Midterm 2

1. The most generic 4th order tensor containing elastic constants, C_{ijkl}, has 81 components.
   a. Explain why the maximum number of independent constants in C_{ijkl} is 21.
   [3 points]
   b. Given that a general isotropic 4th order tensor is written as $\underset{\sim}{C} = \left(\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}\right)e_i \otimes e_j \otimes e_k \otimes e_l$, explain how an isotropic linear elastic material requires only two elastic constants. Write constitute equation for such a material.
   [3 points]

Midterm #2 - Answers

(1)

(a)

Since $E_{ij} = E_{ji}$

$$W = \frac{1}{2} C_{ijkl} E_{ij} E_{kl} = \frac{1}{2} C_{jikl} E_{ji} E_{kl} = \frac{1}{2} C_{ijlk} E_{ij} E_{lk}$$

$$= \frac{1}{2} C_{jilk} E_{ji} E_{lk} \quad \Rightarrow_{+} \quad \begin{array}{l} C_{ijkl} = C_{jikl} \\ C_{ijkl} = C_{ijlk} \end{array}$$

Also, since $W = \frac{1}{2} C_{ijkl} E_{ij} E_{kl} = \frac{1}{2} C_{klij} E_{kl} E_{ij}$

$$\Rightarrow \quad C_{ijkl} = C_{klij}$$

$\Rightarrow$ Number of constants goes from $81 \longrightarrow 21$.

(b)

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \nu \delta_{il} \delta_{jk}$$

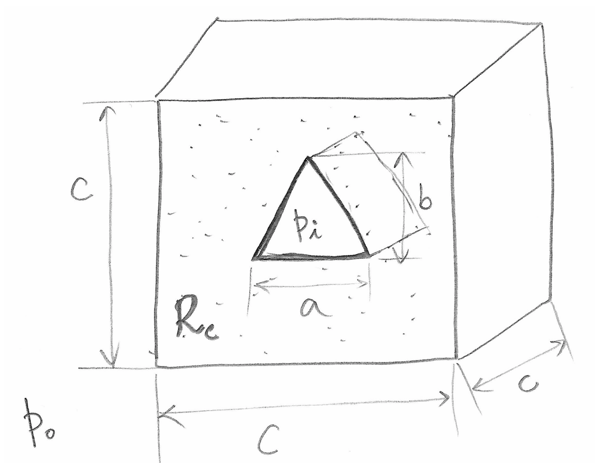
Since, $C_{ijkl} = C_{ijlk} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{il} \delta_{jk} + \nu \delta_{ik} \delta_{jl}$.

$\Rightarrow \quad \nu = \mu$. $\quad \Rightarrow$ 2 elastic constants λ, μ.

$$\Rightarrow \quad T_{ij} = \lambda \delta_{ij} E_{kk} + 2\mu E_{ij}$$

2. A vessel made of a linear isotropic elastic material (with known E, ν) of cubic shape has a prismatic cavity of triangular cross-section, as depicted below. The width, height, and depth of the triangular cavity are a, b, and c, respectively. The cube edge length is c. The outside and inside pressures are p_o *and* p_i, respectively. Calculate the mean stress $\bar{\underset{\sim}{T}}$ in the vessel volume (R_c):

$$\bar{\underset{\sim}{T}} = \frac{1}{vol(R_c)} \iiint_{R_c} \underset{\sim}{T} dV$$ [8 points]



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Given for any volume:

$$\iiint_V \underset{\sim}{T}\, dV = \iint_S \underline{x} \otimes \underline{t}\, ds.$$

<u>For volume R_c:</u>

$\boxed{1}$

$$\overline{\underset{\sim}{T}} = \frac{1}{vol(R_c)} \left(\iint_{S_i} \underline{x} \otimes \underline{t}\, ds + \iint_{S_i} \underline{x} \otimes \underline{t}\, ds \right) \quad \text{— } ①$$

$\boxed{2}$

$$\iint_{S_i} \underline{x} \otimes \underline{t}\, ds = \iint_{S_i} x_i t_j\, ds = -p_i \iint_{S_i} x_i n_j\, ds$$

$$= -p_i \iiint_{R_i} \frac{\partial x_i}{\partial x_j}\, dV = -p_i\, vol(R_i)\, \delta_{ij}$$

$\boxed{1}$

$$\Rightarrow \iint_{S_i} \underline{x} \otimes \underline{t}\, ds = -p_i\, abc\, \underset{\sim}{I} \quad \text{— } ②$$

Similarly,

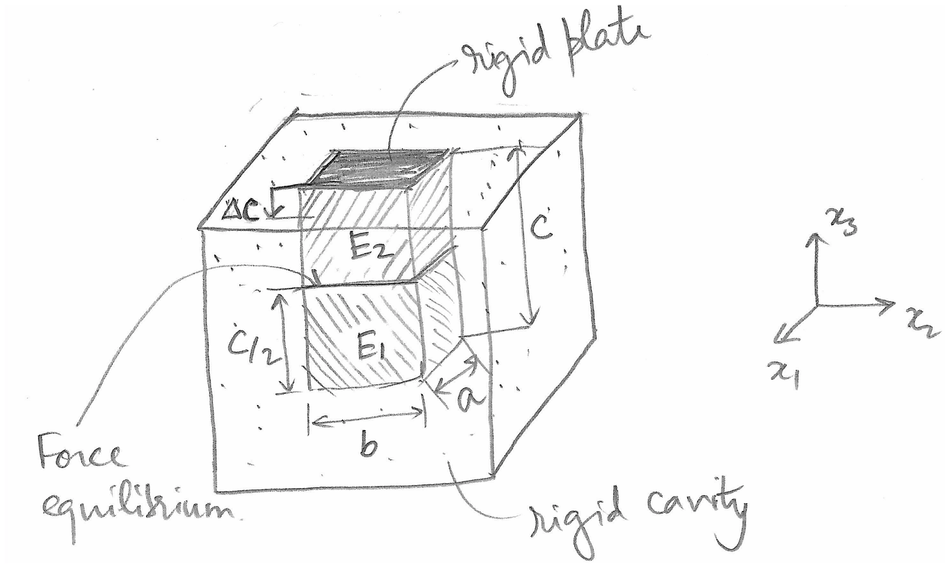
$\boxed{2}$

$$\iint_{S_o} \underline{x} \otimes \underline{t}\, ds = -p_o\, c^3\, \underset{\sim}{I}. \quad \text{— } ③$$

By combining equations ①, ②, ③:

$$\boxed{2} \qquad \frac{T}{\omega} = \frac{1}{(c^3 - abc)}\left[-p_i\, abc - p_o\, c^3\right]\frac{T}{\omega}.$$

3. A parallelepiped of dimensions $a \times b \times c$ is made up of two isotropic linear elastic materials, lower half with E_1, ν and upper half with E_2, ν, which is completely encapsulated within a rigid and smooth cavity. A rigid plate of dimensions $a \times b$ rests on top of the parallelepiped. A downward displacement of "Δc" is applied on the rigid plate. Find the net reaction force exerted onto the plate by the parallelepiped. Assume force equilibrium at the interface between the two materials (E_1 and E_2). [11 points]



(3) Boundary conditions:

$$E_{33}^{(1)} + E_{33}^{(2)} = \frac{-\Delta C}{C} \quad \text{--- (1)} \qquad \boxed{2}$$

$$E_{11}^{(1)} = E_{11}^{(2)} = E_{22}^{(2)} = E_{22}^{(2)} = 0 . \qquad \boxed{1}$$

$T_{11}^{(1)} = T_{22}^{(1)}$ & $T_{11}^{(2)} = T_{22}^{(2)}$ due to symmetry $\boxed{1}$

$T_{33}^{(1)} = T_{33}^{(2)}$ at the interface $\rightarrow$ Force equilibrium $\boxed{1}$

$$E_{11} = E_{22} = 0 \Rightarrow \frac{1}{E}\left(T_{11} - \nu\left(T_{22} + T_{33}\right)\right) = 0 \qquad \boxed{1}$$

$$\Rightarrow T_{11} = T_{22} = \frac{\nu T_{33}}{(1-\nu)}$$ for either material

$$E_{33} = \frac{1}{E}\left(T_{33} - \nu\left(T_{11} + T_{22}\right)\right) = \frac{1}{E}\left(T_{33} - 2\nu T_{11}\right) \qquad \boxed{1}$$

$$\Rightarrow E_{33} = \frac{T_{33}}{E}\left(1 - \frac{2\nu^2}{1-\nu}\right) \longrightarrow$$ Also true for either material

From eq (1)

$$\Rightarrow \left(\frac{T_{33}^{(1)}}{E_1} + \frac{T_{33}^{(2)}}{E_2}\right)\left(\frac{1-\nu-2\nu^2}{1-\nu}\right) = -\frac{\Delta C}{C} \qquad \boxed{1}$$

$$\Rightarrow \boxed{T_{33} = -\frac{\Delta C}{C\left(\frac{1}{E_1} + \frac{1}{E_2}\right)\left(\frac{1-\nu-2\nu^2}{1-\nu}\right)}} \qquad \boxed{2}$$

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Net force on the plate

$$\Rightarrow \boxed{F_p = \frac{\Delta c\, ab}{c\left(\frac{1}{E_1}+\frac{1}{E_2}\right)\left(1-\frac{2\nu^2}{1-\nu}\right)}}$$