Robotics Lecture Notes **Spring 2018**

Lecture 1: Kinematics and Statics of 2R Manipulator

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1.1 Introduction

In this document we introduce some of the fundamental problems in robotics using the simple example of a planar $2R$ robot. The planar $2R$ robot is a serial chain mechanism with 2 links and 2 revolute joints (hence $2R$ is used in the name). A $2R$ robot has two degrees of freedom. Consequently, it can reach any point within its workspace. Although this is a simple robot it is also useful in practice. In many multi-fingered robotic hands, each finger is a $2R$ serial chain connected to a base. We will study both direct and inverse position kinematics and velocity kinematics. Note that we will use basic geometry, trigonometry, and multi-variate calculus to do the derivations. For more complicated spatial chains these tools will be insufficient and we will need to develop more sophisticated tools for solving the position and velocity kinematics problems.

This document is organized as follows: In Section 1.2, we study the position kinematics of a $2R$ robot, in section 1.3 we will study the velocity kinematics problem. A key concept that arises from the velocity kinematics is that of the manipulator Jacobian. In section 1.4, we will show how the manipulator Jacobian also arises in force analysis. The manipulator Jacobian is an instantiation of the more general concept of Jacobian in multi-variable calculus. The Appendix gives the concept of Jacobian in detail.

1.2 Position Kinematics of 2R Robot

In this section we will study the direct and inverse position kinematics for a $2R$ robot.

1.2.1 Direct Kinematics

Given the configuration of the 2R robot, i.e., angles θ_1 and θ_2, find the position of the end effector point A, i.e., (x, y) with respect to the global frame.

$$\begin{aligned} x &= OD + BC \\ x &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \end{aligned} \tag{1.1}$$

$$\begin{aligned} y &= DB + CA \\ y &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{aligned} \tag{1.2}$$

where l_1 and l_2 are the lengths of the two links, OB and BA respectively. For any value of θ_1 and θ_2, we can obtain x and y by substituting the values of θ_1 and θ_2 in Equation (1.1) and (1.2). Note that as discussed in class, in practice there will be joint encoders that will give the values of θ_1 and θ_2. The Equations (1.1) and (1.2) are the *direct position kinematics equations* of 2R robot.

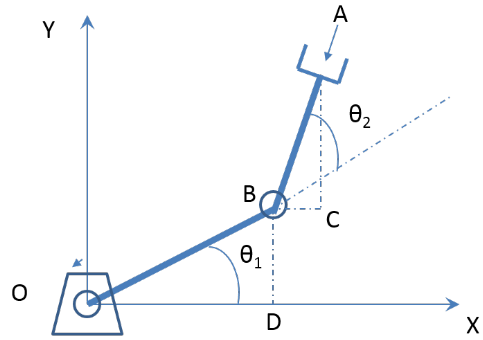

Figure 1.1: Position Kinematics

1.2.2 Inverse Position Kinematics

Given the position, (x, y), of the end effector point A, find the joint angles so that the position (x, y) is reached.
To solve the inverse kinematics, we proceed from Equations (1.1) and (1.2). Squaring and summing Equation (1.1) and (1.2), we get

$$x^2 + y^2 = l_1^2 + l_2^2 + 2l_1 l_2 \cos\theta_1 \cos(\theta_1 + \theta_2) + 2l_1 l_2 \sin\theta_1 \sin(\theta_1 + \theta_2)$$
$$\because \cos^2\theta_1 + \sin^2\theta_1 = 1$$
$$\therefore \frac{(x^2 + y^2 - l_1^2 - l_2^2)}{2l_1 l_2} = \cos(\theta_1 + \theta_2 - \theta_1)$$
$$\therefore \cos\theta_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$
$$\cos\theta_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}$$
$$\sin\theta_2 = \pm\sqrt{1 - \cos^2\theta_2}$$

$$\theta_2 = \operatorname{atan2}(\sin\theta_2, \cos\theta_2) \tag{1.3}$$

Therefore, we obtain two solutions for θ_2. Corresponding to each solution for θ_2, we can obtain a solution for θ_1. One needs to be careful when solving for θ_1 so as to avoid spurious solutions.

1.3 Velocity Kinematics of 2R Robot

In this section we will look at the velocity kinematics of a $2R$ robot. The key concept that will come out of this section is that of the manipulator Jacobian. Solving the velocity kinematics problems is essentially computing the manipulator Jacobian.

1.3.1 Direct Velocity Kinematics

Given the joint angle rate of motion $(\dot\theta_1, \dot\theta_2)$, what is the velocity of the end effector point (v_x, v_y)?

By differentiating Equation (1.1) with respect to time

$$\begin{aligned}
v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} &= \frac{\mathrm{d}}{\mathrm{d}t}(l_1 \cos\theta_1 + l_2\cos(\theta_1+\theta_2)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(l_1\cos\theta_1) + \frac{\mathrm{d}}{\mathrm{d}t}(l_2\cos(\theta_1+\theta_2)) \\
&= l_1(\frac{\partial}{\partial\theta_1}\cos\theta_1)\frac{\mathrm{d}\theta_1}{\mathrm{d}t} + l_2(\frac{\partial}{\partial\theta_1}\cos(\theta_1+\theta_2)\frac{\mathrm{d}\theta_1}{\mathrm{d}t} + \frac{\partial}{\partial\theta_2}\cos(\theta_1+\theta_2)\frac{\mathrm{d}\theta_2}{\mathrm{d}t}) \\
&= -l_1\sin\theta_1\dot{\theta}_1 - l_2\sin(\theta_1+\theta_2)\dot{\theta}_1 - l_2\sin(\theta_1+\theta_2)\dot{\theta}_2
\end{aligned}$$

$$\therefore \dot{x} = (-l_1 sin\theta_1 - l_2\sin(\theta_1+\theta_2))\dot{\theta}_1 - l_2\sin(\theta_1+\theta_2)\dot{\theta}_2 \tag{1.4}$$

Proceeding similarly as above, we get

$$v_y = \dot{y} = (l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2))\dot{\theta}_1 + l_2\cos(\theta_1+\theta_2)\dot{\theta}_2 \tag{1.5}$$

Rewriting the Equations (1.4) and (1.5) in vector-matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \tag{1.6}$$

The matrix above is called Manipulator Jacobian $\boldsymbol{J}(\boldsymbol{\theta})$. So the direct velocity kinematics solution is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \boldsymbol{J} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Remark 1.1 *All manipulators will have a relationship of the form*

$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

where $\boldsymbol{v}$ is the linear/angular velocity of the end effector and $\dot{\boldsymbol{\theta}}$ is the vector of joint angle rates.

1.3.2 Inverse Velocity Kinematics

Given the velocity of the end effector $\boldsymbol{v}$, what should be the joint rates $\dot{\boldsymbol{\theta}}$ so as to achieve the desired velocity $\boldsymbol{v}$?

Provided $\boldsymbol{J}^{-1}(\boldsymbol{\theta})$ exist, the joint rates is

$$\dot{\theta} = \boldsymbol{J}^{-1}(\boldsymbol{\theta})\boldsymbol{v} \tag{1.7}$$

If $\boldsymbol{J}^{-1}(\boldsymbol{\theta})$ does not exist, $\det(\boldsymbol{J}(\boldsymbol{\theta})) = 0$. For 2R manipulator

$$\boldsymbol{J}(\boldsymbol{\theta}) = \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix}$$

$$\det(\boldsymbol{J}(\boldsymbol{\theta})) = -l_1 l_2\sin\theta_1\cos(\theta_1+\theta_2) - l_2^2\sin(\theta_1+\theta_2)\cos(\theta_1+\theta_2) + l_1 l_2\cos\theta_1\sin(\theta_1+\theta_2) + l_2^2\sin(\theta_1+\theta_2)\cos(\theta_1+\theta_2)$$

$$\begin{aligned}
\det(\boldsymbol{J}(\boldsymbol{\theta})) = 0 \quad &\Longleftrightarrow \quad l_1 l_2(\cos\theta_1\sin(\theta_1+\theta_2) - \sin\theta_1\cos(\theta_1+\theta_2)) = 0 \\
&\text{or} \quad l_1 l_2\sin(\theta_1+\theta_2-\theta_1) = 0 \\
&\text{or} \quad l_1 l_2\sin\theta_2 = 0 \\
&\text{or} \quad \sin\theta_2 = 0 \\
&\because \quad l_1 \neq 0, l_2 \neq 0 \\
&\therefore \quad \theta_2 = 0 \text{ or } \pi
\end{aligned}$$

When $\theta_2 = 0$ or π, $\boldsymbol{J}(\boldsymbol{\theta})$ is not invertible and the values of joint angles for which the Jacobian is singular are called singular configurations of the manipulator. At a singular configuration, a manipulator loses one (or more) degrees of freedom, i.e., there are directions in which the end effector of the manipulator cannot move. In the case of the $2R$ manipulator (if $l_2 < l_1$), the singualrities are always at the boundaries of the workspace of the robot. What happens if $l_2 > l_1$?

1.4 Statics of 2R Robot

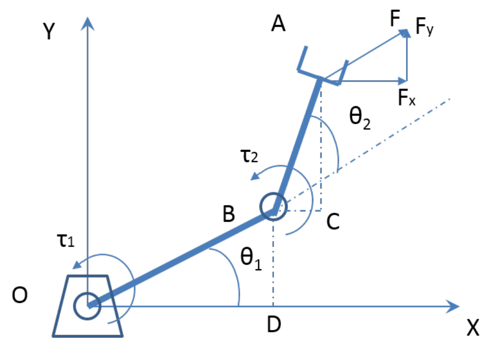

Figure 1.2: Statics Equilibrium

Consider the 2R manipulator in the absence of gravity, let τ_1 and τ_2 be the torques at the two joints and F the force applied at the end effector. If the 2R manipulator is in static equilibrium, we can find following relationship between force at end effector and joint torques

$$\begin{aligned}\tau_2 &= -F_x l_2 \sin(\theta_1+\theta_2) + F_y l_2 \cos(\theta_1+\theta_2)\\ \tau_1 &= -F_x(l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2)) + F_y(l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2))\end{aligned}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}-l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2)\\ -l_2\sin(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2)\end{bmatrix}\begin{bmatrix}F_x\\ F_y\end{bmatrix} \tag{1.8}$$

$$\therefore \tau = J^T F$$

Note the *duality* in the relationship between joint angle rates and end effector velocity and joint torques and end effector forces. The fact that the end effector forces are related to the joint torques through the Jacobian transpose is of fundamental importance in robotics.

Appendix: Concept of Jacobian in Multi-Variable Calculus

In this section, we will consider the concept of the Jacobian in multi-variable calculus. Note that our concept of manipulator Jacobian is essentially the use of the more general concept of Jacobian. You can verify that by using the definition of the Jacobian given below you can write down the manipulator Jacobian.

Let $\boldsymbol{x} \in \mathbb{R}^n$ be an $n-$dimensional vector, where $\mathbb{R}^n$ is the $n-$dimensional Euclidean space[1]. As an example, $n = 1$, i.e., $\mathbb{R}$ denotes the real line, $n = 2$, i.e., $\mathbb{R}^2$ denotes the two-dimensional plane, and $n = 3$, i.e., $\mathbb{R}^3$

[1] We will define a Euclidean space more formally later.

denotes the three-dimensional space. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a mapping or function from $\mathbb{R}^n$ to $\mathbb{R}$. An equivalent way of writing the function is $\boldsymbol{x} \mapsto f(\boldsymbol{x})$ or $(x_1, x_2, \ldots, x_n) \mapsto f(x_1, x_2, \ldots, x_n)$. Note that $\boldsymbol{x} \mapsto f(\boldsymbol{x})$ should be read as $\boldsymbol{x}$ maps to $f(\boldsymbol{x})$.

Now let $\boldsymbol{x}$ vary with time t.

$$\boldsymbol{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \qquad \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \begin{bmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}t} \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}x_n}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}$$

What is $\frac{\mathrm{d}f}{\mathrm{d}t}$?

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{\partial f}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}t} + \cdots + \frac{\partial f}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}t} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}$$

Now let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$, $\boldsymbol{x} \in \mathbb{R}^n$, be a *vector function* (a function that maps a vector of dimension n to a vector of dimension m), i.e.,

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

Since $\boldsymbol{x}$ is a function of time t, we have

$$\frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}t} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} \tag{1.9}$$

The matrix above is called **Jacobian matrix** or simply **Jacobian**. The dimension of the Jacobian is $m \times n$.

Remark 1.2 *For the 2R robot $m = 2$ and $n = 2$ and f_1 and f_2 are the equations relating the x and y coordinates of the end effector to the joint angles given by Equation* (1.1) *and* (1.2). *Using the definition of the Jacobian in* (1.9), *one can verify that one obtains the manipulator Jacobian. This is true not only for the 2R manipulator but for any manipulator.*

Appendix: The atan2() function

The atan2() function is defined in terms of the ordinary tan inverse (arctan()) as:

$$atan2(y, x) = \begin{cases} \arctan\frac{y}{x} & x > 0 \\ \arctan\frac{y}{x} + \pi & y \geq 0, x < 0 \\ \arctan\frac{y}{x} - \pi & y < 0, x < 0 \\ \frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases} \tag{1.10}$$

The atan2() function pays attention to the signs of the numerator and denominator and helps to find the appropriate solution based on the signs. The range of the angle returned by the atan2() function is between $(-\pi, \pi)$.