Power of a Dual Quaternion

Let $\sigma = p + \epsilon q$ be a rigid body displacement in dual quaternion form.

Since p is the unit quaternion corresponding to the rotation part, ~~define~~ we know that

$$p = \cos\frac{\theta}{2} + \hat{l}\,\sin\frac{\theta}{2}$$

where θ is the angle of rotation about axis $\hat{l}$.

Let us define

$$d = (2\,q\,p^{*}) \cdot \hat{l}$$

Note: The vector $\hat{l}$ is in dual quaternion rotation i.e. a 4×1 element with first component as 0.

Define $$m = \frac{1}{2}\left(t \times \hat{l} + (t - d\hat{l})\cot\frac{\theta}{2}\right)$$

Then the rigid body transformation σ can also be written as

$$\sigma = \cos\left(\frac{\theta+\epsilon d}{2}\right) + \sin\left(\frac{\theta+\epsilon d}{2}\right)(\hat{l}+\epsilon m)$$

Defining $\bar{\theta} = \theta + \epsilon d$ ← Dual Angle

and $\bar{\hat{l}} = \hat{l} + \epsilon m$

we can write

$$\sigma = \cos\frac{\bar{\theta}}{2} + \bar{\hat{l}}\,\sin\frac{\bar{\theta}}{2}$$

$$\therefore\ \boxed{\sigma^{\tau} = \cos\frac{\tau\bar{\theta}}{2} + \bar{\hat{l}}\,\sin\frac{\tau\bar{\theta}}{2}}$$

Note that: $$\cos\left(\frac{\theta+\epsilon d}{2}\right) = \cos\frac{\theta}{2} - \epsilon\frac{d}{2}\sin\frac{\theta}{2}$$

$$\sin\left(\frac{\theta+\epsilon d}{2}\right) = \sin\frac{\theta}{2} + \epsilon\frac{d}{2}\cos\frac{\theta}{2}$$