p172

Rule: Cross Product Calculated by a Determinant

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors. Then the cross product $\mathbf{u} \times \mathbf{v}$ is given by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}.$$

p170

Theorem 2.7: Magnitude of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors, and let θ be the angle between them. Then, $\| \mathbf{u} \times \mathbf{v} \| = \| \mathbf{u} \| \cdot \| \mathbf{v} \| \cdot \sin\theta$.

Example 2.35

p169

p171

Calculating the Cross Product

$i = \langle 1,0,0 \rangle$
$j = \langle 0,1,0 \rangle$
$k = \langle 0,0,1 \rangle$

$0i + 4j + 0k = 4j$

Use **Properties of the Cross Product** to find the magnitude of the cross product of $\mathbf{u} = \langle 0, 4, 0 \rangle$ and $\mathbf{v} = \langle 0, 0, -3 \rangle$. $-3k$

$\|i\times k\|$ is defined to be the signed area of this parallelogram (it's a unit square in this case)

$$u \times v = (4j) \times (-3k) = -12(j \times k)$$
$$= -12(i)$$
$$= \boxed{-12i}$$

$j \times k = i$

$k \times j = -i$

remember the right hand rule

came directly from right hand rule

p174

Theorem 2.8: Area of a Parallelogram

If we locate vectors $\mathbf{u}$ and $\mathbf{v}$ such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\| \mathbf{u} \times \mathbf{v} \|$ (**Figure 2.57**).

we started our definition of cross product here so we'll call this Definition 2.8

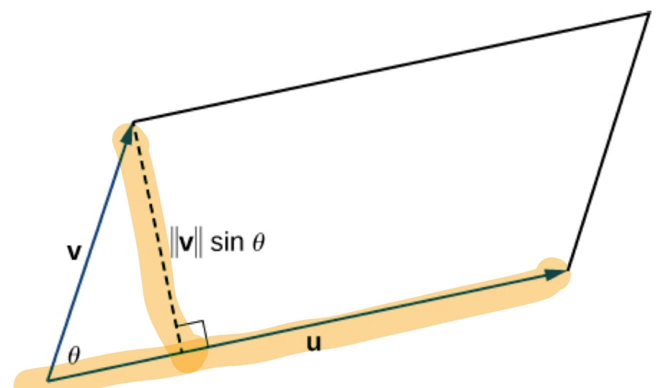


Figure 2.57 The parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ has base $\| \mathbf{u} \|$ and height $\| \mathbf{v} \| \sin\theta$.

$$|u \times v| = |u|\,|v|\,|\sin\theta|$$

p 174

Example 2.39

Finding the Area of a Triangle

$R-P=\vec{PR}$

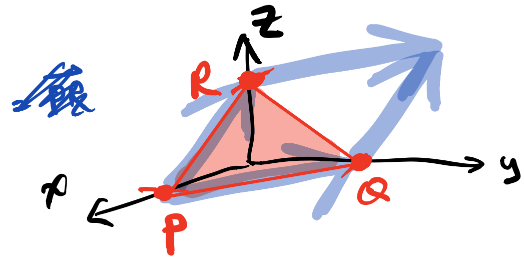
solve for $P+\vec{PR}=R$

Let $P=(1,0,0)$, $Q=(0,1,0)$, and $R=(0,0,1)$ be the vertices of a triangle (**Figure 2.58**). Find its area.

~~Let's use~~

$\vec{PR} = R-P = (0,0,1)-(1,0,0) = \langle -1,0,1\rangle = -i+k$

$\vec{PQ} = Q-P = (0,1,0)-(1,0,0) = \langle -1,1,0\rangle = -i+j$



$|\vec{PR}\times\vec{PQ}|$ = area of parallelogram

divide by 2 to get area of triangle

$$\vec{PR}\times\vec{PQ} = \langle -1,0,1\rangle\times\langle -1,1,0\rangle = (-i+k)\times(-i+j)$$

$$= -i\times(-i+j) + k\times(-i+j)$$

$$= -i\times -i + -i\times j + k\times -i + k\times j$$

$$= 0 - k - k\times i - i$$

$$= -k - j - i$$

Aside

$a\times a = -a\times a$

so $a\times a = 0$

$$|\vec{PR}\times\vec{PQ}| = |\langle -1,-1,-1\rangle|$$

$$= \sqrt{1^2+1^2+1^2} = \sqrt{3} = \text{area of parallelogram}$$

$$\boxed{\text{area of } \Delta \text{ is } \sqrt{3}/2}$$

(Textbook uses trig)

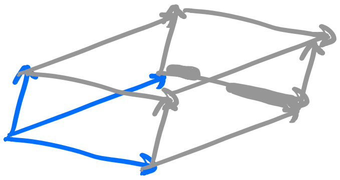
p179

Example 2.42

Using the Triple Scalar Product

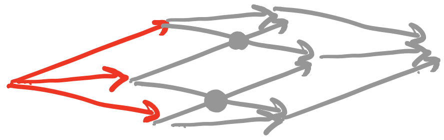
Use the triple scalar product to show that vectors $\mathbf{u} = \langle 2, 0, 5 \rangle$, $\mathbf{v} = \langle 2, 2, 4 \rangle$, and $\mathbf{w} = \langle 1, -1, 3 \rangle$ are coplanar—that is, show that these vectors lie in the same plane.

not coplaner



vol $\neq 0$.

coplaner



volume of parallelpiped $= 0$

$$\begin{matrix} 2 & 0 & 5 \\ 2 & 2 & 4 \\ 1 & -1 & 3 \end{matrix}$$

signed vol of uvw parallelpiped $= \begin{vmatrix} u \\ v \\ w \end{vmatrix}$

$$\begin{vmatrix} 2 & 0 & 5 \\ 2 & 2 & 4 \\ 1 & -1 & 3 \end{vmatrix} = 2(6 - -4) - 0(\quad) + 5(-2 - 2)$$

$$= 2(10) \quad 0 \quad + 5(-4)$$

$$= 20 - 20 =$$

$$= 0$$

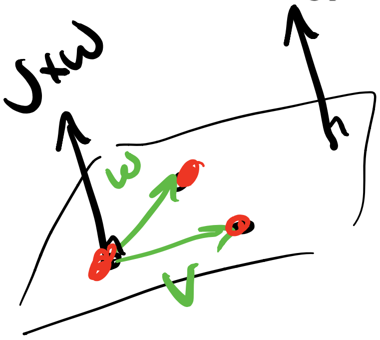
Yes these vectors lie on the same plane.

Example 2.43

Finding an Orthogonal Vector

p 180

Only a single plane can pass through any set of three noncolinear points. Find a vector orthogonal to the plane containing points $P = (9,\ -3,\ -2)$, $Q = (1,\ 3,\ 0)$, and $R = (-2,\ 5,\ 0)$.



"Easy" way to find a vector perpendicular to 2 others — cross product

How do we make v & w using P, Q, R

vectors "on" or "parallel to" plane

$$R - Q = (-2,5,0) - (1,3,0) = \langle -3, 2, 0 \rangle = v$$

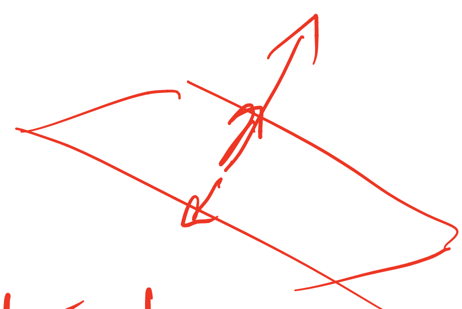
$$P - Q = (9,-3,-2) - (1,3,0) = \langle 8, -6, -2 \rangle = w$$

$$v \times w = \begin{vmatrix} i & j & k \\ -3 & 2 & 0 \\ 8 & -6 & -2 \end{vmatrix}$$

$$= i(-4-0) - j(6-0) + k(18-16)$$

$$= \boxed{-4i - 6j + 2k}$$

$$= -2(2i + 3j - k)$$



$$\boxed{2i + 3j - k}$$ also $\perp$

Example 2.43

Finding an Orthogonal Vector

Only a single plane can pass through any set of three noncolinear points. Find a vector orthogonal to the plane containing points $P = (9, -3, -2)$, $Q = (1, 3, 0)$, and $R = (-2, 5, 0)$.

Solution

The plane must contain vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$:

$$\overrightarrow{PQ} = \langle 1-9, 3-(-3), 0-(-2) \rangle = \langle -8, 6, 2 \rangle$$
$$\overrightarrow{QR} = \langle -2-1, 5-3, 0-0 \rangle = \langle -3, 2, 0 \rangle.$$

The cross product $\overrightarrow{PQ} \times \overrightarrow{QR}$ produces a vector orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$. Therefore, the cross product is orthogonal to the plane that contains these two vectors:

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{QR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & 6 & 2 \\ -3 & 2 & 0 \end{vmatrix} \\ &= 0\mathbf{i} - 6\mathbf{j} - 16\mathbf{k} - (-18\mathbf{k} + 4\mathbf{i} + 0\mathbf{j}) \\ &= -4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}.\end{aligned}$$

We have seen how to use the triple scalar product and how to find a vector orthogonal to a plane. Now we apply the cross product to real-world situations.

Sometimes a force causes an object to rotate. For example, turning a screwdriver or a wrench creates this kind of rotational effect, called torque.

Definition

Torque, τ (the Greek letter *tau*), measures the tendency of a force to produce rotation about an axis of rotation. Let $\mathbf{r}$ be a vector with an initial point located on the axis of rotation and with a terminal point located at the point where the force is applied, and let vector $\mathbf{F}$ represent the force. Then torque is equal to the cross product of $\mathbf{r}$ and $\mathbf{F}$:

$$\tau = \mathbf{r} \times \mathbf{F}.$$

See **Figure 2.61**.

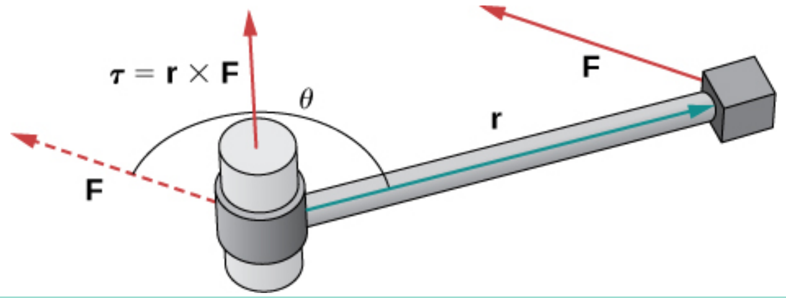


Figure 2.61 Torque measures how a force causes an object to rotate.

Think about using a wrench to tighten a bolt. The torque τ applied to the bolt depends on how hard we push the wrench (force) and how far up the handle we apply the force (distance). The torque increases with a greater force on the wrench at a greater distance from the bolt. Common units of torque are the newton-meter or foot-pound. Although torque is dimensionally equivalent to work (it has the same units), the two concepts are distinct. Torque is used specifically in the context of rotation, whereas work typically involves motion along a line.

Example 2.44

Evaluating Torque

A bolt is tightened by applying a force of 6 N to a 0.15-m wrench (**Figure 2.62**). The angle between the wrench and the force vector is $40°$. Find the magnitude of the torque about the center of the bolt. Round the answer to two decimal places.

Recall

$|u \times v| = |u||v||\sin\theta|$

p174

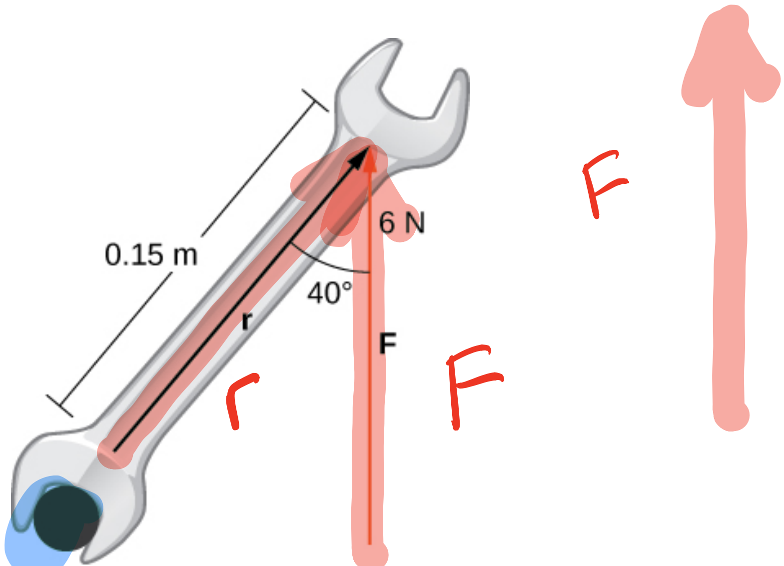


Figure 2.62 Torque describes the twisting action of the wrench.

torque is a vector: magnitude of twisting force & direction screw is pointing

Solution

Substitute the given information into the equation defining torque:

$$\|\tau\| = \|\mathbf{r} \times \mathbf{F}\| = |r||F||\sin\theta|$$

(definition)

$$= 0.15\text{m} * 6\text{N} * \sin 40° = \boxed{0.9 \sin 40° \text{ Nm}}$$

2.42 Calculate the force required to produce $15\ \text{N} \cdot \text{m}$ torque at an angle of $30\,°$ from a 150-cm rod.

2.5 | Equations of Lines and Planes in Space

Learning Objectives

2.5.1 Write the vector, parametric, and symmetric of a line through a given point in a given direction, and a line through two given points.
2.5.2 Find the distance from a point to a given line.
2.5.3 Write the vector and scalar equations of a plane through a given point with a given normal.
2.5.4 Find the distance from a point to a given plane.
2.5.5 Find the angle between two planes.

By now, we are familiar with writing equations that describe a line in two dimensions. To write an equation for a line, we must know two points on the line, or we must know the direction of the line and at least one point through which the line passes. In two dimensions, we use the concept of slope to describe the orientation, or direction, of a line. In three dimensions, we describe the direction of a line using a vector parallel to the line. In this section, we examine how to use equations to describe lines and planes in space.

Equations for a Line in Space

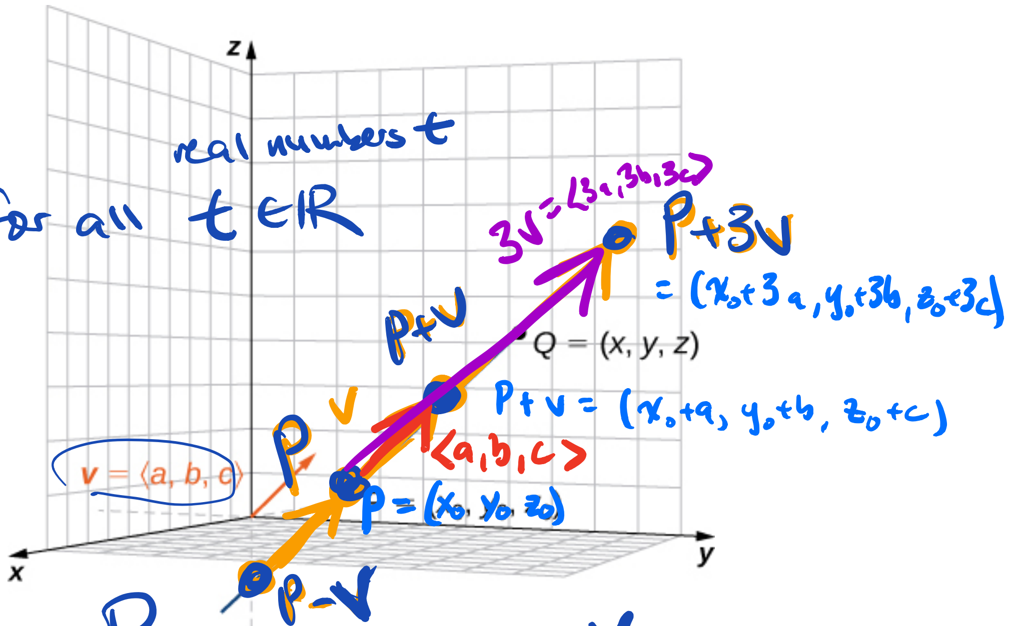
Let's first explore what it means for two vectors to be parallel. Recall that parallel vectors must have the same or opposite directions. If two nonzero vectors, $\mathbf{u}$ and $\mathbf{v}$, are parallel, we claim there must be a scalar, k, such that $\mathbf{u} = k\mathbf{v}$. If $\mathbf{u}$ and $\mathbf{v}$ have the same direction, simply choose $k = \frac{\|\mathbf{u}\|}{\|\mathbf{v}\|}$. If $\mathbf{u}$ and $\mathbf{v}$ have opposite directions, choose $k = -\frac{\|\mathbf{u}\|}{\|\mathbf{v}\|}$. Note that the converse holds as well. If $\mathbf{u} = k\mathbf{v}$ for some scalar k, then either $\mathbf{u}$ and $\mathbf{v}$ have the same direction $(k > 0)$ or opposite directions $(k < 0)$, so $\mathbf{u}$ and $\mathbf{v}$ are parallel. Therefore, two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} = k\mathbf{v}$ for some scalar k. By convention, the zero vector $\mathbf{0}$ is considered to be parallel to all vectors.

As in two dimensions, we can describe a line in space using a point on the line and the direction of the line, or a parallel vector, which we call the **direction vector** (Figure 2.63). Let L be a line in space passing through point $P(x_0, y_0, z_0)$. Let $\mathbf{v} = \langle a, b, c \rangle$ be a vector parallel to L. Then, for any point on line $Q(x, y, z)$, we know that $\overrightarrow{PQ}$ is parallel to $\mathbf{v}$. Thus, as we just discussed, there is a scalar, t, such that $\overrightarrow{PQ} = t\mathbf{v}$, which gives

$$\begin{aligned} \overrightarrow{PQ} &= t\mathbf{v} \\ \langle x - x_0, y - y_0, z - z_0 \rangle &= t\langle a, b, c \rangle \\ \langle x - x_0, y - y_0, z - z_0 \rangle &= \langle ta, tb, tc \rangle . \end{aligned} \quad (2.11)$$

$\boxed{(x,y,z) = P + t\mathbf{v}}$ for all $t \in \mathbb{R}$ real numbers t

describes a line.



Equati

Figure 2.63 Vector $\mathbf{v}$ is the direction vector for $\overrightarrow{PQ}$.

$$(x, y, z) = \underbrace{(x_0, y_0, z_0)}_{P} + t\underbrace{\langle a, b, c \rangle}_{\mathbf{v}}$$

$$\boxed{(x, y, z) = (x_0 + ta,\ y_0 + tb,\ z_0 + tc)}$$

three equations in one

Using vector operations, we can rewrite **Equation 2.11** as

$$\begin{aligned}\langle x - x_0, y - y_0, z - z_0 \rangle &= \langle ta, tb, tc \rangle \\ \langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle &= t \langle a, b, c \rangle \\ \langle x, y, z \rangle &= \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle .\end{aligned}$$

$$\begin{cases} x = x_0 + ta, \\ y = y_0 + tb, \\ z = z_0 + tc \end{cases}$$

Setting $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, we now have the **vector equation of a line**:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}. \tag{2.12}$$

Equating components, **Equation 2.11** shows that the following equations are simultaneously true: $x - x_0 = ta$, $y - y_0 = tb$, and $z - z_0 = tc$. If we solve each of these equations for the component variables x, y, and z, we get a set of equations in which each variable is defined in terms of the parameter t and that, together, describe the line. This set of three equations forms a set of **parametric equations of a line**:

$$x = x_0 + ta \quad y = y_0 + tb \quad z = z_0 + tc.$$

If we solve each of the equations for t assuming a, b, and c are nonzero, we get a different description of the same line:

$x - x_0 = ta$ $\quad t = \dfrac{y - y_0}{b}$ $\quad t = \dfrac{z - z_0}{c}$

$\dfrac{x - x_0}{a} = t$

Because each expression equals t, they all have the same value. We can set them equal to each other to create **symmetric equations of a line**:

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.}$$

symmetric equ of line

We summarize the results in the following theorem.

Theorem 2.11: Parametric and Symmetric Equations of a Line

A line L parallel to vector $\mathbf{v} = \langle a, b, c \rangle$ and passing through point $P(x_0, y_0, z_0)$ can be described by the following parametric equations:

$$x = x_0 + ta, \ y = y_0 + tb, \ \text{and } z = z_0 + tc. \tag{2.13}$$

If the constants a, b, and c are all nonzero, then L can be described by the symmetric equation of the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{2.14}$$

The parametric equations of a line are not unique. Using a different parallel vector or a different point on the line leads to a different, equivalent representation. Each set of parametric equations leads to a related set of symmetric equations, so it follows that a symmetric equation of a line is not unique either.

Example 2.45

Equations of a Line in Space

Find parametric and symmetric equations of the line passing through points $(1, 4, -2)$ and $(-3, 5, 0)$.

Solution

First, identify a vector parallel to the line:

p187

Example 2.45

Equations of a Line in Space

Find parametric and symmetric equations of the line passing through points $(1, 4, -2)$ and $(-3, 5, 0)$.

start on Monday

Announcements

- reading is due ~~2.21~~ 2.28
- lab is due 2.28
- I haven't posted webwork yet, webwork will NOT be due ~~2.21~~ then