Robotics Lecture Notes **Spring 2018**

Lecture 4: Forward Kinematics of Serial Manipulators

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Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

4.1 Introduction

In this lecture, we will study the forward (or direct) kinematics of serial chain manipulators using the products of exponential formula. Before, we move to the forward kinematics problem, we first remind ourselves about rigid body transformations.

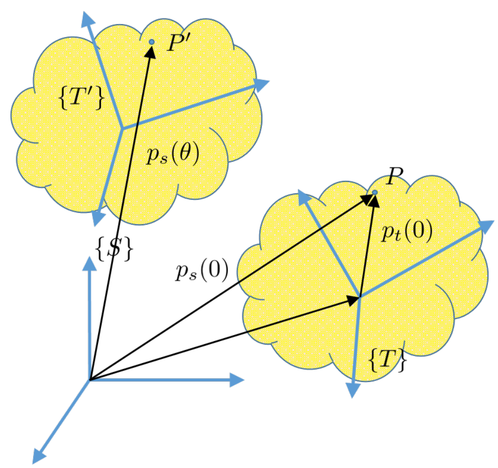

Figure 4.1: Figure showing rigid body transformation of an object. The points P is transformed to point P'.

Figure 4.1, where $\{S\}$ is the world frame, shows the initial configuration of the body represented by the body frame $\{T\}$ and the configuration after a rigid body transformation shown by the frame $\{T'\}$. Let P be a point on the body. The homogeneous coordinates of P in the frame $\{T\}$ is denoted by $\mathbf{p}_t(0)$, where the argument 0 is for initial position. Let $g_{st}(0)$ be the transformation matrix of frame $\{T\}$ with respect to frame $\{S\}$. The homogeneous coordinates of P in the world frame $\{S\}$ at the initial configuration is

$$\mathbf{p}_s(0) = g_{st}(0)\mathbf{p}_t(0)$$

Let $\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ \mathbf{0} & 0 \end{bmatrix}$ represent the twist corresponding to the rigid body transformation and θ be the angle of the rotation about the rotation axis ω. Let $\mathbf{p}_s(\theta)$ be the position of point P in the frame $\{S\}$ after the rigid body transformation. Then

$$\mathbf{p}_s(\theta) = e^{\hat{\xi}\theta}\mathbf{p}_s(0) = e^{\hat{\xi}\theta}g_{st}(0)\mathbf{p}_t(0)$$

Therefore the transformation between the world frame $\{S\}$ and the frame $\{T'\}$, which is written as $g_{st}(\theta)$, is

$$\boxed{g_{st}(\theta) = e^{\hat{\xi}\theta}g_{st}(0)} \tag{4.1}$$

4.2 Rotation of a Rigid Link about an Axis

In this section we will look at the rigid body transformation generated by the rotation of a rigid link about an axis. Let $\{S\}$ be the world frame. The rigid link rotates about the axis ω (a unit vector). Let $\mathbf{p}_s(0)$ be

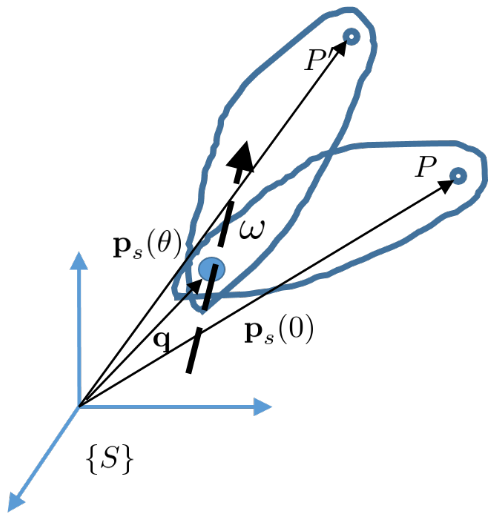

Figure 4.2: Figure showing a rigid body transformation generated by the rotation of the link about the axis ω. The points P is transformed to point P'.

the initial position of the point P on the link expressed in the frame $\{S\}$. Let $\mathbf{p}_s(\theta)$ be the position of the point P on the link expressed in the frame $\{S\}$ after rotation by an angle θ about the axis ω. Let $\mathbf{q}$ be the position vector of a point on the axis of rotation. Then

$$\mathbf{p}_s(\theta) = \mathbf{q} + e^{\hat{\omega}\theta}(\mathbf{p}_s(0) - \mathbf{q}) \tag{4.2}$$

Equation (4.2) can be rewritten in matrix form using homogeneous coordinates as

$$\begin{bmatrix} \mathbf{p}_s(\theta) \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{\omega}\theta} & (\mathbf{I} - e^{\hat{\omega}\theta})\mathbf{q} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{p}_s(0) \\ 1 \end{bmatrix} \tag{4.3}$$

Therefore the rigid body transformation matrix due to the rotation of the link is

$$\boxed{g = \begin{bmatrix} e^{\hat{\omega}\theta} & (\mathbf{I} - e^{\hat{\omega}\theta})\mathbf{q} \\ \mathbf{0} & 1 \end{bmatrix}}$$

Now, we know that, in general, the rigid body transformation can be expressed as $e^{\hat{\xi}\theta}$, where ξ is twist coordinates corresponding to the rigid body transformation. We will now show that for the transformation produced by rotation of a rigid link about an axis, the twist coordinates and twist are

$$\boxed{\xi = \begin{bmatrix} -\omega \times \mathbf{q} \\ \omega \end{bmatrix}, \qquad \hat{\xi} = \begin{bmatrix} \hat{\omega} & -\omega \times \mathbf{q} \\ \mathbf{0} & 0 \end{bmatrix}}$$

For a *translational joint*, the twist coordinates, twist, and rigid body transformation are given by

$$\boxed{\xi = \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix}, \qquad \hat{\xi} = \begin{bmatrix} \mathbf{0} & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix}, \qquad g = \begin{bmatrix} \mathbf{I} & \mathbf{v} \\ \mathbf{0} & 1 \end{bmatrix}}$$

4.3 Product of Exponentials Formula for Forward Kinematics

Let us now consider a serial chain manipulator where rigid links are joined by revolute (R) or prismatic joints (P). Irrespective of whether the joint is revolute or prismatic, we will use θ to denote the relative displacement between the two links at the joint. For any n-DoF manipulator, the unit twist corresponding to the ith revolute joint is

$$\xi_i = \begin{bmatrix} -\omega_i \times \mathbf{q}_i \\ \omega_i \end{bmatrix} \tag{4.4}$$

where ω_i is the unit vector representing the axis of the ith joint and $\mathbf{q}_i$ is any point on the axis of the ith joint. The unit twist corresponding to the jth prismatic joint is

$$\xi_j = \begin{bmatrix} \mathbf{v}_j \\ \mathbf{0} \end{bmatrix} \tag{4.5}$$

where $\mathbf{v}_j$ is the unit vector corresponding to the direction of translation of the prismatic joint.

Definition 4.1 (Joint Space) *The set of all possible joint displacements (angular or linear) of a manipulator is called the joint space of the manipulator. For a serial chain manipulator since the specification of the joint variables form a minimum cardinality description of the set of rigid links forming the manipulator, the joint space is also called the configuration space of the manipulator.*

Let $\mathcal{Q}$ denote the joint space of a manipulator. Let $\{S\}$ be the world frame and $\{T\}$ be the tool frame or end-effector frame of the manipulator. The forward kinematics problems is then stated as follows:

Definition 4.2 (Forward Kinematics Problem) *Given a configuration of the robot $\mathbf{q} \in \mathcal{Q}$, find the configuration of the tool frame with respect to the base frame, $g_{st}(\mathbf{q}) \in SE(3)$, where*

$$g_{st} : \mathcal{Q} \rightarrow SE(3)$$

is the mapping from joint space to end effector space (or task space). We are usually interested in obtaining the whole mapping g_{st}, instead of its value at a particular joint configuration.

4.3.1 Procedure to Compute Forward Kinematics Solution

We will now present the procedure to compute the forward kinematics solution for any n-DoF serial manipulator. We assume that the link lengths of the manipulator are known. Let L_i, $i = 1, 2, \ldots, n$ be the link lengths of the manipulator. The joint axis, ω_i for R joints and $\mathbf{v}_i$ for P joints, will be provided by the manipulator manufacturer with respect to some frame fixed to the base of the manipulator. The following steps are to be used.

1. Choose a *base or reference configuration* of the manipulator. Choose a base (world) frame $\{S\}$ and a tool frame $\{T\}$ for the manipulator. Usually choosing the frame $\{S\}$ as the one provided by the robot manufacturer is convenient (however, you are free to choose a different one). Of course, the frame $\{S\}$ should be chosen such that the motion of the manipulator does not move the frame. The links and joints are numbered starting at 1 for the base joint and increasing towards the tool frame. The tool frame is selected on the end effector of the manipulator. Without loss of generality, we will assume that at the reference configuration of all the joint variables are 0.

2. In the reference configuration, using the geometry of the manipulator (i.e., link lengths) compute $g_{st}(0)$.

3. Compute the twists corresponding to each joint ξ_i. For revolute joints, this involves choosing a point $\mathbf{q}_i$ on the frame of each robot and then using Equation (4.4). For a prismatic joint, we simply use the unit vector corresponding to the axis and use Equation (4.5).

4. For any value of the joint angles or linear displacements, $\mathbf{q} = (\theta_1, \theta_2, \ldots, \theta_n)$, the configuration of the tool frame with respect to the base frame, $g_{st}(\mathbf{q})$ is given by

$$\boxed{g_{st}(\mathbf{q}) = e^{\hat{\xi_1}\theta_1} e^{\hat{\xi_1}\theta_2} \ldots e^{\hat{\xi_n}\theta_n} g_{st}(0) = \Pi_{i=1}^{n} e^{\hat{\xi_i}\theta_i} g_{st}(0)} \tag{4.6}$$