Robotics Lecture Notes **Spring 2018**

Lecture 2: Rigid Body Rotation

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2.1 Introduction

We will now study in more detail about orientation of rigid bodies, which can be represented by the rotation matrix $\mathbf{R}$. The nine elements of the rotation matrix can be used to represent the intuitive notion of orientation of a rigid body. However, as we know, there are only three degrees of freedom in orientation, whereas, the 3×3 rotation matrix has 9 parameters. There are 6 constraints (three corresponding to the fact that the columns of rotation matrices are unit vectors and three corresponding to the fact that the columns form an orthogonal reference frame). Therefore, it should be possible to represent rotations with three independent parameters. However, as we will see later, all three parameter representation of rotations suffer from *representational singularity* and in the context of robotics, a good way to represent the rotation is with 4 parameters. There are a number of classes of three parameter representations for rotations. We will study three classes, namely the axis-angle representation (also known as the canonical representation), the fixed angle representation, and the Euler angle representation. We will also study the most practically used representation of rotations, which is the quaternion representation (also called Euler parameters). To understand the axis-angle representation, we will first need to understand some algebraic properties of the set of all rotation matrices and the set of all skew-symmetric matrices, which we do in the next section.

Notation: Throughout these lecture notes, we will use the following notations:

- $\mathbb{R}^n$: Vector space of dimension n with real entries. Each element of $\mathbb{R}^n$ will be written as a $n \times 1$ column vector (by convention). When $n = 1$, we have the real line, when $n = 2$, we have the plane and for $n = 3$, we have three dimensional space.
- $\mathbb{R}^{m \times n}$: Set of $m \times n$ matrices with real entries. We will write matrices in the standard way with a two-dimensional array of $m \times n$ numbers.
- For a square matrix $\mathbf{M}$, $det(\mathbf{M})$ will represent the determinant of $\mathbf{M}$ and $trace(\mathbf{M})$ will represent the trace of $\mathbf{M}$, i.e., sum of the diagonal elements of $\mathbf{M}$. $\mathbf{M}^T$ is the transpose of $\mathbf{M}$.

2.2 Rotation Matrix and its Properties

Let $\{B\}$ and $\{W\}$ be two reference frames. We know that the rotation matrix of frame $\{B\}$ with respect to frame $\{W\}$ is given by

$$ {}^W_B\mathbf{R} = \begin{bmatrix} {}^W\mathbf{X}_B & {}^W\mathbf{Y}_B & {}^W\mathbf{Z}_B \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} {}^B\mathbf{X}_W^T \\ {}^B\mathbf{Y}_W^T \\ {}^B\mathbf{Z}_W^T \end{bmatrix} \tag{2.1} $$

where ${}^W\mathbf{X}_B$, ${}^W\mathbf{Y}_B$, and ${}^W\mathbf{Z}_B$ are the unit vectors of frame $\{B\}$ expressed in frame $\{W\}$ and ${}^B\mathbf{X}_W$, ${}^B\mathbf{Y}_W$, and ${}^B\mathbf{Z}_W$ are the unit vectors of frame $\{W\}$ expressed in frame $\{B\}$. Note that the equation above is the definition of a rotation matrix.

2.2.1 Properties of Rotation Matrix

Let $\mathbf{R} \in \mathbb{R}^{3\times3}$ be a rotation matrix and $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ be its columns. Any, rotation matrix has the following properties.

1. Each column and row in a rotation matrix is a unit vector.
2. The columns (rows) in a rotation matrix are orthonormal.
$$\mathbf{r}_i^T\mathbf{r}_j = \begin{cases} 0 & i \neq j \\ \\ 1 & i = j \end{cases}$$
3. The inverse of a rotation matrix is its transpose.
$$\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}_{3\times3} \Longrightarrow \mathbf{R}^{-1} = \mathbf{R}^T$$
4. The determinant of a rotation matrix is $\det(\mathbf{R}) = +1$.
$$\det(\mathbf{R}) = \mathbf{r}_1^T(\mathbf{r}_2 \times \mathbf{r}_3) = \mathbf{r}_1^T\mathbf{r}_1 = +1$$
5. The eigenvalues of $\mathbf{R}$ are 1, $e^{i\theta}$, and $e^{-i\theta}$, where $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and $i = \sqrt{-1}$. Note that the magnitude of each eigenvalue is 1. This implies that if a vector is multiplied by the rotation matrix then its magnitude remains unchanged.
6. The trace of the rotation matrix is $1 + 2\cos(\theta)$.
7. Rotation preserves dot product. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two vectors and $\{A\}$ and $\{B\}$ be two reference frames. Then,
$${}^A\mathbf{V}_1^T\,{}^A\mathbf{V}_2 = {}^B\mathbf{V}_1^T({}^A_B\mathbf{R}^T\,{}^A_B\mathbf{R})\,{}^B\mathbf{V}_2 = {}^B\mathbf{V}_1^T\,{}^B\mathbf{V}_2$$

2.3 Rotation Group, $SO(3)$

Rotation matrices are 3×3 matrices that satisfy $det(R) = 1$. This set of matrices is denoted by $SO(3)$, Special Orthogonal group of dimension 3.
$$SO(3) = \{R \in \mathbb{R}^{3\times3}\colon \underbrace{RR^T = I}_{Orthogonal},\ \underbrace{det(R) = +1}_{Special}\}$$
In general $SO(n)$ can be defined in a similar way as follows,
$$SO(n) = \{R \in \mathbb{R}^{n\times n}\colon RR^T = I,\ det(R) = +1\}$$
We are interested in $n = 2$ and $n = 3$.

Furthermore, the rotation group $SO(3)$ is a Lie group (which we will define later in class). The axis-angle representation of rotations comes from the fact that the rotation matrix group is a Lie group (although we will not go to the general detailed mathematics of proving this fact).

A key property of rotation matrices is that the product of two rotation matrices is a rotation matrix. Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be two rotation matrices. Let $\mathbf{R} = \mathbf{R}_1\mathbf{R}_2$. Then,
$$\mathbf{R}\mathbf{R}^T = (\mathbf{R}_1\mathbf{R}_2)(\mathbf{R}_1\mathbf{R}_2)^T = \mathbf{R}_1\mathbf{R}_2\mathbf{R}_2^T\mathbf{R}_1)^T = \mathbf{R}_1)\mathbf{I}\mathbf{R}_1^T = \mathbf{I}.$$

Furthermore,

$$det(\mathbf{R}) = det(\mathbf{R}_1\mathbf{R}_2) = det(\mathbf{R}_1)det(\mathbf{R}_2) = 1.$$

Therefore $\mathbf{R}$ satisfies the properties of an element of $SO(3)$. Hence $\mathbf{R} = \mathbf{R}_1\mathbf{R}_2$ is a rotation matrix. From here, it is straightforward to deduce that the product of any number of rotation matrices gives a rotation matrix.

2.3.1 Space of all 3×3 skew-symmetric matrices

The set of all 3×3 skew-symmetric matrices is denoted by $so(3)$ (pronounced as small ess-oh-three) and is defined by.

$$so(3) = \{S \in \mathbb{R}^{3\times 3} : S^T = -S\}$$

Please see the Appendix for the interpretation of each element of the set $so(3)$ as a linear operator (or matrix) that is useful for doing cross products. The set $so(3) \subset \mathbb{R}^{3\times 3}$, where $\mathbb{R}^{3\times 3}$ is the set of all real 3×3 matrices is a *vector space* over the reals. In fact, $so(3)$ is the tangent space of the Lie group $SO(3)$ at the identity. The set $so(3)$ also forms a Lie algebra [1].

The standard basis vectors for the vector space $so(3)$ is

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \ \mathbf{e}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \ \mathbf{e}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.3.2 Exponential Coordinates for Rotation (Axis-angle representation)

Let $\omega \in \mathbb{R}^3$ be a unit vector. Let a rigid body **rotate about ω by an angle θ**. Then the rotation matrix is given by :

$$R(\omega, \theta) = e^{\hat{\omega}\theta} \equiv exp(\hat{\omega}, \theta) \tag{2.2}$$

Thus, there is an exponential map that maps an element of $so(3)$ to an element of $SO(3)$. More formally,

$$exp : so(3) \rightarrow SO(3).$$

To show the above we need to show the following: (a) $e^{\hat{\omega}\theta}$ is an element of $SO(3)$ (see Lemma 2.1) and (b) For every element, R, of $SO(3)$ there is an element of $so(3)$ and a θ such that $R = e^{\hat{\omega}\theta}$ (see Lemma 2.2). Our proofs will be constructive. We will first derive a closed form expression for the matrix exponential $e^{(\hat{\omega}\theta)}$, where ω is a unit vector.

$$R(\omega, \theta) = e^{\hat{\omega}\theta} = I + \theta\hat{\omega} + \frac{\theta^2}{2!}\hat{\omega}^2 + \frac{\theta^3}{3!}\hat{\omega}^3 + ... \tag{2.3}$$

For any skew-symmetric matrix, $\hat{a} \in so(3)$ the following relations hold (Proof by direct calculation) :

$$\left\{ \begin{array}{r} (1) \quad \hat{a}^2 = aa^T - \|a\|^2 I \\ \text{If} \|a\| = 1, \text{then } \hat{a}^2 = aa^T - I \\ (2) \quad \hat{a}^3 = -\|a\|^2\hat{a} \end{array} \right\}$$

[1] We will see the formal definitions of the terms used here later in class. The formal relationship between $SO(3)$ and $so(3)$ stated here is the reason behind the existence of an exponential map between so(3) and SO(3) that allows us to form the axis-angle representation. However, we will not use the Lie group theoretic formalism to form the axis-angle representation. We will use direct computation to show the existence of the axis-angle representation.

Using the above two results with $a = \omega\theta$, $\|\omega\| = 1$, (you will fill in the gaps in the next assignment)

$$\begin{aligned} R(\omega, \theta) = e^{\hat{\omega}\theta} = I + (\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...)\hat{\omega} + (\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + ...)\hat{\omega}^2 \\ \boxed{e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta)} \end{aligned} \tag{2.4}$$

where, we use the following series expansion of $\sin\theta$ and $\cos\theta$.

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...$$

The boxed equation above is called Rodrigues formula. *Rodrigues' formula gives a way of computing the rotation matrix for rotation by any angle θ about an axis represented by a unit vector ω.* If ω is not a unit vector, i.e., $\|\omega\| \neq 1$, then

$$e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\theta) + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|\theta))$$

Lemma 2.1 *Exponentials of skew-symmetric matrices are orthogonal. Given $\hat{\omega} \in so(3)$ and $\theta \in \mathbb{R}$, $e^{\hat{\omega}\theta} \in SO(3)$.*

Proof: Define $R \equiv e^{\hat{\omega}\theta}$, $\theta \in \mathbb{R}$

$$\begin{aligned} [e^{\hat{\omega}\theta}]^{-1} &= e^{-\hat{\omega}\theta} \quad \text{(From property (2) of matrix exponentials)} \\ e^{-\hat{\omega}\theta} &= e^{\hat{\omega}^T\theta} (\because \hat{\omega} = -\hat{\omega}^T \text{since } \hat{\omega} \text{ is skew symmetric)} \\ e^{\hat{\omega}^T\theta} &= [e^{\hat{\omega}\theta}]^T \quad \text{From property (5) of matrix exponentials} \end{aligned} \tag{2.5}$$

Therefore $RR^T = R^TR = I$, which implies that $det(R) = \pm 1$. Now, note that if $\theta = 0$, $det(e^{\hat{\omega}\theta}) = det(e^0) = 1$ Since det is a continuous function of matrix entries and the exponential map is a continuous function $\therefore$ any perturbation of θ from 0 should give a value of θ in the vicinity of 1. Now $det(R)$ can be either $+1$ or -1. Therefore $det(R) = 1$. Thus, $e^{\hat{\omega}\theta} \in SO(3)$. ■

We now show that every rotation matrix can be represented as matrix exponential of some skew-symmetric matrix. This is formally stated as the lemma below and its proof is constructive, i.e., given a rotation matrix, we give a way to compute ω and θ, which will give the rotation matrix through the exponential map.

Lemma 2.2 *Given $R \in SO(3)$, $\exists\omega \in \mathbb{R}^3$, $\|\omega\| = 1$ and $\theta \in \mathbb{R}$ such that*

$$R = e^{\hat{\omega}\theta}$$

Proof: We will show the above by directly computing a ω and a θ for any given rotation matrix.

Let $R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$

Let $C_\theta = \cos\theta$, $S_\theta = \sin\theta$, $v_\theta = 1 - \cos\theta$

Let $\omega = \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}$

By direct calculation using Rodrigues formula :

$$R = e^{\hat{\omega}\theta} = \begin{bmatrix} \omega_1^2 v_\theta + C_\theta & \omega_1\omega_2 v_\theta - \omega_3 S_\theta & \omega_1\omega_3 v_\theta + \omega_2 S_\theta \\ \omega_1\omega_2 v_\theta + \omega_3 S_\theta & \omega_2^2 v_\theta + C_\theta & \omega_2\omega_3 v_\theta - \omega_1 S_\theta \\ \omega_1\omega_3 v_\theta - \omega_2 S_\theta & \omega_2\omega_3 v_\theta + \omega_1 S_\theta & \omega_3^2 v_\theta + C_\theta \end{bmatrix} \tag{2.6}$$

From Equation (2.6), we know that

$$\text{Trace}(\mathbf{R}) = \omega_1^2 v_\theta + C_\theta + \omega_2^2 v_\theta + C_\theta + \omega_3^2 v_\theta + C_\theta = (\omega_1^2 + \omega_2^2 + \omega_3^2)(1 - C_\theta) + 3C_\theta = 1 + 2\cos\theta$$

where we used $v_\theta = 1 - \cos\theta$ and $\omega_1^2 + \omega_2^2 + \omega_3^2 = 1$. Therefore,

$$r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$$

$$\therefore \theta = \arccos\left(\frac{Trace(R) - 1}{2}\right) \tag{2.7}$$

Note : There are multiple values of θ, i.e. $\theta \pm 2\pi n$, $-\theta \pm 2\pi n$
Now we have to find ω. From the off-diagonal terms

$$r_{32} - r_{23} = 2\omega_1 \sin\theta$$

$$r_{13} - r_{31} = 2\omega_2 \sin\theta$$

$$r_{21} - r_{12} = 2\omega_3 \sin\theta$$

$$\therefore \omega = \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12}\end{bmatrix} \tag{2.8}$$

■

Once θ and ω are computed, the three parameter representation of rotation is given by $(\theta\omega_1,\ \theta\omega_2,\ \theta\omega_3)$, where ω_1, ω_2, and ω_3 are the components of the unit vector ω.

1. Thus, given an axis and an angle by which we rotate a body we can obtain the rotation matrix and vice-versa, i.e., given a rotation matrix we can find an equivalent axis and an angle by which the object should be rotated.

2. Although the problem of finding the axis and angle has multiple solutions in many applications in robotics that will not cause a problem.

3. The fact that the axis of rotation ω cannot be computed when $sin(\theta) = 0$ or $\theta = \pi$ is known as *representation singularity.*

4. Exponential coordinates are called canonical coordinates of the rotation group.

2.3.3 Some examples of rotation matrices using exponential coordinates

Let $x = \begin{bmatrix}1\\0\\0\end{bmatrix}$ and the rotation angle be θ, what is the rotation matrix?

In other words, what is the rotation matrix for rotation about the x-axis by an angle θ?

$$\hat{x} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix} \qquad \hat{x}^2 = \begin{bmatrix}0 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}$$

$$\therefore R_x(\theta) = e^{\hat{x}\theta} = I + \begin{bmatrix}0 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{bmatrix}\sin\theta + \begin{bmatrix}0 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}(1-\cos\theta) = \begin{bmatrix}1 & 0 & 0\\0 & C_\theta & -S_\theta\\0 & S_\theta & C_\theta\end{bmatrix}$$

Let $y = \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad z = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$R(y,\theta) = R_y(\theta) = \begin{bmatrix}C_\theta & 0 & S_\theta\\0 & 1 & 0\\-S_\theta & 0 & C_\theta\end{bmatrix}$$

and $R_z(\theta) = \begin{bmatrix}C_\theta & -S_\theta & 0\\S_\theta & C_\theta & 0\\0 & 0 & 1\end{bmatrix}$

Thus far, we have seen the representation of an element of $SO(3)$, i.e., a rigid body rotation as (a) the elements of a rotation matrix, $\mathbf{R}$ and (b) a three-parameter axis-angle representation or exponential coordinate representation. Other popular three parameter representations for rotation are fixed angle representations and Euler angle representations.

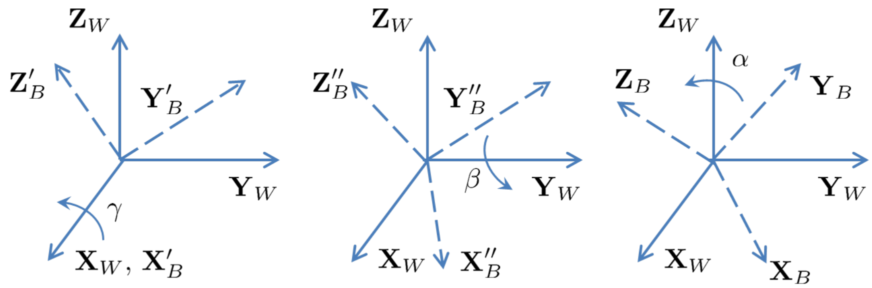
2.4 Fixed Angle and Euler Angle Representation

Any rotation, $\mathbf{R}$, can be represented by 3 consecutive *elementary rotations* about either 3 fixed axes or 3 moving axes (all of which cannot be the same). When the elemenatry rotations are about fixed axes, we have a **fixed angle** representation. When the elementary rotations are about moving axes (or axes transformed by the rotation), we have an **Euler angle** representation.

There are 12 fixed angle representations and 12 Euler angle representations. Howeve, the fixed angle and Euler angle representations are not independent. Corresponding to each fixed angle representation there is an Euler angle representation and viceversa. To see that there are 12 fixed angle representations, consider the first rotation to be about the X axis. Then we can have the 4 following sequences of rotation, $X-Y-Z$), $(X-Z-Y)$, $(X-Y-X)$, and $(X-Z-X)$. Now for each of the first rotation about the Y and Z axis there are 4 possible combinations of elementary rotations analogous to the one described before. Thus, in total, there are 12 possible combinations. A similar reasoning will show that there are 12 possible combinations for Euler angles.

We will now look at an example of fixed angle rotation and Euler angle rotation and see the relationship between the two.

2.4.1 Roll-Pitch-Yaw representation of rotation



(a) Roll (b) Pitch (c) Yaw

Figure 2.1: Roll-Pitch-Yaw fixed angle representation

The roll-pitch-yaw or the $X - Y - Z$ fixed angle representation is a standard representation for rotation in aerospace engineering. Let $\{B\}$ and $\{W\}$ be two reference frames with their origins coincident, with $\{W\}$ being the fixed frame. We assume that $\{B\}$ and $\{W\}$ were originally aligned and one gets to the current $\{B\}$ by the following elementary rotations applied in sequence (please see Figure 2.1):

1. Rotate $\{B\}$ about $\mathbf{X}_W$ by an angle γ (roll, see Figure 2.1(a)). $\mathbf{X}'_B, \mathbf{Y}'_B, \mathbf{Z}'_B$ is the frame after rotation. Note that since the frames are coinicident initially and rotation is about $X-$ axis $\mathbf{X}_W$ and $\mathbf{X}'_B$ are identical. Let $\mathbf{R}_1$ be the rotation matrix associated with this rotation.
2. Rotate $\{B\}$ about $\mathbf{Y}_W$ by an angle β (pitch). $\mathbf{X}''_B, \mathbf{Y}''_B, \mathbf{Z}''_B$ is the frame after rotation. Let $\mathbf{R}_2$ be the rotation matrix associated with this rotation.
3. Rotate $\{B\}$ about $\mathbf{Z}_W$ by an angle α (yaw). $\mathbf{X}_B, \mathbf{Y}_B, \mathbf{Z}_B$ is the frame after rotation. Let $\mathbf{R}_3$ be the rotation matrix associated with this rotation.

The rotation matrices $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$ are as follows:

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} \quad \mathbf{R}_2 = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \quad \mathbf{R}_3 = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let the representation of a vector $\mathbf{v}$ in frames $\{B\}$ and $\{W\}$ be ${}^B\mathbf{v}$ and ${}^W\mathbf{v}$ respectively. Note that just like the orientation of $\{B\}$ can be modeled as a sequence of elementary rotations starting from the assumption that $\{B\}$ is aligned with $\{W\}$, we can think of the vector $\mathbf{v}$ being transformed by the elementary rotations. After the first elementary rotation, the description of $\mathbf{v}$ in $\{W\}$ becomes $\mathbf{R}_1{}^B\mathbf{v}$. After the second rotation, the description of $\mathbf{v}$ in $\{W\}$ becomes $\mathbf{R}_2(\mathbf{R}_1{}^B\mathbf{v})$ and so on. Therefore, we finally get

$${}^W\mathbf{v} = \mathbf{R}_3(\mathbf{R}_2(\mathbf{R}_1{}^B\mathbf{v})) = \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1{}^B\mathbf{v}$$

Therefore, the rotation matrix of frame $\{B\}$ with respect to frame $\{W\}$ is given by

$${}^W_B\mathbf{R} = \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1 = \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix} \tag{2.9}$$

Thus given the roll-pitch-yaw angles γ, β, α, we can obtain the rotation matrix $\mathbf{R}$ using Equation (2.9) above.

Remark 2.3 *The roll-pitch-yaw angles can be potentially used for determining the orientation of a flying robot, e.g., a quadcopter.* **How can you obtain the roll, pitch, yaw angle of a quadrotor, i.e., what sensor(s) should you use to estimate the roll, pitch, and yaw angle?**

2.4.2 Z-Y-X Euler angles

Once again we consider two frames $\{B\}$ and $\{W\}$. Assuming that $\{B\}$ and $\{W\}$ are initially coincident, one can go from $\{W\}$ to $\{B\}$ by the following sequence of elementary rotations.

1. Rotate $\{B\}$ about $\mathbf{Z}_B$ (which is the same as $\mathbf{Z}_W$) by an angle α. The axes of new frame, $\{B'\}$, are $\mathbf{X}'_B, \mathbf{Y}'_B, \mathbf{Z}'_B$.
2. Now rotate about $\mathbf{Y}'_B$ by an angle β. The axes of new frame $\{B''\}$ are $\mathbf{X}''_B$, $\mathbf{Y}''_B$ (which is same as $\mathbf{Y}'_B$, $\mathbf{Z}''_B$.
3. Now rotate about $\mathbf{X}''_B$ by an angle α to get to the final frame $\{B\}$.

Note that the elementary rotation matrices are:

$$ {}^{B''}_{B}\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix} \quad {}^{B'}_{B''}\mathbf{R} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \quad {}^{W}_{B'}\mathbf{R} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

In this case, the rotation matrix ${}^{W}_{B}\mathbf{R}$

$$ {}^{W}_{B}\mathbf{R} = {}^{W}_{B'}\mathbf{R} {}^{B'}_{B''}\mathbf{R} {}^{B''}_{B}\mathbf{R} = \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix} \tag{2.10} $$

Thus, given the Z-Y-X Euler angles, we can obtain the rotation matrix using Equation (2.10).

Remark 2.4 *Note that the rotation matrix obtained by the X-Y-Z fixed angles is same as the rotation matrix obtained by the Z-Y-X Euler angles. For rotations about moving axis reversing the sequence of rotations gives rise to the same rotation matrix. The fact that there is a fixed axis representation that is equivalent to a moving axis representation is true for all the 12 possible scenarios. Thus, there are 12 distinct three angle representations for rotations.*

Remark 2.5 *Although we have given two examples, the general principal in these examples hold. If we are given three fixed angles, we multiply the matrices in the order of the rotation to obtain the rotation matrix. If we are given three Euler angles, we multiply the matrices in the reverse order of the rotations to obtain the rotation matrix.*

2.4.3 Obtaining the three angles from the Rotation matrix

Thus far, we have seen that given three angles, we can obtain a rotation matrix. Our gaol is to now obtain the three angle representation from a rotation matrix. We will use the Z-Y-X Euler angle as an example. We assume that we are given the rotation matrix

$$ \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

From the expression for $\mathbf{R}$ above and Equation (2.10)

$$ \cos\beta = \sqrt{r_{11}^2 + r_{21}^2}, \quad \sin\beta = -r_{31}, \quad \therefore \boxed{\beta = atan2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})} $$

$$\cos\alpha = \frac{r_{11}}{\cos\beta}, \quad \sin\alpha = \frac{r_{21}}{\cos\beta}, \quad \therefore \boxed{\alpha = atan2\left(\frac{r_{21}}{\cos\beta}, \frac{r_{11}}{\cos\beta}\right)}$$

$$\sin\gamma = \frac{r_{32}}{\cos\beta}, \quad \cos\gamma = \frac{r_{33}}{\cos\beta}, \quad \therefore \boxed{\gamma = atan2\left(\frac{r_{32}}{\cos\beta}, \frac{r_{33}}{\cos\beta}\right)}$$

Note that when $\cos\beta = 0$, i.e., $\beta = \frac{\pi}{2}$, we cannot compute α and γ separately. We can only compute $\alpha + \gamma$ or $\alpha - \gamma$. The fact that there are some rotation matrices for which we cannot determine the Euler angles is called a *representation singularity*. Representation singularity arises because of our choice of the parameterization of the rotation group. For any three angle parameterization (Euler angle or fixed angle) there is always a representational singularity, although the representation singularities can be for different rotations. In fact it is a fundamental fact that there is no three-parameter representation of $SO(3)$ that does not suffer from representation singularity. Therefore, we will now study a 4 parameter representation of $SO(3)$, namely quaternions, that do not suffer from singularities.

2.5 Quaternions

Quaternions are generalizations of complex numbers. A quaternion is defined as

$$Q = q_0 + q_1 i + q_2 j + q_3 k \tag{2.11}$$

where $q_0 \in \mathbb{R}$ is a scalar and $\mathbf{q} = [q_1, q_2, q_3]^T \in \mathbb{R}^3$ is a vector with

$$ii = jj = kk = ijk = -1 \quad \text{and} \quad ij = -ji = k, \quad jk = -kj = i \quad ki = -ik = j$$

Often we will use the notation $(q_0, \mathbf{q})$ to represent a quaternion. Any scalar $a \in \mathbb{R}$ can be represented in quaternion notation with the scalar part $q_0 = a$ and the vector part $\mathbf{q} = \mathbf{0}$. Thus, a scalar a is written as $(a, \mathbf{0})$, where $\mathbf{0}$ is a 3×1 vector of zeroes. Analogously, any vector, $\mathbf{v} \in \mathbb{R}^3$ can be represented in quaternion notation using $q_0 = 0$ and $\mathbf{q} = \mathbf{v}$. Thus a vector $\mathbf{v}$ is written as $(0, \mathbf{v})$.

Remark 2.6 Note about Notations: *In this class we will be using capital letters to denote quaternions, small letters to denote scalars, bold small letters to denote vectors and bold capital letters to denote matrices. Whenever, we depart from this convention it should be clear from the context what the variable represents.*

2.5.1 Quaternion Product

The conjugate of a quaternion $Q = (q_0, \mathbf{q})$ is defined as $Q^* = (q_0, -\mathbf{q})$. The product of two quaternions, $P = (p_0, \mathbf{p})$ and $Q = (q_0, \mathbf{q})$ is given by

$$Q \otimes P = (q_0 p_0 - \mathbf{q} \cdot \mathbf{p}, q_0 \mathbf{p} + p_0 \mathbf{q} + \mathbf{p} \times \mathbf{q}) \tag{2.12}$$

Note that the definition of quaternion product defined above is different from treating a quaternion as a 4-dimensional vector and using the usual dot product to multiply vectors. The multiplication of two quaternions gives a quaternion. The dot product and cross product on the right hand side are for three dimensional vectors. A key aspect of this multiplication operation is that it allows the definition of an inverse of a quaternion and thus allows division in the set of quaternions (note that this is very different from the usual product operations with vectors, where there is no notion of inverse of a vector). Although quaternions can be thought of as a 4-dimensional vector and it is sometimes useful and convenient to do so, the key properties of the set of quaternions comes from the definition of multiplication operation in

Equation (2.12). The multiplication of two quaternions can also be written in matrix form. For a quaternion Q, we define the left hand and right hand compound operator respectively as

$$Q^{+} = \begin{bmatrix} q_0 & -\boldsymbol{q}^T \\ \boldsymbol{q} & q_0\mathbf{1} - \hat{\boldsymbol{q}} \end{bmatrix} \quad \text{and} \quad Q^{\oplus} = \begin{bmatrix} q_0 & -\boldsymbol{q}^T \\ \boldsymbol{q} & q_0\mathbf{1} + \hat{\boldsymbol{q}} \end{bmatrix} \tag{2.13}$$

where $\mathbf{1}$ is a 3×1 column vector with each entry as 1, and $\hat{\boldsymbol{q}}$ is the skew-symmetric matrix associated with the vector $\boldsymbol{q}$. The product of two quaternions $Q \otimes P$ is defined in matrix vector form using the left hand compound operator as

$$Q \otimes P = Q^{+}P = \begin{bmatrix} q_0 & -\boldsymbol{q}^T \\ \boldsymbol{q} & q_0\mathbf{1} - \hat{\boldsymbol{q}} \end{bmatrix} \begin{bmatrix} p_0 \\ \boldsymbol{p} \end{bmatrix} \tag{2.14}$$

Alternatively, product of two quaternions $Q \otimes P$ is defined in matrix vector form using the right hand compound operator as

$$Q \otimes P = P^{\oplus}Q = \begin{bmatrix} p_0 & -\boldsymbol{p}^T \\ \boldsymbol{p} & p_0\mathbf{1} + \hat{\boldsymbol{p}} \end{bmatrix} \begin{bmatrix} q_0 \\ \boldsymbol{q} \end{bmatrix} \tag{2.15}$$

The magnitude of a quaternion is given by

$$\|Q\|^2 = Q \otimes Q^* = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

This can be obtained by direct algebraic computation using Equation (2.12). The inverse of a quaternion can be thus defined as

$$Q^{-1} = \frac{Q^*}{\|Q\|^2}$$

The quaternion $Q = (1, \mathbf{0})$ is the identity element for quaternion multiplication.

2.5.2 Unit Quaternions and Rotations

Any rotation can be represented by a unit quaternion. However, the representation is not unique as we will discuss later. It can be shown that the set of all unit quaternions form a group under the binary operation of quaternion product defined by Equation (2.12) with $(1, \mathbf{0})$ as the identity element. **Can you prove this?** Fora unit quaternion, the inverse Q^{-1} is the same as the conjugate Q^*.

The composition of two rotations is given by the quaternion product. Let Q be the unit quaternion corresponding to a rotation. The rotation of a vector $\boldsymbol{v}$ using quaternion algebra is given by

$$\boldsymbol{v}' = Q \otimes \begin{bmatrix} 0 \\ \boldsymbol{v} \end{bmatrix} \otimes Q^{-1} = Q^{+}(Q^{-1})^{\oplus} \begin{bmatrix} 0 \\ \boldsymbol{v} \end{bmatrix} \tag{2.16}$$

2.5.3 Summary of Quaternion Operations

Let $Q = (q_0, \mathbf{q})$ and $P = (p_0, \mathbf{p})$ be two quaternions. Then, we have the following:
Scalar Multiplication: $sQ = (sq_0, s\mathbf{q})$, for any $s \in \mathbb{R}$.
Quaternion Addition: $Q + P = (q_0 + p_0, \mathbf{q} + \mathbf{p})$.
Quaternion Multiplication: $Q \otimes P = (q_0p_0 - \mathbf{q} \cdot \mathbf{p}, q_0\mathbf{p} + p_0\mathbf{q} + \mathbf{p} \times \mathbf{q})$.
Conjugate of a Quaternion: $Q^* = (q_0, -\mathbf{q})$.
Magnitude of a Quaternion: $\|Q\| = QQ^*$.

2.5.4 Conversion between exponential coordinates and unit quaternions

Let ω be the axis and θ be the angle for a rotation matrix $\mathbf{R}$, i.e.,

$$R = e^{\hat{\omega}\theta}, \quad \theta \in \mathbb{R}, \quad \omega \in \mathbb{R}^3, \quad \|\omega\| = 1.$$

Then the corresponding unit quaternion is defined by

$$\boxed{Q = \left(\cos\frac{\theta}{2}, \omega \sin\frac{\theta}{2}\right)} \tag{2.17}$$

Note that $-\theta$ and $-\omega$ gives the same rotation. Conversely, if we are given a unit quaternion $Q = (q_0, \mathbf{q})$, then

$$\theta = 2\arccos q_0 \quad \omega = \frac{\mathbf{q}}{\sin\frac{\theta}{2}} \tag{2.18}$$

2.5.5 Conversion between rotation matrix and unit quaternions

Above, we saw the conversion between the exponential coordinates of rotation and the unit quaternion for rotation. In this section, we will study the direct conversion between quaternions and rotation matrices and see that the unit quaternion representation of rotations is indeed singularity free. Given a unit quaternion $Q = (q_0, \mathbf{q})$, where $\mathbf{q} = [q_1\ q_2\ q_3]^T$, (i.e., q_1, q_2, q_3 are the components of the vector part of the quaternion) we have the following:

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & 1 - 2(q_1^2 + q_3^2) & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix} \tag{2.19}$$

Equation (2.19) can be obtained by noting the fact that any unit quaternion can be written in the form of Equation (2.17) and using Equation(2.12) from Lecture notes 2 (**Exercise: Derive the Equation** (2.19)). Note that since Q is an unit quaternion, by substituting $1 = q_0^2 + q_1^2 + q_2^2 + q_3^2$ in the diagonal elements, we can write Equation (2.19) as

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 + q_2^2 - q_1^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 + q_3^2 - q_1^2 - q_2^2 \end{bmatrix} \tag{2.20}$$

Equation (2.19) or (2.20) can be used to convert a unit quaternion to a rotation matrix. To perform the inverse operation, i.e., convert a given rotation matrix to a unit quaternion, we have to be more careful. In essence, there are 4 possible ways. We can find any one of q_0, q_1, q_2 or q_3 from the diagonal elements and use them as follows:

Solution 1: If we know q_0, then we can obtain q_1, q_2, and q_3 in terms of q_0.

$$q_1 = \frac{r_{32} - r_{23}}{4q_0}, \quad q_2 = \frac{r_{13} - r_{31}}{4q_0}, \quad q_3 = \frac{r_{21} - r_{12}}{4q_0}$$

Solution 2: If we know q_1, then we can obtain q_0, q_2, and q_3 in terms of q_1.

$$q_0 = \frac{r_{32} - r_{23}}{4q_1}, \quad q_2 = \frac{r_{12} + r_{21}}{4q_1}, \quad q_3 = \frac{r_{13} + r_{31}}{4q_1}$$

Solution 3: If we know q_2, then we can obtain q_0, q_1, and q_3 in terms of q_2.

$$q_0 = \frac{r_{13} - r_{31}}{4q_2}, \quad q_1 = \frac{r_{12} + r_{21}}{4q_2}, \quad q_3 = \frac{r_{23} + r_{32}}{4q_2}$$

Solution 4: If we know q_3, then we can obtain q_0, q_1, and q_2 in terms of q_3.

$$q_0 = \frac{r_{21} - r_{12}}{4q_3}, \quad q_1 = \frac{r_{13} + r_{31}}{4q_3}, \quad q_2 = \frac{r_{23} + r_{32}}{4q_3}$$

Note that we can use any of the solutions above but each of these solution methods becomes numerically unstable as the value of the q_i, $i = 0, 1, 2, 3$, goes to 0. However, note that since the quaternion is of unit magnitude all of the q_i's cannot go to 0 simultaneously (or even become near 0 simultaneously). In fact, there has to be at least one q_i with value greater than 0.5. Thus, we can avoid the representational singularity if we can compute the maximum value of q_i and use the corresponding solution method. To compute the maximum q_i, note that from the 3 diagonal elements in Equation (2.20) we have the following:

$$\begin{aligned} q_0^2 + q_1^2 - q_2^2 - q_3^2 &= r_{11} \\ q_0^2 + q_2^2 - q_1^2 - q_3^2 &= r_{22} \\ q_0^2 + q_3^2 - q_1^2 - q_2^2 &= r_{33} \\ q_0^2 + q_3^2 + q_1^2 + q_2^2 &= 1 \end{aligned} \tag{2.21}$$

Rewriting Equation (2.21) in vector matrix form, we have

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} q_0^2 \\ q_1^2 \\ q_2^2 \\ q_3^2 \end{bmatrix} = \begin{bmatrix} r_{11} \\ r_{22} \\ r_{33} \\ 1 \end{bmatrix} \tag{2.22}$$

Since the rotation matrix is given, r_{11}, r_{22}, and r_{33} are known. Therfore, we can obtain

$$\begin{bmatrix} q_0^2 \\ q_1^2 \\ q_2^2 \\ q_3^2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r_{11} \\ r_{22} \\ r_{33} \\ 1 \end{bmatrix} \tag{2.23}$$

From the above, we can obtain the maximum of q_0^2, q_1^2, q_2^2 and q_3^2. If q_0^2 is maximum then we can use the Solution 1 from above, if q_1^2 is maximum then we can use the Solution 2 from above and so on. Note that you can use the positive value of the square root without any loss of generality.

Remark 2.7 *Please note that you cannot use the values of q_i that you obtain directly from Equation (2.23), since you will have two possible signs for each q_i that you cannot resolve from these equations alone.*

Appendix: Basic Notions of Group Theory

A set G together with a binary operation, denoted by $\circ$, defined on elements of G is called a group if it satisfies the following properties:

1. *Closure* : If $g_1, g_2 \in G$, then $g_1 \circ g_2 \in G$.
2. *Identity* : $\exists$ an identity element, e, such that $g \circ e = e \circ g = g \; \forall \; g \in G$.
3. *Inverse* : For each $g \in G$, there exists a unique $g^{-1} \in G$ such that $g \circ g^{-1} = g^{-1} \circ g = e$.
4. *Associativity* : If $g_1, g_2, g_3 \in G$, then $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

Common Examples of Groups

1. Set of real numbers, $\mathbb{R}$ with the binary operation of addition is a group with 0 as the identity element, and negative as the inverse.
2. Set of real numbers, $\mathbb{R}$ with the binary operation of multiplication is a group with 1 as the identity element, and the reciprocal as the inverse.
3. The set of all rotation matrices in three-dimensional space, i.e., $SO(3)$, with matrix multiplication as the binary operation forms a group, with the (3×3) identity matrix I as the identity element.
4. The set of all planar rotation matrices, i.e., $SO(2)$, with matrix multiplication as the binary operation forms a group, with the (2×2) identity matrix I as the identity element.
5. The set of all rigid body transformations, i.e., $SE(3)$, with matrix multiplication as the binary operation forms a group, with the (4×4) identity matrix I as the identity element.

Appendix: Matrix Exponentials

The matrix exponential is a generalization of ordinary exponential function to square matrices. Let X be a $n \times n$ square matrix with real or complex entries. Then

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k = I + X + \frac{1}{2!} X^2 + \frac{1}{3!} X^3 + ... \tag{2.24}$$

where I is an $n \times n$ identity matrix. Note that e^X is a matrix of the same dimensions as X. We now look at some properties of matrix exponentials that is useful in proving Rodrigues theorem.

Properties of Matrix Exponentials

Let X and Y be two $n \times n$ matrices. Then the following holds :

1. $e^{\mathbf{0}} = I$, where $\mathbf{0}$ is an $n \times n$ matrix of zeros and I is the $n \times n$ identity matrix.
2. $e^{aX} e^{bX} = e^{(a+b)X}$, where a, b are scalars.
   The above implies that $e^X e^{-X} = I \quad$ (from (1) and (2)).
3. If XY=YX, i.e. the product of two matrices is commutative, then,
   $$e^X e^Y = e^Y e^X = e^{X+Y}$$
   Note : $e^X e^Y \neq e^{X+Y}$ in general. This is different from your intuition of scalar exponentials.
4. $e^{YXY^{-1}} = Y e^X Y^{-1}$, if Y is invertible. To prove this, use expand the left hand side using Equation (2.24) and then show that $(YXY^{-1})^n = Y^{-1} X^n Y$ by method of induction. The base case is for $n = 2$.
5. $(e^X)^T = (e^{X^T})$. If X is symmetric, e^X is also symmetric. This can be again shown by expanding the left hand side using Equation (2.24) and noting that $X^{n^T} = X^{T^n}$ for a square matrix X.

Appendix:Cross Product as a Linear Operation (or matrix vector multiplication)

For any two vectors $\mathbf{a}$ and $\mathbf{b} \in \mathbb{R}^3$, the cross product is defined by

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Where,

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Cross product by a vector $\mathbf{a}$ can be thought of as a linear operator, *i.e.* , multiplication of a vector by a matrix.

$$a \times b = \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{\hat{a}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$a \times b = \hat{a}b$$

where $\hat{\mathbf{a}}$ is a (3×3) skew-symmetric matrix and $\mathbf{b}$ is a (3×1) vector.

<u>Some Useful Results</u>:

1. $R(\hat{w})R^T = (\widehat{Rw})$. To prove this, one can use direct calculation of left and right hand side to see that they are equivalent.

2. The rotation matrix $\mathbf{R}$ preserves distances.In other words for any two points $q, p \in \mathbb{R}^3$,
   The above can be proved as follows (we use the fact $R^TR = I$):

$$\|Rq - Rp\| = (R(q-p))^T R(q-p) = (q-p)^T \underbrace{R^TR}_{=I}(q-p) = (q-p)^T(q-p) = \|q-p\|.$$

3. The rotation matrix $\mathbf{R}$ preserves orientation. In other words, $R(v \times w) = (Rv) \times (Rw)$. This can be seen as follows:

$$(Rv) \times (Rw) = (\widehat{Rv})(Rw) = R\hat{v}\underbrace{R^TR}_{=I}w = R(\hat{v}w) = R(v \times w).$$

4. A rotation R $\in SO(3)$ is a rigid body transformation since it preserves both distance and orientation.