Announcements.

- new instructor starting Monday
- exam / open on webwork.
- lab 3 will be next week
- today: Conditional Probability.

outcome of "at least 5". You would not expect exactly 2/6. The long-term relative frequency of obtaining this result would approach the theoretical probability of 2/6 as the number of repetitions grows larger and larger.

This important characteristic of probability experiments is the known as the **Law of Large Numbers**: as the number of repetitions of an experiment is increased, the relative frequency obtained in the experiment tends to become closer and closer to the theoretical probability. Even though the outcomes don't happen according to any set pattern or order, overall, the long-term observed relative frequency will approach the theoretical probability. (The word **empirical** is often used instead of the word observed.) The Law of Large Numbers will be discussed again in Chapter 7.

It is important to realize that in many situations, the outcomes are not equally likely. A coin or die may be **unfair**, or **biased** . Two math professors in Europe had their statistics students test the Belgian 1 Euro coin and discovered that in 250 trials, a head was obtained 56% of the time and a tail was obtained 44% of the time. The data seem to show that the coin is not a fair coin; more repetitions would be helpful to draw a more accurate conclusion about such bias. Some dice may be biased. Look at the dice in a game you have at home; the spots on each face are usually small holes carved out and then painted to make the spots visible. Your dice may or may not be biased; it is possible that the outcomes may be affected by the slight weight differences due to the different numbers of holes in the faces. Gambling casinos have a lot of money depending on outcomes from rolling dice, so casino dice are made differently to eliminate bias. Casino dice have flat faces; the holes are completely filled with paint having the same density as the material that the dice are made out of so that each face is equally likely to occur. Later in this chapter we will learn techniques to use to work with probabilities for events that are not equally likely.

"OR" Event:
An outcome is in the event A OR B if the outcome is in A or is in B or is in both A and B. For example, let $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7, 8\}$. $A\ OR\ B = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Notice that 4 and 5 are NOT listed twice.

"AND" Event:
An outcome is in the event A AND B if the outcome is in both A and B at the same time. For example, let A and B be $\{1, 2, 3, 4, 5\}$ and $\{4, 5, 6, 7, 8\}$, respectively. Then A AND B = $\{4, 5\}$.

The **complement** of event A is denoted A' (read "A prime"). A' consists of all outcomes that are **NOT** in A. Notice that $P(A) + P(A') = 1$. For example, let $S = \{1, 2, 3, 4, 5, 6\}$ and let $A = \{1, 2, 3, 4\}$. Then, $A' = \{5, 6\}$. $P(A) = \frac{4}{6}$, $P(A') = \frac{2}{6}$, and $P(A) + P(A') = \frac{4}{6} + \frac{2}{6} = 1$

The **conditional probability** of A given B is written $P(A|B)$. $P(A|B)$ is the probability that event A will occur given that the event B has already occurred. **A conditional reduces the sample space**. We calculate the probability of A from the reduced sample space B. The formula to calculate $P(A|B)$ is

$$P(A|B) = \frac{P(AANDB)}{P(B)}$$

S = {1,2,3,4,5,6}
A = {2,3}
B = {2,4,6}

$\boxed{P(A|B) = \frac{1}{3}}$

where $P(B)$ is greater than 0.

For example, suppose we toss one fair, six-sided die. The sample space $S = \{1, 2, 3, 4, 5, 6\}$. Let A = face is 2 or 3 and B = face is even (2, 4, 6). To calculate $P(A|B)$, we count the number of outcomes 2 or 3 in the sample space $B = \{2, 4, 6\}$. Then we divide that by the number of outcomes in B (and not S).

We get the same result by using the formula. Remember that S has 6 outcomes.

$$P(A|B) = \frac{\text{P(A and B)}}{\text{P(B)}} = \frac{\text{(the number of outcomes that are 2 or 3 and even in S) / 6}}{\text{(the number of outcomes that are even in S) / 6}} = \frac{1/6}{3/6} = \boxed{\frac{1}{3}}$$

Roll a die.

Let A = "you get a prime"

B = "you get a odd number"

C = "you get a perfect square

So $S = \{1,2,3,4,5,6\}$

$A = \{2,3,5\}$ $\quad$ not $A = \{1,4,6\}$

$B = \{1,3,5\}$ $\quad$ not $B = \{2,4,6\}$

$C = \{1,4\}$ $\quad$ not $C = \{2,3,5,6\}$

Find

$P(A) = 3/6$

$P(B) = 3/6$ $\quad$ $P(\text{not } B) = 1 - 3/6 = 3/6$

$P(C) = 2/6$ $\quad$ $P(\text{not } C) = 1 - 2/6 = 4/6$

$$\boxed{P(\underset{\text{prime}}{A} \mid \underset{\text{odd}}{B}) = \frac{2}{3}} \quad \begin{matrix} \leftarrow \{3,5\} \\ \leftarrow \{1,3,5\} \end{matrix}$$

Roll a die. I tell you it's odd.

What's the chance it's also prime?

$$\boxed{P(A \mid \text{not } B) = \frac{1}{3}} \quad \begin{matrix} \leftarrow \{2\} \\ \leftarrow \{2,4,6\} \end{matrix}$$

Chance of prime given not odd

Notice that we had $P(A) = \frac{3}{6} = \frac{1}{2}$

$P(A|B) = \frac{2}{3}$

Since $P(A|B) \neq P(A)$ in this example,

we say A, B dependent.

Find

$P(C) = \frac{2}{6}$ ← $\{1, 4\}$ ← $\{1,2,3,4,5,6\}$

$P(C|B) = \frac{1}{3}$ ← $\{1\}$ ← $\{1,3,5\}$

"chance of square given odd"

Here $P(C) = \frac{2}{6} = \frac{1}{3}$

$P(C|B) = \frac{1}{3}$

So we say, C, B are independent

Just to clarify.

$$P(C) = \frac{\#C}{\#S}$$

intersection

$$P(C|B) = \frac{\#(C \text{ and } B)}{\#B}$$

for example

$C = \{1, 4\}$
$B = \{1, 3, 5\}$
$C \text{ and } B = \{1\}$

In the text we have an equivalent formula for $P(C|B)$.

Derivation

$$P(C|B) = \boxed{\frac{\#C \text{ and } B \,/\, \#S}{\#B \,/\, \#S}} = \frac{P(C \text{ and } B)}{P(B)}$$

model example

$$\boxed{\frac{1/6}{3/6}}$$

$\leftarrow P(C \text{ and } B)$
$\leftarrow P(B)$

$C = \{1, 4\}$ $\quad \#C = 2$
$B = \{1, 3, 5\}$ $\quad \#B = 3$
$C \text{ and } B = \{1\}$ $\quad \# C \text{ and } B = 1$
$S = \{1, 2, 3, 4, 5, 6\}$ $\quad \#S = 6$

$P(C) = \#C / \#S$
$= 2/6$

Conditional Probability

Formula: $P(A|B) = \frac{\# \text{ A and B}}{\# \text{ B}}$

easier to use if we can list all the outcomes

Alternate Formula: $P(A|B) = \frac{P(\text{A and B})}{P(B)}$

may have to use if just given probabilities.

Independence means $P(A|B) = P(A)$ Definition of independence

so $\frac{P(\text{A and B})}{P(B)} = P(A)$

alternate definition.

so $\boxed{P(\text{A and B}) = P(A)P(B)}$

Only if A, B independent

Usually $P(A|B) \neq P(A)$

usually $P(A|B) \neq P(B|A)$

But if $P(A|B) = P(A)$ then $P(B|A) = P(B)$

In general, $P(A|B) = \frac{P(\text{A and B})}{P(B)}$

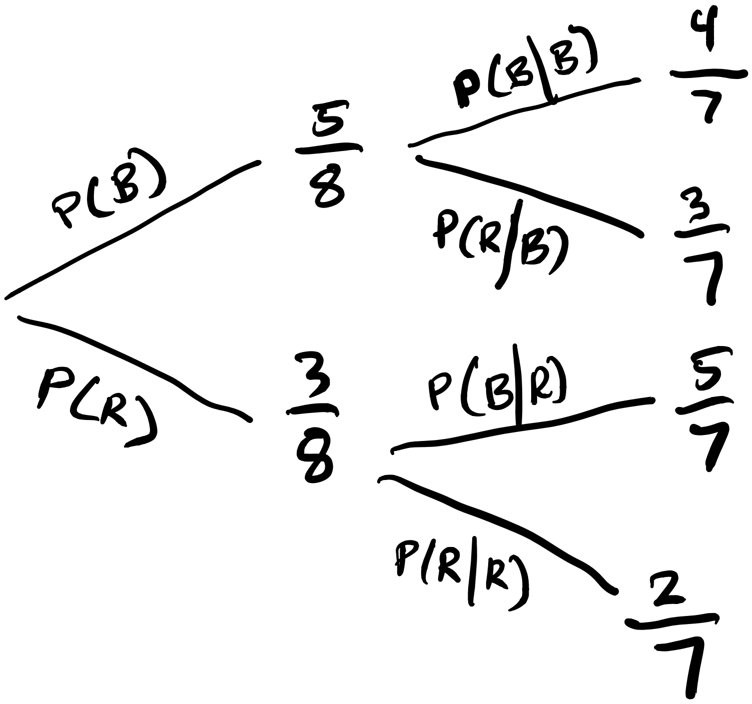
so

In general $\boxed{P(B)P(A|B) = P(\text{A and B})}$

probability tree formula

Probability trees & Multi step experiments.

You have an urn with 5 blue 3 red balls. Draw 2 balls. Whats the chance you draw one of each color.



$P(B \underset{\text{then}}{\text{and}} B) = P(B)P(B|B) = \frac{5}{8} \cdot \frac{4}{7} = \frac{20}{56}$

$P(B \underset{\text{then}}{\text{and}} R) = \frac{5}{8} \cdot \frac{3}{7} = \frac{15}{56}$

$P(R \underset{\text{then}}{\text{and}} B) = \frac{3}{8} \cdot \frac{5}{7} = \frac{15}{56}$

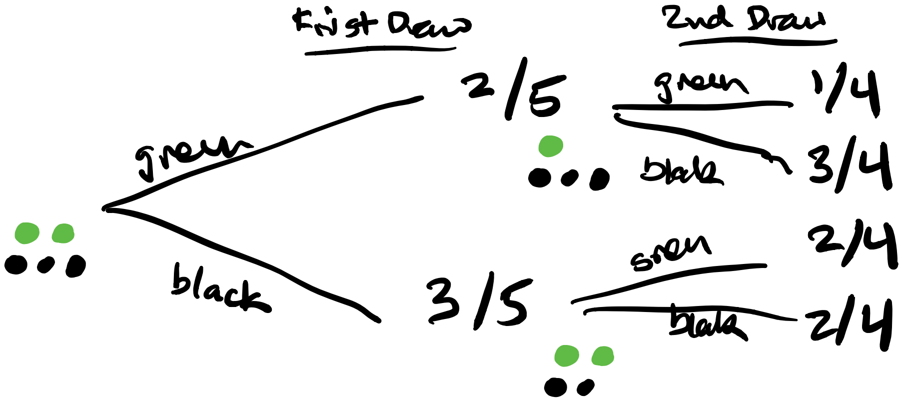
$P(R \underset{\text{then}}{\text{and}} R) = \frac{3}{8} \cdot \frac{2}{7} = \frac{6}{56}$

$P(\text{one of each color}) = \frac{15}{56} + \frac{15}{56} = \boxed{\frac{30}{56}}$

implied "without replacement" (maybe draw them at same time)

Draw 2 balls from urn with 2 green 3 black. Find the chance you draw both black.

- Multiply along branch.
- Add up leaves.



$P(GG) = \frac{2}{5} \cdot \frac{1}{4} = \frac{2}{20}$

$P(GB) = \frac{2}{5} \frac{3}{4} = \frac{6}{20}$

$P(BG) = \frac{3}{5} \frac{2}{4} = 6/20$

$P(BB) = \frac{3}{5} \frac{2}{4} = \boxed{6/20}$

Roll a die. $S = \{1,2,3,4,5,6\}$

Flip a coin. $S = \{H, T\}$

Draw a card. $S = \{$

$\diamondsuit$A	$\clubsuit$A	$\heartsuit$A	$\spadesuit$A
$\diamondsuit$2	$\clubsuit$2	$\heartsuit$2	$\spadesuit$2
$\diamondsuit$3	$\clubsuit$3	$\heartsuit$3	$\spadesuit$3
$\diamondsuit$4	$\clubsuit$4	$\heartsuit$4	$\spadesuit$4
$\diamondsuit$5	$\clubsuit$5	$\heartsuit$5	$\spadesuit$5
$\diamondsuit$6	$\clubsuit$6	$\heartsuit$6	$\spadesuit$6
$\diamondsuit$7	$\clubsuit$7	$\heartsuit$7	$\spadesuit$7
$\diamondsuit$8	$\clubsuit$8	$\heartsuit$8	$\spadesuit$8
$\diamondsuit$9	$\clubsuit$9	$\heartsuit$9	$\spadesuit$9
$\diamondsuit$10	$\clubsuit$10	$\heartsuit$10	$\spadesuit$10
$\diamondsuit$J	$\clubsuit$J	$\heartsuit$J	$\spadesuit$J
$\diamondsuit$Q	$\clubsuit$Q	$\heartsuit$Q	$\spadesuit$Q
$\diamondsuit$K	$\clubsuit$K	$\heartsuit$K	$\spadesuit$K $\}$

Sample space has 52 outcomes.

There are
- 2 colors
- 4 suits
- 13 numbers.

Draw 2 cards (without replacement).

Find the chance they're both spades.

$$P(\spadesuit\spadesuit) = \frac{\cancel{13}}{\underset{4}{\cancel{52}}} \cdot \frac{\overset{3}{\cancel{12}}}{\underset{17}{\cancel{51}}} = \boxed{\frac{1}{17}}$$

$$\underset{\text{any order}}{P(\spadesuit\heartsuit)} = \boxed{\underbrace{\frac{13}{52}}_{\spadesuit} \cdot \underbrace{\frac{13}{51}}_{\heartsuit} + \underbrace{\frac{13}{52}}_{\heartsuit} \cdot \underbrace{\frac{13}{51}}_{\spadesuit}} = \frac{2 \cdot 13 \cdot 13}{52 \cdot 51}$$

put the card back in

Draw 2 cards with replacement. Find $P(\spadesuit\spadesuit)$

$$P(\spadesuit\spadesuit) = \frac{\cancel{13}^{\,1}}{\cancel{52}_{\,4}} \cdot \frac{\cancel{13}^{\,1}}{\cancel{52}_{\,4}} = \boxed{\frac{1}{16}}$$

Notice

It's slightly harder to draw 2 spades without replacement (1/17 chance) than with replacement (1/16 chance).

harder $\downarrow$ lower probability

Suppose 80% of my flowers are violets,
8% are white violets.
83% are either violets or not white.

given.

What percent of my flowers are either white or not violets?
What percent of my white flowers are not violets?

$$P(V) = 0.80$$
$$P(W \text{ and } V) = 0.08$$
$$P(V \text{ or not } W) = 0.83$$

$$P(W \text{ or not } V) = ?$$
$$P(\text{not } V \mid W) = ?$$

	not V	V	
not W	0.03	0.72	
W	0.17	0.08	0.25
		+ 0.80	

$0.80 - 0.08 = 0.72$

$x + 0.72 + 0.08 = 0.83$

$1 - 0.83 = 0.17$

We've solved the venn diagram. Let's find the requested probabilities

$$P(W \text{ or not } V) = 0.03 + 0.17 + 0.08 = \boxed{0.28}$$

$$P(\text{not } V \mid W) = \frac{P(\text{not } V \text{ and } W)}{P(W)} = \frac{0.17}{0.25} = \boxed{0.68}$$

3.8 Summary of Formulas[8]

Formula 3.1: Complement
If A and A' are complements then $P(A) + P(A') = 1$

Formula 3.2: Addition Rule
$P(A \text{ OR } B) = P(A) + P(B) - P(A \text{ AND } B)$

Formula 3.3: Mutually Exclusive
If A and B are mutually exclusive then $P(A \text{ AND } B) = 0$; so $P(A \text{ OR } B) = P(A) + P(B)$.

Formula 3.4: Multiplication Rule

- $P(A \text{ AND } B) = P(B)P(A|B)$
- $P(A \text{ AND } B) = P(A)P(B|A)$

Formula 3.5: Independence
If A and B are independent then:

- $P(A|B) = P(A)$
- $P(B|A) = P(B)$
- $P(A \text{ AND } B) = P(A)P(B)$

[8]This content is available online at <http://cnx.org/content/m16843/1.5/>.