Lecture 1a: Introduction to Configuration Space

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1.1 Introduction

We will now study the concept of configuration space of robots, which is a fundamental concept used in robot motion planning. We will assume that robots are made up of rigid bodies that are connected by joints. Before proceeding further, we will recall the geometric concept of point, and the phyiscal concepts of point mass and rigid body.

Notion of Point: Intuitively, we understand that a *point* is a geometric entity that has zero dimensions. One key aspect to realize is that the concept of point is a **mathematical abstraction**. There is no physical object that is a point object. However, *modeling an object as a point* is useful in applications as we shall see later in class.

Another way that we use the notion of point while not appealing to geometry is that a point is an *element of a set*. This is actually a more general way to think about a point and the reader should note that the geometric definition is a special case of the definition above where the underlying set is the two dimensional (Euclidean) plane or the three dimensional (Euclidean) space around us. If you are not used to thinking of a point as an element of a set, we will see plenty of examples in the class, where you will realize this.

Point Mass: A *point mass* is an object whose dimensions are considered to be zero but it has non-zero mass. You should once again realize that there is no physical object that is a point mass. Nevertheless, a point mass is a very useful *mathematical model of objects* that is valid in many applications. Recall that when we study Newton's law in high school, we implicitly assume that the object is a point mass.

Rigid Body: A *rigid body* is a set of points in which the *distance between any two given points remain the same* irrespective of the external forces acting on it. A rigid body is an *idealization of a solid*. There is no physical object that is a rigid body, but *assuming an object to be a rigid body is a good mathematical model* of a physical object in many applications.

Sometimes we will distinguish between *planar rigid bodies* and *spatial rigid bodies*. A planar rigid body is rigid body with no thickness. However, as you can see, it is a mathematical abstraction, since no actual physical object is planar. A rigid body (with not necessarily zero thickness) can be mathematically modeled as a planar rigid body for the purposes of kinematics, if its motion is confined to a plane. A spatial rigid body is a general rigid body that is not constrained to move in a plane. The concept of planar rigid bodies is useful to study planar mechanisms or planar robots, i.e., mechanisms or robots whose motion is constrained to be on parallel planes (or the motion of the *end effector* of the mechanism or robot is constrained to be in a plane).

We will consider a robot to be made up of a collection of rigid bodies (called *links*) whose motion is constrained by *joints*. A *joint* is an entity that constrains the relative motion between two rigid bodies. Two basic joints that we will consider are Revolute (R) joints and prismatic (P) joints. We are not going to delve into the details of the different types of joints here. The assumption is that you have already seen this in an undergraduate class of mechanisms or machine design. One aspect to realize though is that the concepts of links and joints are also mathematical abstractions! There are different physical realizations of these mathematical notions!

1.2 Configuration and Degrees of Freedom

Below we will study the notions of configuration and degrees of freedom of an object. In the discussion below, an object may be a point mass, a rigid body, or a collection of rigid bodies.

Definition 1.1 *The configuration of an object is the description of the set of all points on the object.*

when we specify the configuration of an object, we specify the position of all points in the object. Note that although it may seem from the definition that it takes infinitely many parameters to describe a rigid body or a collection of rigid bodies (since there are infinitely many points), a rigid body (or a collection of rigid bodies) may be described by finitely many parameters. This leads us to the concept of degrees of freedom.

Definition 1.2 Degrees of Freedom*: The degrees of freedom (DoF) of an object is the minimum number of parameters required to describe it.*

The configuration of a point mass is its position. For example, a point mass in two dimensional plane has DoF 2, since there are two independent parameters (the x and y coordinates) required to describe the position of the point anywhere in the plane. Similarly, a point mass in three dimensional space has DoF 3, since it requires three independent parameters to describe its position.

A planar rigid body has DoF 3, and a spatial rigid body has DoF 6. In other words, it requires 3 independent parameters to describe the position of all points on a planar rigid body and it takes 6 independent parameters to describe the position of all points on a spatial rigid body. For a proof of this fact see Chapter 2, Section 2.1 of the book by Lynch and Park.

To compute the DoF of a robot where rigid links (or rigid bodies) are connected by joints, we can use the Gruebler-Kutzhbach (GK) formula. Here again, we will not go into the derivation of this formula, since I presume, you have seen this formula before (at least for planar mechanisms). For a planar robot the GK formula states that the DoF of a robot is

$$\text{DoF(planar)} = 3(L-1) - 2J_1 - J_2$$

where L is the total number of links including the ground link, J_1 is the number of 1 DoF joints, and J_2 is the number of 2 DoF joints. For spatial robots, the GK formula states that the DoF is given by

$$\text{DoF(spatial)} = 6(L-1) - 5J_1 - 4J_2 - 3J_3 - 2J_4 - J_5$$

where as before L is the number of links including the ground link and J_i, $i = 1, 2, \ldots, 5$, is the number of joints that allows i DoF (or constrains $6 - i$ DoF) between two links. Recall that the above formulas are derived by using the fact that the DoF = Total number of degrees of freedom of all the links - Total number of constraints imposed by all the joints.

Note that revolute and prismatic joints are 1 DoF joints. Revolute joints allows 1 direction of relative rotation between the two links and prismatic joints allow 1 direction of relative translation between the two links.

Example 1: For the $2R$ manipulator studied in class, to show that its DoF is 2, we note that $L = 3$, $J_1 = 2$, and $J_i = 0$, for $i = 2, 3, 4, 5$. Now using the spatial GK formula, you can verify that the DoF of the manipulator is 2. Since the $2R$ manipulator is a planar manipulator, you could also use the GK formula for planar manipulators to get the same result.

Example 2: A serial chain manipulator with n 1 DoF joints (either prismatic or revolute or a combination of them) has DoF n. Note that, here $L = n + 1$, $J_1 = n$, $J_i = 0$, for $i = 2, 3, 4, 5$. Therefore

$$DoF = 6(n + 1 - 1) - 5n = n$$

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1.3 Configuration Space

The set of all possible configurations of a robot (or an object) is called the configuration space of the robot (object). The dimension of the configuration space of the robot (object) is the DoF of the robot (object). Some examples of configuration space are as follows:

1. For a point robot in a two dimensional world, the configuration space consists of all points in the plane (denoted by $\mathbb{R}^2$, the two-dimensional Euclidean plane). A circular wheeled mobile robot that can move in any direction in the plane instantaneously (also called holonomic mobile robot) can be modeled as a point robot with a two dimensional configuration space. In fact, for kinematics purposes, any holonomic mobile robot in the plane (irrespective of the shape) can be modeled as a point robot in the plane.
2. For a point robot in a three dimensional world, the configuration space consists of all points in $3D$ world (denoted by $\mathbb{R}^3$, the three-dimensional Euclidean space).
3. For the $2R$ manipulator that we discussed in class, the configuration space consists of all pairs of the two joint angles. The configuration space is two dimensional. However, in general, it is not the same as the configuration space of a holonomic point robot. Intuitively, we can see this by noting that the two parameters here represent angles, whereas for the point robot in the first example, the two parameters represent its x, y coordinates with respect to a reference frame fixed in the world.
4. For a $n-$DoF serial chain manipulator with all revolute joints, the configuration space consists of all $n-$tuples of joint angles. The dimension of the the configuration space is n. For manipulators, the configuration space is also called the *joint space* of the manipulator.

Remark 1.3 *Note that the above discussion implies that any robot (i.e., the configuration of any robot) can be viewed as a point in its configuration space. This mathematical abstraction is crucial in development of motion planning algorithms for robots. We will see that thinking of a point as an element of a set and getting comfortable with this general notion of point is very useful.*

Remark 1.4 *The topological and geometric properties of the configuration space is extremely important for the purposes of motion planning and control. We will further explore this topic when we study motion planning.*

1.4 Configuration Space of Planar Rigid Bodies

For defining the configuration space of planar rigid bodies, one needs to do a little more work. We will first need to choose a reference frame attached to the body and moving with the body (call it *body frame*). The configuration space is defined as the set of all positions (x, y) of the origin of the body frame and the orientation of the body frame $\mathcal{B}$ with respect to the global frame, $\mathcal{G}$, (denoted by θ). In other words, the configuration space consists of all tuples (x, y, θ). The origin of the body frame is denoted by O_B and the origin of the global frame is denoted by O_G.

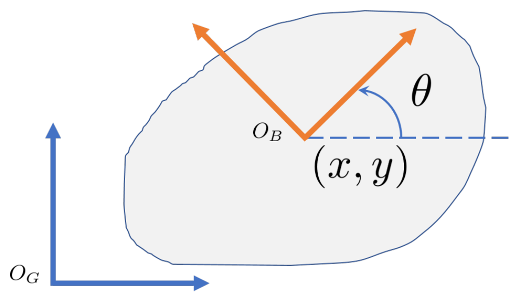

Figure 1.1: Configuration of a planar rigid body. The reference frame with origin O_B is the body frame and the reference frame with origin O_G is the global frame.

We will now look at a more formal presentation of the configuration space of a planar rigid body. Let $\mathbf{q} = (q_x, q_y)$ be the coordinates of a point on the planar rigid body in the body frame $\mathcal{G}$. Let (x', y') be the coordinates of the point in the frame $\mathcal{B}$. Therefore,

$$\begin{aligned} q_x &= x + x' \cos\theta + y' \cos\left(\frac{\pi}{2} + \theta\right) = x + x' \cos\theta - y' \sin\theta \\ q_y &= y + x' \sin\theta + y' \sin\left(\frac{\pi}{2} + \theta\right) = y + x' \sin\theta + y' \cos\theta \end{aligned} \tag{1.1}$$

The above system of equations can be written in matrix form as

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{1.2}$$

The set of all matrices of the form shown above, i.e., $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is denoted by $SO(2)$ and is called the special orthogonal group of dimension 2. You may note that the determinant of this matrix is always 1 (that is why we call the set special) and if we consider each column of the matrix as a vector then the dot product between the vectors is 0 (i.e., the columns are orthogonal, and that is why we use orthogonal in the name). This will be elaborated upon in the next class. Another way of writing the Equation (1.2) in matrix form is

$$\begin{bmatrix} q_x \\ q_y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \tag{1.3}$$

Formally, the configuration space of a planar rigid body can be defined as all 3×3 matrices of the form shown in Equation (1.3). This set is denoted by $SE(2)$, and read as special Euclidean group of dimension 2. The 2 comes from the fact that the motion of the rigid body is in a two-dimensional planar world. The actual dimension of the configuration space is 3.

Remark 1.5 *In the above discussion, it seems that we used some algebraic trickery to define a set of matrices and then abstractly defined the set, $SO(2)$ and $SE(2)$ without any motivation. The motivation and importance of this abstract definition will be made clear in the next couple of classes, in the general context of spatial rigid bodies. The discussion above will then become a special case of what we will study. The reason for presenting these here is to make you familiar with the notations and the terminology.*

1.5 Configuration Space of Spatial Rigid Bodies

A robot is usually modeled as a collection of rigid bodies whose motion is constrained through joints. Furthermore, many of the objects in the environment that a robot has to manipulate can also be modeled as rigid bodies. To methodically study motions of robots or objects manipulated by robots, we first need to have a proper understanding of the configuration space of rigid bodies. Intuitively we want to understand the following: (a) How do we describe the *configuration*) of a rigid body with respect to a chosen reference frame? (b) If a rigid body is moved from one configuration to another, how do we describe the relationship between the two configurations (more precisely, how do we describe the rigid body transformation)?

To pose the questions above more precisely and develop tools for understanding them, we will first look at description of position of points (or particles) in different reference frames and the mapping that relates the position of a point in different reference frames. We will note that the underlying mathematics describing configuration of a rigid body as well as rigid body transformation is same as that of the mapping that changes description of a point between two reference frames.

1.5.1 Changing Description of a point between two reference frames

Let Q be a point in the three dimensional Euclidean space (or $\mathbb{R}^3$). We know that the position of a point can be described by three coordinates if we choose an appropriate reference frame and the coordinates depend on the reference frame chosen. Let $\{B\}$ and $\{W\}$ be the two reference frames shown in Figure 1.2. Let ${}^B\mathbf{p}_Q$ and ${}^W\mathbf{p}_Q$ be the position of point Q in reference frames $\{B\}$ and $\{W\}$ respectively. **In this section, our goal is to establish the relationship between ${}^B\mathbf{p}_Q$ and ${}^W\mathbf{p}_Q$.**

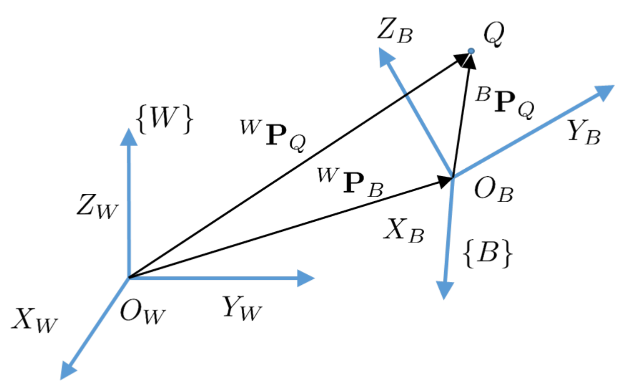

Figure 1.2: Transformation of points between two reference frames

In Figure 1.2, let ${}^Wp_B$ be the position vector of the origin of frame $\{B\}$ in frame $\{W\}$. From knowledge of vector addition we can write:

$$ {}^W\mathbf{p}_Q = {}^W\mathbf{p}_B +^W ({}^B\mathbf{p}_Q) \tag{1.4} $$

where ${}^W({}^B\mathbf{p}_Q)$ is the vector ${}^B\mathbf{p}_Q$ expressed in the frame $\{W\}$. Now, we will compute the vector ${}^W({}^B\mathbf{p}_Q)$. Let

$$ {}^B\mathbf{p}_Q = x_B\mathbf{X_B} + y_B\mathbf{Y_B} + z_B\mathbf{Z_B} \tag{1.5} $$

where $\mathbf{X}_B$, $\mathbf{Y}_B$, $\mathbf{Z}_B$ are the unit basis vectors of frame $\{B\}$ written in frame $\{B\}$. Therefore from Equation (1.5), we have

$$ {}^W({}^B\mathbf{p}_Q) = x_B^W\mathbf{X_B} + y_B^W\mathbf{Y_B} + z_B^W\mathbf{Z_B} \tag{1.6} $$

where ${}^W\mathbf{X}_B$, ${}^W\mathbf{Y}_B$, ${}^W\mathbf{Z}_B$ are the unit vectors $\mathbf{X}_B$, $\mathbf{Y}_B$, $\mathbf{Z}_B$, of frame $\{B\}$ written in the frame $\{W\}$. Let

$$ \begin{aligned} {}^W\mathbf{X}_B &= r_{11}\mathbf{X}_W + r_{21}\mathbf{Y}_W + r_{31}\mathbf{Z}_W \\ {}^W\mathbf{Y}_B &= r_{12}\mathbf{X}_W + r_{22}\mathbf{Y}_W + r_{32}\mathbf{Z}_W \\ {}^W\mathbf{Z}_B &= r_{13}\mathbf{X}_W + r_{23}\mathbf{Y}_W + r_{33}\mathbf{Z}_W \end{aligned} $$

Substituting the above in Equation (1.6) we obtain

$$ \begin{aligned} {}^W({}^B\mathbf{p}_Q) &= x_B(r_{11}\mathbf{X}_W + r_{21}\mathbf{Y}_W + r_{31}\mathbf{Z}_W) \\ &\quad + y_B(r_{12}\mathbf{X}_W + r_{22}\mathbf{Y}_W + r_{32}\mathbf{Z}_W) \\ &\quad + z_B(r_{13}\mathbf{X}_W + r_{23}\mathbf{Y}_W + r_{33}\mathbf{Z}_W) \\ &= (r_{11}x_B + r_{12}y_B + r_{13}z_B)\mathbf{X}_W \\ &\quad + (r_{21}x_B + r_{22}y_B + r_{23}z_B)\mathbf{Y}_W \\ &\quad + (r_{31}x_B + r_{32}y_B + r_{33}z_B)\mathbf{Z}_W \end{aligned} \tag{1.7} $$

Thus, in component form the vector ${}^W({}^B\mathbf{p}_Q)$ can be written as

$$ {}^W({}^B\mathbf{p}_Q) = \begin{bmatrix} r_{11}x_B + r_{12}y_B + r_{13}z_B \\ r_{21}x_B + r_{22}y_B + r_{23}z_B \\ r_{31}x_B + r_{32}y_B + r_{33}z_B \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} $$

Substituting the above in Equation (1.4) we obtain

$$ \boxed{{}^W\mathbf{p}_Q = {}^W\mathbf{p}_B + {}^W_B\mathbf{R} \cdot {}^B\mathbf{p}_Q} \tag{1.8} $$

where ${}^W_B\mathbf{R}$ is called the rotation matrix of frame $\{B\}$ with respect to frame $\{W\}$ and

$$ {}^W_B\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

Thus, if we know ${}^W_B\mathbf{R}$ and ${}^W\mathbf{p}_B$, where ${}^W\mathbf{p}_B$ is the position vector of the origin of the frame $\{B\}$ in frame $\{W\}$, we can change the description of any point Q in reference frame $\{B\}$ to frame $\{W\}$. Therefore, generally speaking, we say that the tuple $(\mathbf{p}, \mathbf{R})$ where $\mathbf{p}$ is a vector and $\mathbf{R}$ is a rotation matrix describes the relationship of two reference frames.

Now, also note that

$$ \begin{array}{lll} r_{11} = \mathbf{X}_B \cdot \mathbf{X}_W & r_{12} = \mathbf{Y}_B \cdot \mathbf{X}_W & r_{13} = \mathbf{Z}_B \cdot \mathbf{X}_W \\ r_{21} = \mathbf{X}_B \cdot \mathbf{Y}_W & r_{22} = \mathbf{Y}_B \cdot \mathbf{Y}_W & r_{23} = \mathbf{Z}_B \cdot \mathbf{Y}_W \\ r_{31} = \mathbf{X}_B \cdot \mathbf{Z}_W & r_{22} = \mathbf{Y}_B \cdot \mathbf{Z}_W & r_{23} = \mathbf{Z}_B \cdot \mathbf{Z}_W \end{array} $$

From the definition of the rotation matrix entries given above we note that

$$ {}^W_B\mathbf{R} = \left[\ {}^W\mathbf{X}_B \quad {}^W\mathbf{Y}_B \quad {}^W\mathbf{Z}_B\ \right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} {}^B\mathbf{X}_W^T \\ {}^B\mathbf{Y}_W^T \\ {}^B\mathbf{Z}_W^T \end{bmatrix} \tag{1.9} $$

In words,

- Columns of ${}^W_B\mathbf{R}$ are unit vectors of $\{B\}$ expressed in frame $\{W\}$.
- Rows of ${}^W_B\mathbf{R}$ are unit vectors of $\{W\}$ expressed in reference $\{B\}$.

Therefore ${}^B_W\mathbf{R} = {}^W_B\mathbf{R}^T$ or ${}^W_B\mathbf{R}^{-1} = {}^W_B\mathbf{R}^T$.

Remark 1.6 *Note that not only points but (free) vectors can also be transformed between two reference frames. To transform (free) vectors one needs to multiply the vector by the rotation matrix. Let* ${}^B\mathbf{v}$ *and* ${}^W\mathbf{v}$ *be the description of vector* $\mathbf{v}$ *in the frames* $\{B\}$ *and* $\{W\}$ *respectively. Then* ${}^W\mathbf{v} = {}^W_B\mathbf{R}\,{}^B\mathbf{v}$.

Remark 1.7 *If* ${}^W\mathbf{p}_Q$ *is given and we want to find* ${}^B\mathbf{p}_Q$*, then we have*

$${}^B\mathbf{p}_Q = {}^W_B\mathbf{R}^T({}^W\mathbf{p}_Q - {}^W\mathbf{p}_B)$$

Be cautious that this formula is valid when we are given origin of frame $\{B\}$ *in frame* $\{W\}$, ${}^W\mathbf{p}_B$.

Exercise: What would be the formula for ${}^B\mathbf{p}_Q$, if you are given position vector of origin of frame $\{W\}$ in frame $\{B\}$, ${}^B\mathbf{p}_Q$, and ${}^W_B\mathbf{R}$?

1.5.2 Configuration of Spatial Rigid Body

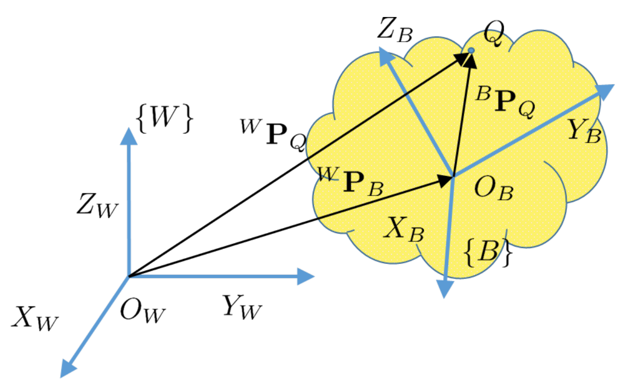

Figure 1.3: Spatial description of a rigid body

Figure 1.3 shows a rigid body with $\{B\}$ being a frame that is attached to the body and moves with the body. The origin of frame $\{B\}$ is O_B. The world frame is $\{W\}$. From the discussion of the previous section, it is clear that all points on the rigid body can be uniquely determined in the world frame if we know the tuple $({}^W\mathbf{p}_B, {}^W_B\mathbf{R})$. Thus, the **configuration** of the rigid body can be described by the tuple $({}^W\mathbf{p}_B, {}^W_B\mathbf{R})$. Note that configuration is the generalization of the notion of location for points to rigid bodies (that consist of infinite number of points). In general, the configuration of any rigid body in a reference frame can be described by a tuple $(\mathbf{p}, \mathbf{R})$. Thus we see that the underlying description of relationship between reference frames in Section 1.5.1 and configuration of a rigid body are the same.

The *configuration space* of a rigid body is the space of all tuples $(\mathbf{p}, \mathbf{R})$. We will define it formally and more precisely in one of the subsequent lectures.

Appendix: Cross Product as a Linear Operation (or matrix vector multiplication)

For any two vectors **a** and **b** $\in \mathbb{R}^3$, the cross product is defined by

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Where,

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Cross product by a vector **a** can be thought of as a linear operator, *i.e.* , multiplication of a vector by a matrix.

$$a \times b = \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{\hat{a}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$a \times b = \hat{a}b$$

where $\hat{\mathbf{a}}$ is a (3×3) skew-symmetric matrix and **b** is a (3×1) vector.

<u>Some Useful Results</u>:

1. R $(v \times w)$ = (Rv) $\times$ (Rw)
2. R $(\hat{w})R^T = (\hat{Rw})$
3. A rotation R $\in SO(3)$ is a rigid body transformation
   (a) R preserves distances,i.e.,
   $$\|Rq - Rp\| = \|q - p\|, \forall, q, p \in \mathbb{R}^3$$
   (b) R preserves orientation, i.e.,
   $$\text{R } (v \times w) = Rv \times Rw, \forall, v, w \in \mathbb{R}^3$$