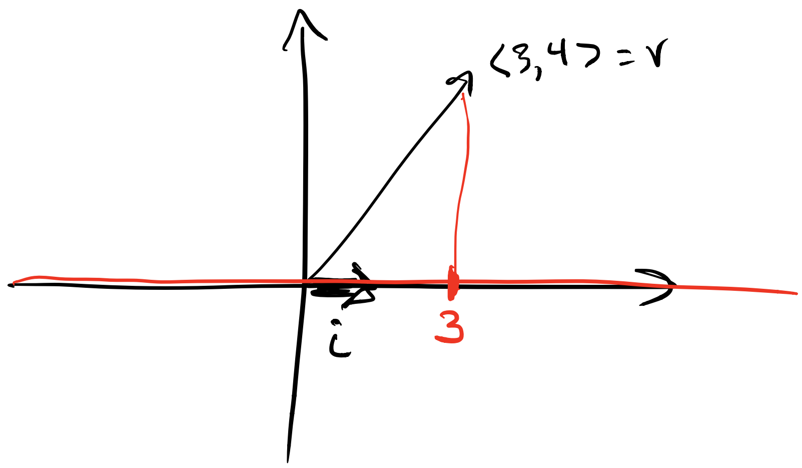
In 2D space



$v = \langle 3,4\rangle$

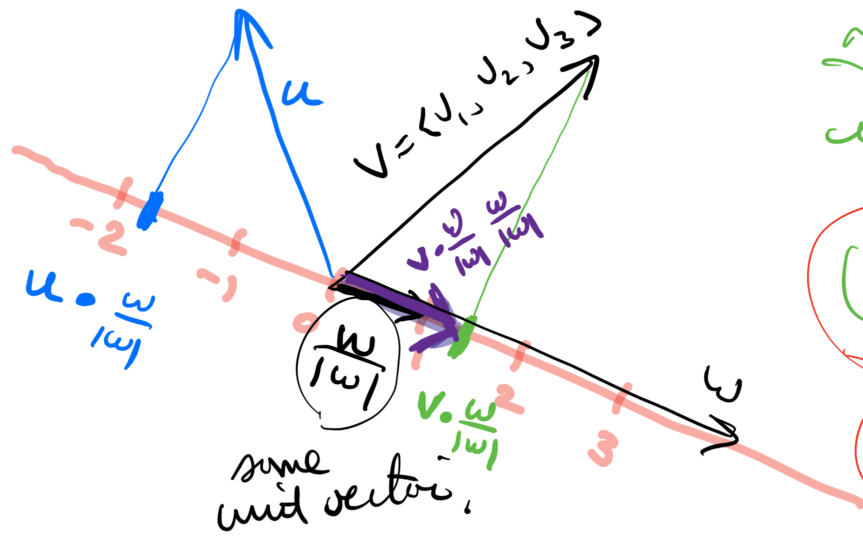
$i = \langle 1,0\rangle$

$v \cdot i = 3$

There's a sort of projection onto i going.

Projection means "casting a shadow"

It turns out that the same thing happens for $(\,—\,) \cdot$ unit vectors for any unit vector.



The dot product is secretly a "projection"

$(v) \cdot \frac{w}{|w|}$ scalar

$\underbrace{v \cdot \frac{w}{|w|}}_{\text{length}}\ \underbrace{\frac{w}{|w|}}_{\text{direction}}$ vector

$\boxed{\mathrm{proj}_w v :=}\ \mathrm{proj}_{\frac{w}{|w|}} v := \boxed{v \cdot \frac{w}{|w|}\ \frac{w}{|w|}}$

projection of v onto $\frac{w}{|w|}$

The unit vector pointing in direction of ω $= \frac{w}{|w|} = \frac{\langle 3,4\rangle}{5} = \langle 0.6, 0.8\rangle$

$v = \langle 10,7\rangle$

$w = \langle 3,4\rangle$

$|w| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

Practice $\quad v = \langle 10, 7\rangle \quad w = \langle 3, 4\rangle$

find $\text{proj}_w v = v \cdot \frac{w}{|w|}\,\frac{w}{|w|} =$

$= \langle 10,7\rangle \cdot \langle 0.6, 0.8\rangle\ \langle 0.6, 0.8\rangle$

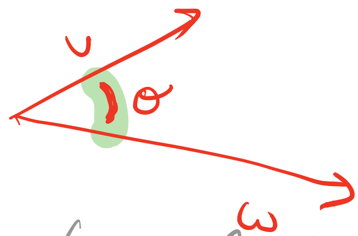
$= 11.6\,\langle 0.6, 0.8\rangle$

$\langle 10,7\rangle \cdot \langle 0.6, 0.8\rangle = 6 + 5.6 = 11.6$

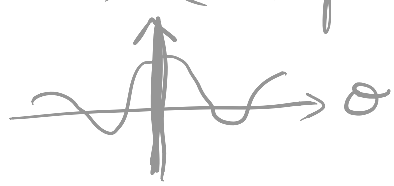
$$\boxed{\text{proj}_w v = \langle 6.96,\ 9.28\rangle}$$

Explaining the angle formula

$$\theta = \arccos\left(\frac{v}{|v|} \cdot \frac{w}{|w|}\right)$$



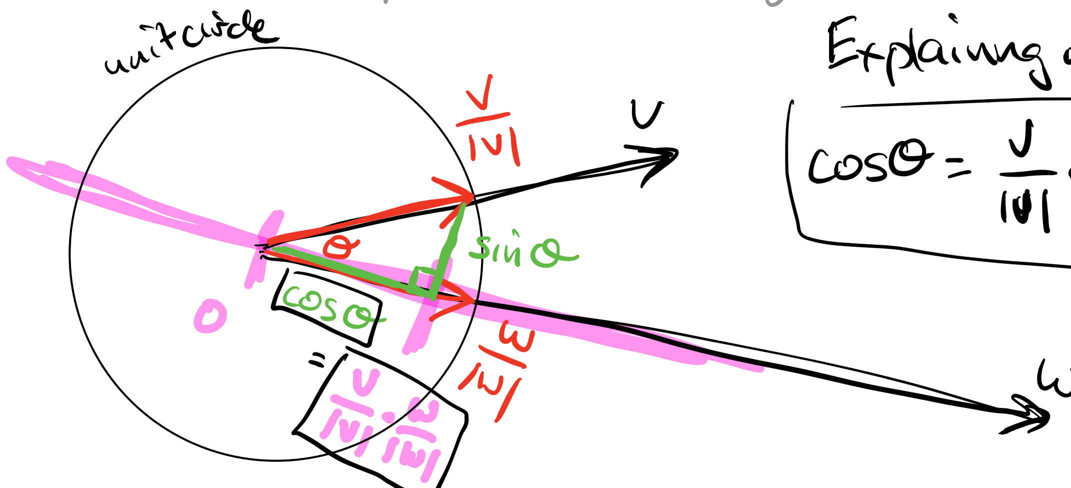
Remember it's cosine because $\cos(\theta) = \cos(-\theta)$ (even function)



$\cos 30^\circ =$ $\qquad$ $\cos(-30^\circ) =$

$= \frac{w \cdot v}{|w||v|} \quad = \quad = \frac{v \cdot w}{|v||w|}$

since dot product is symmetric.



Explaining why

$$\boxed{\cos\theta = \frac{v}{|v|} \cdot \frac{w}{|w|}}$$

2.3 | The Dot Product

Learning Objectives

2.3.1 Calculate the dot product of two given vectors.
2.3.2 Determine whether two given vectors are perpendicular.
2.3.3 Find the direction cosines of a given vector.
2.3.4 Explain what is meant by the vector projection of one vector onto another vector, and describe how to compute it.
2.3.5 Calculate the work done by a given force.

If we apply a force to an object so that the object moves, we say that *work* is done by the force. In **Introduction to Applications of Integration (http://cnx.org/content/m53638/latest/)** on integration applications, we looked at a constant force and we assumed the force was applied in the direction of motion of the object. Under those conditions, work can be expressed as the product of the force acting on an object and the distance the object moves. In this chapter, however, we have seen that both force and the motion of an object can be represented by vectors.

In this section, we develop an operation called the *dot product*, which allows us to calculate work in the case when the force vector and the motion vector have different directions. The dot product essentially tells us how much of the force vector is applied in the direction of the motion vector. The dot product can also help us measure the angle formed by a pair of vectors and the position of a vector relative to the coordinate axes. It even provides a simple test to determine whether two vectors meet at a right angle.

The Dot Product and Its Properties

We have already learned how to add and subtract vectors. In this chapter, we investigate two types of vector multiplication. The first type of vector multiplication is called the dot product, based on the notation we use for it, and it is defined as follows:

$\langle a,b,c\rangle \cdot \langle d,e,f\rangle = ad+be+cf$

Definition

The **dot product** of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by the sum of the products of the components

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{2.3}$$

Note that if **u** and **v** are two-dimensional vectors, we calculate the dot product in a similar fashion. Thus, if $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

When two vectors are combined under addition or subtraction, the result is a vector. When two vectors are combined using the dot product, the result is a scalar. For this reason, the dot product is often called the *scalar product*. It may also be called the *inner product*.

Example 2.21

Calculating Dot Products

a. Find the dot product of $\mathbf{u} = \langle 3, 5, 2 \rangle$ and $\mathbf{v} = \langle -1, 3, 0 \rangle$.

b. Find the scalar product of $\mathbf{p} = 10\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$ and $\mathbf{q} = -2\mathbf{i} + \mathbf{j} + 6\mathbf{k}$.

Solution

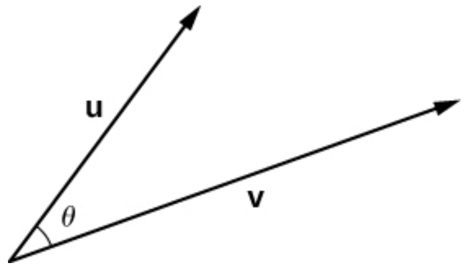


Figure 2.44 Let θ be the angle between two nonzero vectors **u** and **v** such that $0 \le \theta \le \pi$.

Theorem 2.4: Evaluating a Dot Product

The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta. \tag{2.4}$$

$\frac{u}{|u|} \cdot \frac{v}{|v|} = \cos\theta$

$\boxed{\arccos\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right) = \theta}$

Proof

Place vectors **u** and **v** in standard position and consider the vector $\mathbf{v} - \mathbf{u}$ (**Figure 2.45**). These three vectors form a triangle with side lengths $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{v} - \mathbf{u}\|$.

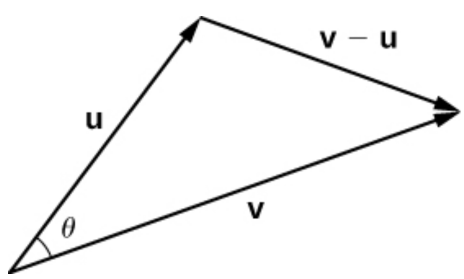


Figure 2.45 The lengths of the sides of the triangle are given by the magnitudes of the vectors that form the triangle.

Recall from trigonometry that the law of cosines describes the relationship among the side lengths of the triangle and the angle θ. Applying the law of cosines here gives

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta.$$

The dot product provides a way to rewrite the left side of this equation:

$$\begin{aligned}
\|\mathbf{v} - \mathbf{u}\|^2 &= (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) \\
&= (\mathbf{v} - \mathbf{u}) \cdot \mathbf{v} - (\mathbf{v} - \mathbf{u}) \cdot \mathbf{u} \\
&= \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} \\
&= \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{u} \\
&= \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2.
\end{aligned}$$

Substituting into the law of cosines yields

$$\begin{aligned}
\|\mathbf{v} - \mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta \\
\|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta \\
-2\mathbf{u} \cdot \mathbf{v} &= -2\|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta \\
\mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta.
\end{aligned}$$

□

We can use this form of the dot product to find the measure of the angle between two nonzero vectors. The following equation rearranges **Equation 2.3** to solve for the cosine of the angle:

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}. \tag{2.5}$$

Using this equation, we can find the cosine of the angle between two nonzero vectors. Since we are considering the smallest angle between the vectors, we assume $0° \le \theta \le 180°$ (or $0 \le \theta \le \pi$ if we are working in radians). The inverse cosine is unique over this range, so we are then able to determine the measure of the angle θ.

Example 2.23

Finding the Angle between Two Vectors

Find the measure of the angle between each pair of vectors.

a. $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$

b. $\langle 2, 5, 6 \rangle$ and $\langle -2, -4, 4 \rangle$

Solution

a. To find the cosine of the angle formed by the two vectors, substitute the components of the vectors into **Equation 2.5**:

$$\begin{aligned}\cos\theta &= \frac{(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(2\mathbf{i}-\mathbf{j}-3\mathbf{k})}{\|\mathbf{i}+\mathbf{j}+\mathbf{k}\| \cdot \|2\mathbf{i}-\mathbf{j}-3\mathbf{k}\|}\\ &= \frac{1(2)+(1)(-1)+(1)(-3)}{\sqrt{1^2+1^2+1^2}\sqrt{2^2+(-1)^2+(-3)^2}}\\ &= \frac{-2}{\sqrt{3}\sqrt{14}} = \frac{-2}{\sqrt{42}}.\end{aligned}$$

Therefore, $\theta = \arccos\frac{-2}{\sqrt{42}}$ rad.

b. Start by finding the value of the cosine of the angle between the vectors:

$$\begin{aligned}\cos\theta &= \frac{\langle 2, 5, 6\rangle \cdot \langle -2, -4, 4\rangle}{\|\langle 2, 5, 6\rangle\| \cdot \|\langle -2, -4, 4\rangle\|}\\ &= \frac{2(-2)+(5)(-4)+(6)(4)}{\sqrt{2^2+5^2+6^2}\sqrt{(-2)^2+(-4)^2+4^2}}\\ &= \frac{0}{\sqrt{65}\sqrt{36}} = 0.\end{aligned}$$

Now, $\cos\theta = 0$ and $0 \le \theta \le \pi$, so $\theta = \pi/2$.

$|b| = \sqrt{2^2+4^2+1^2} = \sqrt{21}$

$|a| = \sqrt{1^2+2^2+0^2} = \sqrt{5}$

2.23 Find the measure of the angle, in radians, formed by vectors $\mathbf{a} = \langle 1, 2, 0 \rangle$ and $\mathbf{b} = \langle 2, 4, 1 \rangle$. Round to the nearest hundredth.

$a \cdot b = 2 + 8 + 0 = 10$

The angle between two vectors can be acute $(0 < \cos\theta < 1)$, obtuse $(-1 < \cos\theta < 0)$, or straight $(\cos\theta = -1)$. If $\cos\theta = 1$, then both vectors have the same direction. If $\cos\theta = 0$, then the vectors, when placed in standard position, form a right angle (**Figure 2.46**). We can formalize this result into a theorem regarding orthogonal (perpendicular) vectors.

$\theta = \arccos\left(\frac{a}{|a|}\cdot\frac{b}{|b|}\right) = \boxed{\arccos\left(\frac{10}{\sqrt{5}\sqrt{21}}\right)} = \arccos\frac{10}{\sqrt{105}}$

$= \boxed{0.22 \text{ RAD}}$

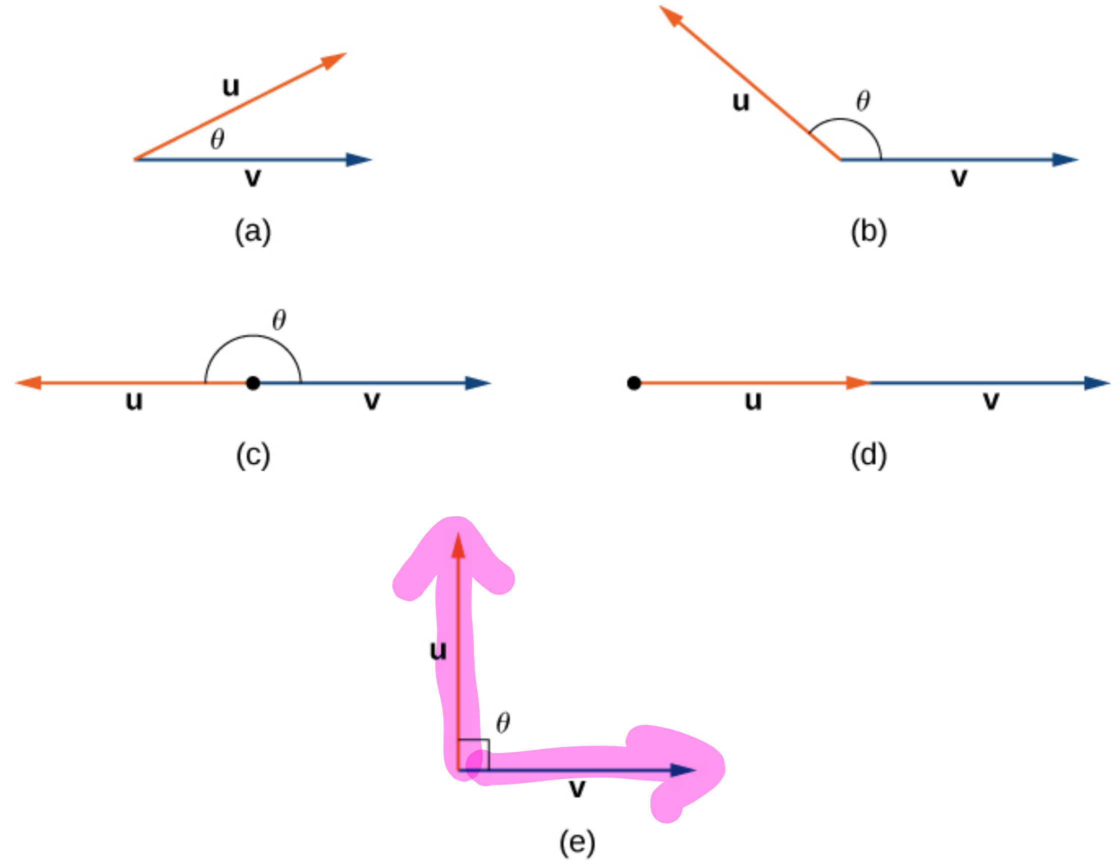


Figure 2.46 (a) An acute angle has $0 < \cos\theta < 1$. (b) An obtuse angle has $-1 < \cos\theta < 0$. (c) A straight line has $\cos\theta = -1$. (d) If the vectors have the same direction, $\cos\theta = 1$. (e) If the vectors are orthogonal (perpendicular), $\cos\theta = 0$.

Theorem 2.5: Orthogonal Vectors

The nonzero vectors **u** and **v** are **orthogonal vectors** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Proof

Let **u** and **v** be nonzero vectors, and let θ denote the angle between them. First, assume $\mathbf{u} \cdot \mathbf{v} = 0$. Then

$$\|\mathbf{u}\| \ \|\mathbf{v}\| \cos\theta = 0.$$

However, $\|\mathbf{u}\| \neq 0$ and $\|\mathbf{v}\| \neq 0$, so we must have $\cos\theta = 0$. Hence, $\theta = 90°$, and the vectors are orthogonal.

Now assume **u** and **v** are orthogonal. Then $\theta = 90°$ and we have

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \ \|\mathbf{v}\| \cos\theta = \|\mathbf{u}\| \ \|\mathbf{v}\| \cos 90° = \|\mathbf{u}\| \ \|\mathbf{v}\| (0) = 0.$$

□

The terms *orthogonal*, *perpendicular*, and *normal* each indicate that mathematical objects are intersecting at right angles. The use of each term is determined mainly by its context. We say that vectors are orthogonal and lines are perpendicular. The term *normal* is used most often when measuring the angle made with a plane or other surface.

Example 2.24 2.3 (Dot Product)

Identifying Orthogonal Vectors

Determine whether $\mathbf{p} = \langle 1, 0, 5 \rangle$ and $\mathbf{q} = \langle 10, 3, -2 \rangle$ are orthogonal vectors.

$p \cdot q = 10 + 0 - 10 = \boxed{0}$ so yes, p & q are orthogonal p156

Definition

The **vector projection** of $\mathbf{v}$ onto $\mathbf{u}$ is the vector labeled $\text{proj}_{\mathbf{u}}\mathbf{v}$ in Figure 2.50. It has the same initial point as $\mathbf{u}$ and $\mathbf{v}$ and the same direction as $\mathbf{u}$, and represents the component of $\mathbf{v}$ that acts in the direction of $\mathbf{u}$. If θ represents the angle between $\mathbf{u}$ and $\mathbf{v}$, then, by properties of triangles, we know the length of $\text{proj}_{\mathbf{u}}\mathbf{v}$ is $\|\text{proj}_{\mathbf{u}}\mathbf{v}\| = \|\mathbf{v}\|\cos\theta$. When expressing $\cos\theta$ in terms of the dot product, this becomes

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scalar projection

$$\begin{aligned}\|\text{proj}_{\mathbf{u}}\mathbf{v}\| &= \|\mathbf{v}\|\cos\theta\\ &= \|\mathbf{v}\|\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)\\ \boxed{\text{comp}_u v =} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}. \quad \boxed{v\cdot\frac{u}{|u|}}\end{aligned}$$

We now multiply by a unit vector in the ~~direction of $\mathbf{u}$ to get~~ $\text{proj}_{\mathbf{u}}\mathbf{v}$:

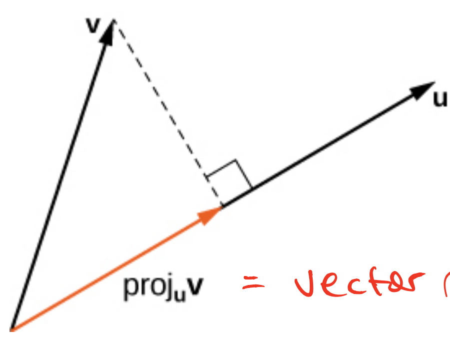
$$= \boxed{v\cdot\frac{u}{|u|}\ \frac{u}{|u|}} \;\; = \boxed{\frac{v\cdot u}{u\cdot u}\,u}$$

vector projection

$$\boxed{\text{proj}_{\mathbf{u}}\mathbf{v}} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}\left(\frac{1}{\|\mathbf{u}\|}\mathbf{u}\right) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}. \tag{2.6}$$

The length of this vector is also known as the **scalar projection** of $\mathbf{v}$ onto $\mathbf{u}$ and is denoted by

$$\|\text{proj}_{\mathbf{u}}\mathbf{v}\| = \text{comp}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}. \tag{2.7}$$



$= \text{vector projection} = \text{proj}_u v = v\cdot\frac{u}{|u|}\,\frac{u}{|u|}$

Figure 2.50 The projection of $\mathbf{v}$ onto $\mathbf{u}$ shows the component of vector $\mathbf{v}$ in the direction of $\mathbf{u}$.

Example 2.27

Finding Projections

Find the projection of $\mathbf{v}$ onto $\mathbf{u}$.

a. $\mathbf{v} = \langle 3, 5, 1\rangle$ and $\mathbf{u} = \langle -1, 4, 3\rangle$

b. $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{u} = \mathbf{i} + 6\mathbf{j}$

Solution

a. Substitute the components of $\mathbf{v}$ and $\mathbf{u}$ into the formula for the projection:

Example 2.28

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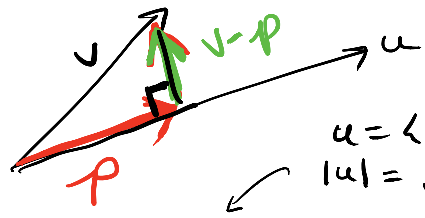
Resolving Vectors into Components

$v = (p) + (v - p)$

Express $\mathbf{v} = \langle 8, -3, -3 \rangle$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\mathbf{u} = \langle 2, 3, 2 \rangle$.



Solution

$u = \langle 2,3,2 \rangle$

$|u| = \sqrt{2^2+3^2+2^2} = \sqrt{17}$

Let $\mathbf{p}$ represent the projection of $\mathbf{v}$ onto $\mathbf{u}$:

$$\begin{aligned}
\boxed{\mathbf{p} = \text{proj}_{\mathbf{u}}\mathbf{v}} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} \\
&= \frac{\langle 2, 3, 2 \rangle \cdot \langle 8, -3, -3 \rangle}{\|\langle 2, 3, 2 \rangle\|^2}\langle 2, 3, 2 \rangle \\
&= \frac{16 - 9 - 6}{2^2 + 3^2 + 2^2}\langle 2, 3, 2 \rangle \\
&= \frac{1}{17}\langle 2, 3, 2 \rangle \\
&= \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle.
\end{aligned}$$

$= v \cdot \frac{u}{|u|}\frac{u}{|u|} = \frac{v \cdot u}{|u||u|}u = \boxed{\frac{v \cdot u}{u \cdot u}u}$

$v \cdot u$, $\sqrt{17}$, $\sqrt{17}$, 17

$= \frac{1}{17}\langle 2,3,2 \rangle$

$u \cdot u = |u|^2$

$v \cdot u = 16 - 9 - 6 = 1$

$p = \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle$

$8 - \frac{2}{17} = \frac{136-2}{17} = \frac{134}{17}$

$-3 - \frac{3}{17} = \frac{-51-3}{17} = \frac{-54}{17}$

$-3 - \frac{2}{17} = \frac{-51-2}{17} = \frac{-53}{17}$

Then,

v − p

$$\mathbf{q} = \mathbf{v} - \mathbf{p} = \langle 8, -3, -3 \rangle - \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle = \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle.$$

To check our work, we can use the dot product to verify that $\mathbf{p}$ and $\mathbf{q}$ are orthogonal vectors:

$$\mathbf{p}\cdot\mathbf{q} = \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle \cdot \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle = \frac{268}{17} - \frac{162}{17} - \frac{106}{17} = 0.$$

Then,

$$\mathbf{v} = \mathbf{p} + \mathbf{q} = \boxed{\left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle + \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle}$$

final answer.

these add up to v, which is ⟨8, −3, −3⟩

AND These are orthogonal

2.27 Express $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\mathbf{u} = 4\mathbf{i} + 2\mathbf{j}$.

p, q orthogonal means $p \cdot q = 0$

Example 2.29

Scalar Projection of Velocity

A container ship leaves port traveling $15°$ north of east. Its engine generates a speed of 20 knots along that path (see the following figure). In addition, the ocean current moves the ship northeast at a speed of 2 knots. Considering both the engine and the current, how fast is the ship moving in the direction $15°$ north of east? Round the answer to two decimal places.

$$\begin{aligned}
\|\mathbf{u}\times\mathbf{v}\|^2 &= (u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2\\
&= u_2^2v_3^2-2u_2u_3v_2v_3+u_3^2v_2^2+u_3^2v_1^2-2u_1u_3v_1v_3+u_1^2v_3^2+u_1^2v_2^2-2u_1u_2v_1v_2+u_2^2v_1^2\\
&= u_1^2v_1^2+u_1^2v_2^2+u_1^2v_3^2+u_2^2v_1^2+u_2^2v_2^2+u_2^2v_3^2+u_3^2v_1^2+u_3^2v_2^2+u_3^2v_3^2\\
&\quad -\left(u_1^2v_1^2+u_2^2v_2^2+u_3^2v_3^2+2u_1u_2v_1v_2+2u_1u_3v_1v_3+2u_2u_3v_2v_3\right)\\
&= \left(u_1^2+u_2^2+u_3^2\right)\left(v_1^2+v_2^2+v_3^2\right)-(u_1v_1+u_2v_2+u_3v_3)^2\\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2-(\mathbf{u}\cdot\mathbf{v})^2\\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2-\|\mathbf{u}\|^2\|\mathbf{v}\|^2\cos^2\theta\\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2\left(1-\cos^2\theta\right)\\
&= \|\mathbf{u}\|^2\|\mathbf{v}\|^2\left(\sin^2\theta\right).
\end{aligned}$$

Taking square roots and noting that $\sqrt{\sin^2\theta}=\sin\theta$ for $0\le\theta\le 180°$, we have the desired result:

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\ \|\mathbf{v}\|\sin\theta.$$

□

This definition of the cross product allows us to visualize or interpret the product geometrically. It is clear, for example, that the cross product is defined only for vectors in three dimensions, not for vectors in two dimensions. In two dimensions, it is impossible to generate a vector simultaneously orthogonal to two nonparallel vectors.

Example 2.35

Calculating the Cross Product

Use **Properties of the Cross Product** to find the magnitude of the cross product of $\mathbf{u}=\langle 0, 4, 0\rangle$ and $\mathbf{v}=\langle 0, 0, -3\rangle$.

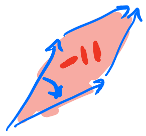
Solution

We have

$$\begin{aligned}
\|\mathbf{u}\times\mathbf{v}\| &= \|\mathbf{u}\|\cdot\|\mathbf{v}\|\cdot\sin\theta\\
&= \sqrt{0^2+4^2+0^2}\cdot\sqrt{0^2+0^2+(-3)^2}\cdot\sin\frac{\pi}{2}\\
&= 4(3)(1)=12.
\end{aligned}$$

2.34 Use **Properties of the Cross Product** to find the magnitude of $\mathbf{u}\times\mathbf{v}$, where $\mathbf{u}=\langle -8, 0, 0\rangle$ and $\mathbf{v}=\langle 0, 2, 0\rangle$.

$\begin{vmatrix} 3 & 5 \\ 4 & 3 \end{vmatrix} = 9-20 = -11$

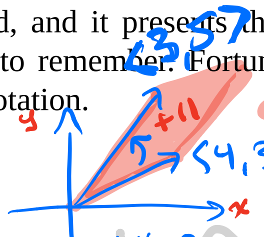


Determinants and the Cross Product

Using **Equation 2.9** to find the cross product of two vectors is straightforward, and it presents the cross product in the useful component form. The formula, however, is complicated and difficult to remember. Fortunately, we have an alternative. We can calculate the cross product of two vectors using **determinant** notation.

A 2×2 determinant is defined by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2-b_1a_2.$$



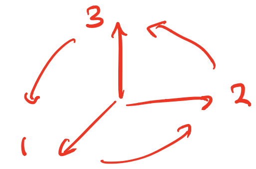
signed area

find the determinant

$\begin{vmatrix} 4 & 3 \\ 3 & 5 \end{vmatrix} = 20-9 = \boxed{11}$

For example,

$$\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 3(1) - 5(-2) = 3 + 10 = 13.$$



A 3×3 determinant is defined in terms of 2×2 determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \quad (2.10)$$

Equation 2.10 is referred to as the *expansion of the determinant along the first row*. Notice that the multipliers of each of the 2×2 determinants on the right side of this expression are the entries in the first row of the 3×3 determinant. Furthermore, each of the 2×2 determinants contains the entries from the 3×3 determinant that would remain if you crossed out the row and column containing the multiplier. Thus, for the first term on the right, a_1 is the multiplier, and the 2×2 determinant contains the entries that remain if you cross out the first row and first column of the 3×3 determinant. Similarly, for the second term, the multiplier is a_2, and the 2×2 determinant contains the entries that remain if you cross out the first row and second column of the 3×3 determinant. Notice, however, that the coefficient of the second term is negative. The third term can be calculated in similar fashion.

Example 2.36

Using Expansion Along the First Row to Compute a 3×3 Determinant

tells you signed volume

Evaluate the determinant $\begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix}$. $= 2(4-9) - 5(-4+6)$ $+ -1(-3 - -2)$

$= 2(-5) - 5(2) - 1(-1)$

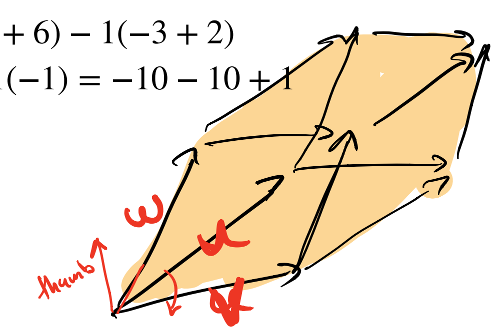
$= -10 - 10 + 1 = \boxed{-19}$

Solution

We have

u
v
w

$$\begin{aligned} \begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix} &= 2\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} - 5\begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} \\ &= 2(4-9) - 5(-4+6) - 1(-3+2) \\ &= 2(-5) - 5(2) - 1(-1) = -10 - 10 + 1 \\ &= -19. \end{aligned}$$



2.35 Evaluate the determinant $\begin{vmatrix} 1 & -2 & -1 \\ 3 & 2 & -3 \\ 1 & 5 & 4 \end{vmatrix}$.

opposite the right hand rule, So it has negative signed volume

Technically, determinants are defined only in terms of arrays of real numbers. However, the determinant notation provides a useful mnemonic device for the cross product formula.

Rule: Cross Product Calculated by a Determinant

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors. Then the cross product $\mathbf{u} \times \mathbf{v}$ is given by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}.$$