac{\gamma}{16}\right)^2}\psi + B\sin\sqrt{p^2 - \left(\tfrac{\gamma}{16}\right)^2}\psi\right]$$

Example: $\gamma = 5,\ \psi = 2\pi\ =$ one rotor revolution

$$e^{-\frac{5}{16}(2\pi)} = .14 \qquad 86\%\ \text{reduction}$$

Forced Response: $\theta = \theta_o + \theta_s \sin\psi + \theta_c \cos\psi$

Harmonic Balance: $\beta = \beta_o + \beta_s \sin\psi + \beta_c \cos\psi$

$$\overset{*}{\beta} = +\beta_s \cos\psi - \beta_c \sin\psi$$

$$n = \frac{\gamma}{8} \qquad \overset{**}{\beta} = -\beta_s \sin\psi - \beta_c \cos\psi$$

$$\begin{bmatrix} p^2 & 0 & 0 \\ 0 & p^2-1 & -n \\ 0 & n & p^2-1 \end{bmatrix} \begin{Bmatrix} \beta_o \\ \beta_s \\ \beta_c \end{Bmatrix} = n \begin{Bmatrix} \theta_o - \phi \\ \theta_s \\ \theta_c \end{Bmatrix}$$

$$\beta_o = \frac{n}{p^2}\left(\theta_o - \phi\right)$$

$$\begin{Bmatrix} \beta_s \\ \beta_c \end{Bmatrix} = \frac{n}{\left(p^2-1\right)^2 + n^2} \begin{bmatrix} p^2-1 & n \\ -n & p^2-1 \end{bmatrix} \begin{Bmatrix} \theta_s \\ \theta_c \end{Bmatrix}$$

$$\beta_s = \frac{n\left(p^2-1\right)\theta_s + n^2\theta_c}{\left(p^2-1\right)^2 + n^2}$$

$$\beta_c = \frac{-n^2\theta_s + n\left(p^2-1\right)\theta_c}{\left(p^2-1\right)^2 + n^2}$$

# EQUATIONS FOR RADIAL INFLOW

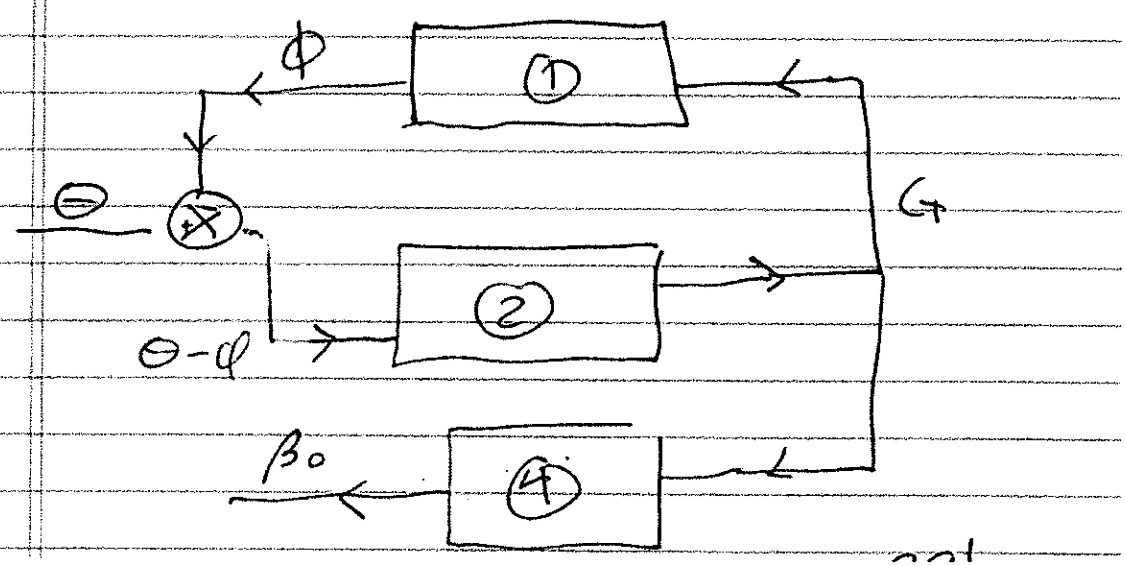
$\nu = \phi r \qquad \varphi = \text{constant}$

1.) Momentum: $C_T = \phi^2$ (no loss)

2.) Blade Element: $\frac{C_T}{\sigma a} = \frac{B^3}{6}(\theta - \phi)$

3.) Blade Dynamics: $\beta_o = \frac{\gamma B^4}{8p^2}(\theta - \phi)$

4.) $\beta_o = \frac{3B\gamma}{4\sigma a p^2} C_T$



# PERTURBATIONS

$$\Delta C_T = 2\phi\Delta\phi$$

$$\frac{\Delta C_T}{\sigma a} = \frac{B^3}{6}\left(\Delta\theta - \Delta\phi\right)$$

$$\Delta\beta_o = \frac{\gamma B^4}{8p^2}\left(\Delta\theta - \Delta\phi\right)$$

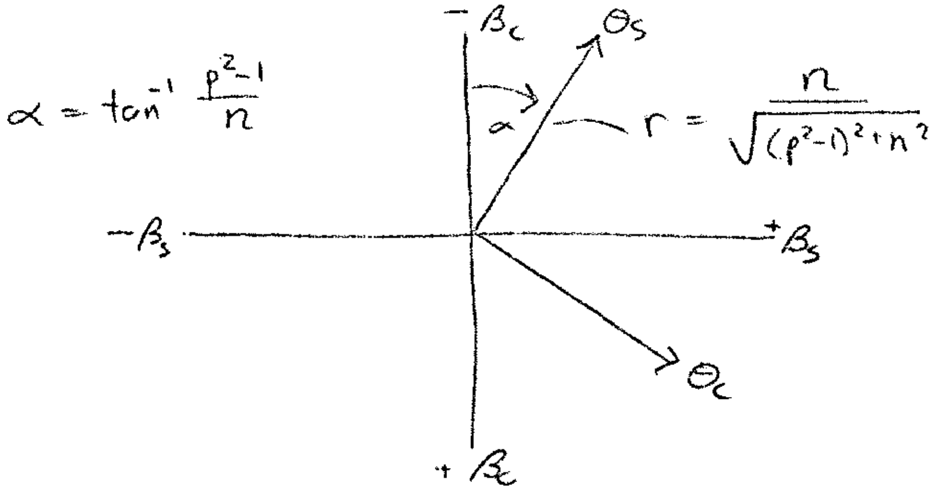
With inflow feedback:

$$\frac{\Delta C_T}{\sigma a} = \frac{1}{\left(1 + \frac{\sigma a}{8V}\right)}\frac{B^3}{6}\Delta\theta$$

$$\Delta\beta_o = \frac{\gamma B^4}{\left(1 + \frac{\sigma a}{8V}\right)}\frac{\Delta\theta}{8p^2}$$

$$\gamma_t^* = \frac{B^4\gamma}{\left(1 + \frac{\sigma a}{8V}\right)} \quad V = \tfrac{3}{2}\phi$$

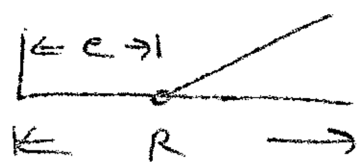
$$\beta_s^2 + \beta_c^2 = \frac{n^2\left(\theta_s^2 + \theta_c^2\right)}{\left(p^2 - 1\right)^2 + n^2}$$




# Phase Shift of Pilot Control

## Effect of Hinge Offset

$$\epsilon \equiv \frac{e}{R}$$

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$$\overset{**}{\beta} + \frac{\gamma}{8}\left[\frac{1 - \frac{8}{3}\epsilon + 2\epsilon^2 - \frac{1}{3}\epsilon^4}{(1-\epsilon)^3}\right]^{*} \overset{*}{\beta} + \left[1 + \frac{K_\beta}{\Omega^2 I_y} + \overbrace{\frac{3\epsilon}{2(1-\epsilon)}}\right]\beta$$

$$= \frac{\gamma}{8}\left[\frac{\left(1 - \frac{4}{3}\epsilon + \frac{1}{3}\epsilon^4\right)}{(1-\epsilon)^3}\right]^{**}(\theta_o - \phi)$$

most important $\epsilon$ term since it affects

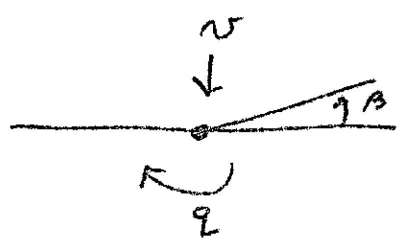
$$p^2 = 1 + \frac{K_\beta}{\Omega^2 I_y} + \frac{3\epsilon}{2(1-\epsilon)}$$

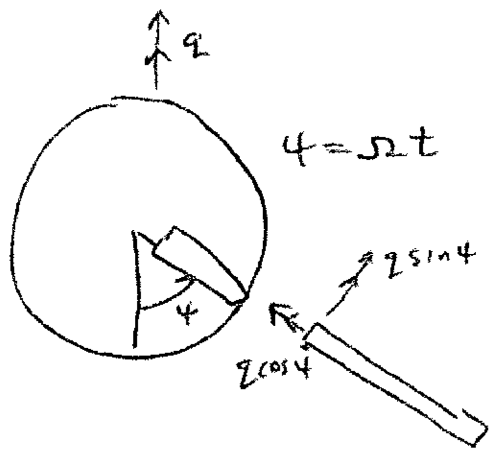
$$[\ \ ]^{**} = \left[\frac{1 + \frac{2}{3}\epsilon + \frac{1}{3}\epsilon^2}{1-\epsilon}\right]$$



$$^{*}[\ \ ] = [1 + \epsilon/3]$$

# Response to Roll Rate




$$V = \frac{1}{2} K_\beta \beta^2$$

$$T = \tfrac{1}{2} \int_0^R m \left[ (\Omega \cos\beta + q \cos\psi \sin\beta)^2 + \left(\dot\beta^2 - q \sin\psi\right)^2 \right] x^2 \mathrm{d}x$$

$$T = \tfrac{1}{2} I_\beta \left[ \Omega^2 \cos^2\beta + 2\Omega q \cos\beta \sin\beta \cos\psi + q^2 \cos^2\psi \sin^2\beta \right.$$
$$\left. + \dot\beta^2 - 2\dot\beta q \sin\psi + q^2 \sin^2\psi \right]$$

$$\frac{\partial T}{\partial \dot\beta} = I_\beta \left[ \dot\beta - q \sin\psi \right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot\beta} = I_\beta \left[ \ddot\beta - \dot q \sin\psi - q\Omega \cos\psi \right]$$

$$\frac{\partial T}{\partial \beta} = I_\beta \left[ -\Omega^2 \sin\beta \cos\beta + \Omega q \cos\psi \left( \cos^2\beta - \sin^2\beta \right) \right.$$
$$\left. + q^2 \cos^2\psi \sin\beta \cos\beta \right]$$

## Linearized Equations (Vacuum)

$$I_\beta \ddot\beta + \left( I_\beta \Omega^2 + K_\beta \right) \beta = I_\beta \dot q \sin\psi + 2 I_\beta q \Omega \cos\psi$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot\beta} - \frac{dT}{\partial \beta} + \frac{\partial V}{\partial \beta} = 0$$

# Nondimensionalize and add aerodynamics

$$\overline{q} = \frac{q}{\Omega},\ \lambda = \frac{v}{\Omega R} = -\overline{q} r \sin\psi,\ \phi = -\overline{q}\sin\psi$$

$$\overset{**}{\beta} + \tfrac{\gamma}{8}\overset{*}{\beta} + p^2\beta = \overset{*}{\overline{q}}\sin\psi + 2\overline{q}\cos\psi + \tfrac{\gamma}{8}\overline{q}\sin\psi$$

$$\beta_c = \frac{2\left(p^2-1\right)-n^2}{\left(p^2-1\right)^2+n^2}\overline{q} - \frac{n}{\left(p^2-1\right)^2+n^2}\overset{*}{\overline{q}}$$

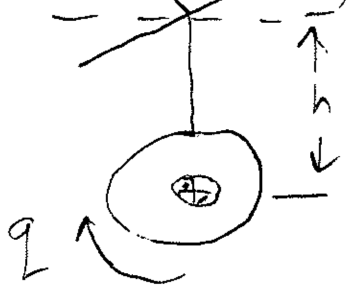
$$\beta_s = \frac{n\left(p^2+1\right)}{\left(p^2-1\right)^2+n^2}\overline{q} + \frac{p^2-1}{\left(p^2-1\right)^2+n^2}\overset{*}{\overline{q}}$$

$$p = 1,\ \dot{\overline{q}} = 0,\ \beta_c = -\overline{q},\ \beta_s = \tfrac{2}{n}\overline{q}$$

Loading...

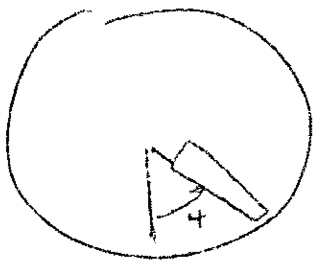
roll damping

Roll Moment Due to hub height




$$\frac{C_L}{\sigma a} = -\beta_s \frac{h}{R}\frac{C_T}{\sigma a}$$

# Pitch and Roll Moments (Average)



$$L = \frac{b}{2\pi} \int_0^{2\pi} (\beta K_\beta)(-\sin\psi)\,\mathrm{d}\psi$$
$$\beta = \beta_o + \beta_s \sin\psi + \beta_c \cos\psi$$
$$L = -\tfrac{b}{2}\beta_s K_\beta$$

$$C_L = \frac{L}{\rho\pi R^2 (\Omega^2 R^2) R} = \frac{-bK_\beta}{2\rho\pi R^5 \Omega^2}\beta_s = \frac{-\sigma a}{2\gamma}\left(p^2 - 1\right)\beta_s$$

$$\frac{C_L}{\sigma a} = \frac{-\beta_s\left(p^2 - 1\right)}{2\gamma}, \quad \frac{C_M}{\sigma a} = \frac{-\beta_c\left(p^2 - 1\right)}{2\gamma}$$

$$\left(\frac{C_L}{\sigma a}\right)_{\text{aerodynamic}} = \text{same}$$

$$\frac{C_L}{\sigma a} = -\beta_s\left[\frac{(p^2 - 1)}{2\gamma} + \frac{C_T}{\sigma a}\frac{h}{R}\right]$$
$$\frac{C_M}{\sigma a} = -\beta_c\left[\frac{(p^2 - 1)}{2\gamma} + \frac{C_T}{\sigma a}\frac{h}{R}\right]$$

Total Moments

# IV. Blade dynamics (axial flow)

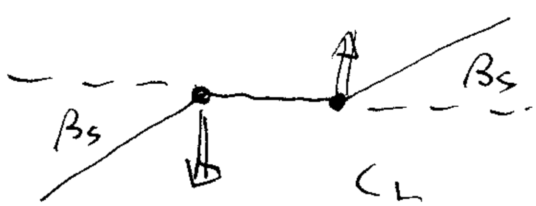
$$\overset{**}{\beta} + \frac{\gamma}{8}\overset{*}{\beta} + p^2\beta = \frac{\gamma}{8}(\theta - \phi) + \overset{*}{\overline{q}}\sin\psi + 2\overline{q}\cos\psi + \frac{\gamma}{8}\overline{q}\sin\psi$$

$$\gamma = \frac{\rho a c R^4}{I_\beta} \qquad p^2 = 1 + \frac{K_\beta}{\Omega^2 I_\beta} + \frac{3}{2}\frac{e}{R-e}$$

$$\overline{q} = \frac{q}{\Omega} \quad \text{(roll rate)}\ \theta = \theta_o + \theta_s \sin\psi + \theta_c \cos\psi$$

Roll Moment $$\frac{C_L}{\sigma a} = -\beta_s \frac{(p^2 - 1)}{2\gamma}$$

$$\frac{C_M}{\sigma a} = -\beta_c \frac{(p^2 - 1)}{2\gamma}$$




$$\frac{C_L}{\sigma a} = -\beta_s \left[\frac{3\,\epsilon}{2(1-\epsilon)}\right]$$

# Summary Thus Far

## I. Momentum Theory (axial flow)

$$\frac{\mathrm{d}C_T}{\mathrm{d}r} = 4(\eta+\nu)\nu r = 4\lambda\nu r = 4\phi\nu r^2$$

$$\lambda = \eta+\nu,\ \phi = \tfrac{\eta+\nu}{r}$$

## II. Blade element theory (axial flow)

$$\frac{\mathrm{d}C_T}{\mathrm{d}r} = \tfrac{\sigma a}{2}\left(r^2\theta - r\lambda\right) = \tfrac{\sigma a}{2}r^2(\theta-\phi)$$

$$\frac{\mathrm{d}C_P}{\mathrm{d}r} = \tfrac{\sigma a}{2}\left[r^3\tfrac{C_d}{a} + r^2\theta\lambda - r\lambda^2\right] = \tfrac{\sigma a}{2}r^3\left[\tfrac{C_d}{a} + (\theta-\phi)\phi\right]$$

## III. Tip-loss correction

$$\frac{\mathrm{d}C_T}{\mathrm{d}r} = 4(\eta+\nu)\nu r k$$

$$k = \tfrac{2}{\pi}\cos^{-1}\left(e^{-f}\right) \quad f = \tfrac{1}{2}\tfrac{b(1-r)}{\sin\phi_A}$$

$$\phi_A = \phi(r=1)\ \text{ w/o tip-loss}$$

Poor-man'sapproximation:

$$B = 1 - \tfrac{2\ln(2)\sin\phi_A}{b}$$

# IV. Blade dynamics (axial flow)

$$\overset{**}{\beta} + \tfrac{\gamma}{8}\overset{*}{\beta} + p^2\beta = \tfrac{\gamma}{8}(\theta - \phi)$$
$$+ \overset{*}{\overline{q}}\sin\psi + 2\overline{q}\cos\psi + \tfrac{\gamma}{8}\overline{q}\sin\psi$$

$$\gamma = \frac{\rho a c R^4}{I_\beta} \qquad p^2 = 1 + \frac{K_\beta}{\Omega^2 I_\beta} + \frac{3}{2}\frac{e}{R-e}$$

$$\overline{q} = \frac{q}{\Omega} \quad \text{(roll rate)}\ \theta = \theta_o + \theta_s \sin\psi + \theta_c \cos\psi$$

Roll Moment $\dfrac{C_L}{\sigma a} = -\beta_s \dfrac{(p^2-1)}{2\gamma}$

$$\frac{C_M}{\sigma a} = -\beta_c \frac{(p^2-1)}{2\gamma}$$

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$$\gamma_t = \frac{\rho a c R^4 B^4}{I_\beta}$$

$$\gamma_t^* = \frac{\gamma_t}{1 + \frac{\sigma a}{8V}}$$

$$V = \eta + 2\nu$$

$$\frac{C_L}{\sigma a} = -\beta_s \left[ \frac{(p^2 - 1)}{2} \gamma + \frac{C_T}{\sigma a} \frac{h}{R} \right]$$

$$\frac{C_M}{\sigma a} = -\beta_c \left[ \frac{(p^2 - 1)}{2} \gamma + \frac{C_T}{\sigma a} \frac{h}{R} \right]$$

$$\begin{bmatrix} p^2 - 1 & -n \\ n & p^2 - 1 \end{bmatrix} \begin{Bmatrix} \beta_s \\ \beta_c \end{Bmatrix} = n \begin{Bmatrix} \theta_s \\ \theta_c \end{Bmatrix} + \overset{*}{\overline{q}} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + \overline{q} \begin{Bmatrix} n \\ 2 \end{Bmatrix}$$

$$\begin{aligned} \begin{Bmatrix} \beta_s \\ \beta_c \end{Bmatrix} = & \frac{n}{(p^2 - 1)^2 + n^2} \begin{bmatrix} p^2 - 1 & n \\ -n & p^2 - 1 \end{bmatrix} \begin{Bmatrix} \theta_s \\ \theta_c \end{Bmatrix} \\ & + \frac{1}{(p^2 - 1)^2 + n^2} \begin{bmatrix} p^2 - 1 & n \\ -n & p^2 - 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \overset{*}{\overline{q}} \\ & + \frac{1}{(p^2 - 1)^2 + n^2} \begin{bmatrix} p^2 - 1 & n \\ -n & p^2 - 1 \end{bmatrix} \begin{Bmatrix} n \\ 2 \end{Bmatrix} \overline{q} \end{aligned}$$

$$= \frac{1}{(p^2 - 1)^2 + n^2} \begin{Bmatrix} n (p^2 - 1) \theta_s + (p^2 - 1) \overset{*}{\overline{q}} + n (p^2 + 1) \overline{q} + n^2 \theta_c \\ -n^2 \theta_s + (p^2 - 1) \theta_c n - n \overset{*}{\overline{q}} + [2 (p^2 - 1) - n^2] \overline{q} \end{Bmatrix}$$

$$\gamma_t = \frac{\rho a c R^4 B^4}{I_y}$$ tip-loss Lock Number

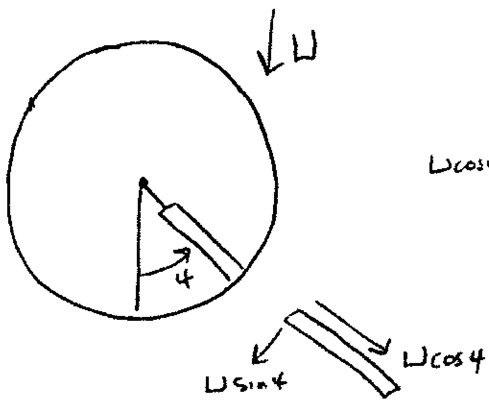
$$\gamma^* = \frac{\gamma_t}{1 + \frac{\sigma a}{8V}}$$ equivalent Lock Number

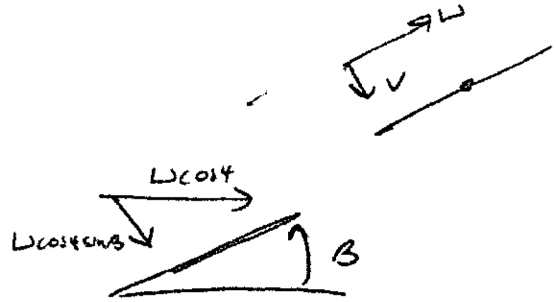
$$V = \eta + 2\nu$$

$$\frac{C_T}{\sigma a} = \frac{B^3}{6}(\theta - \phi)$$

$$B = 1 - \frac{2\ln(2)\sin\phi}{b}$$

# Forward Flight




$$\sqcup_T = \Omega x + \sqcup \sin\psi, \quad \sqcup_P = V + \dot{\beta} x + \sqcup \cos\psi \beta$$

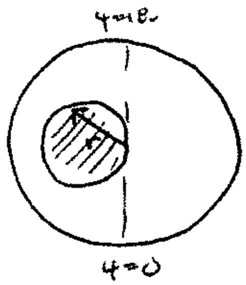
$$\begin{aligned} I_y \ddot{\beta} + \left(\Omega^2 I_y + K_\beta\right)\beta = \tfrac{1}{2}\rho a c \int_0^R & \left[(\Omega x + \sqcup \sin\psi)^2 \theta \right. \\ & \left. - (\Omega x + \sqcup \sin\psi)\left(V + \dot{\beta} x + \sqcup \cos\psi\beta\right)\right] x \mathrm{d}x \end{aligned}$$

$$V = V_o + V_1 \tfrac{x}{R}, \quad \theta = \theta_o + \theta_s \sin\psi + \theta_c \cos\psi$$

$$\mu = \frac{\sqcup}{\Omega R}, \quad \lambda = \frac{V_o}{\Omega R}, \quad \phi = \frac{V_1}{\Omega R}$$

$$\begin{aligned} & \overset{**}{\beta} + \tfrac{\gamma}{8}\left(1 + \tfrac{4}{3}\mu \sin\psi\right)\overset{*}{\beta} + \left[p^2 + \tfrac{\gamma}{8}\left(\tfrac{4}{3} + 2\mu\sin\psi\right)\mu\cos\psi\right]\beta \\ & \quad = \tfrac{\gamma}{8}\left(1 + \tfrac{8}{3}\mu\sin\psi + 2\mu^2\sin^2\psi\right)\left(\theta_o + \theta_s\sin\psi + \theta_c\cos\psi\right) \\ & \qquad - \tfrac{\gamma}{8}\left(1 + \tfrac{4}{3}\mu\sin\psi\right)\phi - \tfrac{\gamma}{6}\left(1 + \tfrac{3}{2}\mu\sin\psi\right)\lambda \\ & + \tfrac{\gamma}{8}\left(1 + \tfrac{4}{3}\mu\sin\psi\right)\overline{q}\sin\psi + \overset{*}{\overline{q}}\sin\psi + 2\overline{q}\cos\psi \end{aligned}$$

# Reversed Flow



$$r + \mu \sin \psi < 0$$
$$r < -\mu \sin \psi$$
$$\int_0^{r+\mu \sin \psi} -L_x \mathrm{d}x + \int_{r+\mu \sin \psi}^{1} +L_x \mathrm{d}x$$

order $\mu^4$ in Flapping

# Harmonic Balance

$$\beta = a_o + a_1 \cos \psi + b_1 \sin \psi + b_2 \sin 2\psi + a_2 \cos 2\psi + \cdots$$

## 1st harmonics only:

$$
\begin{aligned}
&(-a_1 \cos \psi - b_1 \sin \psi) + n\,(-a_1 \sin \psi \qquad + b_1 \cos \psi) \\
&\quad + \tfrac{4}{3} n\mu \left(-\tfrac{1}{2} a_1 + \tfrac{1}{2} a_1 \cos 2\psi + \tfrac{b_1}{2} \sin 2\psi\right) \\
&\quad + p^2 \left(a_o + a_1 \cos \psi + b_1 \sin \psi\right) \\
&\quad + n\mu \left(\tfrac{4}{3} \cos \psi + \mu \sin 2\psi\right) \left(a_o + a_1 \cos \psi + b_1 \sin \psi\right) \\
&\quad = n \left(1 + \tfrac{8}{3}\mu \sin \psi + \mu^2 - \mu^2 \cos 2\psi\right) \left(\theta_o + \theta_s \sin \psi + \theta_c \cos \psi\right) \\
&\qquad\qquad - n\phi \left(1 + \tfrac{4}{3}\mu \sin \psi\right)
\end{aligned}
$$

$$\begin{bmatrix} p^2 & 0 & 0 \\ \frac{4}{3}n\mu & p^2-1 & n\left(1+\frac{\mu^2}{2}\right) \\ 0 & -n\left(1-\frac{\mu^2}{2}\right) & p^2-1 \end{bmatrix} \begin{Bmatrix} a_o \\ a_1 \\ b_1 \end{Bmatrix} =$$

$$\begin{bmatrix} n\left(1+\mu^2\right) & \frac{4}{3}n\mu & 0 & -n \\ 0 & 0 & n\left(1+\frac{1}{2}\mu^2\right) & 0 \\ \frac{8}{3}n\mu & n\left(1+\frac{3}{2}\mu^2\right) & 0 & -\frac{4}{3}n\mu \end{bmatrix} \begin{Bmatrix} \theta_o \\ \theta_s \\ \theta_c \\ \phi \end{Bmatrix}$$

$$a_o = \frac{n}{p^2}\left(1+\mu^2\right)\theta_o + \frac{4}{3}\frac{n\mu}{p^2}\theta_s - \frac{n}{p^2}\phi$$

$$\begin{bmatrix} p^2-1 & n\left(1+\frac{\mu^2}{2}\right) \\ -n\left(1-\frac{\mu^2}{2}\right) & p^2-1 \end{bmatrix} \begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix} =$$

$$\begin{bmatrix} -\frac{4}{3}\frac{n^2\mu}{p^2}\left(1+\mu^2\right) & -\frac{4}{3}\frac{4}{3}n^2\mu^2\frac{1}{p^2} & n\left(1+\frac{1}{2}\mu^2\right) & +\frac{4}{3}\frac{n^2\mu}{p^2} \\ \frac{8}{3}n\mu & n\left(1+\frac{3}{2}\mu^2\right) & 0 & -\frac{4}{3}n\mu \end{bmatrix} \begin{Bmatrix} \theta_o \\ \theta_s \\ \theta_c \\ \phi \end{Bmatrix}$$

$$\text{Determinant} = \left(p^2 - 1\right)^2 + n^2\left(1 - \tfrac{\mu^4}{4}\right)$$

$$\text{Det} = 0 \text{ when } \mu = \sqrt{2}\left[1 + \tfrac{\left(p^2-1\right)^2}{n^2}\right]^{\frac{1}{4}}$$

Neglect $\mu^4$ if reversed flow neglected

$$\mathrm{C} \equiv \frac{8\left(p^2-1\right)}{\gamma} = \frac{p^2-1}{n}$$

$$\begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix} = \frac{1}{1+\mathrm{C}^2}\begin{bmatrix} \mathrm{C} & -\left(1-\frac{\mu^2}{2}\right) \\ \left(1-\frac{\mu^2}{2}\right) & \mathrm{C} \end{bmatrix} \cdot$$

$$\begin{bmatrix} -\frac{4}{3}\frac{n\mu}{p^2}\left(1+\mu^2\right) & -\frac{16}{9}\frac{n\mu^2}{p^2} & \left(1+\frac{1}{2}\mu^2\right) & \frac{4}{3}\frac{n\mu}{p^2} \\ \frac{8}{3}\mu & \left(1+\frac{3}{2}\mu^2\right) & 0 & -\frac{4}{3}\mu \end{bmatrix}\begin{Bmatrix} \theta_o \\ \theta_s \\ \theta_c \\ \phi \end{Bmatrix}$$

$$\frac{a_1}{\theta_o} = \frac{-\frac{4}{3}\frac{n\mu}{p^2}\left(1+\mu^2\right)\mathrm{C} - \frac{8}{3}\mu\left(1+\frac{\mu^2}{2}\right)}{1+\mathrm{C}^2} \approx \frac{-\frac{8}{3}\mu\left(1+\frac{n\mathrm{C}}{2p^2}\right)}{1+\mathrm{C}^2}$$

$$\frac{a_1}{\theta_s} = \frac{-\frac{16}{9}\frac{n\mu^2}{p^2}\mathrm{C} - \left(1+2\mu^2\right)}{1+\mathrm{C}^2} \approx \frac{-\left(1+2\mu^2\right) - \frac{16}{9}\frac{n\mu^2}{p^2}\mathrm{C}}{1+\mathrm{C}^2}$$

$$\frac{a_1}{\theta_c} = \frac{\left(1+\frac{1}{2}\mu^2\right)\mathrm{C}}{1+\mathrm{C}^2}$$

$$\frac{a_1}{\phi} = \frac{\frac{4}{3}\frac{n\mu}{p^2}\mathrm{C} + \frac{4}{3}\mu\left(1+\frac{\mu^2}{2}\right)}{\mathrm{C}^2+1}$$

$$\frac{b_1}{\theta_o} = \frac{-\frac{4}{3}\frac{n\mu}{p^2}\left(1-\frac{\mu^2}{2}\right)+\frac{8}{3}\mu\mathrm{C}}{1+\mathrm{C}^2} \approx \frac{\frac{8}{3}\mu\left[\mathrm{C}-\frac{n}{2p^2}\right]}{1+\mathrm{C}^2}$$

$$\frac{b_1}{\theta_s} = \frac{-\frac{16}{9}\frac{n\mu^2}{p^2}\left(1-\frac{\mu^2}{2}\right)+\mathrm{C}\left(1+\frac{3}{2}\mu^2\right)}{1+\mathrm{C}^2}$$

$$\frac{b_1}{\theta_c} = \frac{1}{1+\mathrm{C}^2}$$

$$\frac{b_1}{\phi} = \frac{\frac{4}{3}\frac{n\mu}{p^2}\left(1-\frac{\mu^2}{2}\right)-\frac{4}{3}\mathrm{C}\mu}{1+\mathrm{C}^2}$$

$$\frac{C_T}{\sigma a} = \frac{1}{6}\left(1+\frac{3}{2}\mu^2\right)\theta_o - \frac{1}{6}\phi + \frac{1}{4}\mu\theta_s$$

$$\begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix} = \begin{bmatrix} a_1/\theta_o & a_1/\theta_s & a_1/\theta_c & a_1/\phi \\ b_1/\theta_o & b_1/\theta_s & b_1/\theta_c & b_1/\phi \end{bmatrix} \begin{Bmatrix} \theta_0 \\ \theta_s \\ \theta_c \\ \phi \end{Bmatrix}$$

The terms are partial derivatives – components of cyclic flapping.

# Forward Flight Loads with Inflow

$$\frac{C_T}{\sigma a} = \tfrac{1}{6}\left(1+\tfrac{3}{2}\mu^2\right)\theta_o - \tfrac{1}{4}\nu_o + \tfrac{1}{4}\mu\theta_s - \tfrac{1}{8}\mu\nu_s$$

$$\frac{C_L}{\sigma a} = -\tfrac{1}{16}\left[\theta_s\left(1+\tfrac{3}{2}\mu^2\right) + \tfrac{8}{3}\mu\theta_o - \nu_s - 2\mu\nu_o + \beta_c\left(1-\tfrac{\mu^2}{2}\right)\right]$$

$$\frac{C_M}{\sigma a} = -\tfrac{1}{16}\left[\theta_c\left(1+\tfrac{1}{2}\mu^2\right) - \nu_c - \beta_s\left(1+\tfrac{\mu^2}{2}\right) - \tfrac{4}{3}\mu\beta_o\right]$$

Alternate Form:

$$\frac{C_T}{\sigma a} = \frac{4p^2}{3}\frac{1}{\gamma}\beta_o$$

$$\frac{C_L}{\sigma a} = \frac{-\left(p^2-1\right)}{2\gamma}\beta_s$$

$$\frac{C_M}{\sigma a} = \frac{-\left(p^2-1\right)}{2\gamma}\beta_c$$

$$\begin{Bmatrix}\nu_o\\ \nu_s\\ \nu_c\end{Bmatrix} = \frac{1}{\mu}\begin{bmatrix}\frac{1}{2} & 0 & \frac{15\pi}{64}\\ 0 & -4 & 0\\ \frac{15\pi}{64} & 0 & 0\end{bmatrix}\begin{Bmatrix}C_T\\ C_L\\ C_M\end{Bmatrix} \quad \text{(edgewise)}$$

$$\frac{15\pi}{64} \approx \frac{3}{4}$$

Homework: add $\nu_o, \nu_s, \nu_c$ formulas
to forward-flight harmonic balance.
Use $\beta_0, \beta_s, \beta_c,$ formulas.

$$\frac{\text{Root Moment}}{\sigma a} = \tfrac{1}{8}\left(1+\mu^2\right)\theta_o - \tfrac{1}{6}\nu_o + \tfrac{1}{6}\mu\theta_s - \tfrac{1}{12}\mu\nu_s$$

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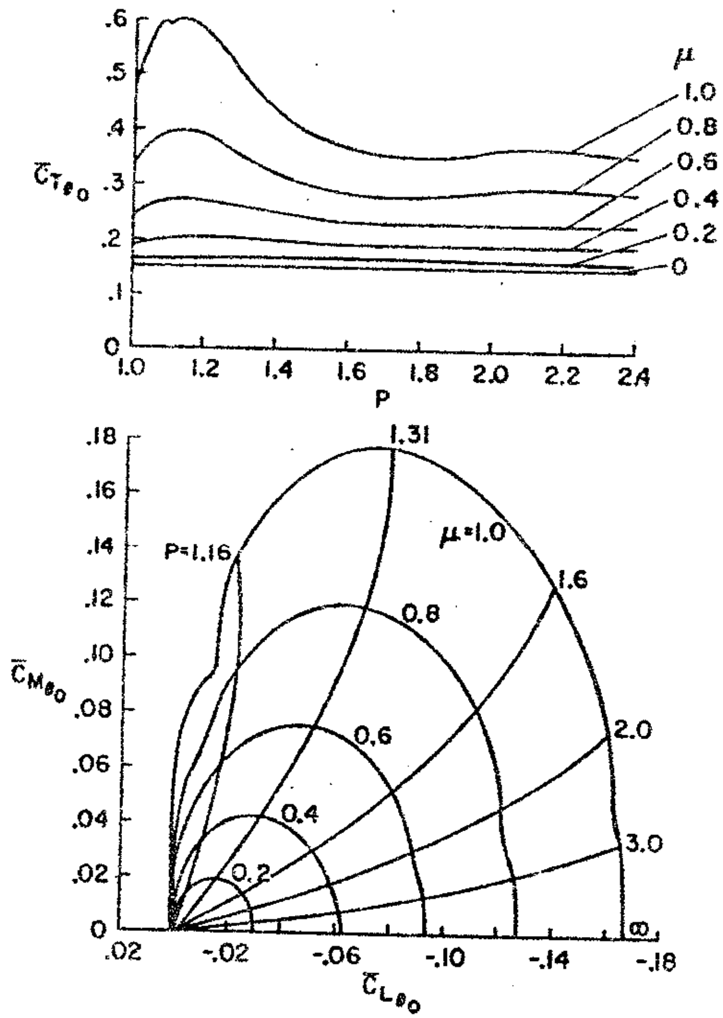



Figure 22.— Response derivatives with respect to $\theta_0$, $\gamma = 5$, $\theta_{q_1} = 0$.

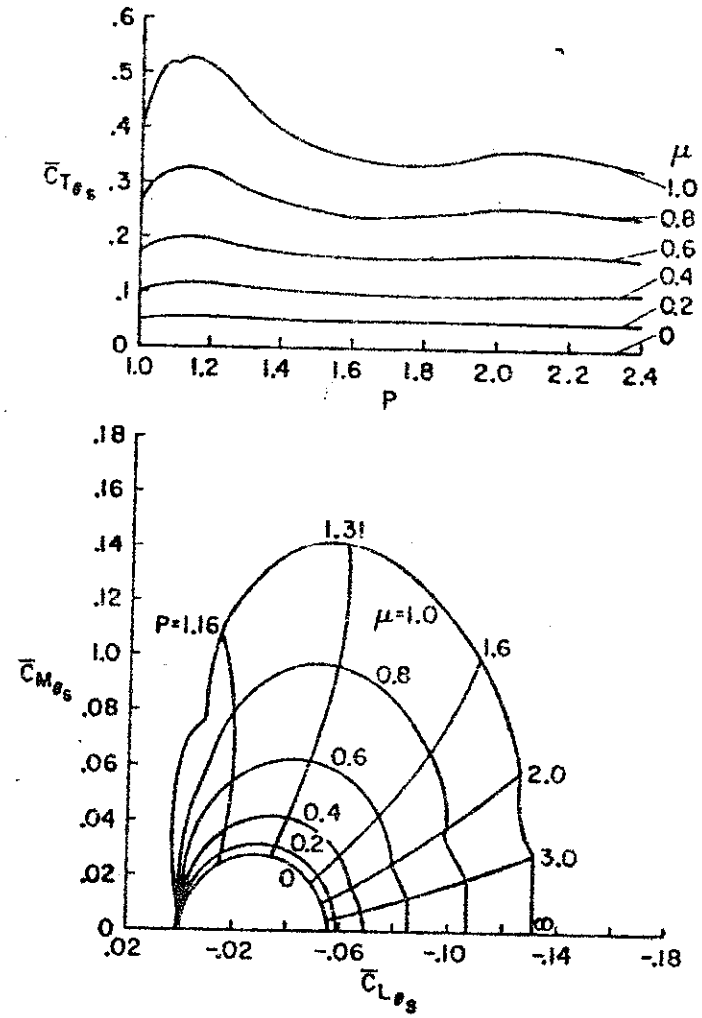



Figure 23.— Response derivatives with respect to $\theta_s$, $\gamma = 5$, $\theta_{q_1} = 0$.

# Transient Solutions

Method #1 PerturbationMethods
Method #2 FourierMethods
Method #3 IntegrationMethods

# Convert to Matthieu

$$\overset{**}{\beta} + n\left(1 + \tfrac{4}{3}\mu \sin\psi\right)\overset{*}{\beta} + \left[p^2 + n\left(\tfrac{4}{3}\mu\cos\psi + \mu^2\sin 2\psi\right)\right]\beta = 0$$

$$\overset{**}{\beta} + C(\psi)\overset{*}{\beta} + K(\psi)\beta = 0$$

Let $\beta = e^{-\frac{1}{2}\int_0^{\psi} C(\psi)\mathrm{d}\psi} x(\psi)$

$$\overset{*}{\beta} = -\tfrac{1}{2}C(\psi)\, e^{-\int \mathrm{d}\psi} x(\psi) + e^{-\int \mathrm{d}\psi}\overset{*}{x}(\psi)$$

$$\overset{**}{\beta} = -C(\psi)\, e^{-\int \mathrm{d}\psi}\overset{*}{x}(\psi) + \tfrac{1}{4}C^2(\psi)\, e^{-\int \mathrm{d}\psi} x(\psi) + e^{-\int \mathrm{d}\psi}\overset{**}{x}(\psi) - \tfrac{1}{2}\overset{*}{C}(\psi)\, e^{-\int \mathrm{d}\psi} x(\psi)$$

$$\overset{**}{x} - C(\psi)\overset{*}{x}(\psi) + \tfrac{1}{4}C^2(\psi)\,x(\psi) - \tfrac{1}{2}C^2(\psi)\,x(\psi) - \tfrac{1}{2}\overset{*}{C}(\psi)\,x(\psi) + C(\psi)\overset{*}{x}(\psi) + Kx(\psi) = 0$$

$$\boxed{\overset{**}{x} + \left[K - \tfrac{1}{2}\overset{*}{C}(\psi) - \tfrac{1}{4}C^2(\psi)\right] x(\psi) = 0}$$

$$
\begin{aligned}
K - \tfrac{1}{4}C^2 - \tfrac{1}{2}\overset{*}{C} = p^2 &+ n\left(\tfrac{4}{3}\mu\cos\psi + \mu^2\sin 2\psi\right) \\
&- \tfrac{1}{4}n^2\left(1 + \tfrac{8}{3}\mu\sin\psi + \tfrac{16}{9}\mu^2\sin^2\psi\right) \\
&- \tfrac{1}{2}n\tfrac{4}{3}\mu\cos\psi
\end{aligned}
$$

$$
\begin{aligned}
= p^2 - \tfrac{1}{4}n^2 - \tfrac{2}{9}n^2\mu^2 + \tfrac{2}{3}n\mu\cos\psi - \tfrac{2}{3}n^2\mu\sin\psi \\
+ n\mu^2\sin 2\psi + \tfrac{2}{9}n^2\mu^2\cos 2\psi
\end{aligned}
$$

As an approximation, neglect $\mu^2$ terms

Let $p^2 - \frac{1}{4}n^2 \equiv \omega^2$

Let $\sqrt{\frac{4}{9}n^2\mu^2 + \frac{4}{9}n^4\mu^2} = \frac{2}{3}n\mu\sqrt{1+n^2} \equiv \epsilon$

$$
\boxed{\ddot{x} + \left(\omega^2 + \epsilon\cos t\right)x = 0}
$$

equivalent equation $t = \psi$

$$
\beta = \underbrace{e^{-\frac{n}{2}\psi}}_{\text{damping}} \quad \underbrace{e^{+\frac{2}{3}n\mu\cos\psi}}_{\text{periodic}} \quad \underbrace{x(\psi)}_{\text{change due to periodicity}}
$$

# Why Multiple Time Scales?

Example where we know answer :

$$\dot{x} + (2+\epsilon)\,x = 0 \qquad \epsilon << 1$$

$$x = Ce^{st} \qquad s + (2+\epsilon) = 0$$

$$s = -\,(2+\epsilon) \qquad x = Ce^{-(2+\epsilon)t}$$

$$x = Ce^{-\epsilon t}e^{-2t}$$

Try an expansion.

$$x = x_o + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

$$\dot{x}_o + \epsilon \dot{x}_1 + \epsilon^2 \dot{x}_2$$
$$+ 2x_o + 2\epsilon x_1 + 2\epsilon^2 x_2$$
$$+ \epsilon x_o + \epsilon^2 x_1 = 0$$

Constant terms :

$$(1)\ \dot{x}_o + 2x_o = 0 \qquad x_o = C_o e^{-2t}$$
$$(\epsilon)\ \dot{x}_1 + 2x_1 = -x_o = -C_o e^{-2t}$$
$$\left(\epsilon^2\right)\ \dot{x}_2 + 2x_2 = -x_1$$

$$(s+2)\,x_1 = -\frac{C_o}{s+2}$$
$$x_1(s) = -\frac{C_o}{(s+2)^2}$$
$$x_1(t) = -C_o t e^{-2t}$$

$$x = x_o + \epsilon x_1 = C_o(1-\epsilon t)\,e^{-2t}$$

Compare $e^{-\epsilon t} \sim 1 - \epsilon t$

But you can't tell whether

$$1 - \epsilon t \Rightarrow e^{-\epsilon t}$$

or

$$1 - \epsilon t \Rightarrow 1 - \sin(\epsilon t)$$

So you don't know if damping or frequency changed.

<u>Time scales</u>

$$x = Ce^{-\epsilon t} e^{-2t} = Ce^{t_1} e^{-2t_o}$$

$$t_o = t \qquad t_1 = \epsilon t \qquad t_2 = \epsilon^2 t$$

$$x(t_o, t_1, t_2, \cdots)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}t_o}{\mathrm{d}t}\frac{\partial x}{\partial t_0} + \frac{\mathrm{d}t_1}{\mathrm{d}t}\frac{\partial x}{\partial t_1} + \frac{\mathrm{d}t_2}{\mathrm{d}t}\frac{\partial x}{\partial t_2}$$

$$= \frac{\partial x}{\partial t_0} + \epsilon\frac{\partial x}{\partial t_1} + \epsilon^2\frac{\partial x}{\partial t_2} \cdots$$

$$\left(\frac{\partial}{\partial t_0}+\epsilon\frac{\partial}{\partial t_1}\right)(x_o+\epsilon x_1)$$
$$+2(x_o+\epsilon x_1)+\epsilon(x_o+\epsilon x_1)=0$$

$$(1)\qquad \frac{\partial x_o}{\partial t_0}+2x_o=0 \qquad x_o=C_o e^{-2t_o}$$
$$(\epsilon)\qquad \frac{\partial x_1}{\partial t_0}+2x_1=-x_o-\frac{\partial x_o}{\partial t_1}$$

$C_o$ is a function of $t_1$!

$$-C_o e^{-2t_o}-\frac{\partial C_o}{\partial t_1}e^{-2t_o}\equiv 0$$

So that no $te^{-2t}$ in $x_1$

$$\frac{\partial C_o}{\partial t_1}+C_o=0$$
$$C_o=\left(e^{-t_1}\right)C_1$$

$x_o = C_1 e^{-t_1} e^{-2t_o}$

$C_1$ is a function of $t_2$

and $x_1 \equiv 0$

$x_o = C_1 e^{-\epsilon t} e^{-2t}$

exact solution!

$$\ddot{x} + \left[\omega^2 + \epsilon \cos t\right] x = 0 \qquad\qquad x \equiv y$$

## Method#1 Multiple Time Scale

$$x = x_o + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t_o} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \cdots$$

$$t_o = t, \qquad t_1 = \epsilon t, \qquad t_2 = \epsilon^2 t \cdots$$

$$(1) \qquad \frac{\partial^2 y_o}{\partial t_o^2} + \omega^2 y_o = 0$$

$$(\epsilon) \qquad \frac{\partial^2 y_1}{\partial t_o^2} + \omega^2 y_1 = -y_o \cos t_o - 2\frac{\partial^2}{\partial t_o \partial t_1} y_o$$

$$(\epsilon^2) \qquad \frac{\partial^2 y_2}{\partial t_o^2} + \omega^2 y_2 = -y_1 \cos t_o - 2\frac{\partial^2}{\partial t_o \partial t_2} y_o - \frac{\partial^2}{\partial t_1^2} y_o - 2\frac{\partial^2}{\partial t_o \partial t_1} y_1$$

$y_o$ solution

$$y_o = A \cos \omega t_o + B \sin \omega t_o \qquad\qquad (\omega \neq 0)$$

$$A(t_1, t_2), \quad B(t_1, t_2)$$

$y_1$ solution

$$\begin{aligned}\frac{\partial^2 y_1}{\partial t_o^2} + \omega^2 y_1 = & -\tfrac{1}{2}\left[A \cos(\omega - 1) t_o + A \cos(\omega + 1) t_o\right] \\ & -\tfrac{1}{2}\left[B \sin(\omega - 1) t_o + B \sin(\omega + 1) t_o\right] \\ & + 2\frac{\partial A}{\partial t_1}\omega \sin \omega t_o - 2\frac{\partial B}{\partial t_1}\omega \cos \omega t_o\end{aligned}$$

No Secular terms $\Rightarrow \dfrac{\partial A}{\partial t_1} = 0, \ \dfrac{\partial B}{\partial t_1} = 0 \qquad\qquad \omega \neq \tfrac{1}{2}, \ \omega^2 \neq \tfrac{1}{4}$

$$\Rightarrow \frac{\partial y}{\partial t_1} = 0$$

$$y_1 = -\frac{A}{2}\left[\frac{\cos(\omega-1)t_o}{\omega^2-(\omega-1)^2}+\frac{\cos(\omega+1)t_o}{\omega^2-(\omega+1)^2}\right]-\frac{B}{2}\left[\frac{\sin(\omega+1)t_o}{\omega^2-(\omega+1)^2}+\frac{\sin(\omega-1)t_o}{\omega^2-(\omega-1)^2}\right]$$

$$y_1 = \frac{A}{2}\left[\frac{\cos(\omega-1)t_o}{1-2\omega}+\frac{\cos(\omega+1)t_o}{1+2\omega}\right]+\frac{B}{2}\left[\frac{\sin(\omega+1)t_o}{1+2\omega}+\frac{\sin(\omega-1)t_o}{1-2\omega}\right]$$

$$y_1\cos t_o = \frac{A}{4}\left[\frac{\cos\omega t_o+\cos(\omega-2)t_o}{1-2\omega}+\frac{\cos(\omega+2)+\cos\omega t_o}{1+2\omega}\right] + \frac{B}{4}\left[\frac{\sin(\omega+2)t_o+\sin\omega t_o}{1+2\omega}+\frac{\sin\omega t_o+\sin(\omega-2)t_o}{1-2\omega}\right]$$

## $y_2$ solution

To remove secular terms $\qquad \omega \neq 1$

$$(\cos\omega t)\qquad -\frac{A}{4}\left[\frac{1}{1-2\omega}+\frac{1}{1+2\omega}\right]-2\frac{\partial B}{\partial t_2}\omega=0$$

$$(\sin\omega t)\qquad -\frac{B}{4}\left[\frac{1}{1+2\omega}+\frac{1}{1-2\omega}\right]+2\frac{\partial A}{\partial t_2}\omega=0$$

$$\frac{\partial B}{\partial t_2}+\frac{1}{4\omega}\frac{A}{(1-4\omega^2)}=0$$

$$\frac{\partial A}{\partial t_2}-\frac{1}{4\omega}\frac{B}{(1-4\omega^2)}=0$$

$$\frac{\partial^2 A}{\partial t_2^2}+\frac{1}{16\omega^2}\frac{1}{(1-4\omega^2)^2}A=0$$

$$A = \sin\left[\frac{1}{4\omega(1-4\omega^2)}t_2\right] \Rightarrow \quad \omega\neq 0 \quad \omega\neq\frac{1}{2} \quad \omega\neq 1$$

$$Y_o = \cos(\omega t)\sin\left[\frac{\epsilon^2}{4\omega(1-4\omega^2)}t\right]$$

$\Rightarrow$ Frequency Shift $\quad \omega \pm \dfrac{\epsilon^2}{4\omega(1-4\omega^2)}$

Special Case $\omega = \frac{1}{2}$

$$\frac{\partial^2 y_1}{\partial t_o^2} + \frac{1}{4} y_1 = -\frac{1}{2}\left[A\cos\tfrac{1}{2}t_o + A\cos\tfrac{3}{2}t_o\right]$$
$$-\frac{1}{2}\left[-B\sin\tfrac{1}{2}t_o + B\sin\tfrac{3}{2}t_o\right]$$
$$+\,2\frac{\partial A}{\partial t_1}\frac{1}{2}\sin\tfrac{1}{2}t_o - 2\frac{\partial B}{\partial t_1}\frac{1}{2}\cos\tfrac{1}{2}t_o$$

No secular terms $\Rightarrow$

cos $\to$ $$+\frac{1}{2}A + \frac{\partial B}{\partial t_1} = 0 \qquad \frac{\partial^2 A}{\partial t_1^2} - \frac{1}{4}A = 0$$

sin $\to$ $$\frac{1}{2}B + \frac{\partial A}{\partial t_1} = 0 \qquad A = e^{\pm\frac{1}{2}t_1} = e^{\pm\frac{1}{2}\epsilon t}$$

damping shift

Special Case $\omega = 1$

No secular terms

$$(\cos t) \qquad -\frac{A}{4}\left[\frac{1+1}{1-2} + \frac{1}{1+2}\right] - 2\frac{\partial B}{\partial t_2} = 0$$
$$(\sin t) \qquad -\frac{B}{4}\left[\frac{1}{1+2} + \frac{1-1}{1-2}\right] - 2\frac{\partial B}{\partial t_2} = 0$$
$$\frac{\partial B}{\partial t_2} + \frac{A}{8}\left[-2 + \frac{1}{3}\right] = 0$$
$$\frac{\partial A}{\partial t_2} + \frac{B}{8}\left[\frac{1}{3}\right] = 0$$
$$\frac{\partial^2 A}{\partial t_2^2} - \frac{1}{24}\left(\frac{5}{3}\right)A = 0 \qquad A = e^{\pm\frac{1}{6}\sqrt{\frac{5}{2}}\epsilon^2 t}$$

damping shift

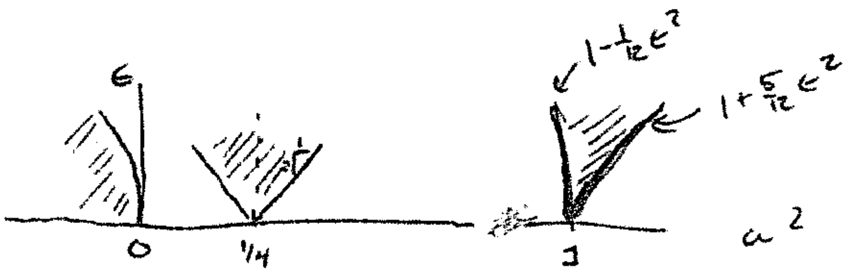
Near $\omega = \frac{1}{2},\ \omega^2 = \frac{1}{4}$ Find neutral point (no scales)

Let $\omega^2 = \frac{1}{4} + \Delta_1 \epsilon + \Delta_2 \epsilon^2 + \cdots$

$$\begin{aligned}\frac{\partial^2 y_1}{\partial t^2} + \frac{1}{4} y_1 = &-\frac{1}{2}\left[A\cos\left(\omega - 1\right)t + A\cos\left(\omega + 1\right)t\right] \\ &-\frac{1}{2}\left[B\sin\left(\omega - 1\right)t + B\sin\left(\omega + 1\right)t\right] \\ &-\Delta_1\left[A\cos\omega t + B\sin\omega t\right] \qquad \Delta_1 = \pm\tfrac{1}{2}\end{aligned}$$

$\cos \rightarrow \quad -\frac{1}{2}A - \Delta_1 A = 0$

$\sin \rightarrow \quad +\frac{1}{2}B - \Delta_1 B = 0$




Near $\omega = 1,\ \omega^2 = 1, \qquad \Delta_1 = 0$

cosine $\qquad -\frac{A}{4}\left[-2 + \frac{1}{3}\right] - \Delta_2 A = 0 \qquad \Delta_2 = +\frac{5}{12}$

sin $\qquad -\frac{B}{4}\left[\frac{1}{3}\right] - \Delta_2 B = 0 \qquad \Delta_2 = -\frac{1}{12}$

Near $\omega = 0 \quad \omega^2 = 0 \quad \Delta_1 = 0$

$\Delta_2 = -\frac{1}{2}, \quad \omega^2 = -\frac{1}{2}\epsilon^2$

# SUMMARY SHEET FLAPPING

## Only two parameters define eigenvalues :

$$\omega^2 \equiv p^2 - n^2/4, \quad \epsilon \equiv \tfrac{2}{3} n\mu\sqrt{1+n^2}$$

$$\text{where } \; n \equiv \gamma/8$$

Eigenvalue formula depends on region

Region"0" : $\omega^2 < -\frac{1}{2}\epsilon^2 < 0$

$$\text{freq.} = 0, \quad \text{damping} = \frac{n}{2} \pm \sqrt{\frac{n^2}{4} - p^2} \mp \frac{\epsilon^2}{4\sqrt{n^2/4 - p^2}} \left(\frac{1}{1-4\omega^2}\right)$$

Region"$\frac{1}{2}$" : $\frac{1}{4} - \frac{1}{2}\epsilon < \omega^2 < \frac{1}{4} + \frac{1}{2}\epsilon$

$$\text{freq.} = 0.5, \quad \text{damping} = \frac{n}{2} \pm \frac{1}{2}\epsilon$$

Region"1" : $1 - \frac{1}{12}\epsilon^2 < \omega^2 < 1 + \frac{5}{12}\epsilon^2$

$$\text{freq.} = 1.0, \quad \text{damping} = \frac{n}{2} \pm \frac{1}{6}\sqrt{\frac{5}{2}}\epsilon^2$$

All Other Region :

$$\text{freq.} = \omega \pm \frac{\epsilon^2}{4\omega\left(1-4\omega^2\right)}, \quad \text{damping} = \frac{n}{2}$$

$$\epsilon^2 << 1$$

# Method #2 Hill's Infinite Determinant

$$\beta = a_o + a_1 \cos\psi + b_1 \sin\psi + a_2 \cos 2\psi + b_2 \sin 2\psi + \cdots$$

$$a_o(\psi),\ a_1(\psi),\ a_2(\psi),\cdots$$

$$\overset{*}{\beta} = \overset{*}{a}_o + \overset{*}{a}_1 \cos\psi - a_1 \sin\psi + \overset{*}{b}_1 \sin\psi + b_1 \cos\psi + \cdots$$

$$\overset{**}{\beta} = \overset{**}{a}_o + \overset{**}{a}_1 \cos\psi + \overset{**}{b}_1 \sin\psi$$

$$- 2\overset{*}{a}_1 \sin\psi + 2\overset{*}{b}_1 \cos\psi - a_1 \cos\psi - b_1 \sin\psi + \cdots$$

Collect like harmonics $\left(\overset{**}{x} + \left[\omega^2 + \epsilon \sin t\right] x = 0\right)$

$$\begin{Bmatrix} \overset{**}{a}_o \\ a_1 \\ b_1 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix} \begin{Bmatrix} \overset{*}{a}_o \\ a_1 \\ b_1 \end{Bmatrix} + \begin{bmatrix} \omega^2 & 0 & \frac{1}{2}\epsilon \\ 0 & \omega^2 - 1 & 0 \\ \epsilon & 0 & \omega^2 - 1 \end{bmatrix} \begin{Bmatrix} a_o \\ a_1 \\ b_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Laplace Transform $\rightarrow$ Characteristic Equation

$$\left(s^2 + \omega^2\right)\left[\left(s^2 + \omega^2 - 1\right) + 4s^2\right] - \tfrac{1}{2}\epsilon^2 \left(s^2 + \omega^2 - 1\right) = 0$$

$\epsilon = 0$ (constant coefficient)

$s = \pm i\omega \qquad s = \pm i\,(1 \pm \omega) \qquad$ (6 roots)

Hill's Infinite Determinant (Revised)

$$\ddot{x} + \left[\omega^2 + \epsilon \cos t\right] x = 0$$

$$x = a_o + a_1 \cos t + b_1 \sin t + a_2 \cos(2t) + b_2 \sin(2t)$$

$$\begin{aligned} \dot{x} = \dot{a}_o + \dot{a}_1 \cos t + \dot{b}_1 \sin t + \dot{a}_2 \cos(2t) + \dot{b}_2 \sin(2t) \\ - a_1 \sin t + b_1 \cos t - 2a_2 \sin(2t) + 2b_2 \cos(2t) \end{aligned}$$

$$\begin{aligned} \ddot{x} = \ddot{a}_o + \ddot{a}_1 \cos t + \ddot{b}_1 \sin t + \ddot{a}_2 \cos(2t) + \ddot{b}_2 \sin(2t) \\ - 2\dot{a}_1 \sin t + 2\dot{b}_1 \cos t - 4\dot{a}_2 \sin(2t) + 4\dot{b}_2 \cos(2t) \\ - a_1 \cos t - b_1 \sin t - 4a_2 \cos(2t) - 4b_2 \sin(2t) \end{aligned}$$

$$\begin{aligned} x \cos t = a_o \cos t + \frac{a_1}{2}(1 + \cos 2t) + \frac{b_1}{2} \sin(2t) \\ + \frac{a_2}{2}(\cos t + \cos 3t) + \frac{b_2}{2}(\sin t - \sin 3t) \end{aligned}$$

## Collect Like Harmonics

const

$\cos t$

$\sin t$

$\cos 2t$

$\sin 2t$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \ddot{\begin{Bmatrix} a_o \\ a_1 \\ b_1 \\ a_2 \\ b_2 \end{Bmatrix}} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix} \dot{\begin{Bmatrix} a_o \\ a_1 \\ b_1 \\ a_2 \\ b_2 \end{Bmatrix}}$$

$$+ \begin{bmatrix} \omega^2 & \frac{\epsilon}{2} & 0 & 0 & 0 \\ \epsilon & \omega^2 - 1 & 0 & \frac{\epsilon}{2} & 0 \\ 0 & 0 & \omega^2 - 1 & 0 & \frac{\epsilon}{2} \\ 0 & \frac{\epsilon}{2} & 0 & \omega^2 - 4 & 0 \\ 0 & 0 & \frac{\epsilon}{2} & 0 & \omega^2 - 4 \end{bmatrix} \begin{Bmatrix} a_o \\ a_1 \\ b_1 \\ a_2 \\ b_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$[M]\{\ddot{a}\} + [C]\{\dot{a}\} + [K]\{a\} = \{0\}$$

Method #1 : $\quad \text{Det}\left[Ms^2 + Cs + K\right] = 0$

solve for $s$

Method #2 : $\quad$ Take Eigenvalues of $[D]$

$$\begin{Bmatrix} \ddot{a} \\ \dot{a} \end{Bmatrix} = \begin{bmatrix} -C & -K \\ I & 0 \end{bmatrix} \begin{Bmatrix} \dot{a} \\ a \end{Bmatrix} = [D] \begin{Bmatrix} \dot{a} \\ a \end{Bmatrix}$$

Special Case $a_o\, a_1\, b_1 \quad (3x3)$

$$\text{Det} = \left(s^2+\omega^2\right)\left[\left(s^2+\omega^2-1\right)+4s^2\right]-\tfrac{1}{2}\epsilon^2\left(s^2+\omega^2-1\right)=0$$

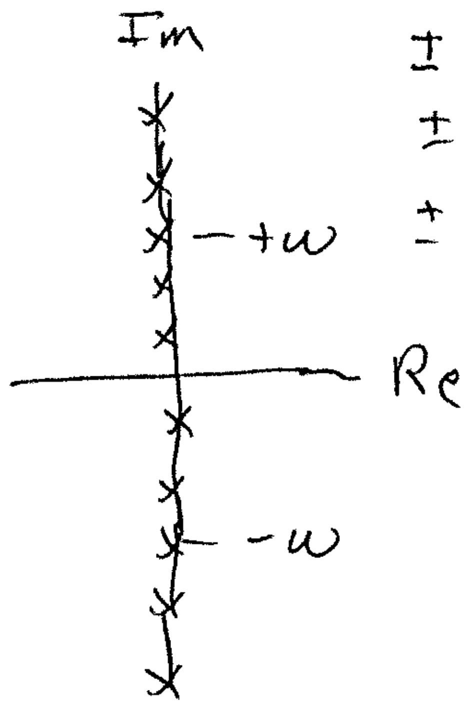
6 roots

$\epsilon = 0$ (constant coefficient)

$s = \pm i\omega \qquad s = \pm i\,(1 \pm \omega) \qquad a_o\, a_1\, b_1$

$s = \pm i\,(2 \pm \omega) \qquad a_2\, b_2$

10 roots




$\pm\omega$

$\pm\omega\pm1$

$\pm\omega\pm2$

# ADVANCED METHODS

pp. 58 – 62 Blank

# FLOQUET THEORY

$$\{\dot{x}\} = [D(t)]\{x\}$$
$$[D(t+T)] = [D(t)]$$

First-order: $d = \underbrace{d_o}_{\text{mean}} + \underbrace{d_p(t)}_{\text{zero average}}$

$$\dot{x} = d(t)x, \qquad x(0) \text{ given}$$
$$x(t) = \underbrace{e^{d_o t}}_{\text{exponential}} \cdot \underbrace{e^{\int d_p(t)\mathrm{d}t}}_{\text{periodic}} x(0)$$

## Transition Matrix

$$\left[\dot{\Phi}\right] = [D(t)]\,[\Phi]$$
$$[\Phi(0)] = [\mathrm{I}]$$

$$\{x(t)\} = [\Phi(t)]\{x(0)\}$$

$$\{x(t+T)\} = [\Phi(t+T)]\{x(0)\}$$

$$= [\Phi(t)][\Phi(T)]\{x(0)\}$$

$$[\Phi(t+T)] = [\Phi(t)][\Phi(T)]$$

Eigenvalues and Eigenvectors

$$[A_o]^{-1}[Q][A_o] = \begin{bmatrix} \ddots & & \\ & \Lambda_i & \\ & & \ddots \end{bmatrix}$$

$$[Q] \equiv [\Phi(T)]$$

Let: $[\psi(t)] = [\Phi(t)][A_o]$

$$[\psi(t+T)] = [\Phi(t)][\Phi(T)][A_o]$$

$$= [\psi(t)]\begin{bmatrix} \ddots & & \\ & \Lambda_i & \\ & & \ddots \end{bmatrix}$$

$$\Lambda_i \equiv e^{\eta_i T} \qquad \eta_i = \frac{1}{T}\ln(\Lambda_i)$$

Form of Solution

$$\{x(t)\} = [\Phi(t)]\{x(0)\} =$$

$$\underbrace{[\Phi(t)][A_o]\begin{bmatrix} \ddots & & \\ & e^{-\eta_i t} & \\ & & \ddots \end{bmatrix}}_{A(t)}\begin{bmatrix} \ddots & & \\ & e^{\eta_i t} & \\ & & \ddots \end{bmatrix}[A_o]^{-1}\{x(0)\}$$

$$[A(t+T)] = [\Phi(t)][\Phi(T)][A_o]\begin{bmatrix} \ddots & & \\ & e^{-\eta_i (t+T)} & \\ & & \ddots \end{bmatrix}$$

$$= [\Phi(t)][A_o][A_o]^{-1}[\Phi(T)][A_o]\begin{bmatrix} \ddots & & \\ & \frac{1}{\Lambda_i} & \\ & & \ddots \end{bmatrix}\begin{bmatrix} \ddots & & \\ & e^{-\eta_i t} & \\ & & \ddots \end{bmatrix}$$

$$[A(t+T)] = [\Phi(t)][A_o]\begin{bmatrix} \ddots & & \\ & e^{-\eta_i t} & \\ & & \ddots \end{bmatrix} = [A(t)]$$

$$\{x\} = [A(t)]\begin{bmatrix} \ddots & & \\ & e^{\eta_i t} & \\ & & \ddots \end{bmatrix}[A_o]^{-1}\{x(0)\}$$

$$[A(0)] \equiv [A_o]$$

$$\Lambda_j = a_j + b_j i \qquad\qquad i = \sqrt{-1}$$

$$\eta_j = \frac{1}{T}\ln\left(a_j + ib_j\right)$$

$$= \frac{1}{T}\ln\left[\sqrt{a_j^2 + b_j^2}\, e^{i\tan^{-1}(b_j/a_j)}\right]$$

$$\eta_j = \frac{1}{T}\ln\left[\sqrt{a_j^2 + b_j^2}\right] + i\left[\frac{1}{T}\tan^{-1}\frac{b_j}{a_j} + \frac{2\pi n}{T}\right]$$

$$n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots$$

$\sqrt{a^2 + b^2} > 1$ unstable

$\sqrt{a^2 + b^2} < 1$ stable

Note a negative real $\Lambda_j$ has an $\eta_j$ with an imaginary part $\dfrac{\pi}{T}$

Sturm - Liouville Theorem

$$\sum \eta_j = \frac{1}{T} trace\left[D(t)\right] \mathrm{d}t$$

Example:

$$\ddot{\beta} + C(\psi)\dot{\beta} + K(\psi)\beta = 0$$

$$x_1 = \beta, \qquad x_2 = \dot{\beta}$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -K(\psi) & -C(\psi) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

$$\sum \eta_j = \eta_1 + \eta_2 = \frac{1}{2\pi} \int_0^{2\pi} -C(\psi)\mathrm{d}\psi$$

$$C(\psi) = \eta\left(1 + \tfrac{4}{3}\mu \sin\psi\right)$$

$$\eta_1 + \eta_2 = -\tfrac{\gamma}{8}$$

Methods of Solution based on Floquet Form

1.Modified Hill Method (Crimi, Multiblade coordinates)

$\{x\} = \sum_{n=-\infty}^{+\infty} \{y\}_n e^{(ni+t)\frac{2\pi}{T}} \Rightarrow$ infinite number ofconstant coefficient equations, $\{y(t)\}_n$

Assume $\{y\} = e^{\eta t}$, indinite determinant for $\eta$
$\Rightarrow$ infinite number of roots. (basic $\eta \pm (2\pi i/T)\, n)$.

2. Direct integration
$a$)Numerically integrate the equation $n$ times to obtain $\phi(t)$. [Once for each set of initial conditions $x_j = 1 \;\; x_i = 0 \;\; (i \neq j) j = 1, n$.]

Eigenvalues of $[\phi(\tau)]$ are $\Lambda_i$

$$\eta_i = \frac{1}{T} \ln \Lambda_i$$

If A(t) desired, eigenvectors of $\phi(\tau)$, $A_o$can be used,

$$[A(t)] = [\phi(t)]\,[A_o] \begin{bmatrix} \ddots & & \\ & e^{-\eta_i t} & \\ & & \ddots \end{bmatrix}$$

Characteristics of Floquet Solution

$$\{\dot{x}\} = [D(t)]\{x\}$$

has solution: $\{x\} = [A(t)]\{\alpha e^{\eta t}\} = [A(t)]\begin{bmatrix} \ddots & & \\ & \alpha e^{\eta t} & \\ & & \ddots \end{bmatrix}[A(0)]^{-1}\{x(0)\}$

$$\Rightarrow [\dot{A}] + [A]\begin{bmatrix} \ddots & & \\ & \eta & \\ & & \ddots \end{bmatrix} = [D][A]$$

$$\text{or } [A]^{-1}[D][A] - \underbrace{[A]^{-1}[\dot{A}]}_{} = \begin{bmatrix} \ddots & & \\ & \eta & \\ & & \ddots \end{bmatrix}$$

changed eigenvalue problem

matrix exponent form of solution

alternate form

$$[A_o]^{-1}[Q][A_o] = \begin{bmatrix} \ddots & & \\ & e^{\eta\tau} & \\ & & \ddots \end{bmatrix} \qquad \ln Q = A_o\begin{bmatrix} \ddots & & \\ & \eta\tau & \\ & & \ddots \end{bmatrix}A_o^{-1}$$

$$e^{\ln[Q]t/\tau} = [A_o]^{-1}\begin{bmatrix} \ddots & & \\ & e^{\eta t} & \\ & & \ddots \end{bmatrix}[A_o]$$

$$\boxed{\{x(t)\} = [P(t)]\,e^{\ln[Q]t/\tau}\{x(0)\}} \qquad \tau = \text{period}$$

$$[P(t)] = [A(t)][A(0)]^{-1} \qquad P(0) = \mathrm{I}$$

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$\Rightarrow \{\dot{x}\} = [D(t)]\{\dot{x}\} + \{f(t)\}$

let $\{x\} = [A(t)]\{y\}$

$$[\dot{A}]\{y\} + [A]\{\dot{y}\} = [D][A]\{y\} + \{f\}$$

$$\{\dot{y}\} - \begin{bmatrix} \ddots & & \\ & \eta & \\ & & \ddots \end{bmatrix}\{y\} = [A]^{-1}\{f\} = \{g\}$$

reduced to constant coefficient

pages 67-68
are blank