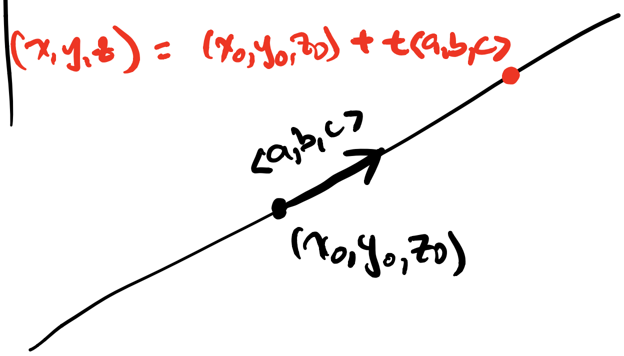
Last time

- line in space

$$\rightarrow \quad \underset{(x,y,z)}{r} = \overset{(x_0,y_0,z_0)}{r_0} + t\,\underset{\langle a,b,c\rangle}{v}$$

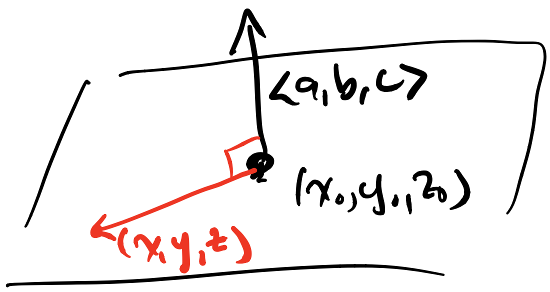
$$\rightarrow \quad \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

$$\rightarrow \quad \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$



- plane

$$0 = \left((x,y,z) - (x_0,y_0,z_0) \right) \cdot \langle a,b,c \rangle$$



$$(r - r_0) \cdot v = 0$$

$$r \cdot v = r_0 \cdot v \qquad \text{point-vector equation of a plane}$$

$$ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{:= d} \qquad \text{also equation of plane.}$$

- Today - find lines & planes
  & distances & nearest points.

- Quadrics

- Next lecture: arclength & speed
  & higher limits

(no lecture on - vector functions
only reading - vector functions' derivatives
& hw - multivariate fns
- partials)

Using vector operations, we can rewrite **Equation 2.11** as

$$\begin{aligned}\langle x - x_0, y - y_0, z - z_0 \rangle &= \langle ta, tb, tc \rangle \\ \langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle &= t \langle a, b, c \rangle \\ \langle x, y, z \rangle &= \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle .\end{aligned}$$

Setting $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, we now have the **vector equation of a line**:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}. \tag{2.12}$$

Equating components, **Equation 2.11** shows that the following equations are simultaneously true: $x - x_0 = ta$, $y - y_0 = tb$, and $z - z_0 = tc$. If we solve each of these equations for the component variables x, y, and z, we get a set of equations in which each variable is defined in terms of the parameter t and that, together, describe the line. This set of three equations forms a set of **parametric equations of a line**:

$$x = x_0 + ta \quad y = y_0 + tb \quad z = z_0 + tc.$$

If we solve each of the equations for t assuming a, b, and c are nonzero, we get a different description of the same line:

$$\frac{x - x_0}{a} = t \quad \frac{y - y_0}{b} = t \quad \frac{z - z_0}{c} = t.$$

Because each expression equals t, they all have the same value. We can set them equal to each other to create **symmetric equations of a line**:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

We summarize the results in the following theorem.

Theorem 2.11: Parametric and Symmetric Equations of a Line

A line L parallel to vector $\mathbf{v} = \langle a, b, c \rangle$ and passing through point $P(x_0, y_0, z_0)$ can be described by the following parametric equations:

$$x = x_0 + ta, \; y = y_0 + tb, \; \text{and } z = z_0 + tc. \tag{2.13}$$

If the constants a, b, and c are all nonzero, then L can be described by the symmetric equation of the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{2.14}$$

The parametric equations of a line are not unique. Using a different parallel vector or a different point on the line leads to a different, equivalent representation. Each set of parametric equations leads to a related set of symmetric equations, so it follows that a symmetric equation of a line is not unique either.

Example 2.45

Equations of a Line in Space

$x_0 \; y_0 \; z_0.$

Find parametric and symmetric equations of the line passing through points $(1, 4, -2)$ and $(-3, 5, 0)$.

$$\begin{array}{r} -3 \;\; 5 \;\; 0 \\ - \;\; \underline{} \\ 4 \;\; -1 \;\; -2 \\ a \;\; b \;\; c \end{array}$$

$$\frac{x+3}{4} = \frac{y-5}{-1} = \frac{z}{-2}$$

Solution

First, identify a vector parallel to the line:

$$\begin{aligned} x &= -3 + 4t \\ y &= 5 - t \\ z &= -2t \end{aligned}$$

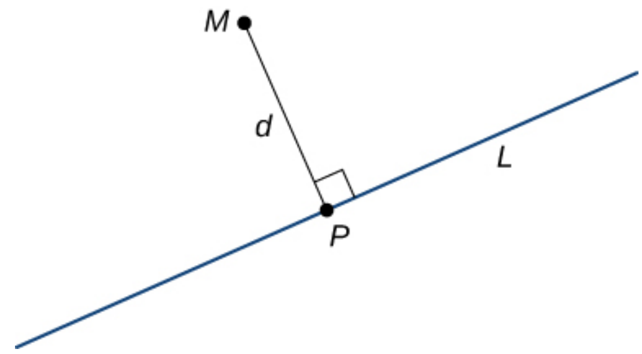


Figure 2.64 The distance from point M to line L is the length of $\overline{MP}$.

When we're looking for the distance between a line and a point in space, **Figure 2.64** still applies. We still define the distance as the length of the perpendicular line segment connecting the point to the line. In space, however, there is no clear way to know which point on the line creates such a perpendicular line segment, so we select an arbitrary point on the line and use properties of vectors to calculate the distance. Therefore, let P be an arbitrary point on line L and let $\mathbf{v}$ be a direction vector for L (**Figure 2.65**).

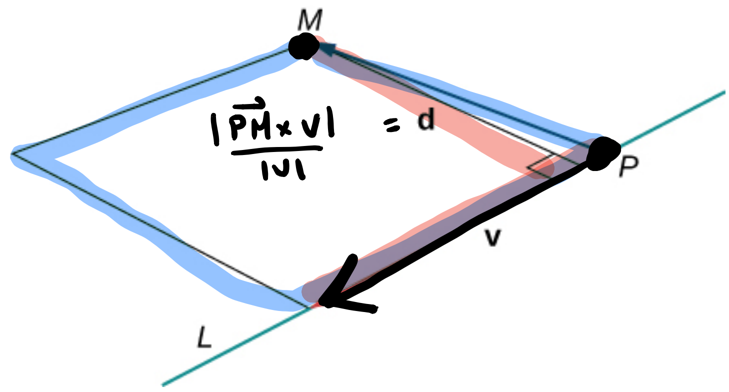


Figure 2.65 Vectors $\overrightarrow{PM}$ and $\mathbf{v}$ form two sides of a parallelogram with base $\|\mathbf{v}\|$ and height d, which is the distance between a line and a point in space.

By **Area of a Parallelogram**, vectors $\overrightarrow{PM}$ and $\mathbf{v}$ form two sides of a parallelogram with area $\|\overrightarrow{PM} \times \mathbf{v}\|$. Using a formula from geometry, the area of this parallelogram can also be calculated as the product of its base and height:

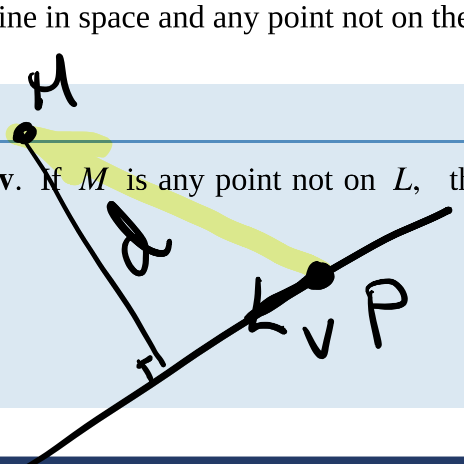
$$\|\overrightarrow{PM} \times \mathbf{v}\| = \|\mathbf{v}\| d.$$

We can use this formula to find a general formula for the distance between a line in space and any point not on the line.

Theorem 2.12: Distance from a Point to a Line

Let L be a line in space passing through point P with direction vector $\mathbf{v}$. If M is any point not on L, then the distance from M to L is

$$d = \frac{\|\overrightarrow{PM} \times \mathbf{v}\|}{\|\mathbf{v}\|}.$$



Example 2.47

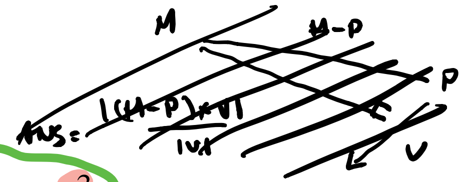
Calculating the Distance from a Point to a Line

M − P =

$\vec{PM} = \langle -2, 2, 0\rangle$

$v = \langle 4, 2, 1\rangle$

P = (3, −1, 3)



Find the distance between the point $M = (1, 1, 3)$ and line $\frac{x-3}{4} = \frac{y+1}{2} = z - 3.$

M − P = (1−3, 1+1, 3−3) = ⟨−2, 2, 0⟩

$\frac{1-3}{4}$ $\frac{1+1}{2}$ $\frac{1-3}{1}$

Solution

From the symmetric equations of the line, we know that vector $\mathbf{v} = \langle 4, 2, 1 \rangle$ is a direction vector for the line. Setting the symmetric equations of the line equal to zero, we see that point $P(3, -1, 3)$ lies on the line. Then,

$$\vec{PM} = \langle 1-3, 1-(-1), 3-3 \rangle = \langle -2, 2, 0 \rangle.$$

To calculate the distance, we need to find $\vec{PM} \times \mathbf{v}$:

ANS = $\frac{|\vec{PM} \times v|}{|v|}$

$$\vec{PM} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 0 \\ 4 & 2 & 1 \end{vmatrix}$$
$$= (2-0)\mathbf{i} - (-2-0)\mathbf{j} + (-4-8)\mathbf{k}$$
$$= 2i + 2j - 12k$$

Therefore, the distance between the point and the line is (**Figure 2.66**)

ANS = $\boxed{d = \frac{\| \vec{PM} \times \mathbf{v} \|}{\| \mathbf{v} \|}}$

$$= \frac{\sqrt{2^2 + 2^2 + 12^2}}{\sqrt{4^2 + 2^2 + 1^2}}$$
$$= \frac{2\sqrt{38}}{\sqrt{21}}.$$

$\vec{PM} \times v = 2i + 2j - 12k$

$|2i + 2j - 12k|$

$= \sqrt{4 + 4 + 144}$

$= \sqrt{152}$

$v = \langle 4, 2, 1\rangle$

$|v| = |\langle 4, 2, 1\rangle| = \sqrt{16+4+1}$

$= \sqrt{21}$

$\boxed{\frac{\sqrt{152}}{\sqrt{21}}}$

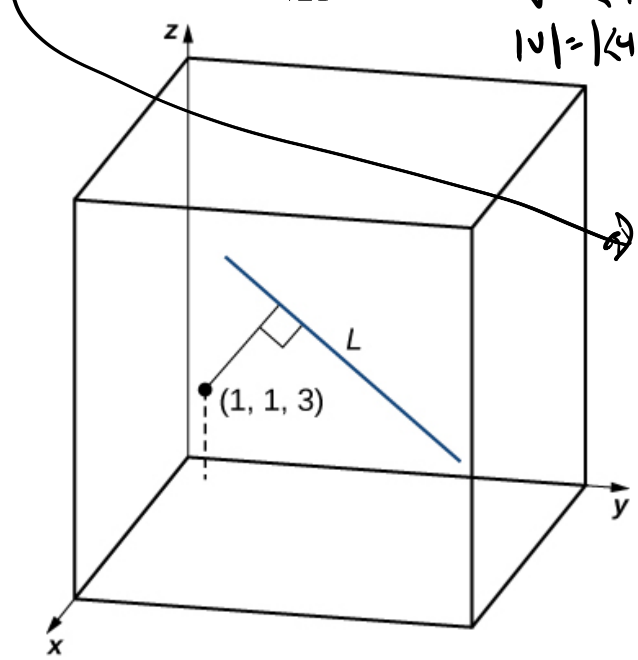


Figure 2.66 Point $(1, 1, 3)$ is approximately 2.7 units from the line with symmetric equations $\frac{x-3}{4} = \frac{y+1}{2} = z - 3.$

2.45 Find the distance between point $(0, 3, 6)$ and the line with parametric equations $x = 1 - t, y = 1 + 2t, z = 5 + 3t.$

Relationships between Lines

Given two lines in the two-dimensional plane, the lines are equal, they are parallel but not equal, or they intersect in a single point. In three dimensions, a fourth case is possible. If two lines in space are not parallel, but do not intersect, then the lines are said to be **skew lines** (Figure 2.67).

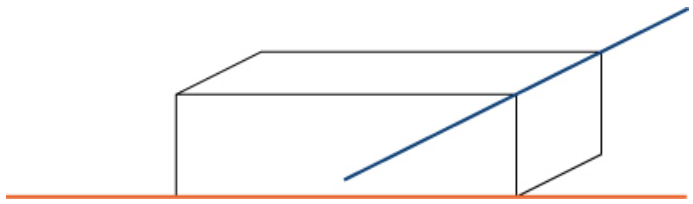


Figure 2.67 In three dimensions, it is possible that two lines do not cross, even when they have different directions.

To classify lines as parallel but not equal, equal, intersecting, or skew, we need to know two things: whether the direction vectors are parallel and whether the lines share a point (Figure 2.68).

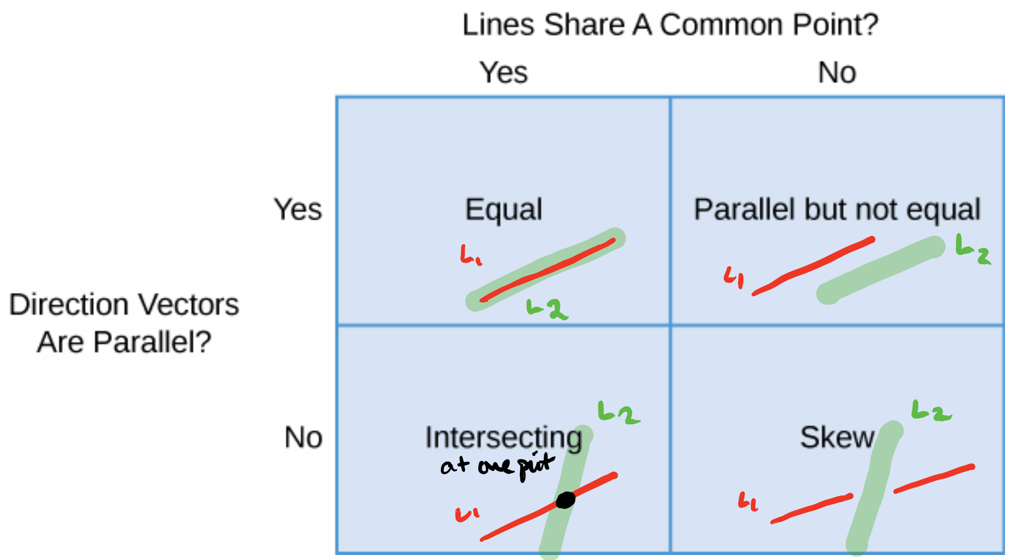


Figure 2.68 Determine the relationship between two lines based on whether their direction vectors are parallel and whether they share a point.

Example 2.48

Classifying Lines in Space

For each pair of lines, determine whether the lines are equal, parallel but not equal, skew, or intersecting.

parametric

a. $L_1 : x = 2s - 1,\ y = s - 1,\ z = s - 4$
$L_2 : x = t - 3,\ y = 3t + 8,\ z = 5 - 2t$

a b c
2 1 1
1 3 -2
not parallel

$2s - 1 = t - 3 \rightarrow 2(3t+9) = t - 3$
$s - 1 = 3t + 8 \rightarrow s = 3t + 9$
$\boxed{s - 4 = 5 - 2t}$
$t = -21/5$, $5t = -21$
$s = 3(-21/5) + 9$
$s = -63/5 + 9$
$\left(-\frac{63}{5}\right) - 4 = 5 - 2\left(-\frac{21}{5}\right)$
$-63 - 20 = 25 - 2(-21)$
$-63 - 20 \neq 25 + 42$
not intersecting $\Rightarrow$ $\boxed{\text{skew}}$ a.

symmetric form

b. $L_1 :\ x = -y = z$
$L_2 : \frac{x-3}{2} = y = z - 2$

a b c
1 -1 1
2 1 1
Not parallel

L_1: $x = 0 + 1s$, $y = 0 - 1s$, $z = 0 + 1s$
L_2: $x = 3 + 2t$, $y = 0 + 1t$, $z = 2 + 1t$

c. $L_1 : x = 6s - 1,\ y = -2s,\ z = 3s + 1$
$L_2 : \frac{x-4}{6} = \frac{y+3}{-2} = \frac{z-1}{3}$

$s = 3 + 2t \rightarrow -t = 3 + 2t \rightarrow -3 = 3t \rightarrow \boxed{t = -1}$
$-s = t$
$s = 2 + t$ $\boxed{s = 1}$
$(1) = 2 + (-1)$? yes ✓✓ check

Solution

b. $\boxed{\text{Thus intersect at one point}}$ — does intersect.

a. Line L_1 has direction vector $\mathbf{v_1} = \langle 2, 1, 1 \rangle$; line L_2 has direction vector $\mathbf{v_2} = \langle 1, 3, -2 \rangle$. Because the direction vectors are not parallel vectors, the lines are either intersecting or skew. To point that does not lie on the line. These characterizations arise naturally from the idea that a plane is determined by three points. Perhaps the most surprising characterization of a plane is actually the most useful.

Imagine a pair of orthogonal vectors that share an initial point. Visualize grabbing one of the vectors and twisting it. As you twist, the other vector spins around and sweeps out a plane. Here, we describe that concept mathematically. Let $\mathbf{n} = \langle a, b, c \rangle$ be a vector and $P = (x_0, y_0, z_0)$ be a point. Then the set of all points $Q = (x, y, z)$ such that $\overrightarrow{PQ}$ is orthogonal to $\mathbf{n}$ forms a plane (**Figure 2.69**). We say that $\mathbf{n}$ is a **normal vector**, or perpendicular to the plane. Remember, the dot product of orthogonal vectors is zero. This fact generates the **vector equation of a plane**: $\mathbf{n} \cdot \overrightarrow{PQ} = 0$. Rewriting this equation provides additional ways to describe the plane:

$$\begin{aligned} \mathbf{n} \cdot \overrightarrow{PQ} &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0. \end{aligned}$$

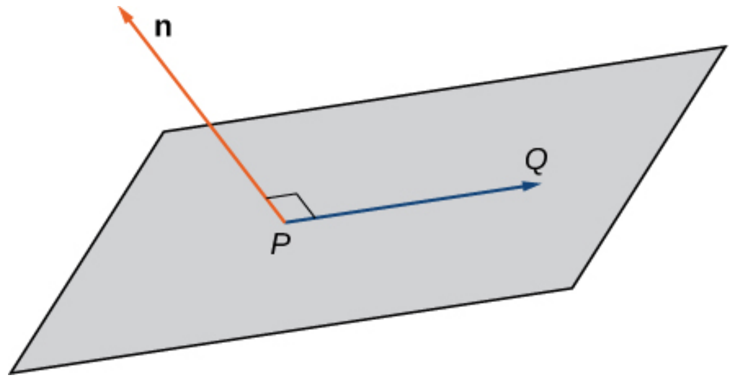


Figure 2.69 Given a point P and vector $\mathbf{n}$, the set of all points Q with $\overrightarrow{PQ}$ orthogonal to $\mathbf{n}$ forms a plane.

Definition

Given a point P and vector $\mathbf{n}$, the set of all points Q satisfying the equation $\mathbf{n} \cdot \overrightarrow{PQ} = 0$ forms a plane. The equation

$$\mathbf{n} \cdot \overrightarrow{PQ} = 0 \tag{2.17}$$

is known as the vector equation of a plane.

The **scalar equation of a plane** containing point $P = (x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{2.18}$$

This equation can be expressed as $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$. This form of the equation is sometimes called the **general form of the equation of a plane**.

As described earlier in this section, any three points that do not all lie on the same line determine a plane. Given three such points, we can find an equation for the plane containing these points.

Example 2.49

Writing an Equation of a Plane Given Three Points in the Plane

Write an equation for the plane containing points $P = (1, 1, -2)$, $Q = (0, 2, 1)$, and $R = (-1, -1, 0)$ in both standard and general forms.

$Q - P = (0,2,1) - (1,1,-2) = \langle -1, 1, 3 \rangle = u$

$R - Q = (-1,-1,0) - (0,2,1) = \langle -1, -3, -1 \rangle = v$

$$u \times v = \begin{vmatrix} i & j & k \\ -1 & 1 & 3 \\ -1 & -3 & -1 \end{vmatrix} = (-1+9)i - (1+3)j + (3+1)k$$
$$= \boxed{8i - 4j + 4k = n}$$

Solution

To write an equation for a plane, we must find a normal vector for the plane. We start by identifying two vectors in the plane:

$$\begin{aligned} \overrightarrow{PQ} &= \langle\, 0-1,\, 2-1,\, 1-(-2)\,\rangle = \langle\, -1,\, 1,\, 3\,\rangle \\ \overrightarrow{QR} &= \langle\, -1-0,\, -1-2,\, 0-1\,\rangle = \langle\, -1,\, -3,\, -1\,\rangle. \end{aligned}$$

The cross product $\overrightarrow{PQ} \times \overrightarrow{QR}$ is orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, so it is normal to the plane that contains these two vectors:

$$\begin{aligned} \mathbf{n} &= \overrightarrow{PQ} \times \overrightarrow{QR} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 3 \\ -1 & -3 & -1 \end{vmatrix} \\ &= (-1+9)\mathbf{i} - (1+3)\mathbf{j} + (3+1)\mathbf{k} \\ &= 8\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}. \end{aligned}$$

Thus, $\mathbf{n} = \langle\, 8,\, -4,\, 4\,\rangle$, and we can choose any of the three given points to write an equation of the plane:

plays the role of P_0

$$\begin{aligned} 8(x-1) - 4(y-1) + 4(z+2) &= 0 \\ 8x - 4y + 4z + 4 &= 0. \end{aligned}$$

$R = (-1, -1, 0)$

$n \cdot R = -8 + 4 + 0 = -4$

$$\boxed{8x - 4y + 4z = -4}$$

The scalar equations of a plane vary depending on the normal vector and point chosen.

Example 2.50

Writing an Equation for a Plane Given a Point and a Line

Find an equation of the plane that passes through point $(1, 4, 3)$ and contains the line given by $x = \dfrac{y-1}{2} = z + 1.$

Solution

Symmetric equations describe the line that passes through point $(0, 1, -1)$ parallel to vector $\mathbf{v}_1 = \langle\, 1,\, 2,\, 1\,\rangle$ (see the following figure). Use this point and the given point, $(1, 4, 3),$ to identify a second vector parallel to the plane:

$$\mathbf{v}_2 = \langle\, 1-0,\, 4-1,\, 3-(-1)\,\rangle = \langle\, 1,\, 3,\, 4\,\rangle.$$

Use the cross product of these vectors to identify a normal vector for the plane:

$$\begin{aligned} \mathbf{n} &= \mathbf{v}_1 \times \mathbf{v}_2 \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & 3 & 4 \end{vmatrix} \\ &= (8-3)\mathbf{i} - (4-1)\mathbf{j} + (3-2)\mathbf{k} \\ &= 5\mathbf{i} - 3\mathbf{j} + \mathbf{k}. \end{aligned}$$

$\text{proj}_n (P-Q) = (P-Q)\cdot\frac{n}{|n|}$ length is this

Q be an arbitrary point in the plane. Then the projection of vector $\vec{QP}$ onto the normal vector describes vector $\vec{RP}$, as shown in Figure 2.70.

Theorem 2.13: The Distance between a Plane and a Point

Suppose a plane with normal vector $\mathbf{n}$ passes through point Q. The distance d from the plane to a point P not in the plane is given by

$$d = \| \text{proj}_{\mathbf{n}} \vec{QP} \| = \left|\text{comp}_{\mathbf{n}} \vec{QP}\right| = \frac{\left|\vec{QP}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|}. \tag{2.19}$$

$(P-Q)\cdot\frac{n}{|n|}$

Example 2.51

Distance between a Point and a Plane

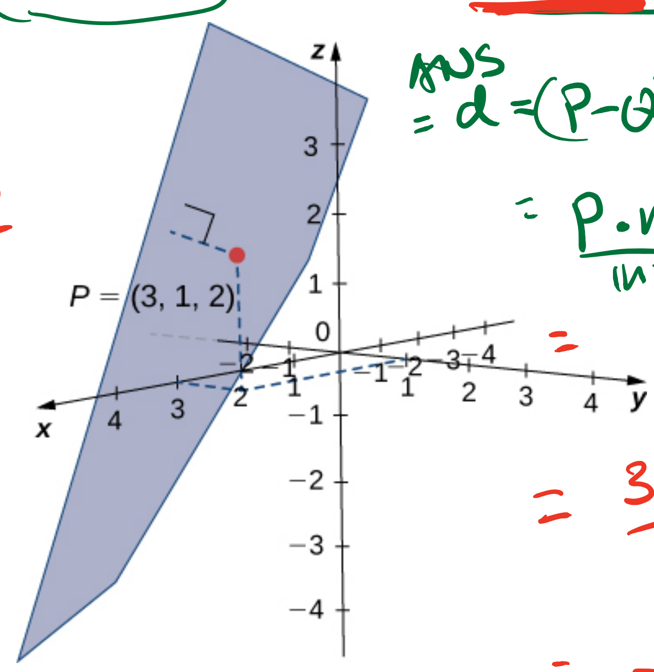
$\langle x,y,z\rangle\cdot n = \underbrace{\langle x_0,y_0,z_0\rangle}_{Q}\cdot n$

Find the distance between point $P = (3,\ 1,\ 2)$ and the plane given by $x - 2y + z = 5$ (see the following figure).

$n = \langle 1,-2,1\rangle$

$P\cdot n = 3-2+2$
$= 3$

$|n| = \sqrt{1+4+1}$
$= \sqrt{6}$



ANS $= d = (P-Q)\cdot\frac{n}{|n|}$

$= \frac{P\cdot n}{|n|} - \frac{Q\cdot n}{|n|}$

$= \frac{3 - 5}{|n|}$

$= \frac{3-5}{\sqrt{6}}$

$= -2/\sqrt{6} \rightarrow$ ANS $\boxed{2/\sqrt{6}}$

Solution

The coefficients of the plane's equation provide a normal vector for the plane: $\mathbf{n} = \langle 1, -2, 1 \rangle$. To find vector $\vec{QP}$, we need a point in the plane. Any point will work, so set $y = z = 0$ to see that point $Q = (5,\ 0,\ 0)$ lies in the plane. Find the component form of the vector from Q to P:

(distances are positive)

$$\vec{QP} = \langle 3-5,\ 1-0,\ 2-0 \rangle = \langle -2,\ 1,\ 2 \rangle.$$

Apply the distance formula from Equation 2.19:

$$
\begin{aligned}
d &= \frac{\left|\overrightarrow{QP} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|} \\
&= \frac{|\langle -2, 1, 2 \rangle \cdot \langle 1, -2, 1 \rangle|}{\sqrt{1^2 + (-2)^2 + 1^2}} \\
&= \frac{|-2 - 2 + 2|}{\sqrt{6}} \\
&= \frac{2}{\sqrt{6}}.
\end{aligned}
$$

2.48 Find the distance between point $P = (5, -1, 0)$ and the plane given by $4x + 2y - z = 3$.

Parallel and Intersecting Planes

We have discussed the various possible relationships between two lines in two dimensions and three dimensions. When we describe the relationship between two planes in space, we have only two possibilities: the two distinct planes are parallel or they intersect. When two planes are parallel, their normal vectors are parallel. When two planes intersect, the intersection is a line (**Figure 2.71**).

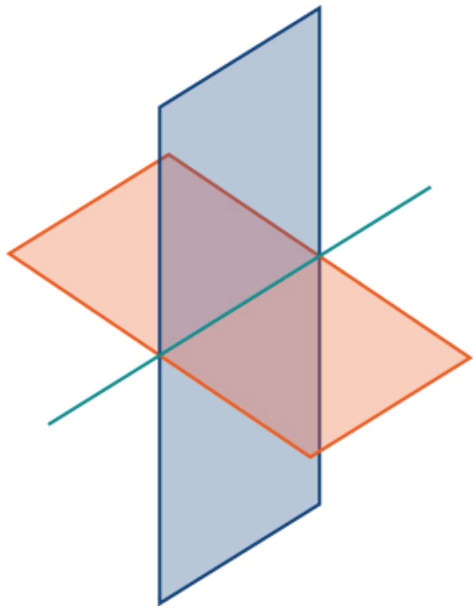


Figure 2.71 The intersection of two nonparallel planes is always a line.

We can use the equations of the two planes to find parametric equations for the line of intersection.

Example 2.52

Finding the Line of Intersection for Two Planes

Find parametric and symmetric equations for the line formed by the intersection of the planes given by $x + y + z = 0$ and $2x - y + z = 0$ (see the following figure).

Parallel planes iff parallel normals.

$n_1 = \langle 1, 1, 1 \rangle$

$n_2 = \langle 2, -1, 1 \rangle$

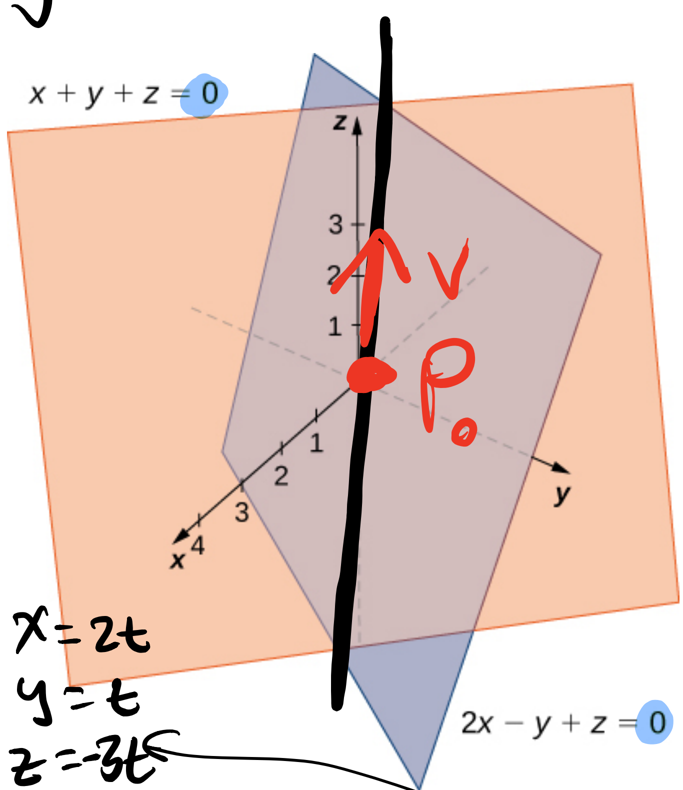
not parallel, so they intersect along a line

Step 1 find v

Step2 find a P_0 on both planes:

$P_0 = (0,0,0)$

$$\boxed{\begin{aligned} x &= 0 + 2t \\ y &= 0 + 1t \\ z &= 0 - 3t \end{aligned}}$$



$v \perp n_1$

$v \perp n_2$

$n_1 = \langle 1,1,1 \rangle$

$n_2 = \langle 2,-1,1 \rangle$

one possible choice of v

$v = n_1 \times n_2$

$$= \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= (1+1)i - (1-2)j + (-1-2)k$$

$x = 2t$

$y = t$

$z = -3t$

$$\boxed{v = 2i + j - 3k}$$

$$\frac{x-0}{2} = \frac{y-0}{1} = \frac{z-0}{-3} \qquad \frac{x}{2} = y = -\frac{z}{3}$$

Solution

Note that the two planes have nonparallel normals, so the planes intersect. Further, the origin satisfies each equation, so we know the line of intersection passes through the origin. Add the plane equations so we can eliminate the one of the variables, in this case, y:

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 0 \\ 2x & - & y & + & z & = & 0 \\ \hline 3x & & & + & 2z & = & 0. \end{array}$$

This gives us $x = -\frac{2}{3}z$. We substitute this value into the first equation to express y in terms of z:

$$\begin{aligned} x + y + z &= 0 \\ -\frac{2}{3}z + y + z &= 0 \\ y + \frac{1}{3}z &= 0 \\ y &= -\frac{1}{3}z. \end{aligned}$$

We now have the first two variables, x and y, in terms of the third variable, z. Now we define z in terms of t. To eliminate the need for fractions, we choose to define the parameter t as $t = -\frac{1}{3}z$. Then, $z = -3t$. Substituting the parametric representation of z back into the other two equations, we see that the parametric equations for the line of intersection are $x = 2t, y = t, z = -3t$. The symmetric equations for the line are $\frac{x}{2} = y = \frac{z}{-3}$.

2.49 Find parametric equations for the line formed by the intersection of planes $x + y - z = 3$ and $3x - y + 3z = 5$.

In addition to finding the equation of the line of intersection between two planes, we may need to find the angle formed by the intersection of two planes. For example, builders constructing a house need to know the angle where different sections

2.49 Find parametric equations for the line formed by the intersection of planes $x + y - z = 3$ and $3x - y + 3z = 5$.

1 1 -1

3 -1 3

$$\begin{vmatrix} i & j & k \\ 3 & -1 & 3 \\ 1 & 1 & -1 \end{vmatrix} = (1-3)i - (-3-3)j + (3+1)k$$

$$= -2i + 6j + 4k = v$$

$x + y - z = 3$

$3x - y + 3z = 5$

set $z = 0$ (choose one variable, x, y, or z, to be zero).

$x + y = 3$

$3x - y = 5$

$+ \quad 4x = 8$

$x = 2$

$2 + y = 3$

$y = 1$

$P_0 = (2, 1, 0)$

$$\boxed{\begin{aligned} x &= 2 + -2t \\ y &= 1 + 6t \\ z &= 0 + 4t \end{aligned}}$$

$$\boxed{\frac{x-2}{-2} = \frac{y-1}{6} = \frac{z-0}{4}}$$

of the roof meet to know whether the roof will look good and drain properly. We can use normal vectors to calculate the angle between the two planes. We can do this because the angle between the normal vectors is the same as the angle between the planes. **Figure 2.72** shows why this is true.

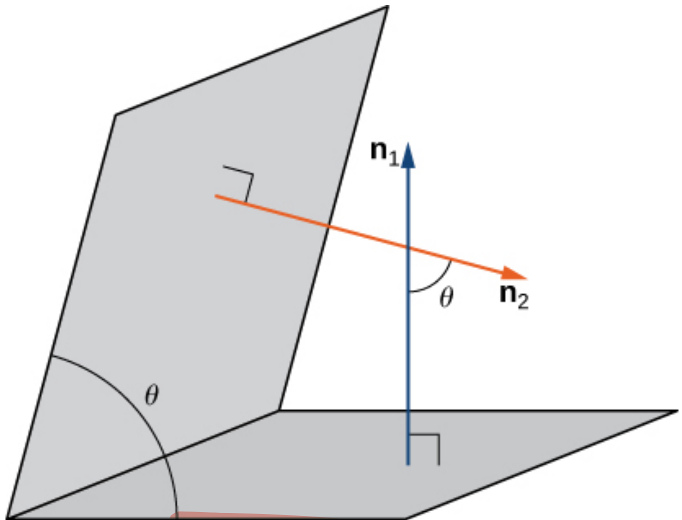


Figure 2.72 The angle between two planes has the same measure as the angle between the normal vectors for the planes.

We can find the measure of the angle θ between two intersecting planes by first finding the cosine of the angle, using the following equation:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \ \|\mathbf{n}_2\|}. \qquad \cos\theta = \frac{n_1}{|n_1|} \cdot \frac{n_2}{|n_2|}$$

We can then use the angle to determine whether two planes are parallel or orthogonal or if they intersect at some other angle.

Example 2.53

Finding the Angle between Two Planes

Determine the angle between them

Determine whether each pair of planes is parallel, orthogonal, or neither. If the planes are intersecting, but not orthogonal, find the measure of the angle between them. Give the answer in radians and round to two decimal places.

same ratio.

1:2 2:-1 2:4 4:-2

$n_1 = \langle 1 \ 2 \ -1 \rangle$ $n_2 = \langle 2 \ 4 \ -2 \rangle$ n_2 is a multiple of n_1, so planes are parallel

a. $x + 2y - z = 8$ and $2x + 4y - 2z = 10$

$x + 2y - z = 5$ → the equations do not intersect

b. $2x - 3y + 2z = 3$ and $6x + 2y - 3z = 1$

$n_1 = \langle 2 \ -3 \ 2 \rangle$ $n_2 = \langle 6 \ 2 \ -3 \rangle$ → not parallel (n_1 & n_2 not multiples of each other)

orthogonal

c. $x + y + z = 4$ and $x - 3y + 5z = 1$

$1 \ 1 \ 1 \quad 1 \ -3 \ 5$

$n_1 \cdot n_2 = 12 - 6 - 6 = 0$

$|n_1| =$

$|n_2| =$

$\cos\theta = \frac{n_1}{|n_1|} \cdot \frac{n_2}{|n_2|} = \frac{n_1 \cdot n_2}{|n_1||n_2|} = 0$

$\cos\theta = 0$

$\boxed{\theta = 90^\circ}$

$= \pi/2$

Solution

a. The normal vectors for these planes are $\mathbf{n}_1 = \langle 1, 2, -1 \rangle$ and $\mathbf{n}_2 = \langle 2, 4, -2 \rangle$. These two vectors are scalar multiples of each other. The normal vectors are parallel, so the planes are parallel.

b. The normal vectors for these planes are $\mathbf{n}_1 = \langle 2, -3, 2 \rangle$ and $\mathbf{n}_2 = \langle 6, 2, -3 \rangle$. Taking the dot product of these vectors, we have

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = \langle 2, -3, 2 \rangle \cdot \langle 6, 2, -3 \rangle = 2(6) - 3(2) + 2(-3) = 0.$$

The normal vectors are orthogonal, so the corresponding planes are orthogonal as well.

c. The normal vectors for these planes are $\mathbf{n}_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{n}_2 = \langle 1, -3, 5 \rangle$:

$\theta = \arccos \frac{\sqrt{3}}{\sqrt{35}}$

$\langle 1, 1, 1 \rangle \cdot \langle 1, -3, 5 \rangle = 1 - 3 + 5 = 3$

$|\langle 1, 1, 1 \rangle| = \sqrt{1+1+1} = \sqrt{3}$

$|\langle 1, -3, 5 \rangle| = \sqrt{1+9+25} = \sqrt{35}$

$\cos\theta = \frac{\cancel{3}}{\cancel{\sqrt{3}}\sqrt{35}} \cdot \frac{\sqrt{3}}{\cancel{\sqrt{3}}} = \frac{\sqrt{3}}{\sqrt{35}}$

$$\begin{aligned}\cos\theta &= \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \, \|\mathbf{n}_2\|} \\ &= \frac{|\langle 1, 1, 1 \rangle \cdot \langle 1, -3, 5 \rangle|}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+(-3)^2+5^2}} \\ &= \frac{3}{\sqrt{105}}.\end{aligned}$$

The angle between the two planes is 1.27 rad, or approximately $73°$.

$\arccos \frac{3}{\sqrt{105}}$

2.50 Find the measure of the angle between planes $x+y-z=3$ and $3x-y+3z=5$. Give the answer in radians and round to two decimal places.

When we find that two planes are parallel, we may need to find the distance between them. To find this distance, we simply select a point in one of the planes. The distance from this point to the other plane is the distance between the planes.

Previously, we introduced the formula for calculating this distance in **Equation 2.19**:

$$d = \frac{\overrightarrow{QP} \cdot \mathbf{n}}{\|\mathbf{n}\|},$$

where Q is a point on the plane, P is a point not on the plane, and $\mathbf{n}$ is the normal vector that passes through point Q. Consider the distance from point (x_0, y_0, z_0) to plane $ax+by+cz+k=0$. Let (x_1, y_1, z_1) be any point in the plane. Substituting into the formula yields

$$\begin{aligned}d &= \frac{|a(x_0-x_1)+b(y_0-y_1)+c(z_0-z_1)|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|ax_0+by_0+cz_0+k|}{\sqrt{a^2+b^2+c^2}}.\end{aligned}$$

We state this result formally in the following theorem.

point in plane

Theorem 2.14: Distance from a Point to a Plane

$n = \langle a, b, c \rangle$ $\quad (x, y, z) \cdot n \quad ax+by+cz = \quad Q \cdot n = -k$

Let $P(x_0, y_0, z_0)$ be a point. The distance from P to plane $ax+by+cz+k=0$ is given by

$P \cdot n$

$$d = \frac{|ax_0+by_0+cz_0+k|}{\sqrt{a^2+b^2+c^2}}.$$

$d = \boxed{(P-Q) \cdot \frac{n}{|n|}}$

see p 198

Example 2.54

Finding the Distance between Parallel Planes

Find the distance between the two parallel planes given by $2x+y-z=2$ and $2x+y-z=8$.

Solution

1) find a pt on plane 1

Point $(1, 0, 0)$ lies in the first plane. The desired distance, then, is

$$\begin{aligned} d &= \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|2(1) + 1(0) + (-1)(0) + (-8)|}{\sqrt{2^2 + 1^2 + (-1)^2}} \\ &= \frac{6}{\sqrt{6}} = \sqrt{6}. \end{aligned}$$

2.51 Find the distance between parallel planes $5x - 2y + z = 6$ and $5x - 2y + z = -3$.