Robotics Lecture Notes **Spring 2018**

Lecture 5: Velocity Kinematics of Serial Manipulators

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5.1 Introduction

The velocity kinematics of serial manipulators relates the joint angle rates of the manipulator to the end effector linear and angular velocity. We consider a n-DoF manipulator. Irrespective of the DoF of the manipulator, the DoF of the end effector is always 6 since the end effector is a rigid body. Let $\mathbf{V} = \begin{bmatrix} \mathbf{v} \\ \omega \end{bmatrix}$ be the velocity twist of the end effector frame with respect to the world frame, where $\mathbf{v}$ is a linear velocity term and **omega** is an angular velocity term. Let $\mathbf{\Theta} = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_n \end{bmatrix}^T$ be the $n \times 1$ vector of joint angles or the configuration of the manipulator and $\dot{\mathbf{\Theta}} = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \cdots & \dot{\theta}_n \end{bmatrix}^T$ be the $n \times 1$ vector of joint angle rates or joint velocity. As we will show below, the relationship between the joint angular rates and velocity of the end effector is given by

$$\mathbf{V} = \mathbf{J}(\mathbf{\Theta})\dot{\mathbf{\Theta}} \tag{5.1}$$

where $\mathbf{J}(\mathbf{\Theta})$ is a $6 \times n$ matrix called the manipulator Jacobian. For a given configuration, the matrix $\mathbf{J}(\mathbf{\Theta})$ is a constant and thus the mapping is a linear mapping (instantaneously). There are two different methods for computing the Jacobian of a manipulator and the resulting Jacobians are called analytic Jacobian and geometric Jacobian. The geometric Jacobian can be further classified as spatial Jacobian or body Jacobian. The Jacobians obtained are different but they are equivalent in the sense that one can go from one to the other.

Analytic Jacobian: The analytic Jacobian is computed by direct symbolic differentiation of the algebraic forward kinematics equations. In this case, the linear velocity, $\mathbf{v}$, is the velocity of the origin of the tool frame with respect to the world frame expressed in the world (or base) frame. The angular velocity is the angular velocity of the tool frame with respect to the base frame expressed in the base frame.

Spatial Jacobian: The spatial Jacobian is a geometric Jacobian and is computed from the exponential formulation of the forward position kinematics and there is no symbolic differentiation involved. The spatial velocity of the end effector is denoted by $\mathbf{V}^s = \begin{bmatrix} \mathbf{v}^s \\ \omega^s \end{bmatrix}$, where ω^s is the angular velocity of the tool frame with respect to the spatial frame, expressed in the spatial frame. The linear velocity $\mathbf{v}^s$ is of a (imaginary) point on the end effector that is located at the origin of the world frame at any particular instant.

Body Jacobian: The body Jacobian is also a geometric Jacobian and is computed from the exponential formulation of the forward position kinematics and there is no symbolic differentiation involved. The body velocity of the end effector is denoted by $\mathbf{V}^b = \begin{bmatrix} \mathbf{v}^b \\ \omega^b \end{bmatrix}$, where ω^b is the angular velocity of the tool frame with respect to the spatial frame, expressed in the tool frame. The linear velocity, $\mathbf{v}^b$, is the velocity of the origin of the tool frame with respect to the world frame expressed in the tool frame.

Note that the main reason for the differences between the Jacobian comes from the choice of the coordinate frame in which the velocity is expressed and the choice of the point on the end effector for which the linear velocity is given. We will study the geometric Jacobian in these notes. The computation of the analytic

Jacobian is as stated in Lecture 1, where we give the formula for computing the Jacobian of a vector function with vector inputs.

5.2 Spatial Jacobian

The relationship between the spatial velocity of the end effector, $\mathbf{V}^s$ and the joint velocity $\dot{\theta}$ is given by

$$\mathbf{V}^s = \mathbf{J}^s(\boldsymbol{\Theta})\dot{\boldsymbol{\Theta}} \tag{5.2}$$

where $\mathbf{J}^s$ is the spatial Jacobian. We will first present the formula for computing the spatial Jacobian and then show the derivation. Let $\xi_1, \xi_2, \ldots, \xi_n$, be the unit twists corresponding to the n joints of the manipulator in the reference configuration (the same twists that we computed for doing the exponential formulation of forward position kinematics). To remind ourselves, $\xi = \begin{bmatrix} -\mathbf{u} \times \mathbf{q} \\ \mathbf{u} \end{bmatrix}$, for revolute joints, where $\mathbf{u}$[1] is the axis of the joint and $\mathbf{q}$ is a point on the axis of the joint. For prismatic joints, $\xi = \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix}$, where $\mathbf{v}$ is the axis of the prismatic joint.

The spatial manipulator Jacobian is given by

$$\mathbf{J}^s(\boldsymbol{\Theta}) = \begin{bmatrix} \xi_1 & \xi_2' & \xi_3' & \cdots & \xi_n' \end{bmatrix}, \quad \text{where,} \quad \xi_i' = Ad_{g_{1,i-1}}\xi_i \quad i = 2, 3, \ldots, n. \tag{5.3}$$

where $Ad_{g_{1,i-1}}$ is the adjoint matrix corresponding to $\mathbf{g}_{1,i-1}$ and

$$\mathbf{g}_{1,i-1} = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}} = \prod_{j=1}^{i-1} e^{\hat{\xi}_j\theta_j} \tag{5.4}$$

Thus, *the ith column of the Jacobian is the ith joint twist in the current manipulator configuration written with respect to the base frame.*

5.2.1 Derivation of the Expression for Spatial Jacobian

We will now derive Equation (5.3). Let $\{S\}$ be the world frame and $\{T\}$ be the end effector frame (or tool frame) of the manipulator. Let $\mathbf{g}_{st} : \mathcal{Q} \to SE(3)$ be the forward kinematics map, where $\mathcal{Q}$ is the configuration (i.e., joint) space of the manipulator and $SE(3)$ is the configuration space of the end effector. For a given configuration of the manipulator, $\boldsymbol{\Theta} = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_n \end{bmatrix}^T$, the configuration of the tool frame with respect to base frame can be written as

$$\mathbf{g}_{st}(\boldsymbol{\Theta}) = \prod_{j=1}^{n} e^{\hat{\xi}_j\theta_j}\mathbf{g}_{st}(0) \tag{5.5}$$

where $\mathbf{g}_{st}(0)$ is the configuration of the tool frame with respect to the base frame in the reference configuration of the robot.

[1]The notation here is changed from ω to $\mathbf{u}$ for the joint axis so as to avoid confusion with angular velocity.

By definition, the spatial velocity of the tool frame is

$$\begin{aligned}\hat{\mathbf{V}}^s_{st} &= \dot{\mathbf{g}}_{st}(\boldsymbol{\Theta})\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta}) \\ &= \left(\sum_{i=1}^{n} \frac{\partial \mathbf{g}_{st}(\boldsymbol{\Theta})}{\partial \theta_i}\dot{\theta}_i\right)\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta}) \quad \text{(By chain rule)} \\ &= \sum_{i=1}^{n} \frac{\partial \mathbf{g}_{st}(\boldsymbol{\Theta})}{\partial \theta_i}\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta})\dot{\theta}_i\end{aligned}$$

Note that, $\frac{\partial \mathbf{g}_{st}(\boldsymbol{\Theta})}{\partial \theta_i}\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta})$, for each $i = 1, \ldots, n$, is a 4×4 matrix which is an element of $se(3)$ as shown in Lemma 5.2 in the Appendix. Since $\dot{\theta}_i$ is a scalar, $\frac{\partial \mathbf{g}_{st}(\boldsymbol{\Theta})}{\partial \theta_i}\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta})\dot{\theta}_i \in se(3)$. The above equation can be written in terms of twist coordinates using Lemma 5.1 in Appendix as

$$\mathbf{V}^s_{st} = \left[\left(\frac{\partial \mathbf{g}_{st}(\boldsymbol{\Theta})}{\partial \theta_1}\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta})\right)^{\vee} \quad \left(\frac{\partial \mathbf{g}_{st}(\boldsymbol{\Theta})}{\partial \theta_2}\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta})\right)^{\vee} \quad \cdots \quad \left(\frac{\partial \mathbf{g}_{st}(\boldsymbol{\Theta})}{\partial \theta_n}\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta})\right)^{\vee}\right]\dot{\boldsymbol{\Theta}} \tag{5.6}$$

where the vee operator when applied to an element of $se(3)$ returns the 6×1 column vector, which is the twist coordinates of a twist. Thus, the spatial Jacobian is the matrix

$$\mathbf{J}^s_{st} = \left[\left(\frac{\partial \mathbf{g}_{st}(\boldsymbol{\Theta})}{\partial \theta_1}\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta})\right)^{\vee} \quad \left(\frac{\partial \mathbf{g}_{st}(\boldsymbol{\Theta})}{\partial \theta_2}\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta})\right)^{\vee} \quad \cdots \quad \left(\frac{\partial \mathbf{g}_{st}(\boldsymbol{\Theta})}{\partial \theta_n}\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta})\right)^{\vee}\right] \tag{5.7}$$

Note that $\mathbf{J}^s_{st}$ is a $6 \times n$ matrix with the ith column given by $\left(\frac{\partial \mathbf{g}_{st}(\boldsymbol{\Theta})}{\partial \theta_i}\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta})\right)^{\vee}$.

From Equation (5.5), we get

$$\begin{aligned}\frac{\partial \mathbf{g}_{st}(\boldsymbol{\Theta})}{\partial \theta_i}\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta}) &= \left(\prod_{j=1}^{i-1} e^{\hat{\xi}_j\theta_j}\right)\left(\frac{\partial}{\partial \theta_i}e^{\hat{\xi}_i\theta_i}\right)\left(\prod_{j=i+1}^{n} e^{\hat{\xi}_j\theta_j}\right)\mathbf{g}_{st}(0)\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta}) \\ &= \left(\prod_{j=1}^{i-1} e^{\hat{\xi}_j\theta_j}\right)\left(\hat{\xi}_i e^{\hat{\xi}_i\theta_i}\right)\left(\prod_{j=i+1}^{n} e^{\hat{\xi}_j\theta_j}\right)\mathbf{g}_{st}(0)\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta}) \\ &= \left(\prod_{j=1}^{i-1} e^{\hat{\xi}_j\theta_j}\right)\left(\hat{\xi}_i\right)\left(\prod_{j=i}^{n} e^{\hat{\xi}_j\theta_j}\right)\mathbf{g}_{st}(0)\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta}) \\ &= \left(\prod_{j=1}^{i-1} e^{\hat{\xi}_j\theta_j}\right)\left(\hat{\xi}_i\right)\left(\prod_{j=i}^{n} e^{\hat{\xi}_j\theta_j}\right)\underbrace{\mathbf{g}_{st}(0)\mathbf{g}^{-1}_{st}(0)}_{I}\left(\prod_{j=n}^{i} e^{-\hat{\xi}_j\theta_j}\right)\left(\prod_{j=i-1}^{1} e^{-\hat{\xi}_j\theta_j}\right) \\ &= \left(\prod_{j=1}^{i-1} e^{\hat{\xi}_j\theta_j}\right)\left(\hat{\xi}_i\right)\underbrace{\left(\prod_{j=i}^{n} e^{\hat{\xi}_j\theta_j}\right)\left(\prod_{j=n}^{i} e^{-\hat{\xi}_j\theta_j}\right)}_{I}\left(\prod_{j=i-1}^{1} e^{-\hat{\xi}_j\theta_j}\right) \\ &= \left(\prod_{j=1}^{i-1} e^{\hat{\xi}_j\theta_j}\right)\left(\hat{\xi}_i\right)\left(\prod_{j=i-1}^{1} e^{-\hat{\xi}_j\theta_j}\right) \\ &= \mathbf{g}_{1,i-1}\hat{\xi}_i\mathbf{g}^{-1}_{1,i-1}\end{aligned}$$

where $\mathbf{g}_{1,i-1} = \prod_{j=1}^{i-1} e^{\hat{\xi}_j\theta_j}$. Therefore, from Lemma 5.3, we can write

$$\left(\frac{\partial \mathbf{g}_{st}(\boldsymbol{\Theta})}{\partial \theta_i}\mathbf{g}^{-1}_{st}(\boldsymbol{\Theta})\right)^{\vee} = Ad_{g_{1,i-1}}\xi_i \tag{5.8}$$

For $i = 1$, it can be easily seen that there will be no term pre-multiplying $\hat{\xi}_1$ in the above derivation. Therefore the first column of the Jacobian will be ξ_1.

5.2.2 Algorithm for Computing Spatial Jacobian

To summarize, the spatial Jacobian can be computed by the following steps:

1. In the reference configuration, write down all the joint twists, ξ_i, $i = 1, \ldots, n$.
2. For $i = 2, \ldots, n$, compute $\mathbf{g}_{1,i-1}$ using Equation (5.4).
3. Compute the adjoint matrix for each $\mathbf{g}_{1,i-1}$, i.e., $Ad_{g_{1,i-1}}$ and subsequently, compute the ith column of the Jacobian, $\xi_i' = Ad_{g_{1,i-1}}\xi_i$, $i = 2, \ldots, n$.
4. Form the Jacobian by concatenating the columns $\xi_1, \xi_2', \ldots, \xi_n'$.

5.3 Body Jacobian

Analogous to the spatial Jacobian, we can define the body Jacobian as the matrix that relates the joint velocity to the body velocity of the end effector by

$$\mathbf{V}^b = \mathbf{J}^b(\boldsymbol{\Theta})\dot{\boldsymbol{\Theta}} \tag{5.9}$$

where $\mathbf{J}^b(\boldsymbol{\Theta})$ is called the body Jacobian. The body manipulator Jacobian is given by

$$\mathbf{J}^b(\boldsymbol{\Theta}) = \left[\xi_1^\dagger \quad \xi_2^\dagger \quad \xi_3^\dagger \quad \cdots \quad \xi_n^\dagger]\right], \quad \text{where,} \quad \xi_i^\dagger = Ad_{g_{i,n}g_{st}(0)}\xi_i \quad i = 2, 3, \ldots, n. \tag{5.10}$$

where $Ad_{g_{1,i-1}}$ is the adjoint matrix corresponding to $\mathbf{g}_{1,i-1}$ and

$$\mathbf{g}_{i,n}\mathbf{g}_{st}(0) = e^{\hat{\xi}_i\theta_i}e^{\hat{\xi}_{i+1}\theta_{i+1}}\cdots e^{\hat{\xi}_n\theta_n}\mathbf{g}_{st}(0) = \left(\prod_{j=i}^{n} e^{\hat{\xi}_j\theta_j}\right)\mathbf{g}_{st}(0) \tag{5.11}$$

Thus, *the ith column of the Jacobian is the ith joint twist written with respect to the tool frame in the current manipulator configuration.*

5.3.1 Mapping Between Spatial Jacobian and Body Jacobian

The spatial Jacobian and the body Jacobian are related by the following formula;

$$\boxed{\mathbf{J}^s(\boldsymbol{\Theta}) = Ad_{g_{st}(\boldsymbol{\Theta})}\mathbf{J}^b(\boldsymbol{\Theta})} \tag{5.12}$$

The above formula can be derived from the relationship between the spatial velocity and the body velocity of the end effector, which is

$$\begin{aligned} \mathbf{V}_{st}^s &= Ad_{g_{st}(\boldsymbol{\Theta})}\mathbf{V}_{st}^b \\ \Rightarrow \mathbf{J}_{st}^s(\boldsymbol{\Theta})\dot{\boldsymbol{\Theta}} &= Ad_{g_{st}(\boldsymbol{\Theta})}\mathbf{J}^b(\boldsymbol{\Theta})\dot{\boldsymbol{\Theta}} \end{aligned}$$

Since the above has to be true for all values of $\dot{\boldsymbol{\Theta}}$, therefore

$$\mathbf{J}^s(\boldsymbol{\Theta}) = Ad_{g_{st}(\boldsymbol{\Theta})}\mathbf{J}^b(\boldsymbol{\Theta}).$$

5.4 Forward Velocity Kinematics

The forward velocity kinematics of a serial manipulator is defined as follows: *Given the configuration of the robot $\mathbf{\Theta}$, joint velocities $\dot{\mathbf{\Theta}}$, compute the end effector velocity $\mathbf{V}$.*

The forward kinematics problem can be solved by using Equation (5.2) or (5.9) depending on whether we want the spatial or body velocity. Once, we have formed the Jacobian, it is a straightforward matrix-vector multiplication.

Velocity of any point: When we compute the spatial velocity, the point for which the linear velocity is computed, is an (imaginary) point on the end-effector passing through the origin of the spatial frame instantaneously. The linear velocity of any other point on the end effector in the spatial frame can be computed as given below. Let $\mathbf{x}^s$ be the coordinates of a point X attached to the end effector in the base frame and $\mathbf{x}^b$ be the coordinates of the same point in the body frame. The spatial velocity of the point X is

$$\begin{aligned} \mathbf{v}_x^s &= \omega^s \times \mathbf{x}^s + \mathbf{v}^s \\ \Rightarrow \begin{bmatrix} \mathbf{v}_x^s \\ 0 \end{bmatrix} &= \begin{bmatrix} \hat{\omega}^s & \mathbf{v}^s \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}^s \\ 1 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} \mathbf{v}_x^s \\ 0 \end{bmatrix} &= \widehat{\mathbf{V}}^s \begin{bmatrix} \mathbf{x}^s \\ 1 \end{bmatrix} \end{aligned} \tag{5.13}$$

Thus the velocity of the origin of the tool frame in the spatial frame will be

$$\boxed{\begin{bmatrix} \mathbf{v}_t^s \\ 0 \end{bmatrix} = \widehat{\mathbf{V}}^s \begin{bmatrix} \mathbf{p} \\ 1 \end{bmatrix}} \tag{5.14}$$

where, we can obtain $\mathbf{p}$ from the forward kinematics map $\mathbf{g}_{st}(\Theta) = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$.

Relationship between Spatial Jacobian and Analytic Jacobian: As stated before, the analytic Jacobian can be computed by computing the partial derivatives of the forward kinematics equation with respect to the joint angles. However, it is also straightforward to compute the analytic Jacobian from the spatial Jacobian. Let $\mathbf{g}_{st}(\Theta) = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$ be the transformation of the tool frame with respect to the world frame. The relationship between the analytic Jacobian, $\mathbf{J}^a$, and spatial Jacobian, $\mathbf{J}^s$ is given by

$$\mathbf{J}^a(\mathbf{\Theta}) = \begin{bmatrix} \mathbf{I} & -\hat{\mathbf{p}} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{J}^s(\mathbf{\Theta}) \tag{5.15}$$

where $\mathbf{I}$ is a 3×3 identity matrix, $\mathbf{0}$ is 3×3 matrix with all elements 0.

5.5 Inverse Velocity Kinematics

The inverse velocity kinematics of a serial manipulator is defined as follows: *Given the configuration of the robot $\mathbf{\Theta}$, and end effector velocity $\mathbf{V}$, compute the joint velocities $\dot{\mathbf{\Theta}}$.* Note that this problem statement is valid irrespective of whether we are considering the linear velocity portion of the generalized velocity to be the spatial velocity, body velocity, or the velocity of the origin of the toll frame in the spatial frame.

If the Jacobian in Equation 5.1 is invertible, then we can obtain

$$\dot{\mathbf{\Theta}} = \mathbf{J}^{-1}(\mathbf{\Theta})\mathbf{V} \tag{5.16}$$

Now, $\mathbf{J}(\boldsymbol{\Theta})$ is a $m \times n$ matrix, where $m \leq 6$ (usually $m = 3$ for planar robots and $m = 6$ for spatial robots), and n is the DoF of the manipulator. Assuming $m = 6$, since this is the most general case, $\mathbf{J}$ is invertible if $n = 6$, and the manipulator is not at a *singular configuration*.

If $n > m$, the rank of the Jacobian, $\mathbf{J}$ is m. Such manipulators are called *kinematically redundant manipulators*. In this case, the joint angle rates or joint velocity can be obtained by (we are omitting the dependence of $\boldsymbol{\Theta}$ for convenience)

$$\boxed{\dot{\boldsymbol{\Theta}} = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}\mathbf{V}} \tag{5.17}$$

To verify the above formula multiply both sides by $\mathbf{J}$ and we get back Equation (5.1).

If $n < m$, the rank of the Jacobian, $\mathbf{J}$ is n. In this case, the joint angle rates or joint velocity can be obtained by (we are omitting the dependence of $\boldsymbol{\Theta}$ for convenience)

$$\boxed{\dot{\boldsymbol{\Theta}} = (\mathbf{J}^T\mathbf{J})^{-1}\mathbf{J}^T\mathbf{V}} \tag{5.18}$$

To verify the above formula multiply both sides by $\mathbf{J}^T\mathbf{J}$ and we get $\mathbf{J}^T\mathbf{J}\dot{\boldsymbol{\Theta}} = \mathbf{J}^T\mathbf{V}$, which implies Equation (5.1).

5.5.1 More on Redundant Manipulators

As discussed above, for manipulators with kinematic redundancy, i.e., degree-of-freedom greater than 6, we can obtain the velocity inverse kinematics solution using Equation (5.17). However, note that When $n > 6$, Equation (5.1) implies that we have to solve for n variables (the n joint rates) from 6 linear equations. Since $n > 6$, i.e., the number of variables are greater than the number of equations, this system of equations have infinitely many solutions. However, Equation (5.17) gives only one solution. We will now see, how to obtain a general solution for Equation (5.1) that allows us to obtain any of the infinitely many possible solutions.

For a given $\boldsymbol{\Theta}$, $\mathbf{J}(\boldsymbol{\Theta})$ is a constant matrix, which can be thought of as a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^6$. Thus, the vector space $\mathbb{R}^n$ has a non-trivial null space of dimension $n-6$. The vector space $\mathbb{R}^n$ can be decomposed into two subspaces, the null space of $\mathbf{J}$, denoted by $\mathcal{N}(\mathbf{J})$ and the range space of $\mathbf{J}^T$ denoted by $\mathcal{R}(\mathbf{J}^T)$, such that any vector $\dot{\boldsymbol{\Theta}}$ can be written as a sum of two vectors:

$$\dot{\boldsymbol{\Theta}} = \dot{\boldsymbol{\Theta}}_r + \dot{\boldsymbol{\Theta}}_n, \quad \dot{\boldsymbol{\Theta}}_r \in \mathcal{R}(\mathbf{J}^T), \quad \dot{\boldsymbol{\Theta}}_n \in \mathcal{N}(\mathbf{J}).$$

The solution in Equation (5.17) is actually $\dot{\boldsymbol{\Theta}}_r$ which is in the range space of $\mathbf{J}^T$, namely, $\mathcal{R}(\mathbf{J}^T)$. We will now obtain an expression for a vector in the null space of $\mathbf{J}$, i.e., $\mathcal{N}(\mathbf{J})$. Note that by definition $\mathbf{J}\dot{\boldsymbol{\Theta}}_n = \mathbf{0}$. Thus, adding a vector in the null space of $\mathbf{J}$ does not affect the end effector velocity, $\mathbf{V}$.

Since $\dot{\boldsymbol{\Theta}}_r \in \mathcal{R}(\mathbf{J}^T)$, there exists some vector, $\overline{\mathbf{V}}$, in $\mathbb{R}^6$ such that $\dot{\boldsymbol{\Theta}}_r = \mathbf{J}^T\overline{\mathbf{V}}$. Therefore,

$$\begin{aligned}
\dot{\boldsymbol{\Theta}} &= \dot{\boldsymbol{\Theta}}_r + \dot{\boldsymbol{\Theta}}_n \\
\Rightarrow \dot{\boldsymbol{\Theta}} &= \mathbf{J}^T\overline{\mathbf{V}} + \dot{\boldsymbol{\Theta}}_n \\
\Rightarrow \mathbf{J}\dot{\boldsymbol{\Theta}} &= \mathbf{J}\mathbf{J}^T\overline{\mathbf{V}} + \cancelto{0}{\mathbf{J}\dot{\boldsymbol{\Theta}}_n} \\
\Rightarrow \overline{\mathbf{V}} &= (\mathbf{J}\mathbf{J}^T)^{-1}\mathbf{J}\dot{\boldsymbol{\Theta}}.
\end{aligned} \tag{5.19}$$

Since, $\dot{\boldsymbol{\Theta}}_r = \mathbf{J}^T\overline{\mathbf{V}}$, we have

$$\boxed{\dot{\boldsymbol{\Theta}}_r = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}\mathbf{J}\dot{\boldsymbol{\Theta}}} \tag{5.20}$$

Since $\dot{\mathbf{\Theta}} = \dot{\mathbf{\Theta}}_r + \dot{\mathbf{\Theta}}_n$, we have

$$\boxed{\dot{\mathbf{\Theta}}_n = (\mathbf{I} - \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}\mathbf{J})\dot{\mathbf{\Theta}}} \tag{5.21}$$

Note that for any choice of a joint velocity vector, say $\dot{\mathbf{\Theta}}' \in \mathbb{R}^n$, Equation (5.21), gives a vector in the null space of $\mathbf{J}$ (to check this just multiply Equation (5.21) by $\mathbf{J}$). In other words, Equation (5.21) projects any joint velocity vector, $\dot{\mathbf{\Theta}}'$ to the subspace, $\mathcal{N}(\mathbf{J})$. Thus, using Equations (5.20), (5.21), $\dot{\mathbf{\Theta}} = \dot{\mathbf{\Theta}}_r + \dot{\mathbf{\Theta}}_n$, and Equation (5.1), the general solution to the velocity inverse kinematics for redundant manipulators is

$$\boxed{\dot{\mathbf{\Theta}} = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}\mathbf{V} + (\mathbf{I} - \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1}\mathbf{J})\dot{\mathbf{\Theta}}'} \tag{5.22}$$

To verify that the above solution is correct, multiply both sides by $\mathbf{J}$ and you should get back Equation (5.1).

5.6 Statics of Serial Manipulators

In this section, we relate the force and moment acting on the end effector of a manipulator to the torques acting on the joints. We want to understand the following question: *Given, a force and/or moment acting on the end effector, compute the joint torques such that the manipulator remains static.*

Let $\mathbf{F} = \begin{bmatrix} \mathbf{f}_e \\ \tau_e \end{bmatrix}$ be the force-moment pair acting at the end effector and τ be the vector pf joint torques. Then,

$$\tau = \mathbf{J}^T\mathbf{F} \tag{5.23}$$

The relation above can be proven using the concept of virtual work. At a static configuration if there is a virtual displacement of $\Delta\mathbf{\Theta}$ in the joint space, it leads to a displacement of $\Delta\mathbf{X}$ in the end effector space. The work done in the joint space is $\tau \cdot \Delta\mathbf{\Theta}$ and the work done in the end effector space is $\mathbf{F} \cdot \Delta\mathbf{X}$. This work has to be same, since work is a scalar quantity. Thus

$$\begin{aligned} \tau \cdot \Delta\mathbf{\Theta} &= \mathbf{F} \cdot \Delta\mathbf{X} \\ \Rightarrow \tau^T \Delta\mathbf{\Theta} &= \mathbf{F}^T \Delta\mathbf{X} \\ \Rightarrow \tau^T \Delta\mathbf{\Theta} &= \mathbf{F}^T \mathbf{J}(\mathbf{\Theta})\Delta\mathbf{\Theta} \end{aligned} \tag{5.24}$$

The above has to be true for all values of $\Delta\mathbf{\Theta}$. Therefore, $\tau^T = \mathbf{F}^T\mathbf{J}(\mathbf{\Theta})$, which gives Equation (5.23) by taking transpose on both sides.

Note that in the previous sections, we have talked about the analytic Jacobian, the spatial Jacobian, and the body Jacobian. The formula above is valid for any Jacobian. However, we have to be careful about the proper reference frame and the point at which the end-effector force-moment pair is acting. If we use body Jacobian, the end effector force-moment pair acting at the origin of the tool frame and it is expressed in the tool frame. For analytic Jacobian, the force-moment pair is acting at the origin of the tool frame and it is expressed in the base frame. For spatial Jacobian, the force-moment pair is acting at an imaginary point on the end effector that is instantaneously located at the origin of the base frame and force-moment pair is expressed in the base frame.

Appendix

Lemma 5.1 *Suppose* $\hat{\xi}_1 = \begin{bmatrix} \hat{\mathbf{u}}_1 & \mathbf{v}_1 \\ \mathbf{0} & 0 \end{bmatrix}$ *and* $\hat{\xi}_2 = \begin{bmatrix} \hat{\mathbf{u}}_2 & \mathbf{v}_2 \\ \mathbf{0} & 0 \end{bmatrix}$ *be two twists, i.e.,* $\hat{\xi}_1, \hat{\xi}_2 \in se(3)$*, with twist coordinates* $\xi_1 = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{u}_1 \end{bmatrix}$ *and* $\xi_2 = \begin{bmatrix} \mathbf{v}_2 \\ \mathbf{u}_2 \end{bmatrix}$*, then (a)* $\hat{\xi} = \hat{\xi}_1 + \hat{\xi}_2$ *is also a twist, i.e.,* $\hat{\xi} \in se(3)$ *and (b) the twist* *coordinate of* $\hat{\xi}$*, namely,* ξ*, is the sum of the twist coordinates of* $\hat{\xi}_1$ *and* $\hat{\xi}_2$*, i.e.,* $\xi = \xi_1 + \xi_2$*.*

Proof:

$$\hat{\xi} = \hat{\xi}_1 + \hat{\xi}_2 = \begin{bmatrix} \hat{\mathbf{u}}_1 & \mathbf{v}_1 \\ \mathbf{0} & 0 \end{bmatrix} + \begin{bmatrix} \hat{\mathbf{u}}_2 & \mathbf{v}_2 \\ \mathbf{0} & 0 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{u}}_1 + \hat{\mathbf{u}}_2 & \mathbf{v}_1 + \mathbf{v}_2 \\ \mathbf{0} & 0 \end{bmatrix}$$

Now $\hat{\mathbf{u}}_1 + \hat{\mathbf{u}}_2 = (\mathbf{u}_1 + \mathbf{u}_2)^\wedge$ [2], where the notation $(\cdot)^\wedge$ implies that the hat operator is applied to the argument within the parenthesis. Let $\mathbf{u}_1 + \mathbf{u}_2 = \mathbf{u}$ and $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}$. Therefore, $\hat{\xi} = \begin{bmatrix} \hat{\mathbf{u}} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \in se(3)$. and the twist coordinates

$$\xi = \begin{bmatrix} \mathbf{v} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 + \mathbf{v}_2 \\ \mathbf{u}_1 + \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{u}_1 \end{bmatrix} + \begin{bmatrix} \mathbf{v}_2 \\ \mathbf{u}_2 \end{bmatrix} = \xi_1 + \xi_2$$

■

Lemma 5.2 *Let* $\mathbf{\Theta}$ *be the vector of joint angles of an open chain manipulator and let* $\mathbf{g}_{st}(\mathbf{\Theta})$ *be the forward kinematics map. Then* $\frac{\partial \mathbf{g}_{st}(\mathbf{\Theta})}{\partial \theta_i} \mathbf{g}_{st}^{-1}(\mathbf{\Theta}) \in se(3)$*.*

Proof: As shown on page 3, $\frac{\partial \mathbf{g}_{st}(\mathbf{\Theta})}{\partial \theta_i} \mathbf{g}_{st}^{-1}(\mathbf{\Theta}) = \mathbf{g}_{1,i-1} \hat{\xi}_i \mathbf{g}_{1,i-1}^{-1}$, where $\hat{\xi}_i$ is the twist of the ith joint. Thus, if we show that $\mathbf{g}\hat{\xi}\mathbf{g}^{-1} \in se(3)$, for any $\mathbf{g} \in SE(3)$ and $\hat{\mathbf{xi}} \in se(3)$ then we are done. Let $\mathbf{g} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$, $\hat{\xi} = \begin{bmatrix} \hat{\omega} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}$. Then

$$\begin{aligned} \mathbf{g}\hat{\xi}\mathbf{g}^{-1} &= \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{R}^{\mathbf{T}} & -\mathbf{R}^{\mathbf{T}}\mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{R}\hat{\omega}\mathbf{R}^{\mathbf{T}} & -\mathbf{R}\hat{\omega}\mathbf{R}^{\mathbf{T}}\mathbf{p} + \mathbf{R}\mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \quad \text{(by direct multiplication)} \qquad (5.25) \\ &= \begin{bmatrix} \widehat{\mathbf{R}\omega} & -\widehat{\mathbf{R}\omega}\mathbf{p} + \mathbf{R}\mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} \quad \text{(see Appendix of Lecture on rigid body rotation).} \end{aligned}$$

Now, $\widehat{\mathbf{R}\omega} \in so(3)$ and $-\widehat{\mathbf{R}\omega}\mathbf{p} + \mathbf{R}\mathbf{v} \in \mathbb{R}^3$. Therefore, from the definition of $se(3)$, we can conclude that $\mathbf{g}\hat{\xi}\mathbf{g}^{-1} \in se(3)$. ■

Lemma 5.3 *Let* $\mathbf{g} \in SE(3)$ *and* $\hat{\xi} \in se(3)$*, and* $\hat{\xi}' = \mathbf{g}\hat{\xi}\mathbf{g}^{-1}$*. Then* $\xi' = Ad_g \xi$*.*

Proof: Let $\xi' = \begin{bmatrix} \mathbf{v}' \\ \omega' \end{bmatrix}$. Now, from Lemma 5.2, we have that

$$\hat{\xi}' = \begin{bmatrix} \hat{\omega}' & \mathbf{v}' \\ \mathbf{0} & 0 \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{R}\omega} & -\widehat{\mathbf{R}\omega}\mathbf{p} + \mathbf{R}\mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{R}\omega} & \hat{p}\mathbf{R}\omega + \mathbf{R}\mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}$$

Thus, $\omega' = \mathbf{R}\omega$ and $\mathbf{v}' = \hat{p}\mathbf{R}\omega + \mathbf{R}\mathbf{v}$. These two equations can be written in matrix form as

$$\xi' = \begin{bmatrix} \mathbf{v}' \\ \omega' \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \hat{p}\mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \omega \end{bmatrix} = Ad_g \xi \qquad (5.26)$$

■

[2] This can be proven by direct computation