Robotics Lecture Notes **Spring 2018**

Lecture 4: Differential Kinematics of Rigid Body

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4.1 Introduction

In this section, we will learn about the differential kinematics of rigid bodies. We have learned that the configuration space of a rigid body is the group $SE(3)$. Let $g \in SE(3)$ denote a rigid body configuration. We know that g can be thought of as the tuple $\mathbf{p}, \mathbf{R}$, where $\mathbf{p}$ is the position of the origin of the body frame $\mathcal{B}$ in the world frame $\mathcal{A}$ and $\mathbf{R}$ is the rotation matrix of $\mathcal{B}$ with respect to $\mathcal{A}$. Furthermore g can be represented as a 4×4 transformation matrix. The motion of a rigid body is a time sequence of rigid body configurations, which can be written as a function $g(t) = \mathbf{p}(t), \mathbf{R}(t)$. In velocity kinematics of a rigid body, we are usually interested in deriving expressions for the linear velocity of a point on the body and the angular velocity of the body. *Note that there is no notion of linear velocity of a rigid body. Linear velocity is always associated with a point.* In many cases, when we say linear velocity of a rigid body, we implicitly mean the linear velocity of the point which is the origin of the body frame (which is sometimes at the center of mass of the body). The key feature in the differential kinematics of rigid bodies is that unlike positions, for rigid body orientation, there is no representation whose differentiation produces the angular velocities of the rigid body.

4.2 Angular Velocity of a Rigid Body

Let ω_{ab} be the angular velocity of frame $\{B\}$ with respect to frame $\{A\}$. As before, frame $\{B\}$ is a moving frame attached to the body and frame $\{A\}$ is the world fixed frame. We will define two notions of angular velocity:

Definition 4.1 (Spatial Angular Velocity) *The angular velocity of the body $\{B\}$ with respect to $\{A\}$ expressed in the frame $\{A\}$. We denote this by ω^s_{ab}.*

$$\hat{\omega}^s_{ab} = \dot{\mathbf{R}}_{ab}\mathbf{R}^{-1}_{ab} = \dot{\mathbf{R}}_{ab}\mathbf{R}^T_{ab} \tag{4.1}$$

Definition 4.2 (Body Angular Velocity) *The angular velocity of the body $\{B\}$ with respect to $\{A\}$ expressed in the frame $\{B\}$. We denote this by ω^b_{ab}.*

$$\hat{\omega}^b_{ab} = \mathbf{R}^{-1}_{ab}\dot{\mathbf{R}}_{ab} = \mathbf{R}^T_{ab}\dot{\mathbf{R}}_{ab} \tag{4.2}$$

In the above definitions, $\hat{\omega}$ is the skew symmetric matrix corresponding to the 3×1 vector ω, $\dot{\mathbf{R}}$ is the time derivative of the rotation matrix, which means that it is the time derivative of each entry of the rotation matrix. In other words, for

$$\mathbf{R(t)} = \begin{bmatrix} r_{11}(t) & r_{12}(t) & r_{13}(t) \\ r_{21}(t) & r_{22}(t) & r_{23}(t) \\ r_{31}(t) & r_{32}(t) & r_{33}(t) \end{bmatrix}, \quad \dot{\mathbf{R(t)}} = \begin{bmatrix} \dot{r}_{11}(t) & \dot{r}_{12}(t) & \dot{r}_{13}(t) \\ \dot{r}_{21}(t) & \dot{r}_{22}(t) & \dot{r}_{23}(t) \\ \dot{r}_{31}(t) & \dot{r}_{32}(t) & \dot{r}_{33}(t) \end{bmatrix}.$$

Lemma 4.3 *Given $\mathbf{R}(t) \in SO(3)$, the matrices $\dot{\mathbf{R}}(t)\mathbf{R}^{-1}(t)$ and $\mathbf{R}^{-1}(t)\dot{\mathbf{R}}(t)$ are skew-symmetric matrices, i.e., they belong to $so(3)$.*

Proof: For brevity, we will omit the explicit dependence of the rotation matrix $\mathbf{R}$ on time.

$$
\begin{aligned}
& \mathbf{R}\mathbf{R}^T = \mathbf{I} \\
\Rightarrow \quad & \dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T = 0 \\
\Rightarrow \quad & \dot{\mathbf{R}}\mathbf{R}^T = -\mathbf{R}\dot{\mathbf{R}}^T = -(\dot{\mathbf{R}}\mathbf{R}^T)^T
\end{aligned}
$$

Therefore, $\dot{\mathbf{R}}\mathbf{R}^T$ or $\dot{\mathbf{R}}(t)\mathbf{R}^{-1}(t)$ is skew-symmetric. Similarly starting with $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ and differentiating, we can prove that $\mathbf{R}^{-1}(t)\dot{\mathbf{R}}(t)$ is a skew-symmetric matrix. ■

Remark 4.4 *The fact that the derivative of an element of $SO(3)$ belongs to $so(3)$ is significant. Formally, $so(3)$ is the tangent space of $SO(3)$, i.e., the space where all tangents to any point on $SO(3)$ will belong. To help in understanding this, consider the unit sphere in $\mathbb{R}^3$. At any point on the sphere, the tangent will lie in a two dimenisional plane that passes through the point and does not intersect the sphere. Thus, the tangent space here is the two-dimensional plane $\mathbb{R}^2$. If a point is moving along the sphere, the velocity of the point, obtained by the time derivative of the function that gives the position of the point on the sphere, will be in the tangent plane.*

We will now show some justification of the formulas that we defined. Let $\mathbf{q}_b(t)$ be the position of a point in the body frame $\{B\}$ and let $\mathbf{q}_a(t)$ be its position in the world frame $\{A\}$ at time t. As the body undergoes pure rotation, at any instance of time,

$$\mathbf{q}_a(t) = \mathbf{R}_{ab}(t)\mathbf{q}_b(t)$$

Taking the time derivative of the above equation (by noting that $\mathbf{q}_b(t)$ is constant in the body frame), we obtain

$$
\begin{aligned}
\dot{\mathbf{q}}_a(t) &= \dot{\mathbf{R}}_{ab}(t)\mathbf{q}_b = (\dot{\mathbf{R}}_{ab}(t)\mathbf{R}_{ab}^{-1}(t))(\mathbf{R}_{ab}(t)\mathbf{q}_b) \\
&= (\dot{\mathbf{R}}_{ab}(t)\mathbf{R}_{ab}^{-1}(t))\mathbf{q}_a(t) \\
&= \hat{\omega}_{ab}^s(t)\mathbf{q}_a(t) \qquad \text{(by definition)} \\
&= \omega_{ab}^s(t) \times \mathbf{q}_a(t)
\end{aligned}
$$

Thus we obtain our familiar formula for linear velocity of a point on a rotating body, which justifies the used formula.

Relationship between spatial angular velocity and body angular velocity: The body angular velocity and spatial angular velocity of a rigid body is the angular velocity of the body with respect to a world frame expressed either in the body frame $\{B\}$ or the world frame $\{W\}$. Therefore,

$$\boxed{\omega_{ab}^s = \mathbf{R}_{ab}\omega_{ab}^b} \tag{4.3}$$

In terms of the skew-symmetric matrices corresponding to ω_{ab}^s and ω_{ab}^b, we can write

$$\boxed{\hat{\omega}_{ab}^s = \mathbf{R}_{ab}\hat{\omega}_{ab}^b\mathbf{R}_{ab}^T} \tag{4.4}$$

Equation (4.4) can be proven as follows:

$$
\begin{aligned}
\mathbf{R}_{ab}\hat{\omega}_{ab}^b\mathbf{R}_{ab}^T &= \overbrace{\mathbf{R}_{ab}\mathbf{R}_{ab}^T}^{\mathbf{I}}\dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^T \qquad \left(\text{substituting} \quad \hat{\omega}_{ab}^b = \mathbf{R}_{ab}^T\dot{\mathbf{R}}_{ab}\right) \\
&= \dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^T = \hat{\omega}_{ab}^s
\end{aligned}
$$

Note that in the above formulas each of the variables are function of time, t. We do not write it down explicitly for brevity.

Interpretation of $\omega_{ab}^b(t) \times \mathbf{q}_b(t)$: Note that we have derived above that $\omega_{ab}^s(t) \times \mathbf{q}_a(t)$ is the velocity of a point on the rotating body expressed in the spatial frame. We can write an analogous expression using the body angular velocity and the coordinates of the point in body coordinates. Now,

$$\omega_{ab}^b(t) \times \mathbf{q}_b(t) = (\mathbf{R}_{ab}^T \omega_{ab}^s(t)) \times (\mathbf{R}_{ab}^T \mathbf{q}_a(t)) = \mathbf{R}_{ab}^T(\omega_{ab}^s(t) \times \mathbf{q}_a(t)) = \mathbf{R}_{ab}^T \dot{\mathbf{q}}_a(t)$$

Therefore, $\omega_{ab}^b(t) \times \mathbf{q}_b(t)$ is the linear velocity of a point on the rigid body with respect to the spatial frame *expressed in the body frame*. It is NOT the linear velocity of the point with respect to the body frame (which is 0, since the body is a rigid body, hence the point is always stationary with respect to B).

Example: Consider a single link or a 1-DoF manipulator, where the rotation axis is the z-axis, as shown in the Figure. Here

$$\mathbf{R}(t) = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) & 0 \\ \sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore the matrix derivative $\dot{\mathbf{R}}(t)$ is given by

$$\dot{\mathbf{R}}(t) = \begin{bmatrix} -\dot{\theta}(t)\sin\theta(t) & -\dot{\theta}(t)\cos\theta(t) & 0 \\ \dot{\theta}(t)\cos\theta(t) & -\dot{\theta}(t)\sin\theta(t) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The skew-symmetric form of the spatial angular velocity is

$$\hat{\omega}^s = \dot{\mathbf{R}}(t)\mathbf{R}^T(t) = \begin{bmatrix} -\dot{\theta}(t)\sin\theta(t) & -\dot{\theta}(t)\cos\theta(t) & 0 \\ \dot{\theta}(t)\cos\theta(t) & -\dot{\theta}(t)\sin\theta(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta(t) & \sin\theta(t) & 0 \\ -\sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\theta}(t) & 0 \\ \dot{\theta}(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the spatial angular velocity of the manipulator is $\omega^s = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}(t) \end{bmatrix}$.

The skew-symmetric form of the body angular velocity is

$$\hat{\omega}^b = \mathbf{R}^T(t)\dot{\mathbf{R}}(t) = \begin{bmatrix} \cos\theta(t) & \sin\theta(t) & 0 \\ -\sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\dot{\theta}(t)\sin\theta(t) & -\dot{\theta}(t)\cos\theta(t) & 0 \\ \dot{\theta}(t)\cos\theta(t) & -\dot{\theta}(t)\sin\theta(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\theta}(t) & 0 \\ \dot{\theta}(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, the body angular velocity of the manipulator is $\omega^b = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}(t) \end{bmatrix}$.

Note that the body angular velocity and the spatial angular velocity are identical in this case. This is because the rotation is a pure rotation about the z-axis which remains the same. This is true for general planar rigid body rotation where the rotation is always about the z-axis that does not change. Hence, for planar rotation, one does not need to distinguish between body and spatial angular velocity.

4.2.1 Relationship Between Angular Velocity and Quaternion Derivatives

We have seen previously that there are different parameterizations of $SO(3)$, namely fixed angle parameterizations, Euler angle parameterizations and quaternions. It is also useful to relate the time derivatives of these parameterizations to the angular velocities. We will write down the expressions for the quaternion representation here without giving the derivations.

Let $\mathbf{Q}(t) = (q_0(t), q_1(t), q_2(t), q_3(t))$ be the unit quaternion representation of the orientation of a rotating body at any time instant, t. The time derivative of $\mathbf{Q}(t)$ is $\dot{\mathbf{Q}}(t) = (\dot{q}_0(t), \dot{q}_1(t), \dot{q}_2(t), \dot{q}_3(t))$. Then

$$\dot{\mathbf{Q}}(t) = \frac{1}{2}\omega^s(t) \otimes \mathbf{Q}(t) \tag{4.5}$$

where ω^s is the spatial angular velocity of the body. Here, the multiplication on the right hand side is a quaternion multiplication where the angular velocity vector is expressed as a quaternion (i.e., it's real part is 0). Alternately, we can write

$$\omega^s(t) = 2\dot{\mathbf{Q}}(t) \otimes \mathbf{Q}^*(t) \tag{4.6}$$

where again the multiplication on the right hand side is a quaternion multiplication. In matrix form, we can write the above equation as

$$\omega^s = 2\begin{bmatrix} -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{bmatrix}\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = 2J_1(\mathbf{Q})\dot{\mathbf{Q}} \tag{4.7}$$

where $J_1(\mathbf{Q})$ is called the representation Jacobian of the quaternion representation. Note that by using the fact that $J_1 J_1^T = \mathbf{I}$, we can obtain

$$\dot{\mathbf{Q}} = \begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & q_3 & -q_2 \\ -q_3 & q_0 & q_1 \\ q_2 & -q_1 & q_0 \end{bmatrix}\omega^s = \frac{1}{2}J_1^T(\mathbf{Q})\omega^s \tag{4.8}$$

The body angular velocity is given by

$$\omega^b = \mathbf{Q}^* \otimes \omega^s \otimes \mathbf{Q} = 2\mathbf{Q}^* \otimes \dot{\mathbf{Q}} \tag{4.9}$$

4.3 General Rigid Body Motion

A trajectory is a time-parameterized path and a path of rigid body motion is any curve in $SE(3)$. Let $\mathbf{g}_{ab}(t) \in SE(3)$ be the trajectory of motion of a rigid body. More precisely,

$$\mathbf{g}_{ab}(t) = \begin{bmatrix} \mathbf{R}_{ab}(t) & \mathbf{p}_{ab}(t) \\ \mathbf{0} & 1 \end{bmatrix}$$

The time derivative of $\mathbf{g}_{ab}(t)$ is denoted by $\dot{\mathbf{g}}_{ab}(t)$ is

$$\dot{\mathbf{g}}_{ab}(t) = \begin{bmatrix} \dot{\mathbf{R}}_{ab}(t) & \dot{\mathbf{p}}_{ab}(t) \\ \mathbf{0} & 0 \end{bmatrix}$$

A rigid body moving in space has both linear and angular velocity. Let $\mathbf{v}_{ab}$ be the linear velocity and ω_{ab} be the angular velocity of frame $\{B\}$ with respect to frame $\{A\}$. As before, frame $\{B\}$ is a moving frame attached to the body and frame $\{A\}$ is the world fixed frame. Let $\mathbf{V}_{ab}$, which has dimensions 6×1 consist of the concatenated vectors of linear and angular velocity. We will call $\mathbf{V}_{ab}$ the *generalized velocity or simply the velocity* of the rigid body. Thus, $\mathbf{V}_{ab} = \begin{bmatrix} \mathbf{v}_{ab} \\ \omega_{ab} \end{bmatrix}$. We will define two notions of generalized velocity, namely spatial velocity and body velocity.

Definition 4.5 (Spatial Velocity) *The (twist form) of spatial velocity of the body $\{B\}$ with respect to $\{A\}$ is denoted by $\mathbf{V}^s_{ab}$ and is defined by*

$$\boxed{\hat{\mathbf{V}}^s_{ab} = \dot{\mathbf{g}}_{ab}\mathbf{g}^{-1}_{ab} = \begin{bmatrix} \dot{\mathbf{R}}_{ab}\mathbf{R}^T_{ab} & -\dot{\mathbf{R}}_{ab}\mathbf{R}^T_{ab}\mathbf{p}_{ab} + \dot{\mathbf{p}}_{ab} \\ \mathbf{0} & 0 \end{bmatrix}} \tag{4.10}$$

For brevity, in the above formulas, we have dropped the explicit dependence on time, t. As we will see below, in Equation (4.10), the spatial velocity is a twist, i.e., $\hat{\mathbf{V}}^s_{ab} \in se(3)$. The twist coordinates of $\hat{\mathbf{V}}^s_{ab}$ gives the spatial linear velocity and spatial angular velocity of the rigid body. Thus,

$$\boxed{\mathbf{V}^s_{ab} = \begin{bmatrix} \mathbf{v}^s_{ab} \\ \omega^s_{ab} \end{bmatrix} = \begin{bmatrix} -\dot{\mathbf{R}}_{ab}\mathbf{R}^T_{ab}\mathbf{p}_{ab} + \dot{\mathbf{p}}_{ab} \\ (\dot{\mathbf{R}}_{ab}\mathbf{R}^T_{ab})^\vee \end{bmatrix}} \tag{4.11}$$

In the above formula the operator $\vee$, which is pronounced as the *vee* operator is the inverse of the hat operator. The hat operator took a 3×1 vector and returned its corresponding 3×3 skew symmetric matrix form. On the other hand the vee operator takes a 3×3 skew symmetric matrix and returns its corresponding 3×1 vector. In Equation (4.11), $\dot{\mathbf{R}}\mathbf{R}^T$ is a 3×3 skew-symmetric matrix, so $(\dot{\mathbf{R}}\mathbf{R}^T)^\vee$ is the 3×1 vector corresponding to the matrix $\dot{\mathbf{R}}\mathbf{R}^T$ (again we have dropped the subscripts ab for brevity).

Definition 4.6 (Body Velocity) *The (twist form) of body velocity of the body $\{B\}$ with respect to $\{A\}$ is denoted by $\mathbf{V}^b_{ab}$ and is defined by*

$$\boxed{\hat{\mathbf{V}}^b_{ab} = \mathbf{g}^{-1}_{ab}\dot{\mathbf{g}}_{ab} = \begin{bmatrix} \mathbf{R}^T_{ab}\dot{\mathbf{R}}_{ab} & \mathbf{R}^T_{ab}\dot{\mathbf{p}}_{ab} \\ \mathbf{0} & 0 \end{bmatrix}} \tag{4.12}$$

For brevity, in the above formulas, we have dropped the explicit dependence on time, t. As we will see below, in Equation (4.12), the body velocity is a twist, i.e., $\hat{\mathbf{V}}^b_{ab} \in se(3)$. The twist coordinates of $\hat{\mathbf{V}}^b_{ab}$ gives the body linear velocity and body angular velocity of the rigid body. Thus,

$$\boxed{\mathbf{V}^b_{ab} = \begin{bmatrix} \mathbf{v}^b_{ab} \\ \omega^b_{ab} \end{bmatrix} = \begin{bmatrix} \mathbf{R}^T_{ab}\dot{\mathbf{p}}_{ab} \\ (\mathbf{R}^T_{ab}\dot{\mathbf{R}}_{ab})^\vee \end{bmatrix}} \tag{4.13}$$

We will now show that indeed $\dot{\mathbf{g}}_{ab}\mathbf{g}^{-1}_{ab}$ and $\mathbf{g}^{-1}_{ab}\dot{\mathbf{g}}_{ab}$ are skew-symmetric matrices, i.e., they belong to $se(3)$.

Lemma 4.7 *Given $\mathbf{g}(t) \in SO(3)$, the matrices $\dot{\mathbf{g}}(t)\mathbf{g}^{-1}(t)$ and $\mathbf{g}^{-1}(t)\dot{\mathbf{g}}(t)$ are skew-symmetric matrices, i.e., they belong to $se(3)$.*

Proof:

$$\dot{\mathbf{g}}\mathbf{g}^{-1} = \begin{bmatrix} \dot{\mathbf{R}} & \dot{\mathbf{p}} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{R}}\mathbf{R}^T & -\dot{\mathbf{R}}\mathbf{R}^T\mathbf{p} + \dot{\mathbf{p}} \\ \mathbf{0} & 0 \end{bmatrix}$$

As we have noted before $\dot{\mathbf{R}}\mathbf{R}^T$ is a skew symmetric matrix, i.e., $\dot{\mathbf{R}}\mathbf{R}^T \in so(3)$ and $-\dot{\mathbf{R}}\mathbf{R}^T\mathbf{p} + \dot{\mathbf{p}}$ is a 3×1 vector. Therefore from the definition of $se(3)$, we have that $\dot{\mathbf{g}}\mathbf{g}^{-1} \in se(3)$. Now,

$$\mathbf{g}^{-1}\dot{\mathbf{g}} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{R}} & \dot{\mathbf{p}} \\ \mathbf{0} & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{R}^T\dot{\mathbf{R}} & \mathbf{R}^T\dot{\mathbf{p}} \\ \mathbf{0} & 0 \end{bmatrix}$$

Again $\mathbf{R}^T\dot{\mathbf{R}}$ is a skew symmetric matrix, i.e., $\mathbf{R}^T\dot{\mathbf{R}} \in so(3)$ and $-\mathbf{R}^T\dot{\mathbf{p}}$ is a 3×1 vector. Therefore, from the definition of $se(3)$, we have that $\mathbf{g}^{-1}\dot{\mathbf{g}} \in se(3)$. ■

4.3.1 Physical Interpretation of Spatial and Body Velocity

From Equation (4.11) and the discussion on angular velocities, it is clear that $\dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^T$ is the skew-symmetric form of the spatial angular velocity and consequently, $(\dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^T)^\vee$, is the spatial angular velocity, i.e., the angular velocity of frame $\{B\}$ with respect to frame $\{A\}$ expressed in frame $\{A\}$. The spatial linear velocity is the velocity of a (possibly imaginary) point on the rigid body and traveling through the origin of the spatial frame at the time instant t. To see this, note that

$$-\dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^T\mathbf{p}_{ab} + \dot{\mathbf{p}}_{ab} = -\hat{\omega}_{ab}^s\mathbf{p}_{ab} + \dot{\mathbf{p}}_{ab} = \omega_{ab}^s \times (-\mathbf{p}_{ab}) + \dot{\mathbf{p}}_{ab}$$

Since $\mathbf{p}_{ab}$ is the position vector of the origin of frame $\{B\}$ with respect to frame $\{A\}$, $\dot{\mathbf{p}}_{ab}$ is the linear velocity of the origin of frame $\{B\}$ with respect to frame $\{A\}$ (and expressed in frame $\{A\}$). The term $\omega_{ab}^s \times (-\mathbf{p}_{ab})$ is the linear velocity of a point on the rigid body located at the origin at the instant t that arises due to the rotational motion. Therefore, the sum of the two terms gives the net linear velocity of a point on the rigid body and traveling through the origin of the spatial frame at the time instant t.

From Equation (4.13) and the discussion on angular velocities, it is clear that $\mathbf{R}_{ab}^T\dot{\mathbf{R}}_{ab}$ is the skew-symmetric form of the body angular velocity. Hence, $(\mathbf{R}_{ab}^T\dot{\mathbf{R}}_{ab})^\vee$ is the body angular velocity, i.e., the angular velocity of frame $\{B\}$ with respect to frame $\{A\}$ expressed in the frame $\{B\}$. The body linear velocity $\mathbf{R}_{ab}^T\dot{\mathbf{p}}_{ab}$ is the linear velocity of the origin of frame $\{B\}$ with respect to frame $\{A\}$, but expressed in the frame $\{B\}$.

4.3.2 Justification of the Definition of Velocity

Using the definition spatial velocity, we can compute the velocity of a point on the rigid body. Let $\mathbf{q}_b(t)$ be the position of a point in the body frame $\{B\}$ and let $\mathbf{q}_a(t)$ be its position in the world frame $\{A\}$ at time t to represent both.theAs the body undergoes rigid body motion, at any instance of time,

$$\mathbf{q}_a(t) = \mathbf{g}_{ab}(t)\mathbf{q}_b(t)$$

In the above formula, we will overload notation and use $\mathbf{q}_a(t)$ and $\mathbf{q}_b(t)$ to represent both the homogeneous coordinates of the points and the standard 3×1 vector representation. It should be clear from the context which representation is being used. Taking the time derivative of the above equation (by noting that $\mathbf{q}_b(t)$ is constant in the body frame), we obtain

$$\begin{aligned}
\dot{\mathbf{q}}_a(t) &= \dot{\mathbf{g}}_{ab}(t)\mathbf{q}_b = (\dot{\mathbf{g}}_{ab}(t)\mathbf{g}_{ab}^{-1}(t))(\mathbf{g}_{ab}(t)\mathbf{q}_b) \\
&= (\dot{\mathbf{g}}_{ab}(t)\mathbf{g}_{ab}^{-1}(t))\mathbf{q}_a(t) \\
&= \hat{\mathbf{V}}_{ab}^s(t)\mathbf{q}_a(t) \qquad \text{(by definition)} \\
&= \begin{bmatrix} \dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^T & -\dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^T\mathbf{p}_{ab} + \dot{\mathbf{p}}_{ab} \\ \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{q}_a(t) \\ 1 \end{bmatrix} \\
&= \dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^T\mathbf{q}_a(t) + (-\dot{\mathbf{R}}_{ab}\mathbf{R}_{ab}^T\mathbf{p}_{ab} + \dot{\mathbf{p}}_{ab}) \\
&= \omega_{ab}^s(t) \times \mathbf{q}_a(t) + \mathbf{v}_{ab}^s
\end{aligned}$$

Thus we obtain our familiar formula for linear velocity of a point on a rigid body that is rotating and translating, which justifies the used formula.

4.3.3 The Adjoint Matrix

In this section, we want to explore the mapping between spatial velocity of a rigid body and body velocity of a rigid body. Recall that the velocity of a rigid body is a twist, i.e., it belongs to $se(3)$. We will sometimes use spatial velocity twist and body velocity twist to refer to the spatial and body velocities respectively.

The relationship between the spatial velocity and body velocity twists are given by

$$\hat{\mathbf{V}}^b_{ab} = \mathbf{g}^{-1}_{ab}\hat{\mathbf{V}}^s_{ab}\mathbf{g}_{ab} \tag{4.14}$$

$$\hat{\mathbf{V}}^s_{ab} = \mathbf{g}_{ab}\hat{\mathbf{V}}^b_{ab}\mathbf{g}^{-1}_{ab} \tag{4.15}$$

To derive Equation (4.14) note that,

$$\mathbf{g}^{-1}_{ab}\hat{\mathbf{V}}^s_{ab}\mathbf{g}_{ab} = \mathbf{g}^{-1}_{ab}(\dot{\mathbf{g}}_{ab}\underbrace{\mathbf{g}^{-1}_{ab})\mathbf{g}_{ab}}_{I} = \mathbf{g}^{-1}_{ab}\dot{\mathbf{g}}_{ab} = \hat{\mathbf{V}}^b_{ab}$$

To derive Equation (4.15) note that

$$\mathbf{g}_{ab}\hat{\mathbf{V}}^b_{ab}\mathbf{g}^{-1}_{ab} = \underbrace{\mathbf{g}_{ab}(\mathbf{g}^{-1}_{ab}}_{I}\dot{\mathbf{g}}_{ab})\mathbf{g}^{-1}_{ab} = \dot{\mathbf{g}}_{ab}\mathbf{g}^{-1}_{ab} = \hat{\mathbf{V}}^s_{ab}$$

We will now introduce the adjoint matrix that relates $\mathbf{V}^b_{ab} = \begin{bmatrix}\mathbf{v}^b_{ab}\\ \omega^b_{ab}\end{bmatrix}$ and $\mathbf{V}^s_{ab} = \begin{bmatrix}\mathbf{v}^s_{ab}\\ \omega^s_{ab}\end{bmatrix}$. Note that

$$\omega^s_{ab} = \mathbf{R}_{ab}\omega^b_{ab}$$

$$\begin{aligned}
\mathbf{v}^s_{ab} &= -\dot{\mathbf{R}}_{ab}\mathbf{R}^T_{ab}\mathbf{p}_{ab} + \dot{\mathbf{p}}_{ab}\\
&= -\hat{\omega}^s_{ab}\mathbf{p}_{ab} + \dot{\mathbf{p}}_{ab}\\
&= \mathbf{p}_{ab} \times \omega^s_{ab} + \dot{\mathbf{p}}_{ab}\\
&= \mathbf{p}_{ab} \times (\mathbf{R}_{ab}\omega^b_{ab}) + \mathbf{R}_{ab}\mathbf{v}^b_{ab}\\
&= \hat{\mathbf{p}}_{ab}(\mathbf{R}_{ab}\omega^b_{ab}) + \mathbf{R}_{ab}\mathbf{v}^b_{ab}
\end{aligned}$$

Therefore, we can write in matrix form

$$\begin{bmatrix}\mathbf{v}^s_{ab}\\ \omega^s_{ab}\end{bmatrix} = \begin{bmatrix}\mathbf{R}_{ab} & \hat{\mathbf{p}}_{ab}\mathbf{R}_{ab}\\ \mathbf{0} & \mathbf{R}_{ab}\end{bmatrix}\begin{bmatrix}\mathbf{v}^b_{ab}\\ \omega^b_{ab}\end{bmatrix} \tag{4.16}$$

The 6×6 matrix $\begin{bmatrix}\mathbf{R}_{ab} & \hat{\mathbf{p}}_{ab}\mathbf{R}_{ab}\\ \mathbf{0} & \mathbf{R}_{ab}\end{bmatrix}$ that transforms the velocity twist coordinates from one frame to another is called the adjoint transformation. It is formally defined below.

Definition 4.8 *The matrix that converts twists in one reference frame to another is called the adjoint transformation. For any* $\mathbf{g} \in SE(3)$*, we can define the adjoint of* $\mathbf{g}$*, denoted by* Ad_g *as a mapping* $Ad_g : \mathbb{R}^6 \to \mathbb{R}^6$ *given by*

$$\boxed{Ad_g = \begin{bmatrix}\mathbf{R} & \hat{\mathbf{p}}\mathbf{R}\\ \mathbf{0} & \mathbf{R}\end{bmatrix}} \tag{4.17}$$

where $\mathbf{g} = \begin{bmatrix}\mathbf{R} & \mathbf{p}\\ \mathbf{0} & 1\end{bmatrix}$.

The inverse of the adjoint matrix of $\mathbf{g}$ is the same of the adjoint of the inverse of $\mathbf{g}$, i.e., $\mathbf{g}^{-1}$ and is given by

$$\boxed{Ad^{-1}_g = \begin{bmatrix}\mathbf{R}^T & -\mathbf{R}^T\hat{\mathbf{p}}\\ \mathbf{0} & \mathbf{R}^T\end{bmatrix} = Ad_{g^{-1}}} \tag{4.18}$$