HW#3
solutions

HW 3.1

$$v_i = \frac{\partial x_i(X_n, t)}{\partial t}$$

$$v_1 = \frac{1}{3} a X_2 + \frac{2}{9} a^2 X_3 t$$

$$v_2 = \frac{1}{3} a X_3 + \frac{2}{9} a^2 X_1 t$$

$$v_3 = \frac{1}{3} a X_1 + \frac{2}{9} a^2 X_2 t$$

at $(X_1, X_2, X_3) = (l, 2l, 3l)$ and $t = 0$

$$v_1^\circ = \frac{2}{3} la$$

$$v_2^\circ = la$$

$$v_3^\circ = \frac{1}{3} la$$

$$f_i = \frac{\partial v_i(X_n, t)}{\partial t}$$

$$f_1 = \frac{2}{9} a^2 X_3$$

$$f_2 = \frac{2}{9} a^2 X_1$$

$$f_3 = \frac{2}{9} a^2 X_2$$

$$f_1^\circ = \frac{2}{3} la^2$$

$$f_2^\circ = \frac{2}{9} la^2$$

$$f_3^\circ = \frac{4}{9} la^2$$

(2)

$$\underset{\sim}{F} = \begin{bmatrix} (1-x_2/R)\cos(x_1/R) & -\sin(x_1/R) \\ (1-x_2/R)\sin(x_1/R) & \cos(x_1/R) \end{bmatrix}$$

$$\underset{\sim}{\gamma} = \frac{1}{2}(\underset{\sim}{F}^T\underset{\sim}{F} - \underset{\sim}{I}) = \frac{1}{2}\begin{bmatrix} -(x_2/R)(2-x_2/R) & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{E} = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)$$

$$\underset{\sim}{E} = \begin{bmatrix} (1-x_2/R)\cos(x_1/R) - 1 & \{-(x_2/R)\sin(x_1/R)\}/2 \\ \{-(x_2/R)\sin(x_1/R)\}/2 & \cos(x_1/R) - 1 \end{bmatrix}$$

(3)

(i) $\lambda_1, \lambda_2, \lambda_3$

(ii) Same as that for constant volume,

$$dV_x = \det[F] \times dV_y$$

$$\Rightarrow \lambda_1 \lambda_2 \lambda_3 = 1$$

(iii) $\vec{OP} = (1, 1, 1)$

$\underline{p} \equiv$ unit vector along $\vec{OP} = \frac{1}{\sqrt{3}}(\underline{e}_1 + \underline{e}_2 + \underline{e}_3)$

For no change in length : $|\underset{\sim}{F} \cdot \underline{p}| = 1$

$[F]\{p\} \Rightarrow$

HW 3.3

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{Bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{Bmatrix} = \begin{Bmatrix} \lambda_1/\sqrt{3} \\ \lambda_2/\sqrt{3} \\ \lambda_3/\sqrt{3} \end{Bmatrix} = 1$$

$\Rightarrow$ Relation among $\lambda_i$ is: $|\underline{F} \cdot \underline{P}| = 1$

$$\Rightarrow \quad \frac{\lambda_1^2}{3} + \frac{\lambda_2^2}{3} + \frac{\lambda_3^2}{3} = 1$$

$$\Rightarrow \quad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 3$$

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We know that

$$\lambda^2 = C_{rs} A_r A_s$$

$A_r$ → unit vectors along a line at $t=0$.

$\lambda$ → stretch ratio of a line $= \dfrac{l}{l_0}$ ($l$ → final length, $l_0$ → initial length)

Using $2\gamma_{rs} = \underset{\sim}{C} - \underset{\sim}{I}$

$$\lambda^2 - 1 = 2\gamma_{rs} A_r A_s \quad \text{—— (1)}$$

Write 3 unit vectors corresponding to the three sides of the triangle at $t=0$:

$$\overrightarrow{P_1P_2} = (1,0,0) = \underset{\sim}{e}_1$$

$$\overrightarrow{P_1P_3} = \frac{1}{2}\underset{\sim}{e}_1 + \frac{\sqrt{3}}{2}\underset{\sim}{e}_2$$

$$\overrightarrow{P_2P_3} = \frac{\underset{\sim}{e}_1}{2} + \frac{\sqrt{3}}{2}\underset{\sim}{e}_2$$

Use equation (1) for each line:

$$P_1P_2 \Rightarrow \lambda^2 - 1 = \frac{l^2 - l_0^2}{l_0^2} = 2\gamma_{rs} A_r A_s = 2\gamma_{11} \times 1$$

$$\Rightarrow \gamma_{11} = \frac{(2)^2 - (1)^2}{2 \times (1)^2} = \frac{3}{2}$$

Similarly,

For $P_1P_3$: $\frac{l^2 - l_0^2}{2 l_0^2} = \gamma_{rs} A_r A_s$

$$\Rightarrow \frac{(1)^2 - (1)^2}{2(1)^2} = \frac{\gamma_{11}}{4} + \frac{3\gamma_{22}}{4} + \frac{\gamma_{12}\sqrt{3}}{2}$$

$$\Rightarrow \frac{\gamma_{11}}{4} + \frac{3\gamma_{22}}{4} + \frac{\gamma_{12}\sqrt{3}}{2} = 0 \quad \text{—} (2)$$

For $P_2P_3$: $\frac{(1)^2 - (1)^2}{2(1)^2} = \frac{\gamma_{11}}{4} + \frac{3}{4}\gamma_{22} - \frac{\gamma_{12}\sqrt{3}}{2} = 0 \quad (3)$

Summing eqns (2) & (3) and using $\gamma_{11} = \frac{3}{2}$

$$\Rightarrow \frac{3}{4} + \frac{3}{2}\gamma_{22} = 0$$

$$\Rightarrow \gamma_{22} = -\frac{1}{2}$$

And,
$$\frac{\sqrt{3}}{2}\gamma_{12} = -\left(\frac{\gamma_{11}}{4} + \frac{3\gamma_{22}}{4}\right)$$

$$= -\left(\frac{3}{8} + \frac{3}{4}\left(-\frac{1}{2}\right)\right) = 0$$

$$\Rightarrow \underset{\sim}{\gamma} = \begin{bmatrix} 3/2 & 0 \\ 0 & -1/2 \end{bmatrix}$$