HW # 4

Solutions

1.

$$[T] = \begin{pmatrix} 100 & 250 & 0 \\ 250 & 200 & 0 \\ 0 & 0 & 300 \end{pmatrix}$$

(i) $$\underline{t} = \underline{n} \cdot \underline{T} = \frac{1}{2}[1 \ -1 \ 0]\begin{bmatrix} T \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{Bmatrix} -150 \\ 50 \\ 0 \end{Bmatrix}$$

(ii) Find eigenvalues of $[T]$

(iii) $$\frac{1}{3}\,tr[T] = 200$$

(iv) $$S_{ij} = T_{ij} - \frac{T_{kk}\delta_{ij}}{3} = \begin{bmatrix} -100 & 250 & 0 \\ 250 & 0 & 0 \\ 0 & 0 & 100 \end{bmatrix}$$

2.

(i) $$\frac{\partial T_{ij}}{\partial x_i} = 0$$

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = 0$$

$$\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} = 0$$

$$\frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} = 0$$

(2. ii) Unit outward normal $\underline{n}$ on the curved surface

$$x_1^2 + x_2^2 = a^2 \Rightarrow$$

$$n_1 = \cos\theta , \quad n_2 = \sin\theta , \quad n_3 = 0$$

for $0 \le \theta < 2\pi$

Traction on this surface:

$$\underline{t} = \underset{\sim}{T} \cdot \underline{n}$$

$$t_1 = T_{11} n_1 + T_{12} n_2 + T_{13} n_3 = 0$$

$$t_2 = T_{21} n_1 + T_{22} n_2 + T_{23} n_3 = 0$$

$$t_3 = T_{31} n_1 + T_{32} n_2 + T_{33} n_3$$

$$= (-\alpha x_2) \cos\theta + (\alpha x_1) \sin\theta$$

$$= (-\alpha a \sin\theta) \cos\theta + (\alpha a \cos\theta) \sin\theta$$

$$= 0$$

(2. iii) $\underline{n}$ corresponding to the surface at the end $x_3 = 0$,

$$\Rightarrow n_1 = n_2 = 0 , \quad n_3 = 1$$

$$t_i = T_{ij} n_j \Rightarrow$$

$$t_1 = T_{13} n_3 = -\alpha x_2$$

$$t_2 = T_{23} n_3 = \alpha x_1$$

$$t_3 = T_{33} n_3 = \beta + \gamma x_1 + \delta x_2$$

$S_0 \equiv$ surface at the end $x_3 = 0$

Force components on the surface S_0:

$$F_1 = \iint_{S_0} t_1 \, dS = \iint_{S_0} (-\alpha x_2) \, dS = 0$$

(by inspection) since x_2 is equally positive and negative on half of the surface

$$F_2 = \iint_{S_0} t_2 \, dS = \iint_{S_0} (\alpha x_1) \, dS = 0$$

(same reason)

$$F_3 = \iint_{S_0} t_3 \, dS = \iint_{S_0} (\beta + \gamma x_1 + \delta x_2) \, dS$$

$$= \beta \pi a^2$$

$\Rightarrow$ Force on the surface is purely axial in $\underline{e}_3$

$$\underline{F} = \beta \pi a^2 \underline{e}_3$$

③ Assuming $\underline{e}_1$ to be horizontal, the normal vector to the two joints are:

$$n_1 = \cos(30^\circ)\,\underline{e}_1 + \sin(30^\circ)\,\underline{e}_2$$

$$n_2 = \underline{e}_1$$

The stress tensor is $\underset{\sim}{T} = \sigma\, \underline{e}_1 \otimes \underline{e}_1$

Tractions:-

$\Rightarrow$ $\underline{t}_1 = \sigma \cos(30^\circ)\,\underline{e}_1$ → Traction on joint # 1

$\underline{t}_2 = \sigma\,\underline{e}_1$ ⟶ Traction on joint # 2

→ For the first joint :-

Normal component of traction $t_n = \underline{t}_1 \cdot \underline{n}_1$

$$t_n = \sigma \cos^2(30)$$

Shear component of traction $t_t = |\underline{t}_1 - t_n \underline{n}_1|$

$$\Rightarrow t_t = \sigma \cos(30) \left| \left(\underline{e}_1 - \cos(30)\right)\left(\cos(30)\,\underline{e}_1 + \sin(30)\,\underline{e}_2\right) \right|$$

$$t_t = \sigma \sin(30) \cos(30)$$

→ For the 2nd joint,

$$t_n = \sigma$$

$$t_t = 0$$