Wrenches

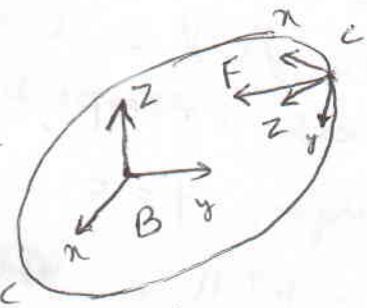
A wrench is a force moment pair

$$F = \begin{bmatrix} f \\ \tau \end{bmatrix} \qquad \underbrace{f \in \mathbb{R}^3}_{\text{Linear Force component}} \qquad \underbrace{\tau \in \mathbb{R}^3}_{\text{Moment}}$$

$\uparrow$
Wrench

Conversion of wrenches between reference frames:

$$F = \begin{bmatrix} f \\ \tau \end{bmatrix}$$



Let $F_C^B \leftarrow$ Wrench acting at point c expressed in reference frame B

$F_C^C \leftarrow$ Wrench acting at point c expressed in reference frame C

$F_B^B \leftarrow$ Equivalent wrench to the wrench at point c acting at point B and expressed in reference frame B.

$$f_b^b = f_c^b = R_{bc} f_c^c$$

$$\tau_b^b = \tau_c^b + r_{bc} \times f_c^b$$

$$= \tau_c^b + r_{bc} \times R_{bc} f_c^c$$

$$= R_{bc} \tau_c^c + r_{bc} \times R_{bc} f_c^c$$

$$\therefore \begin{bmatrix} f_b^b \\ \tau_b^b \end{bmatrix} = \begin{bmatrix} R_{bc} & 0 \\ \hat{r}_{bc} R_{bc} & R_{bc} \end{bmatrix} \begin{bmatrix} f_c^c \\ \tau_c^c \end{bmatrix}$$

Now

$$g_{bc} = \begin{bmatrix} R_{bc} & p_{bc} \\ 0 & 1 \end{bmatrix} \Rightarrow g_{bc}^{-1} = \begin{bmatrix} R_{bc}^T & -R_{bc}^T p_{bc} \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow Ad_{g_{bc}^{-1}} = \begin{bmatrix} R_{bc}^T & -R_{bc}^T \hat{p}_{bc} \\ 0 & R_{bc} \end{bmatrix} \Rightarrow \left(Ad_{g_{bc}^{-1}}\right)^T = \begin{bmatrix} R_{bc} & 0 \\ \hat{p}_{bc} R_{bc} & R_{bc} \end{bmatrix}$$

$$\therefore \boxed{F_B^B = Ad_{g_{bc}^{-1}}^T F_C^C}$$

Screw Coordinates for a wrench :

Poinsot's Theorem: Every wrench is equivalent to a force and a moment applied along the same axis.

Conversion from wrench to screw coordinates:

Let $F = \begin{bmatrix} f \\ \tau \end{bmatrix}$ be a wrench.

We have to find a screw, i.e., a pitch, magnitude and an axis corresponding to the wrench

For pure torque, $f = 0$

$$M = \|\tau\| , \quad h = \infty , \quad l = \lambda \tau, \ \lambda \in \mathbb{R}$$

For $f \neq 0$, i.e, for the general case.

$$h = \frac{f^T \tau}{\|f\|^2} , \quad M = \|f\| , \quad l = \frac{f \times \tau}{\|f\|^2} + \lambda f, \ \lambda \in \mathbb{R}$$

Conversion from screw coordinates to wrench

Let $l = \{ q + \lambda \omega : \lambda \in \mathbb{R} \}, \|\omega\| = 1$ be the screw axis, and h, M be the pitch and magnitude of the screw respectively.

If $h = \infty$, $\quad F = M \begin{bmatrix} 0 \\ \omega \end{bmatrix}$

h finite, $\quad F = \begin{bmatrix} \omega \\ -\omega \times q + h\omega \end{bmatrix}$

Note: The fact that we can associate a twist (or infinitesimal motion) with a screw and a wrench (~~force~~/moment pair) with a screw is very important. It allows us to perform analysis in screw coordinates using geometric notions of dot products and reciprocal screws.

Reciprocal Screws:

Wrenches and Twists are "dual" to each other. This "duality" has a very precise meaning in terms of linear algebra. But here we will be considering an intuitive explanation.

Zero pitch twist $\Rightarrow$ Pure rotation (no linear component)

Zero pitch wrench $\Rightarrow$ Pure force (no angular component)

The dot product between a wrench and a twist is given by F.V which is the instantaneous power. ~~although they are different quantities and~~

A wrench F is reciprocal to a twist V if the instantaneous power is 0, i.e. $F \cdot V = 0$

Reciprocal Screws: Two screws S_1 and S_2 are reciprocal if the twist V about S_1 and wrench F along S_2 are reciprocal.

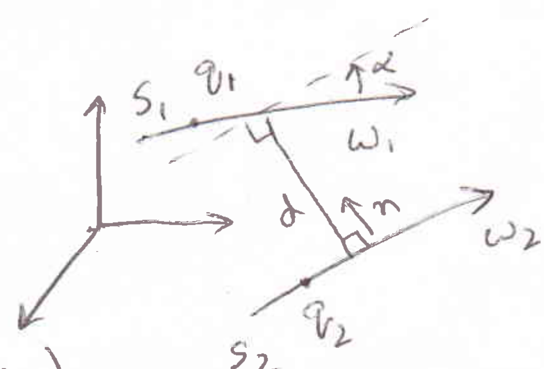
$$\text{let } S_i = (l_i, h_i, M_i)$$

$$\text{where } l_i = \{ q_i + \lambda \omega_i : \lambda \in \mathbb{R} \}$$

Let S_1 and S_2 be two screws.

Let d be the distance between two screws.

$\alpha \leftarrow$ Angle between S_1 and S_2



$$\alpha = \text{atan2}(\omega_1 \times \omega_2 \cdot n, \ \omega_1 \cdot \omega_2)$$

The reciprocal product of two screws is

$$S_1 \odot S_2 = M_1 M_2 ((h_1 + h_2)\cos\alpha - d \sin\alpha)$$

Two screws S_1 and S_2 are reciprocal iff.

$$S_1 \odot S_2 = 0$$

Proof: See Prop. 2.18 of MLS book.
Can be obtained by direct computation.

~~Reciprocal Systems of Screws:~~

<u>Inverse Kinematics of Serial Chain Manipulators</u>

<u>Problem Statement:</u>

Given a configuration $g_{st} \in SE(3)$ of the end effector find the joint angles ~~required~~ such that the ~~of~~ end effector configuration is achieved.

<u>General Algebraic Solution Procedure</u>: Inverse Kinematics (IK) problems are very hard to solve in general and ~~in~~ the problem is usually ~~equivalent to~~ converted to solution of a system of non linear ~~eqn~~ polynomial equations in multiple variables.

~~Although~~ For such a system of nonlinear polynomial equations there are two questions that arise

(1) How many solutions (in real space) does the system of non-linear equations have?

(2) How does one numerically solve ~~for~~ the system of non-linear equations? to find all the solutions?

~~Algebraic In order to solve Problem (1) above be usually these to eliminate variables~~

Answering Problem (1) itself is a ~~very~~ hard question in mathematics and although there are upper bounds that can be computed for the number of solutions it is not possible to compute the exact number of real solutions. <u>Bezout's theorem</u> provides an upper bound for the number of solutions.

Usually the polynomial equations in multiple variables are reduced to a polynomial equation in a single variable by a method called <u>Dialytic</u> <u>elimination</u>.

~~and~~ The degree of the resultant equation serves as an upper bound on the number of solutions.

For a general 6 DoF open chain manipulator there are 16 inverse kinematics solutions.

Numerical Solution: ~~If one is able to~~ Continuation Methods are a popular approach to solve systems of nonlinear polynomial equations. They can be used to find ~~all solutions of~~ one particular numerical solution (or all solutions) to an IK problem.

Geometric Solution:
In general, the difficulty of the IK problem depends on the structure of the manipulator. If the manipulator has number of intersecting axis then it may be possible to solve the IK with geometric reasoning (even for 6 DoF manipulators). We will use this approach in class.

In this method, solution of the IK problem involves reducing the problem into a series of simple geometric (sub) problems. These problems are known as the Paden Kahan subproblems.

These methods have been developed from the insight that although it is hard to solve IK problems in general, for 6 DoF manipulators in which 3 consecutive axes intersect one can find ~~all~~ IK solutions more easily.

Note: To convert a trigonometric equation into a polynomial equation, the following substitutions are usually used.

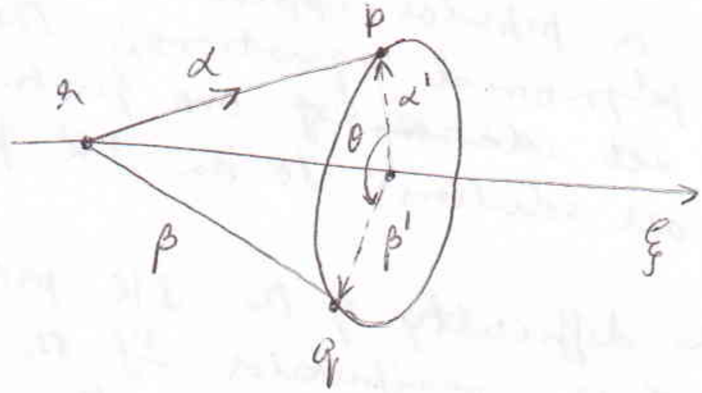
$$\tan\frac{\theta}{2} = u \qquad \cos\theta = \frac{1-u^2}{1+u^2} \qquad \sin\theta = \frac{2u}{1+u^2}$$

Paden Kahan Subproblems:

SP1: Rotation about a single axes:

Let ξ be a zero-pitch twist with unit magnitude and $p, q \in \mathbb{R}^3$ be two points.
Find θ such that $e^{\hat{\xi}\theta} = q$

Note: One solution exists here.



(We are actually showing ω associated with ξ)

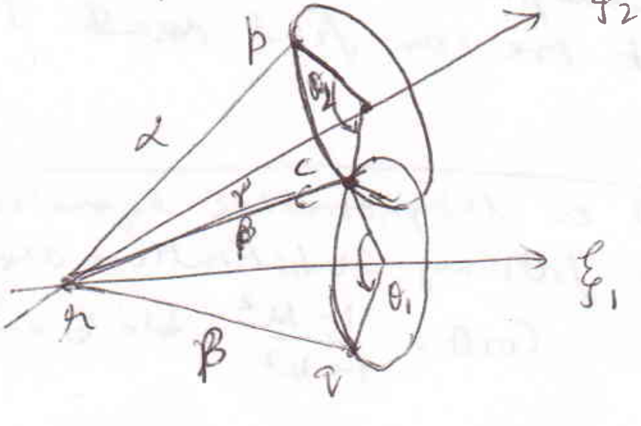
$$\theta = \operatorname{atan2}\left(\omega^T (\alpha' \times \beta'), \alpha'^T \beta'\right)$$

where
$$\alpha' = \alpha - \omega\omega^T \alpha$$
$$\beta' = \beta - \omega\omega^T \beta$$

SP2: Rotation about two subsequent axes that are intersecting:

Let ξ_1 and ξ_2 be two zero-pitch unit magnitude twists with intersecting axes and $p, q \in \mathbb{R}^3$ be two points. Find θ_1 and θ_2 such that

$$e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} p = q$$



$$\gamma = a_1 \omega_1 + a_2 \omega_2 + a_3(\omega_1 \times \omega_2)$$

θ_1 and θ_2 can be obtained from solving

$$e^{\hat{\xi}_2 \theta_2} p = c \quad \text{and} \quad e^{-\hat{\xi}_1 \theta_1} q = c$$

where $c = \gamma - r$, with γ specified in previous page.

The coefficients a_1, a_2, and a_3 are given by

$$a_1 = \frac{(\omega_1^T \omega_2)\, \omega_2^T \alpha - \omega_1^T \beta}{(\omega_1^T \omega_2)^2 - 1}$$

$$a_2 = \frac{(\omega_1^T \omega_2)\, \omega_1^T \beta - \omega_2^T \alpha}{(\omega_1^T \omega_2)^2 - 1}$$

$\alpha = p - r$

$\beta = q - r$

and
$$a_3^2 = \frac{\|\alpha\|^2 - a_1^2 - a_2^2 - 2 a_1 a_2\, \omega_1^T \omega_2}{\|\omega_1 \times \omega_2\|^2}$$

When a solution for a_3 exists, we can find θ_1 and θ_2.

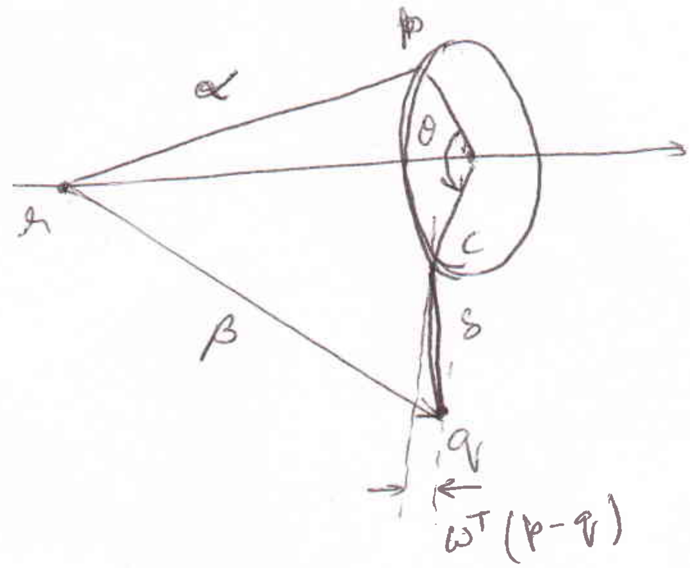
There are either 0 (when the circles do not intersect), 2 solutions (when the circles intersect at two points) and one solution (when the circles intersect at one point).

SP 2': Rotation about two subsequent axes that are non-intersecting.

SP3: Rotation to a given distance:

Let ξ be a zero-pitch unit magnitude twist and $p, q \in \mathbb{R}^3$ be two given points. $\delta > 0$ is a real number. Find θ such that

$$\| q - e^{\hat{\xi}\theta} p \| = \delta$$



$$\alpha = p - r$$

$$\beta = q - r$$

$$\alpha' = \alpha - \omega(\omega^T \alpha)$$

$$\beta' = \beta - \omega(\omega^T \beta)$$

$$\delta'^2 = \delta^2 - |\omega^T(p-q)|^2$$

$$\theta = \theta_0 \pm \cos^{-1}\left(\frac{\|\alpha'\|^2 + \|\beta'\|^2 - \delta'^2}{2\|\alpha'\|\,\|\beta'\|} \right)$$

The above equation has 0, 1 or 2 solutions.

Solving IK Using Sub problems :

Example 10 The general procedure is to use the kinematic ~~of two or more points~~ equations to ~~intersection~~ of two or more special points (at the intersection of two or more axes).

Example 1: IK for a link with spherical joint (or for a one link manipulator with 3 intersecting revolute joints).

Let ξ_1, ξ_2, ξ_3 be zero-pitch twists.

Consider a point p on the axis of joint 3 (or ω_3 / ξ_3) but not on ω_1 and ω_2

$$\therefore \quad e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} = g$$

Note that g is known to us.

Applying both sides of the above equation to the point p

$$e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} p = g p$$

$$\text{or} \quad e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} p = g p \qquad \left[\because e^{\hat{\xi}_3 \theta_3} p = p \right]$$

$\therefore$ By applying SP2 we can solve for θ_1 and θ_2

Once we have θ_1 and θ_2

$$e^{\hat{\xi}_3 \theta_3} = e^{-\hat{\xi}_2 \theta_2} e^{-\hat{\xi}_1 \theta_1} g$$

We can solve for θ_3 from the above using SP1.

Example 2: 3DoF manipulator with two intersecting axis for consecutive joints.

Let ξ_1, ξ_2, ξ_3 be these zero-pitch twists. W.l.o.g. assume that the axes of ξ_1 and ξ_2 intersect.

For simplicity assume $g_{st}(0) = I$ (This is not required).

$$e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} = g$$

Let q be the point of intersection of ξ_1 and ξ_2 (or the axes ω_1 and ω_2 of ξ_1 and ξ_2)

Let $\delta := \| g p - q \|$

$$\therefore \delta = \| e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} p - q \|$$

$$= \| e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{\hat{\xi}_3 \theta_3} p - e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} q \|$$

$$= \| e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} (e^{\hat{\xi}_3 \theta_3} p - q) \|$$

$$= \| e^{\hat{\xi}_3 \theta_3} p - q \|$$

(~~∵ Pure rotation does not~~ ∵ Rigid body transformations do not change distances)

∴ We can compute θ_3 from subproblem SP3.

Once we get θ_3, θ_1 and θ_2 can be obtained from SP2.

Elbow Manipulator (Example 3.5 of book)

Velocity Inverse Kinematics and Singularities

We have shown in the previous class that

$$\underbrace{V^s_{st}}_{6\times 1} = \underbrace{J^s_{st}(\theta)}_{6\times n}\,\underbrace{\dot\theta}_{n\times 1} \quad \& \quad \underbrace{V^b_{st}}_{6\times 1} = \underbrace{J^b_{st}(\theta)}_{6\times n}\,\underbrace{\dot\theta}_{n\times 1}$$

and

$$\underbrace{J^s_{st}(\theta)}_{6\times n} = \underbrace{Ad_{g_{st}(\theta)}}_{6\times 6}\,\underbrace{J^b_{st}(\theta)}_{6\times n}$$

From now, let us denote the manipulator Jacobian by J (we can use either spatial Jacobian or body Jacobian).

~~When n=6, we can compute the inverse~~

Inverse Velocity Kinematics:

Given V^s_{st} (or V^b_{st}) $(\in \mathbb{R}^6)$ compute $\dot\theta$. $(\in \mathbb{R}^n)$

If $n = 6$ and J is invertible then

$$\dot\theta = J^{-1}V$$

However the manipulator Jacobian may not be invertible.

The configurations at which the manipulator ~~loses~~ Jacobian is not invertible are called the singularities of the manipulator.

At a singular configuration the manipulator loses degrees of freedom in certain directions. ~~in space~~ In other words, joint motions cannot generate any velocities of the end effector along those directions.

The eigenvectors corresponding to the zero eigenvalues of the Jacobian span the space where no motion is possible.

Analysis of Singularities:

Analysis of singularities or finding all singular configurations of a manipulator is a hard problem in general. Again in this case the problem involves solution of a highly nonlinear polynomial equation in multiple variables (since $|J| = 0$ for singular configurations).

~~Sing~~ However there are geometric methods that can give insights to singular configurations of manipulator. The problem with geometric methods is that one has to apply it on a case by case basis and it depends on the capability of designer/roboticist to identify all singularities of a manipulator.

Singularities are ~~[illegible]~~ useful to know because even in near singular configurations the magnitude of joint velocities required to move the end effector along certain directions becomes very high. So singularities should be avoided for effective control.

Singularities can be classified as ~~work~~space boundary singularities and interior ~~[illegible]~~ singularities. They can also be classified based on the number of DoF lost at a particular singular configuration.

Geometric methods to identify singularities:

If ~~two or~~ two or more than two columns of ~~a~~ the manipulator Jacobian (for a 6 DoF manipulator) ~~drops rank or~~ are linearly dependent then the Jacobian drops rank or becomes singular.

Below are some conditions under which a 6 DoF manipulator Jacobian drops rank.

(1) Two collinear revolute joints: ~~The Jacobian for a~~

For a 6 DoF manipulator, J is singular if there are two revolute joints with twists

$$\xi_i = \begin{bmatrix} -\omega_i \times q_i \\ \omega_i \end{bmatrix} \quad \text{and} \quad \xi_2 = \begin{bmatrix} -\omega_j \times q_j \\ \omega_j \end{bmatrix}$$

such that

(a) the axes are parallel; i.e., $\omega_i = \pm \omega_j$

and (b) the axes are collinear, i.e.

$$\omega_i \times (q_i - q_j) = 0 \quad \text{and} \quad \omega_j \times (q_i - q_j) = 0$$

(2) Three parallel coplanar revolute joint axes:

Let ξ_1, ξ_2, ξ_3 be the twist of 3 revolute joints with

$$\xi_i = \begin{bmatrix} -\omega_i \times q_i \\ \omega_i \end{bmatrix}, \quad i = 1, 2, 3$$

The J is singular if

(a) the axes are parallel: $\omega_i = \pm \omega_j$; $i, j = 1, 2, 3$, $i \neq j$

(b) the axes are coplanar: $\exists$ a plane with unit normal n s.t. $n^T \omega_i = 0$ and $n^T (q_i - q_j) = 0$; $i, j = 1, 2, 3$, $i \neq j$.

(3) Four intersecting revolute joint axes.

J is singular if $\exists$ 4 revolute joint axes that intersect at a point (say q).

$$\omega_i \times (q_i - q) = 0, \quad i = 1, \dots 4.$$

Exercise 15, 16, 17 of MLS Book (Chapter 3) gives 3 more examples.

Manipulability:

There are global and local measures of manipulability.

Global measures are defined based on workspace (complete, reachable and dextrous)

Local measures are defined ~~as a property~~ at a given configuration. The Jacobian of the manipulator which relates infinitesimal joint motions to infinitesimal workspace motions is used to study local manipulability.

To study local manipulability we ~~first~~ have to first understand the notion of singular values of a matrix.

Let $A \in \mathbb{R}^{p \times n}$. Singular values of A are the square root of the eigen values of $A^T A$.

[Note that singular values and eigen values are not the same except ~~for~~ when A is an $n \times n$ real ~~symmetric~~ symmetric matrix with non-negative eigen values].

Let $\sigma(A)$ be the set of all singular values of a matrix.

If a matrix is singular, i.e. $\text{rank}(A) < \min(p, n)$ then at least one of its singular values is 0.

Maximum singular value of a matrix is

$$\sigma_{max}(A) = \max_{\|x\|_2 = 1} \|Ax\|_2 = \|A\|_2$$

Determinant of a matrix is product of its singular values.

Local Manipulability measures:

(1) $\mu_1(\theta) = \sigma_{min}(J(\theta))$

(2) $\mu_2(\theta) = |J(\theta)|$

(3) $\mu_3(\theta) = \dfrac{\sigma_{min}(J(\theta))}{\sigma_{max}(J(\theta))}$

All these measures are 0 at a singular configuration. The higher the value the better the manipulability.