Summary:

A rigid body ~~motion~~ transformation can be expressed as

$$g_{ab}(\theta) = e^{\hat{\xi}\theta}\, g_{ab}(0) \quad \text{— (1)}$$

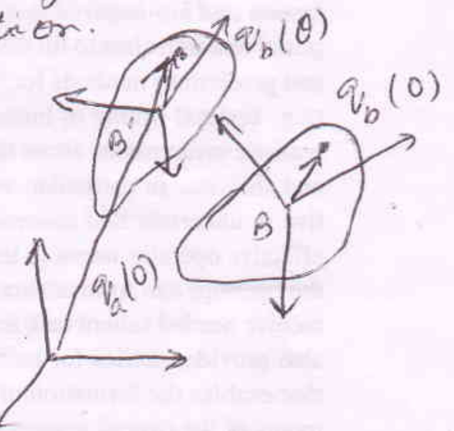
where $\hat{\xi}$ is a unit twist with twist coordinates $\begin{bmatrix} v \\ \omega \end{bmatrix}$ and $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$ is the twist.

$\hat{\xi} \in se(3)$, $g \in SE(3)$.

Equation (1) can be physically interpreted as follows.

To find the coordinates of any point given in body ~~coordinates~~ frame B, first multiply it by $g_{ab}(0)$ (the description of frame B w.r.t A) to convert it into frame A and then multiply it by the rigid body transformation $e^{\hat{\xi}\theta}$ to get the coordinates of the point after the transformation.

i.e., $q_a(\theta) = e^{\hat{\xi}\theta}\, g_{ab}(0)\, q_b(0)$



Application of the above to multiply rigid bodies connected via revolute or translational joints

For a revolute joint $\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}$

For a translation joint $\xi = \begin{bmatrix} v \\ 0 \end{bmatrix}$

where $\omega \leftarrow$ unit vector along axis of rotation

$q \leftarrow$ A point on the axis of rotation expressed in the base frame

$v \leftarrow$ unit vector along axis of translation.

We ~~θ~~ We will denote the ~~displacement~~ motion about (or along) a joint by θ. Thus for revolute joints angular motion is θ, for translational joints translation ~~about~~ along the ~~axis~~ joint axis is θ.

6. For an n-link manipulator (serial chain) let S be the base frame and T be the tool frame. Let $\theta = [\theta_1, \theta_2, \dots, \theta_n]^T$ be the vector of joint angles

$$g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} \cdots e^{\hat{\xi}_n \theta_n} g_{st}(0)$$

where the joints are numbered from the base to the tool in increasing order. $g_{st}(0)$ is the transformation of tool frame w.r.t. base frame at a base configuration, usually assumed to be $\theta = 0$ (i.e., $\theta_1 = \theta_2 = \dots = \theta_n = 0$)

Definitions

For a spatial manipulator if $n > 6$ then the manipulator is kinematically redundant (because only 6 DOF is required to position and orient the end effector in space).

[In the plane if $n > 3$ then the manipulator is kinematically redundant]

If there is one actuator per degree of freedom and $n > 6$ then the manipulator is redundant.

If there are DOF that are not actuated then the manipulator is underactuated.

[underactuated serial chains are used for constructing hand fingers]

Velocity of a rigid body:

SE(3) is the configuration space of a rigid body. A rigid body corresponding to a given configuration is a point in SE(3). or each point (or element) of SE(3) corresponds to a rigid body configuration.

$\therefore$ The motion of a rigid body is a curve in SE(3). Let us represent the curve by $g(t)$.

Pure ~~Rot~~ Rotation:

Let us first consider a subgroup of SE(3) which is the group of pure rotations denoted by SO(3).

~~Let~~ Let ω_{ab} the angular velocity of frame B w.r.t. frame A, where as before frame B is a moving frame attached to the body and frame A is the world fixed frame.

~~Then~~ Let ω^s_{ab} ← Instantaneous Spatial angular velocity as seen from the spatial (A) coordinate frame.

ω^b_{ab} ← Instantaneous body angular velocity in instantaneous body frame.

$$\boxed{\hat{\omega}^s_{ab} = \dot{R}_{ab} R^{-1}_{ab}} \qquad \boxed{\hat{\omega}^b_{ab} = R^{-1}_{ab} \dot{R}_{ab}}$$

We will now look at the motivation of the above formulas. ~~Before we~~

First we show the following:

<u>Lemma</u>: Given $R(t) \in SO(3)$, the matrices $\dot{R}(t) R^{-1}(t)$ and $R^{-1}(t) \dot{R}(t)$ belong to so(3), i.e. they are skew-symmetric.

B

Proof:

$$R(t)\, R(t)^T = I$$

$$\dot{R}(t)\, R(t)^T + R(t)\, \dot{R}(t)^T = 0$$

$$\therefore \dot{R}(t)\, R(t)^T = -R(t)\, \dot{R}(t)^T = -\left[\dot{R}(t)\, R(t)\right]^T$$

Now $R(t)^T = R(t)^{-1}$

$$\therefore \dot{R} R^{-1} = -(\dot{R} R^{-1})^T$$ [Dropping the dependence on t for notational convenience]

$\therefore (\dot{R} R^{-1})$ is skew-symmetric

To prove $R^{-1}\dot{R}$ is skew-symmetric start with $R^T R = I$ and follow the above steps.

$$\dot{R} R^{-1} \in so(3) \qquad R^{-1}\dot{R} \in so(3)$$

Now, $q_a(t) = R_{ab}(t)\, q_b$

Since q_b is constant in the body frame

$$v_{q_a}(t) = \dot{q}_a(t) = \dot{R}_{ab}(t)\, q_b$$

$$= \dot{R}_{ab} R_{ab}^{-1} R_{ab}\, q_b$$

$$= \dot{R}_{ab} R_{ab}^{-1} q_a$$

$$= \hat{\omega}^s_{ab}\, q_a = \omega^s_{ab}(t) \times q(t)$$

For the body velocities just note that

$$v_{q_b}(t) = R_{ab}^T\, v_{q_a}(t)$$

$$\hat{\omega}^b_{ab} = R_{ab}^{-1} \dot{R}_{ab}$$

$$R_{ab}^{-1}\, \hat{\omega}^s_{ab}\, R_{ab} = R_{ab}^{-1}\, \dot{R}_{ab}\, \underbrace{R_{ab}^{-1} R_{ab}}_{I}$$

$$= R_{ab}^{-1} \dot{R}_{ab} = \hat{\omega}^b_{ab}$$

$$\therefore \boxed{\hat{\omega}^b_{ab} = R_{ab}^{-1}\, \hat{\omega}^s_{ab}\, R_{ab}}$$

From the above we can write.

$$\boxed{\omega^b_{ab} = R_{ab}^{-1}\, \omega^s_{ab}} \quad \& \quad \boxed{\omega^s_{ab} = R_{ab}\, \omega^b_{ab}}$$

Proof by direct calculation.

$$v_{a_b}(t) = R_{ab}^{-1} v_{q_a}(t) = R_{ab}^{-1}\left(\omega_{ab}^{s}(t) \times q_a(t)\right)$$

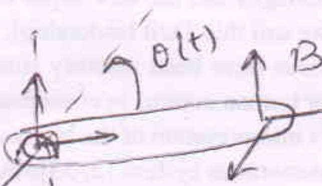
$$= \omega_{ab}^{b}(t) \times q_b$$

Note: v_{a_b} is the linear velocity of a point on the body with respect to the stationary frame expressed in the instantaneous body frame.

It is NOT the linear velocity of the point with respect to the body frame. That linear velocity is 0.

Example:

$$R(t) = \begin{bmatrix} c_\theta(t) & -s_\theta(t) & 0 \\ s_\theta(t) & c_\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



1 DoF manipulator with a revolute joint.

where $c_\theta = \cos\theta$, $s_\theta = \sin\theta$

Spatial velocity is

$$\hat{\omega}^s = \dot{R}R^T = \begin{bmatrix} -\dot\theta\sin\theta & -\dot\theta\cos\theta & 0 \\ \dot\theta\cos\theta & -\dot\theta\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\dot\theta & 0 \\ \dot\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \omega^s = \begin{bmatrix} 0 \\ 0 \\ \dot\theta \end{bmatrix}$$

$$\hat{\omega}^b = R^T\dot{R} = \begin{bmatrix} 0 & -\dot\theta & 0 \\ \dot\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \omega^b = \begin{bmatrix} 0 \\ 0 \\ \dot\theta \end{bmatrix}$$

Rotation is about z-axis which ~~does not~~ remains the same for body frame and spatial frame. Therefore the body and spatial angular velocities are identical in this case.

General Rigid Body Motion:

Let $g_{ab}(t) \in SE(3)$ be the trajectory of the rigid body.

~~A parameterized~~
A trajectory is a time parametrized path. A path of the rigid body motion is any curve in $SE(3)$.

$$g_{ab}(t) = \begin{bmatrix} R_{ab}(t) & p_{ab}(t) \\ 0 & 1 \end{bmatrix}$$

First, let us write the following:

$$\dot{g}_{ab}\, g_{ab}^{-1} = \begin{bmatrix} \dot{R}_{ab} & \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{ab}^T & -R_{ab}^T p_{ab} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{R}_{ab} R_{ab}^T & -\dot{R}_{ab} R_{ab}^T p_{ab} + \dot{p}_{ab} \\ 0 & 0 \end{bmatrix}$$

$$\therefore \dot{g}_{ab}\, g_{ab}^{-1} \in se(3)$$

[Note the analogy to the case of pure rotation]

$\therefore$ We define the spatial velocity $\hat{V}_{ab}^s \in se(3)$ as

$$\boxed{\hat{V}_{ab}^s = \dot{g}_{ab}\, g_{ab}^{-1}} \quad \text{and} \quad V_{ab}^s = \begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix}$$

$$= \begin{bmatrix} -\dot{R}_{ab} R_{ab}^T p_{ab} + \dot{p}_{ab} \\ (\dot{R}_{ab} R_{ab}^T)^\vee \end{bmatrix}$$

where the $\vee$ operator is the inverse of the $\wedge$ operator. It gets the vector that forms a 3×3 skew-symmetric matrix.

Using the spatial velocity we can find the velocity of a point on the rigid body.

Now, $q_a(t) = g_{ab}(t)\, q_b$

$v_{q_a} = \dot{q}_a = \dot{g}_{ab}\, q_b = \dot{g}_{ab}\, g_{ab}^{-1}\, q_a$

$(\dot{g}_{ab}\, g_{ab}^{-1})$ maps the coordinates of a point to its velocity

$$\Rightarrow v_{q_a} = \begin{bmatrix} \dot{R}_{ab} R_{ab}^T & -\dot{R}_{ab} R_{ab}^T p_{ab} + \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_a \\ 1 \end{bmatrix}$$

$$= \dot{R}_{ab} R_{ab}^T q_a + (-\dot{R}_{ab} R_{ab}^T p_{ab} + \dot{p}_{ab})$$

$$= \omega^s_{ab} \times q_a + v^s_{ab}$$

Note: v^s_{ab} is the velocity of a (possibly imaginary) point attached to the rigid body and traveling through the origin of the spatial frame at time t.

$$\hat{V}^b_{ab} = g_{ab}^{-1}\, \dot{g}_{ab} = \begin{bmatrix} R_{ab}^T \dot{R}_{ab} & R_{ab}^T \dot{p}_{ab} \\ 0 & 0 \end{bmatrix}$$

$$V^b_{ab} = \begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix} = \begin{bmatrix} R_{ab}^T \dot{p}_{ab} \\ (R_{ab}^T \dot{R}_{ab})^\vee \end{bmatrix}$$

$$\left[v_{q_b} = g_{ab}^{-1} v_{q_a} = g_{ab}^{-1} \dot{g}_{ab}\, q_b = \hat{V}^b_{ab}(t)\, q_b \right]$$

One can also show that

$$v_{q_b} = R_{ab}^T \dot{R}_{ab}\, q_b + R_{ab}^T \dot{p}_{ab}$$

$$= \omega^b_{ab} \times q_b + \cancel{R_{ab}^T \dot{p}_{ab}}\; v^b_{ab}$$

What is v_{q_b}? Velocity of a point on the rigid body w.r.t. the spatial frame expressed in the body frame B.

Now, $g_{ab}^{-1} \hat{V}^s_{ab} g_{ab}$

$$= g_{ab}^{-1} \dot{g}_{ab} g_{ab}^{-1} g_{ab} = g_{ab}^{-1} \dot{g}_{ab} = \hat{V}^b_{ab}$$

$$\boxed{\hat{V}^b_{ab} = g_{ab}^{-1} \hat{V}^s_{ab} g_{ab}}$$

This is also known as a similarity transformation for matrices.

Note: The above is equivalent to writing $\boxed{V^s_{ab} = Ad_{g_{ab}} V^b_{ab}}$ as we will show below

~~def of Ad~~

~~The Adjoint Matrix:~~

~~From~~ $\cancel{\hat{V}^b_{ab} = g_{ab}^{-1} \hat{V}^s_{ab} g_{ab}}$

~~we can write~~ $\cancel{g_{ab} \hat{V}^b_{ab} g_{ab}^{-1} = \hat{V}^s_{ab}}$

$\cancel{\hat{\omega}^s_{ab} = R_{ab} \hat{\omega}^b_{ab} R^T_{ab} =}$

~~We know that~~ $\cancel{\hat{\omega}^b_{ab} = R_{ab}}$

$\cancel{R_{ab}}$ $\cancel{R^s_{ab}}$

We know that $\hat{\omega}^b_{ab} = R^T_{ab} \dot{R}_{ab}$

$$\hat{\omega}^s_{ab} = \dot{R}_{ab} R^T_{ab}$$

$$\therefore \hat{\omega}^s_{ab} = R_{ab} \hat{\omega}^b_{ab} R^T_{ab}$$ from which we can write:

$$\therefore \boxed{\omega^s_{ab} = R_{ab} \omega^b_{ab}}$$

$$v^s_{ab} = -\dot{R}_{ab} R^T_{ab} p_{ab} + \dot{p}_{ab} = -\hat{\omega}^s_{ab} p_{ab} + \dot{p}_{ab}$$

$$= -\omega^s_{ab} \times p_{ab} + \dot{p}_{ab} = p_{ab} \times \omega^s_{ab} + \dot{p}_{ab}$$

$$= (p_{ab} \times R_{ab} \omega^b_{ab}) + R_{ab} v^b_{ab}$$

$\left[\because R_{ab} v^b_{ab} = R_{ab} R^T_{ab} \dot{p}_{ab} = \dot{p}_{ab}\right]$

$$\therefore \boxed{\begin{aligned} \omega^s_{ab} &= R_{ab}\,\omega^b_{ab} \\ v^s_{ab} &= \hat{p}_{ab} R_{ab}\,\omega^b_{ab} + R_{ab}\, v^b_{ab} \end{aligned}}$$

$$\therefore \begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & \hat{p}_{ab} R_{ab} \\ 0 & R_{ab} \end{bmatrix}}_{6\times 6 \text{ matrix called the adjoint transformation.}} \begin{bmatrix} v^b_{ab} \\ \omega^b_{ab} \end{bmatrix}$$

Defn: ~~We take the~~ The 6×6 matrix that converts twists in one reference frame to another is called the adjoint transformation.

Thus for any $g \in SE(3)$ we can define another which maps one coordinate system to another given by

$$Ad_g : \mathbb{R}^6 \to \mathbb{R}^6 \quad \text{where } g = \begin{bmatrix} R & p \\ & 1 \end{bmatrix}$$

$$Ad_g = \begin{bmatrix} R & \hat{p} R \\ 0 & R \end{bmatrix}$$

Note that we have shown earlier that velocities can be written as twists V^s_{ab}, V^b_{ab} are referred to as twist coordinates of spatial twist (which is also referred to as twist with some abuse in terminology).

Also

$$Ad_g^{-1} = \begin{bmatrix} R^T & (-R^T p)^\wedge R^T \\ 0 & R^T \end{bmatrix} = \begin{bmatrix} R^T & -R^T \hat{p} \\ 0 & R^T \end{bmatrix} = Ad_{g^{-1}}$$

where $(-R^T p)^\wedge$ denotes the skew-symmetric matrix formed from the components of $-R^T p$.

Velocity of a screw motion:

Suppose $g_{ab}(\theta) = e^{\hat{\xi}\theta} g_{ab}(0)$

For a constant twist $\hat{\xi}$

$$\frac{d}{dt}\left(e^{\hat{\xi}\theta}\right) = \hat{\xi}\dot{\theta}\, e^{\hat{\xi}\theta} \quad [\text{Prove this!}]$$

The spatial velocity

$$\hat{V}^s_{ab}(\theta) = \dot{g}_{ab}(\theta)\, g^{-1}_{ab}(\theta)$$

$$= \left(\hat{\xi}\dot{\theta}\, e^{\hat{\xi}\theta} g_{ab}(0)\right)\left(g^{-1}_{ab}(0)\, e^{-\hat{\xi}\theta}\right)$$

$$\left[(AB)^{-1} = B^{-1}A^{-1}\right]$$ for any two invertible matrices A & B.

$$\boxed{\hat{V}^s_{ab}(\theta) = \hat{\xi}\dot{\theta}} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}\dot{\theta} \quad \left[\xi = \begin{bmatrix} v \\ \omega \end{bmatrix}\right]$$

Thus there is a simple expression for the spatial velocity corresponding to rigid body motion.

$$\hat{V}^b_{ab} = g^{-1}_{ab}(\theta)\, \dot{g}_{ab}(\theta)$$

$$= \left(g^{-1}_{ab}(0)\, e^{-\hat{\xi}\theta}\right)\left(\hat{\xi}\dot{\theta}\, e^{\hat{\xi}\theta} g_{ab}(0)\right)$$

$$= \underbrace{\left(g^{-1}_{ab}(0)\,\hat{\xi}\, g_{ab}(0)\right)}_{\text{Proof to be done in assignment.}}\dot{\theta} = \left(Ad_{g^{-1}_{ab}(0)}\,\xi\right)^{\wedge}\dot{\theta}$$

If $g_{ab}(0) = I$, $\hat{V}^b_{ab} = \hat{\xi}\dot{\theta}$

Coordinate Transformations:

Just like rigid body transformations velocities can be transformed between coordinate frames.

Let V^s_{ij} denote the velocity of frame j w.r.t. i expressed in the spatial frame (say S).

For any 3 frames A, B, C we have the following

$$\boxed{\begin{aligned} V^s_{ac} &= V^s_{ab} + Ad_{g_{ab}} V^s_{bc} \\ V^b_{ac} &= Ad_{g^{-1}_{bc}} V^b_{ab} + V^b_{bc} \end{aligned}}$$

Proof of the first claim:

$$\hat{V}^s_{ac} = \dot{g}_{ac}\, g^{-1}_{ac} \quad (1)$$

Now $g_{ac} = g_{ab}\, g_{bc}$

$$\therefore \dot{g}_{ac} = \dot{g}_{ab}\, g_{bc} + g_{ab}\, \dot{g}_{bc} \quad (2)$$

and $g^{-1}_{ac} = (g_{ab}\, g_{bc})^{-1} = g^{-1}_{bc}\, g^{-1}_{ab} \quad (3)$

Substituting (2) and (3) in Eqn. (1)

$$\hat{V}^s_{ac} = (\dot{g}_{ab}\, g_{bc} + g_{ab}\, \dot{g}_{bc})\, g^{-1}_{bc}\, g^{-1}_{ab}$$

$$= \dot{g}_{ab}\, g^{-1}_{ab} + g_{ab}(\dot{g}_{bc}\, g^{-1}_{bc})\, g^{-1}_{ab}$$

$$= \hat{V}^s_{ab} + g_{ab}\, \hat{V}^s_{bc}\, g^{-1}_{ab}$$

$$\therefore V^s_{ac} = V^s_{ab} + Ad_{g_{ab}} V^s_{bc}$$

Proof the second part as an assignment problem

~~Let $\hat{\xi}$ be a twist which represents the motion obtained by applying a rigid body motion~~

~~Let ... and $g \in SE(3)$~~

Useful Stuff for ~~applying~~ applying coordinate transformations to constant twists

Suppose ξ is a twist that represents the motion of a screw (e.g. ξ corresponding to a translation or rotational motion). ξ is a constant twist (i.e., it is not changed by a screw motion about ξ). Now we apply a fixed rigid body transformation $g \in SE(3)$ to ξ.

Then

$$\hat{\xi}' = g\hat{\xi}g^{-1} \quad \text{or} \quad \xi' = Ad_g\,\xi$$

We have noted previously that for screw motion

let $\xi = \begin{bmatrix} v \\ \omega \end{bmatrix}$, then

$$\hat{V}^s_{ab} = \hat{\xi}\dot{\theta}$$

(where θ is the angle of rotation about the screw axis and $\dot{\theta}$ is the corresponding rate)

~~Thus for revolute joints~~

$\cancel{\xi = \begin{bmatrix} 0 \\ \omega \end{bmatrix}}$ $\cancel{\hat{V}^s_{ab} = \begin{bmatrix} \hat{\omega} & 0 \\ 0 & 0 \end{bmatrix}\dot{\theta}}$ $\cancel{\begin{bmatrix} \hat{\omega}\dot{\theta} & 0 \\ 0 & 0 \end{bmatrix}}$

$\cancel{\hat{V}^s_{ab} \quad \hat{\omega}^s_{ab} = \hat{\omega}\dot{\theta}}$

For Revolute Joints:

$$\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}, \quad \hat{V}^s_{ab} = \begin{bmatrix} \hat{\omega} & -\hat{\omega}q \\ 0 & 0 \end{bmatrix}\dot{\theta}$$

$$\begin{bmatrix} v^s_{ab} \\ \omega^s_{ab} \end{bmatrix} = \begin{bmatrix} -\hat{\omega}q\dot{\theta} \\ \omega\dot{\theta} \end{bmatrix} = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix}\dot{\theta} = \xi\dot{\theta}$$

For Translation Joints:

$$\xi = \begin{bmatrix} v \\ 0 \end{bmatrix}, \quad \hat{V}^s = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}\dot{\theta}$$

$$\therefore \begin{bmatrix} v^s \\ \omega^s \end{bmatrix} = \begin{bmatrix} v\dot{\theta} \\ 0 \end{bmatrix} = \xi\dot{\theta}$$

~~Example 2.6 of MLS~~

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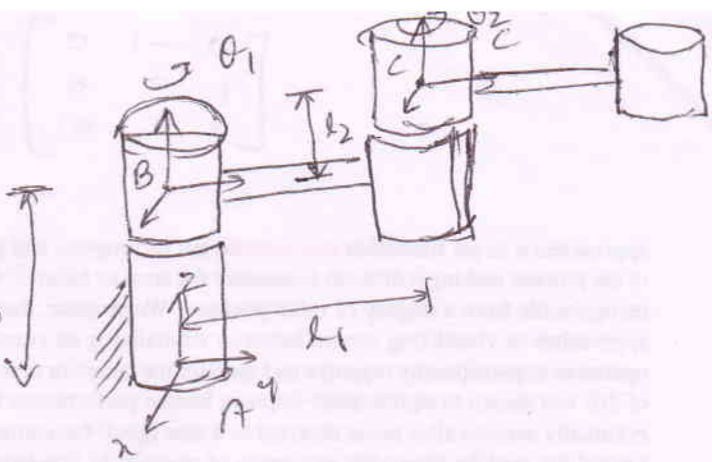
Example 2.6 of MLS:



$$\omega_{ab} = \begin{bmatrix} 0\\0\\1 \end{bmatrix} \qquad \omega^a_{bc} = \begin{bmatrix} 0\\0\\1 \end{bmatrix}$$

$$q_{ab} = \begin{bmatrix} 0\\0\\l_0 \end{bmatrix} \qquad q^a_{bc} = \begin{bmatrix} 0\\l_1\\l_2 \end{bmatrix}$$

$$\xi_{ab} = \xi_1 = \begin{bmatrix} -\omega_{ab} \times q_{ab} \\ q_{ab} \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\0\\0\\1 \end{bmatrix}$$

$$\xi_2 = \begin{bmatrix} -\omega_{bc} \times q_{bc} \\ q_{bc} \end{bmatrix} = \begin{bmatrix} l_1\\0\\0\\0\\0\\1 \end{bmatrix}$$

$$Adg_{ab} = \begin{bmatrix} R_{ab} & \begin{bmatrix} 0 & -l_0 & 0 \\ l_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_{ab} \\ 0 & R_{ab} \end{bmatrix}$$

$$R_{ab} = e^{\hat{\xi}_1 \theta_1} = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$V^s_{ab} = \xi_1 \dot{\theta}_1 = \begin{bmatrix} 0\\0\\0\\0\\0\\1 \end{bmatrix} \dot{\theta}_1 \qquad V^s_{bc} = \begin{bmatrix} l_1\\0\\0\\0\\0\\1 \end{bmatrix} \dot{\theta}_2$$

$$\therefore V^s_{ac} = \begin{bmatrix} 0\\0\\0\\0\\0\\1 \end{bmatrix} \dot{\theta}_1 + \begin{bmatrix} l_1 \cos\theta_1 \\ l_1 \sin\theta_1 \\ 0\\0\\0\\1 \end{bmatrix} \dot{\theta}_2$$

Direct Velocity Kinematics

Problem Statement: Given the joint rates of the ~~end~~ manipulator find the ~~linear~~ linear and angular ~~effector~~ velocity of the manipulator end effector.

Let $g_{st} : Q \rightarrow SE(3)$ be the forward kinematics map.

$Q \leftarrow$ Configuration space of the manipulator

$$\hat{V}^{s}_{st} = \dot{g}_{st}(\theta)\, g^{-1}_{st}(\theta)$$

$$= \sum_{i=1}^{n} \left(\frac{\partial g_{st}(\theta)}{\partial \theta_i} \dot{\theta}_i \right) g^{-1}_{st}(\theta)$$

[By chain rule], where n is the number of joints of the manipulator.

$$= \sum_{i=1}^{n} \underbrace{\left(\frac{\partial g_{st}(\theta)}{\partial \theta_i}\, g^{-1}_{st}(\theta) \right)}_{4\times 4 \text{ matrix}} \dot{\theta}_i$$

$\therefore$ End effector velocity (or twist) is a linear function of the joint rates.

In twist coordinates we can write the above equation as

$$V^{s}_{st} = \underbrace{\left[\left(\frac{\partial g_{st}}{\partial \theta_1} g^{-1}_{st} \right)^{\vee} \cdots \left(\frac{\partial g_{st}}{\partial \theta_n} g^{-1}_{st} \right)^{\vee} \right]}_{6\times n \text{ matrix}} \dot{\theta}$$

This is a column vector of size 6×1

$\dot{\theta}$: $n\times 1$ column vector.

$$\theta = [\theta_1, \theta_2, \ldots, \theta_n]^T$$

$$\therefore V^{s}_{st} = J^{s}_{st}(\theta)\, \dot{\theta}$$

$J^{s}_{st}(\theta)$ is called the spatial manipulator Jacobian.

Note that we had looked at the Jacobian earlier (4)
in the case of the ~~R~~ 2R robot.

We can obtain the ~~spatial~~ Jacobian directly from differentiating the direct kinematics equations relating the representation of the manipulator tool end effector frame in terms of Euler angles (or ~~other~~ quaternions or anything else).

Some people call the Jacobian obtain by direct differentiation of direct kinematics equation as ~~Jacobian~~ ~~However~~ manipulator Jacobian. Some people ~~distinguish~~ call the Jacobian obtained ~~from~~ ~~representation~~ without ~~the~~ differentiation ~~of exp~~ ~~as exponential~~ (as we ~~have~~ will show below ~~written~~) as manipulator Jacobian.

The latter definition is better since it shows the structural geometric properties of the manipulator and is helpful in singularity analysis through geometric reasoning.

Now $g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0)$

where $\hat{\xi}_i$ is a unit twist corresponding to the ith joint

$$\therefore \frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1} = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}} \left(\frac{\partial}{\partial \theta_i} e^{\hat{\xi}_i\theta_i} \right) e^{\hat{\xi}_{i+1}\theta_{i+1}} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0) g_{st}^{-1}$$

$$= \left(e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}} \right) \hat{\xi}_i \left(e^{\hat{\xi}_i\theta_i} \cdots e^{\hat{\xi}_n\theta_n} g_{st}(0) g_{st}^{-1} \right)$$

$$\left(e^{\hat{\xi}_i\theta_i}\right)^{-1} \left(e^{-\hat{\xi}_{i-1}\theta_{i-1}} \cdots e^{-\hat{\xi}_i\theta_i} \right)$$

$$\text{Now } g_{st}^{-1} = g_{st}(0)^{-1} \left(e^{\hat{\xi}_n\theta_n}\right)^{-1} \cdots$$

$$\therefore \frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1} = \left(e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}} \right) \hat{\xi}_i \left(e^{-\hat{\xi}_i\theta_{i-1}} \cdots e^{-\hat{\xi}_1\theta_1} \right)$$

Let $g_{1,i-1} = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_{i-1}\theta_{i-1}}$

$$\therefore \frac{\partial g_{st}}{\partial \theta_i} g_{st}^{-1} = g_{1,i-1} \hat{\xi}_i \, g_{i,i-1}^{-1}$$

$\therefore$ ~~Adg~~ In terms of twist coordinates

$$\left(\left(\frac{\partial g_{st}}{\partial \theta_i} \right) g_{st}^{-1} \right)^{\vee} = Ad_{g_{1,i-1}} \xi_i$$

$\therefore$ The spatial manipulator Jacobian is

$$\boxed{J_{st}^{s}(\theta) = \left[\xi_1 \; \xi_2' \; \cdots \; \xi_n' \right] \quad \text{where } \xi_i' = Ad_{g_{1,i-1}} \xi_i}$$

This is the adjoint of the transformation upto joint $i-1$ from the first joint.

$\therefore$ We can write down the manipulator Jacobian without performing any differentiation.

The ith column of the spatial Jacobian is the ith joint twist, transformed to the current manipulation configuration.

Body Manipulator Jacobian:

Analogous to the spatial relationship we can write.

$$V_{st}^{b} = J_{st}^{b}(\theta)\,\dot{\theta}$$

where $J_{st}^{b}(\theta)$ is called the body manipulator Jacobian.

$$J_{st}^{b}(\theta) = \left[\xi_1^{+} \;\ldots\; \xi_{n-1}^{+} \;\; \xi_n^{+}\right]$$

where $\xi_i^{+} = Ad^{-1}_{g_{i,n}\, g_{st}(0)}\, \xi_i$

where $g_{i,n}\, g_{st}(0) = e^{\hat{\xi}_i \theta_i} \ldots e^{\hat{\xi}_n \theta_n}\, g_{st}(0)$

Columns of J_{st}^{b} correspond to the joint twists written w.r.t. the tool frame at the current configuration.

Note: ① $J_{st}^{s}(\theta) = Ad_{g_{st}(\theta)}\, J_{st}^{b}(\theta)$

② For any point, q, attached to the end effector

$$v_q^s = \hat{V}_{st}^{s}\, q^s = \left(J_{st}^{s}(\theta)\,\dot{\theta}\right)^{\wedge} q^s$$

Spatial Velocity

$$v_q^b = \hat{V}_{st}^{b}\, q^b = \left(J_{st}^{b}(\theta)\,\dot{\theta}\right)^{\wedge} q^b$$

Body Velocity

Proof of ①:

$$V_{st}^{s} = Ad_{g_{st}(\theta)}\, V_{st}^{b}$$

$$\Rightarrow J_{st}^{s}(\theta)\,\dot{\theta} = Ad_{g_{st}(\theta)}\, J_{st}^{b}(\theta)\,\dot{\theta}$$

The above ~~has~~ to be true for all $\dot{\theta}$

$$\therefore \boxed{J_{st}^{s}(\theta) = Ad_{g_{st}(\theta)}\, J_{st}^{b}(\theta)}$$

Example: Jacobian of Stanford arm: (Example 3.9 of MLS book) ①

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \omega_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix} \quad q_2 = \begin{bmatrix} 0 \\ 0 \\ l_0 \end{bmatrix}$$

$$\xi_1 = \begin{bmatrix} -\omega_1 \times q_1 \\ \omega_1 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} -\omega_2 \times q_2 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 0 \\ l_0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \xi_2' = e^{\hat{\xi}_1 \theta_1} \xi_2 = \begin{bmatrix} l_0 \sin\theta_1 \\ -l_0 \cos\theta_1 \\ 0 \\ -\cos\theta_1 \\ -\sin\theta_1 \\ 0 \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \omega_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \xi_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xi_3' = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} \xi_3 = \begin{bmatrix} e^{\hat{z}\theta_1} e^{\hat{x}\theta_2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\theta_1 \cos\theta_2 \\ \cos\theta_1 \cos\theta_2 \\ -\sin\theta_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\omega_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \omega_5 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \omega_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$q_4 = q_5 = q_6 = q_w = \begin{bmatrix} 0 \\ l_1 \\ l_0 \end{bmatrix}$$

$$\therefore \xi_4 = \begin{bmatrix} -\omega_4 \times q_4 \\ \omega_4 \end{bmatrix} \quad \xi_5 = \begin{bmatrix} -\omega_5 \times q_5 \\ \omega_5 \end{bmatrix} \quad \xi_6 = \begin{bmatrix} -\omega_6 \times q_6 \\ \omega_6 \end{bmatrix}$$

$$\therefore \xi_4' = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} \xi_4$$

$$\xi_5' = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} \xi_5$$

$$\xi_6' = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} \cancel{\xi_6} e^{\hat{\xi}_5\theta_5} \xi_6$$

Thus the computations can be done in an iterative fashion which is important for efficient computation of the Jacobian in code.

There are two ways to implement:

(a) Perform a symbolic computation of the Jacobian as a pre-processing step and evaluate the Jacobian numerically when required (usually required in different robot control schemes).

(b) Numerically compute the Jacobian ~~over~~ with an iterative algorithm every time it is required.

[We will write a code for this in the next assignment.]

Note, in the above example

$$J_{st}^{s} = \begin{bmatrix} 0 & -\omega_2' \times q_1 & v_3' & -\omega_4' \times q_\omega' & -\omega_5' \times q_\omega' & -\omega_6' \times q_\omega' \\ \omega_1 & \omega_2' & 0 & \omega_4' & \omega_5' & \omega_6' \end{bmatrix}$$

and $\begin{bmatrix} q' \\ 1 \end{bmatrix} = g \begin{bmatrix} q \\ 1 \end{bmatrix}$

$$\omega_i' = R\,\omega$$