2.2 | Vectors in Three Dimensions

Learning Objectives

2.2.1 Describe three-dimensional space mathematically.
2.2.2 Locate points in space using coordinates.
2.2.3 Write the distance formula in three dimensions.
2.2.4 Write the equations for simple planes and spheres.
2.2.5 Perform vector operations in $\mathbb{R}^3$.

Vectors are useful tools for solving two-dimensional problems. Life, however, happens in three dimensions. To expand the use of vectors to more realistic applications, it is necessary to create a framework for describing three-dimensional space. For example, although a two-dimensional map is a useful tool for navigating from one place to another, in some cases the topography of the land is important. Does your planned route go through the mountains? Do you have to cross a river? To appreciate fully the impact of these geographic features, you must use three dimensions. This section presents a natural extension of the two-dimensional Cartesian coordinate plane into three dimensions.

Three-Dimensional Coordinate Systems

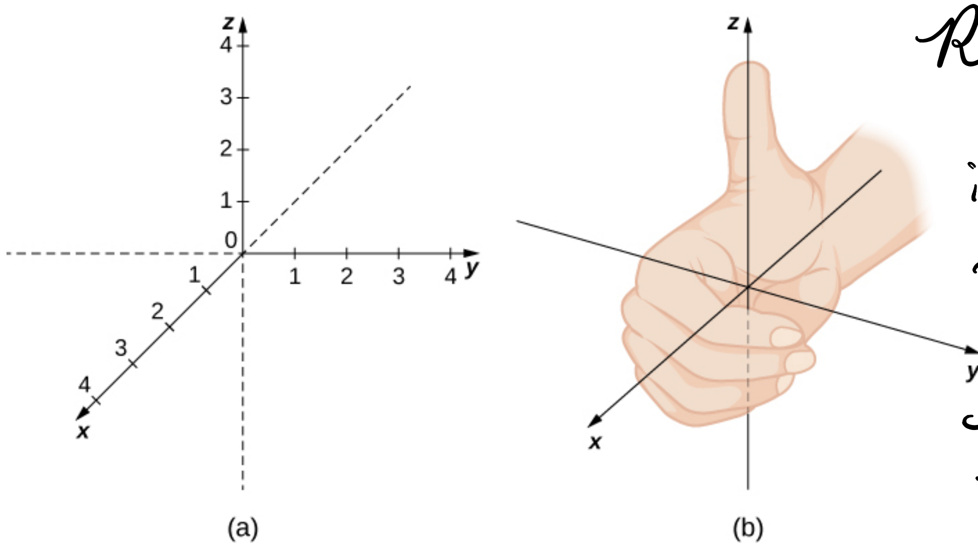
As we have learned, the two-dimensional rectangular coordinate system contains two perpendicular axes: the horizontal *x*-axis and the vertical *y*-axis. We can add a third dimension, the *z*-axis, which is perpendicular to both the *x*-axis and the *y*-axis. We call this system the three-dimensional rectangular coordinate system. It represents the three dimensions we encounter in real life.

Definition

The **three-dimensional rectangular coordinate system** consists of three perpendicular axes: the *x*-axis, the *y*-axis, and the *z*-axis. Because each axis is a number line representing all real numbers in $\mathbb{R}$, the three-dimensional system is often denoted by $\mathbb{R}^3$.

3-dimensional space
3-space

In Figure 2.23(a), the positive *z*-axis is shown above the plane containing the *x*- and *y*-axes. The positive *x*-axis appears to the left and the positive *y*-axis is to the right. A natural question to ask is: How was arrangement determined? The system displayed follows the **right-hand rule**. If we take our right hand and align the fingers with the positive *x*-axis, then curl the fingers so they point in the direction of the positive *y*-axis, our thumb points in the direction of the positive *z*-axis. In this text, we always work with coordinate systems set up in accordance with the right-hand rule. Some systems do follow a left-hand rule, but the right-hand rule is considered the standard representation.



Right-hand rule: if fingers go x-axis to y-axis then thumb points z-axis

By convention:

Figure 2.23 (a) We can extend the two-dimensional rectangular coordinate system by adding a third axis, the *z*-axis, that is perpendicular to both the *x*-axis and the *y*-axis. (b) The right-hand rule is used to determine the placement of the coordinate axes in the standard Cartesian plane.

In two dimensions, we describe a point in the plane with the coordinates (x, y). Each coordinate describes how the point aligns with the corresponding axis. In three dimensions, a new coordinate, z, is appended to indicate alignment with the *z*-axis: (x, y, z). A point in space is identified by all three coordinates (**Figure 2.24**). To plot the point (x, y, z), go x units along the *x*-axis, then y units in the direction of the *y*-axis, then z units in the direction of the *z*-axis.

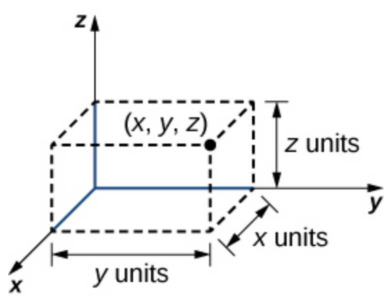
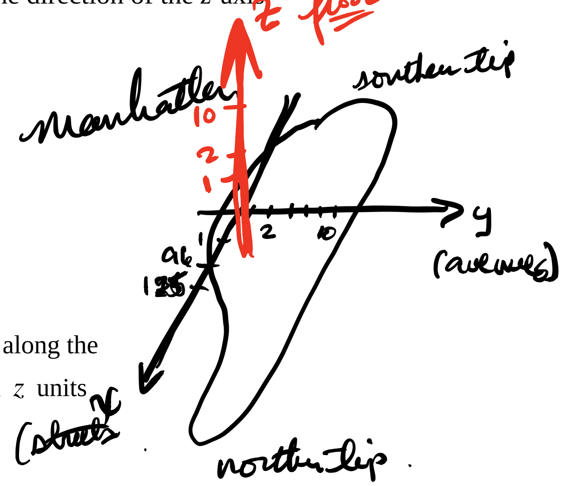


Figure 2.24 To plot the point (x, y, z) go x units along the *x*-axis, then y units in the direction of the *y*-axis, then z units in the direction of the *z*-axis.



Example 2.11

Locating Points in Space

Sketch the point $(1, -2, 3)$ in three-dimensional space.

Solution

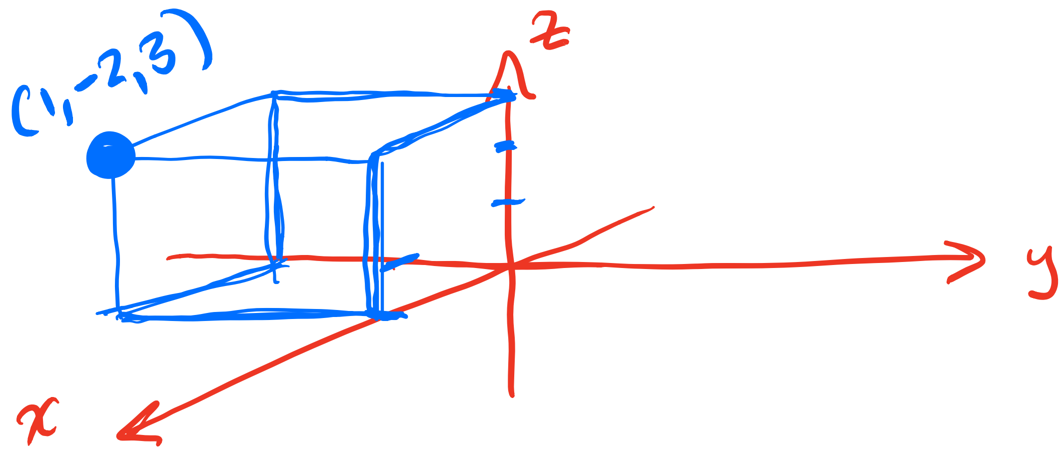
To sketch a point, start by sketching three sides of a rectangular prism along the coordinate axes: one unit in the positive x direction, 2 units in the negative y direction, and 3 units in the positive z direction. Complete the prism to plot the point (**Figure 2.25**).

p124

Example 2.11

Locating Points in Space

Sketch the point $(1, -2, 3)$ in three-dimensional space.



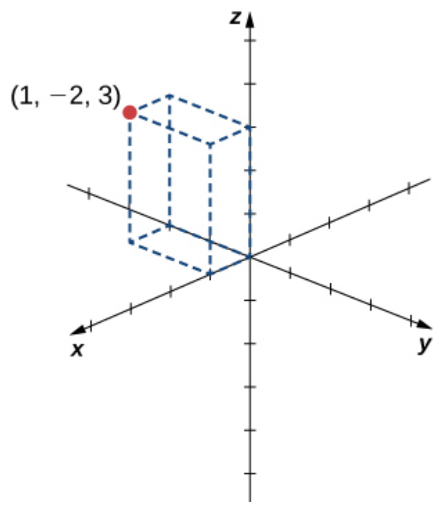


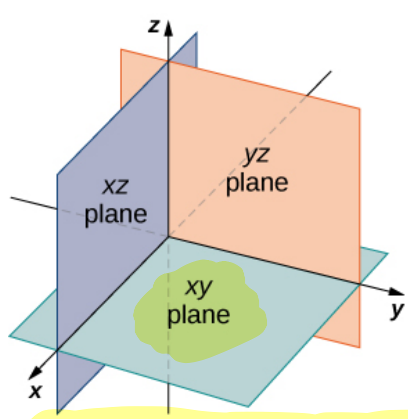
Figure 2.25 Sketching the point $(1, -2, 3)$.

2.11 Sketch the point $(-2, 3, -1)$ in three-dimensional space.

In two-dimensional space, the coordinate plane is defined by a pair of perpendicular axes. These axes allow us to name any location within the plane. In three dimensions, we define **coordinate planes** by the coordinate axes, just as in two dimensions. There are three axes now, so there are three intersecting pairs of axes. Each pair of axes forms a coordinate plane: the *xy*-plane, the *xz*-plane, and the *yz*-plane (Figure 2.26). We define the *xy*-plane formally as the following set: $\{(x, y, 0) : x, y \in \mathbb{R}\}$. Similarly, the *xz*-plane and the *yz*-plane are defined as $\{(x, 0, z) : x, z \in \mathbb{R}\}$ and $\{(0, y, z) : y, z \in \mathbb{R}\}$, respectively.

To visualize this, imagine you're building a house and are standing in a room with only two of the four walls finished. (Assume the two finished walls are adjacent to each other.) If you stand with your back to the corner where the two finished walls meet, facing out into the room, the floor is the *xy*-plane, the wall to your right is the *xz*-plane, and the wall to your left is the *yz*-plane.

Recall in 2D geometry



we have four quadrants.

Figure 2.26 The plane containing the *x*- and *y*-axes is called the *xy*-plane. The plane containing the *x*- and *z*-axes is called the *xz*-plane, and the *y*- and *z*-axes define the *yz*-plane.

In two dimensions, the coordinate axes partition the plane into four quadrants. Similarly, the coordinate planes divide space between them into eight regions about the origin, called **octants**. The octants fill $\mathbb{R}^3$ in the same way that quadrants fill

$\mathbb{R}^2$, as shown in **Figure 2.27**.

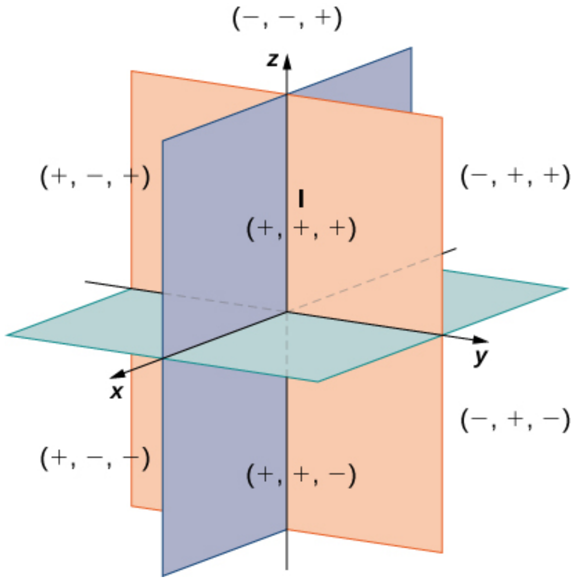


Figure 2.27 Points that lie in octants have three nonzero coordinates.

Most work in three-dimensional space is a comfortable extension of the corresponding concepts in two dimensions. In this section, we use our knowledge of circles to describe spheres, then we expand our understanding of vectors to three dimensions. To accomplish these goals, we begin by adapting the distance formula to three-dimensional space.

If two points lie in the same coordinate plane, then it is straightforward to calculate the distance between them. We that the distance d between two points (x_1, y_1) and (x_2, y_2) in the xy-coordinate plane is given by the formula

In 2-space

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The formula for the distance between two points in space is a natural extension of this formula.

In 3-space

Theorem 2.2: The Distance between Two Points in Space

The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{2.1}$$

The proof of this theorem is left as an exercise. (*Hint:* First find the distance d_1 between the points (x_1, y_1, z_1) and (x_2, y_2, z_1) as shown in **Figure 2.28**.)

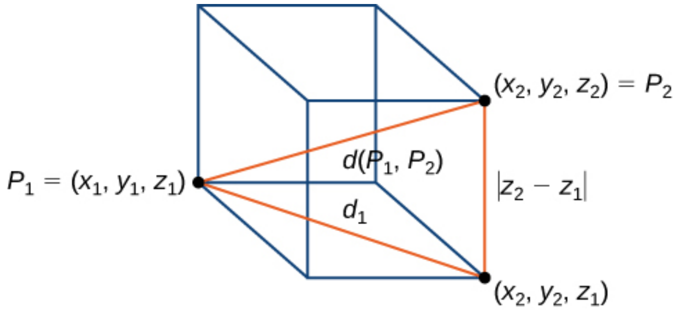


Figure 2.28 The distance between P_1 and P_2 is the length of the diagonal of the rectangular prism having P_1 and P_2 as opposite corners.

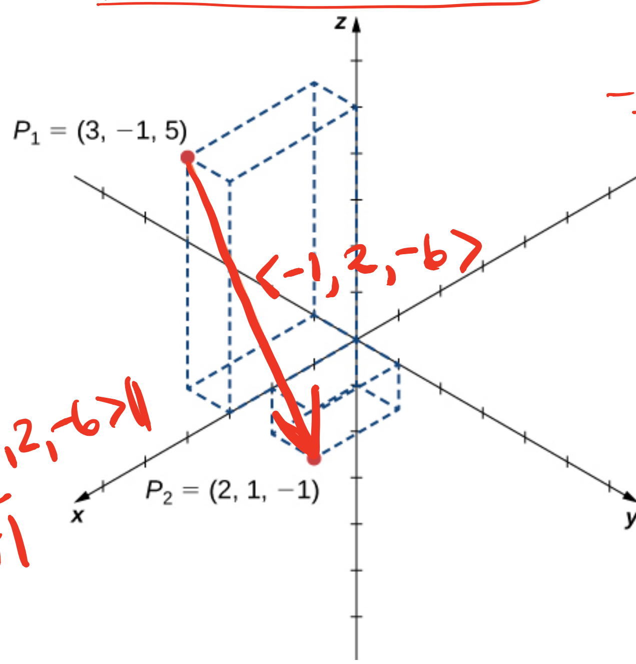
Example 2.12

Distance in Space

Find the distance between points $P_1 = (3, -1, 5)$ and $P_2 = (2, 1, -1)$.

$(2, 1, -1)$
$- (3, -1, 5)$
$\langle -1, 2, -6 \rangle$
$\sqrt{1+4+36}$
$= \boxed{\sqrt{41}}$

$P_2 - P_1$



$\|\langle -1, 2, -6 \rangle\|$
$= \sqrt{41}$

Figure 2.29 Find the distance between the two points.

Solution

Substitute values directly into the distance formula:

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(2-3)^2 + (1-(-1))^2 + (-1-5)^2} \\ &= \sqrt{1^2 + 2^2 + (-6)^2} \\ &= \sqrt{41}. \end{aligned}$$

2.12 Find the distance between points $P_1 = (1, -5, 4)$ and $P_2 = (4, -1, -1)$.

- In $\mathbb{R}^2$ must non-parallel lines intersect? Yes, they must.
- In $\mathbb{R}^3$ must non-parallel lines intersect? No, they might not.

Before moving on to the next section, let's get a feel for how $\mathbb{R}^3$ differs from $\mathbb{R}^2$. For example, in $\mathbb{R}^2$, lines that are not parallel must always intersect. This is not the case in $\mathbb{R}^3$. For example, consider the line shown in **Figure 2.30**. These two lines are not parallel, nor do they intersect.

such as in this example

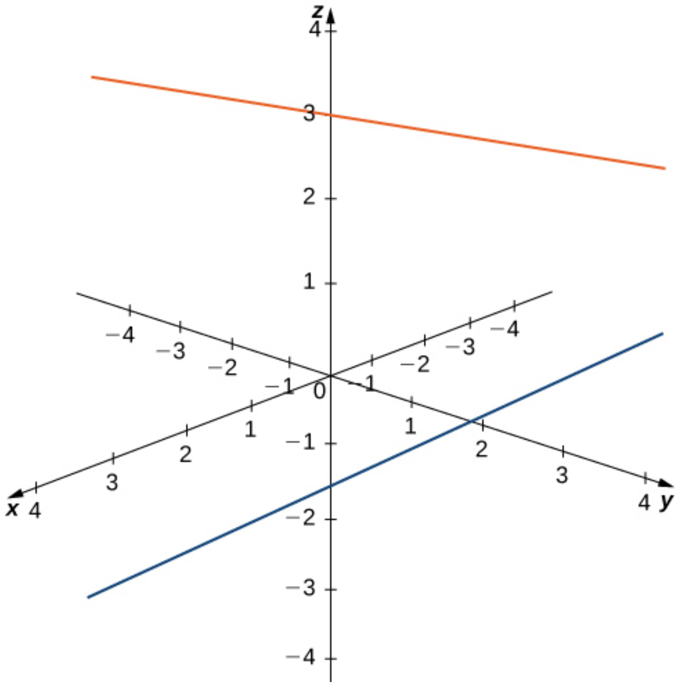


Figure 2.30 These two lines are not parallel, but still do not intersect.

You can also have circles that are interconnected but have no points in common, as in **Figure 2.31**.

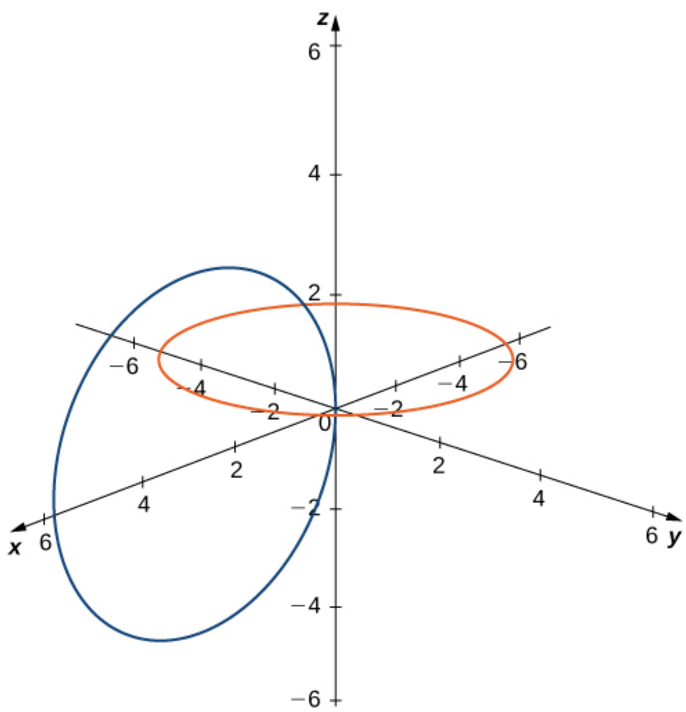


Figure 2.31 These circles are interconnected, but have no points in common.

We have a lot more flexibility working in three dimensions than we do if we stuck with only two dimensions.

Writing Equations in $\mathbb{R}^3$

Now that we can represent points in space and find the distance between them, we can learn how to write equations of geometric objects such as lines, planes, and curved surfaces in $\mathbb{R}^3$. First, we start with a simple equation. Compare the graphs of the equation $x = 0$ in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$ (**Figure 2.32**). From these graphs, we can see the same equation can describe a point, a line, or a plane.

Graph of the equation $x=0$ in $\mathbb{R}$ (the number line) is a point

Graph of the equation $x=0$ in $\mathbb{R}^2$ (it's a straight line)

Graph of the equation $x=0$ in $\mathbb{R}^3$ is a plane (flat & extends infinitely)

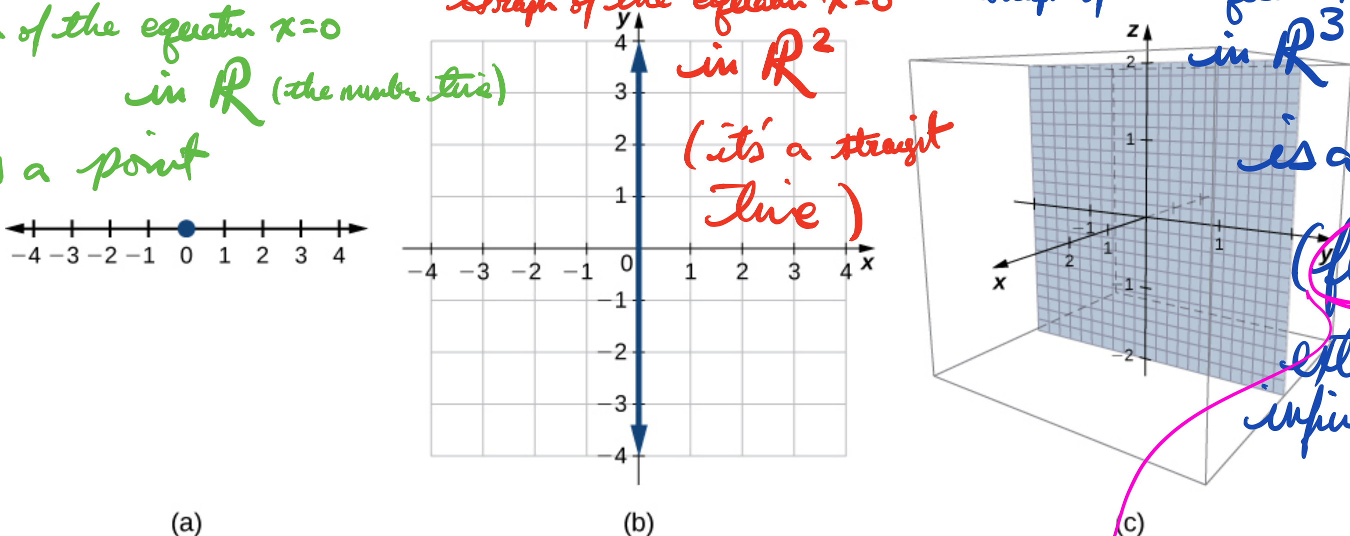


Figure 2.32 (a) In $\mathbb{R}$, the equation $x = 0$ describes a single point. (b) In $\mathbb{R}^2$, the equation $x = 0$ describes a line, the y-axis. (c) In $\mathbb{R}^3$, the equation $x = 0$ describes a plane, the yz-plane.

I mean, not curved like

not a flat surface

In space, the equation $x = 0$ describes all points $(0, y, z)$. This equation defines the yz-plane. Similarly, the xy-plane contains all points of the form $(x, y, 0)$. The equation $z = 0$ defines the xy-plane and the equation $y = 0$ describes the xz-plane (**Figure 2.33**).

all satisfy $y=0$ (so they make up the plane $y=0$)

$z=0$

$(2,0,2)$ $(1,0,1)$ $(-2,0,0)$ $(1,0,-1)$

$y=0$

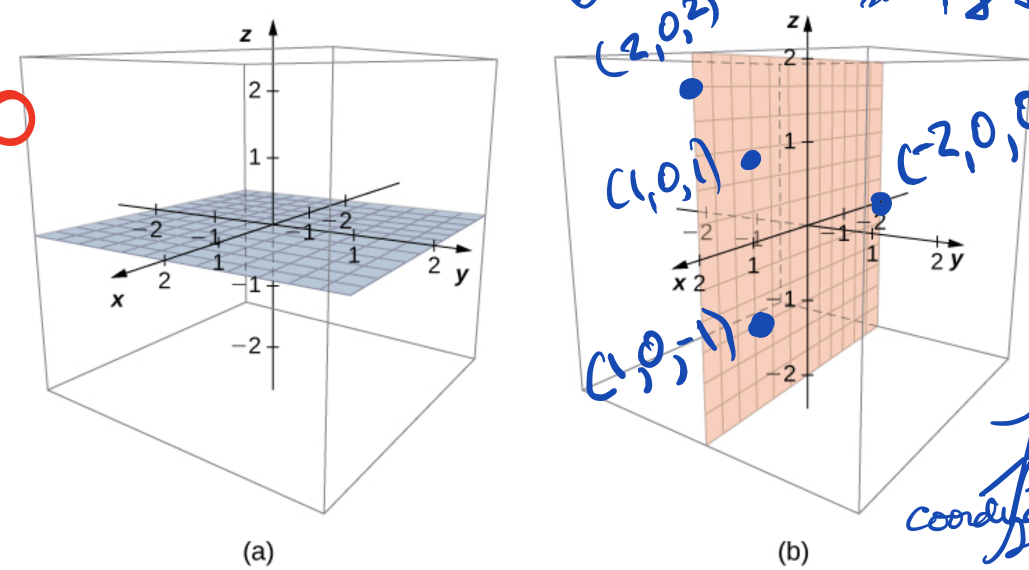


Figure 2.33 (a) In space, the equation $z = 0$ describes the xy-plane. (b) All points in the xz-plane satisfy the equation $y = 0$.

(because all the points in this plane have coordinates that satisfy the equation $y=0$)

Understanding the equations of the coordinate planes allows us to write an equation for any plane that is parallel to one of the coordinate planes. When a plane is parallel to the xy-plane, for example, the z-coordinate of each point in the plane has the same constant value. Only the x- and y-coordinates of points in that plane vary from point to point.

Summary

Rule: Equations of Planes Parallel to Coordinate Planes

1. The plane in space that is parallel to the *xy*-plane and contains point (a, b, c) can be represented by the equation $z = c$.
2. The plane in space that is parallel to the *xz*-plane and contains point (a, b, c) can be represented by the equation $y = b$.
3. The plane in space that is parallel to the *yz*-plane and contains point (a, b, c) can be represented by the equation $x = a$.

Example 2.13

Writing Equations of Planes Parallel to Coordinate Planes

& sketch it

a. Write an equation of the plane passing through point $(3, 11, 7)$ that is parallel to the *yz*-plane.

b. Find an equation of the plane passing through points $(6, -2, 9)$, $(0, -2, 4)$, and $(1, -2, -3)$.

Solution

a. When a plane is parallel to the *yz*-plane, only the *y*- and *z*-coordinates may vary. The *x*-coordinate has the same constant value for all points in this plane, so this plane can be represented by the equation $x = 3$.

b. Each of the points $(6, -2, 9)$, $(0, -2, 4)$, and $(1, -2, -3)$ has the same *y*-coordinate. This plane can be represented by the equation $y = -2$.

2.13 Write an equation of the plane passing through point $(1, -6, -4)$ that is parallel to the *xy*-plane.

As we have seen, in $\mathbb{R}^2$ the equation $x = 5$ describes the vertical line passing through point $(5, 0)$. This line is parallel to the *y*-axis. In a natural extension, the equation $x = 5$ in $\mathbb{R}^3$ describes the plane passing through point $(5, 0, 0)$, which is parallel to the *yz*-plane. Another natural extension of a familiar equation is found in the equation of a sphere.

Definition

A **sphere** is the set of all points in space equidistant from a fixed point, the center of the sphere (**Figure 2.34**), just as the set of all points in a plane that are equidistant from the center represents a circle. In a sphere, as in a circle, the distance from the center to a point on the sphere is called the *radius*.

p130

Example 2.13

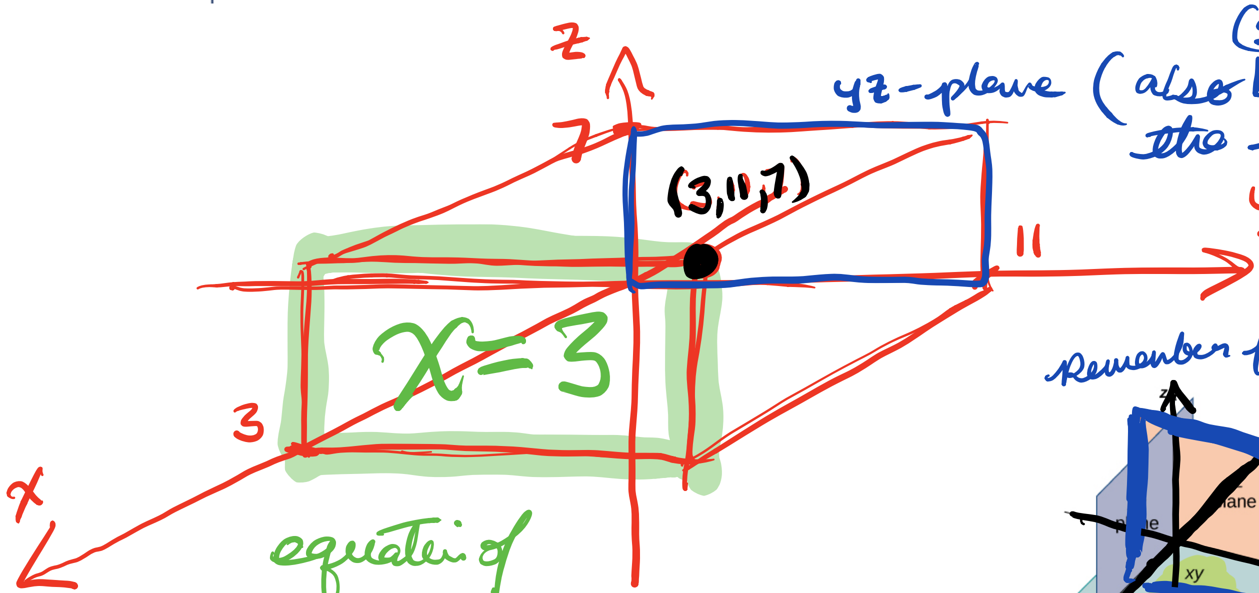
Writing Equations of Planes Parallel to Coordinate Planes

a. Write an equation of the plane passing through point $(3, 11, 7)$ that is parallel to the yz-plane.

& sketch it

(see page 129)

yz-plane (also known as the plane $x=0$)



equation of our plane is $\boxed{x=3}$

Remember from before:

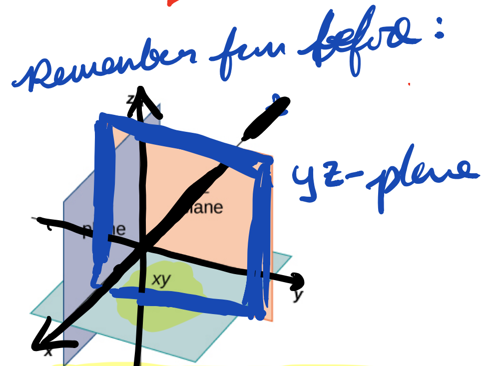


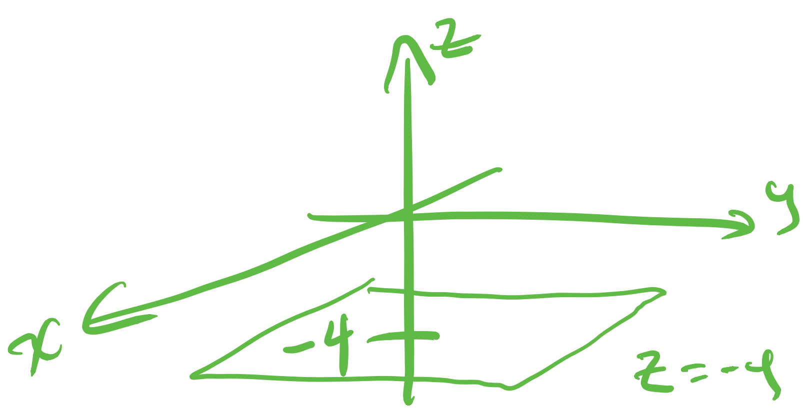
Figure 2.26 The plane containing the x- and y-axes is called the xy-plane. The plane containing the x- and z-axes is called the xz-plane, and the y- and z-axes define the yz-plane.

(z=0)

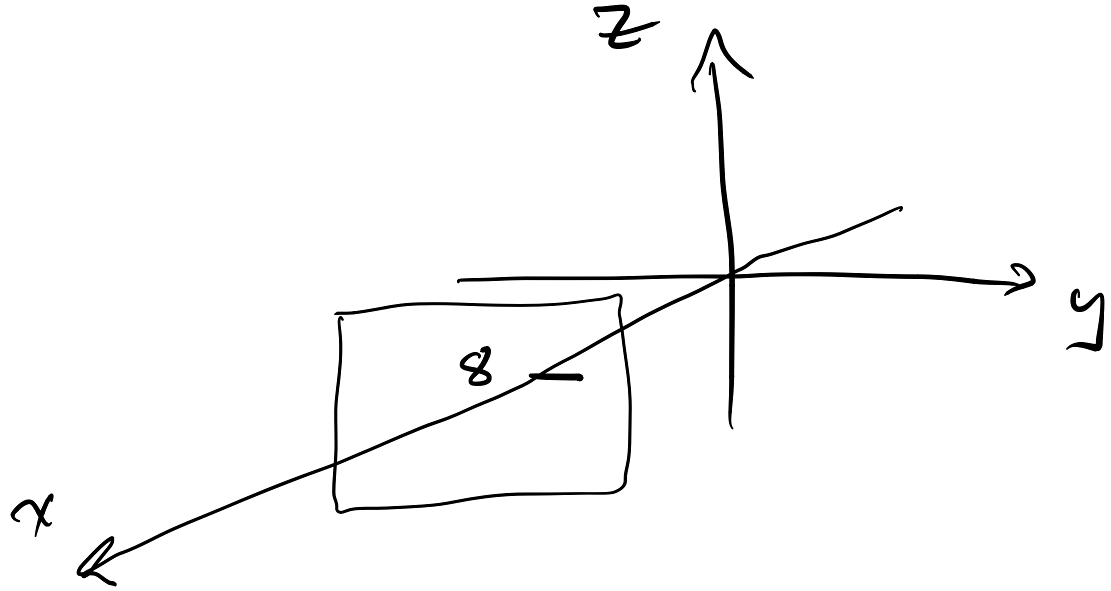
2.13 Write an equation of the plane passing through point $(1, -6, -4)$ that is parallel to the xy-plane.

Answer is $z=-4$

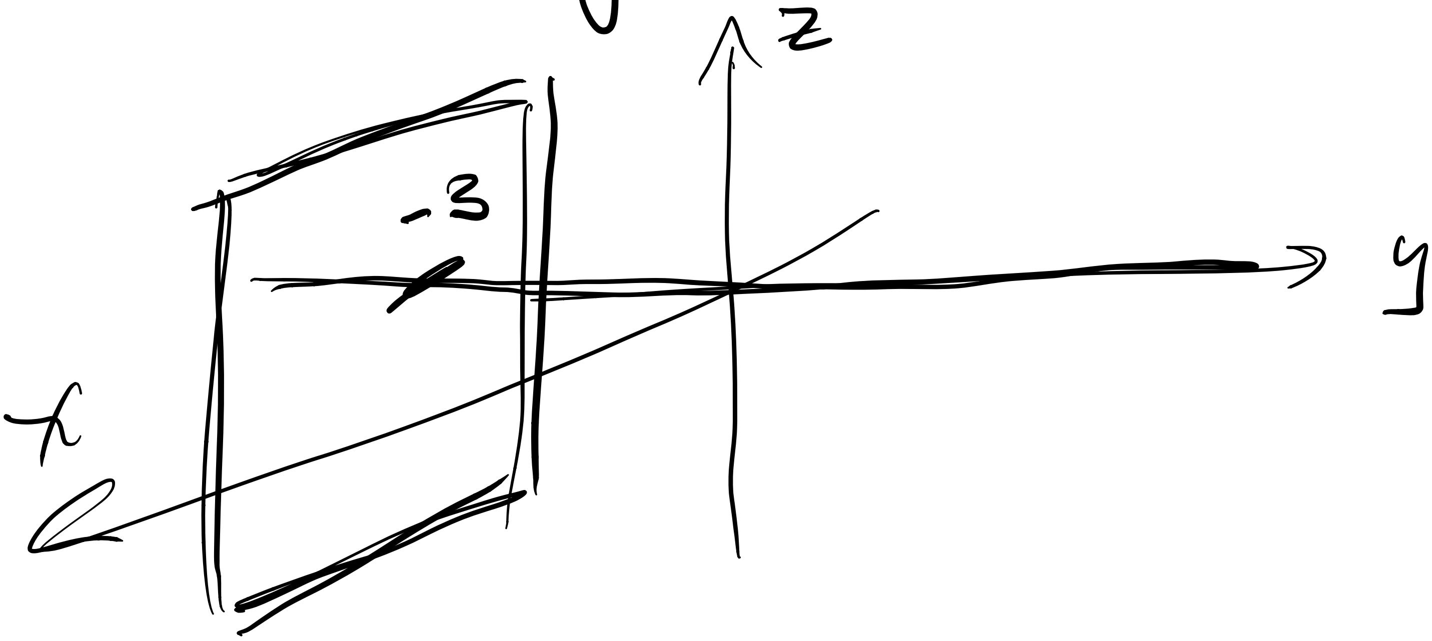
And here's the minimal sketch

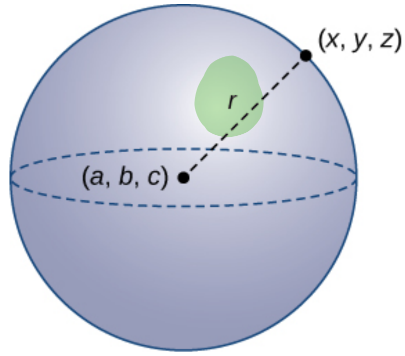


Sketch the plane $x = 8$



Sketch $y = -3$





so r is constant

Figure 2.34 Each point $(x,\ y,\ z)$ on the surface of a sphere is r units away from the center $(a,\ b,\ c)$.

The equation of a circle is derived using the distance formula in two dimensions. In the same way, the equation of a sphere is based on the three-dimensional formula for distance.

Rule: Equation of a Sphere

The sphere with center $(a,\ b,\ c)$ and radius r can be represented by the equation

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2. \tag{2.2}$$

This equation is known as the **standard equation of a sphere**.

(it's derived from the distance formula)

Example 2.14

Finding an Equation of a Sphere

on the (surface of the) sphere

Find the standard equation of the sphere with center $(10, 7, 4)$ and point $(-1, 3, -2)$, as shown in **Figure 2.35**.

r = radius of the sphere

1st find r.

10 7 4
−1 3 −2
11 4 6
121 16 36

$\boxed{r = \sqrt{173}}$

any other point on sphere (x, y, z)

same $r = \sqrt{173}$

$\sqrt{(x-10)^2+(y-7)^2+(z-4)^2} = \sqrt{173}$

$\boxed{(x-10)^2+(y-7)^2+(z-4)^2 = 173}$

173 is r^2 ($\sqrt{173}$ is r)

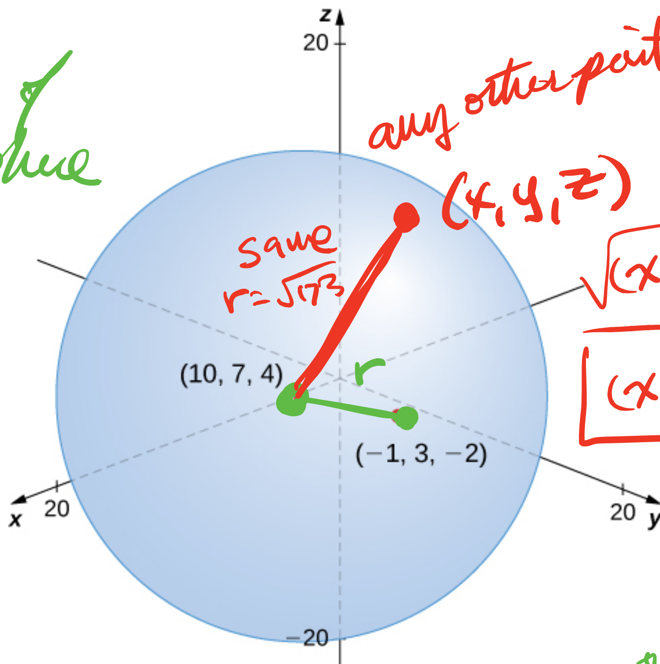


Figure 2.35 The sphere centered at $(10, 7, 4)$ containing point $(-1, 3, -2)$.

the sphere is just the surface
the surface "contains" the point means the point is on the surface

Solution

Use the distance formula to find the radius r of the sphere:

$$\begin{aligned} r &= \sqrt{(-1-10)^2 + (3-7)^2 + (-2-4)^2} \\ &= \sqrt{(-11)^2 + (-4)^2 + (-6)^2} \\ &= \sqrt{173}. \end{aligned}$$

The standard equation of the sphere is

$$(x-10)^2 + (y-7)^2 + (z-4)^2 = 173.$$

2.14 Find the standard equation of the sphere with center $(-2, 4, -5)$ containing point $(4, 4, -1)$.

Example 2.15

Finding the Equation of a Sphere

Let $P = (-5, 2, 3)$ and $Q = (3, 4, -1)$, and suppose line segment PQ forms the diameter of a sphere

Solution

We must have either $x - 4 = 0$ or $z - 2 = 0$, so the set of points forms the two planes $x = 4$ and $z = 2$ (**Figure 2.37**).

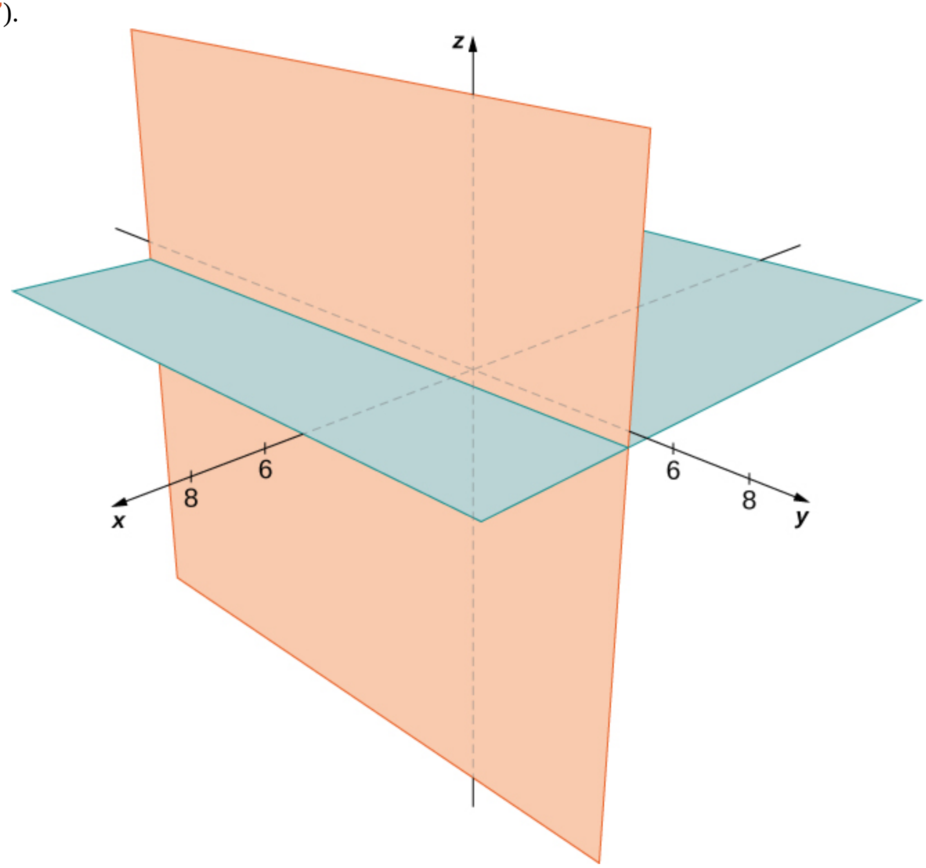


Figure 2.37 The set of points satisfying $(x - 4)(z - 2) = 0$ forms the two planes $x = 4$ and $z = 2$.

2.16 Describe the set of points that satisfies $(y + 2)(z - 3) = 0$, and graph the set.

Example 2.17

Graphing Other Equations in Three Dimensions

in R^2 this a circle

Describe the set of points in three-dimensional space that satisfies $(x - 2)^2 + (y - 1)^2 = 4$, and graph the set.

Solution

in R^3 this is a cylinder

The x- and y-coordinates form a circle in the xy-plane of radius 2, centered at $(2, 1)$. Since there is no restriction on the z-coordinate, the three-dimensional result is a circular cylinder of radius 2 centered on the line with $x = 2$ and $y = 1$. The cylinder extends indefinitely in the z-direction (**Figure 2.38**).

(because the equation is missing z, it extends arbrarily in z direction)

$(x-2)^2+(y-1)^2=4$

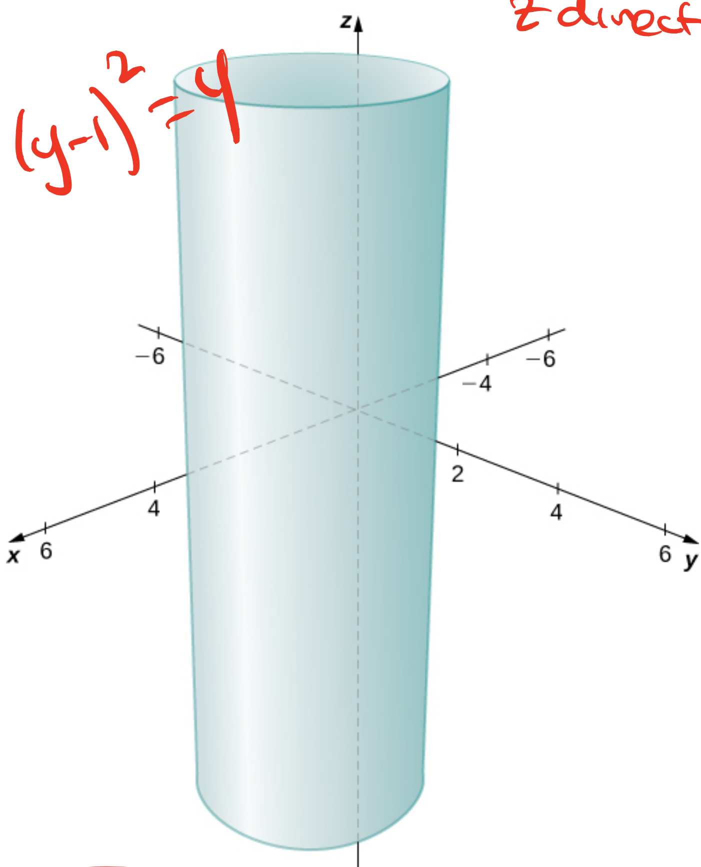


Figure 2.38 The set of points satisfying $(x-2)^2+(y-1)^2=4$. This is a cylinder of radius 2 centered on the line with $x=2$ and $y=1$.

2.17 Describe the set of points in three dimensional space that satisfies $x^2+(z-2)^2=16$, and graph the surface.

Working with Vectors in $\mathbb{R}^3$

Just like two-dimensional vectors, three-dimensional vectors are quantities with both magnitude and direction, and they are represented by directed line segments (arrows). With a three-dimensional vector, we use a three-dimensional arrow.

Three-dimensional vectors can also be represented in component form. The notation $\mathbf{v}=\langle x, y, z\rangle$ is a natural extension of the two-dimensional case, representing a vector with the initial point at the origin, $(0, 0, 0)$, and terminal point (x, y, z). The zero vector is $\mathbf{0}=\langle 0, 0, 0\rangle$. So, for example, the three dimensional vector $\mathbf{v}=\langle 2, 4, 1\rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$ (**Figure 2.39**).

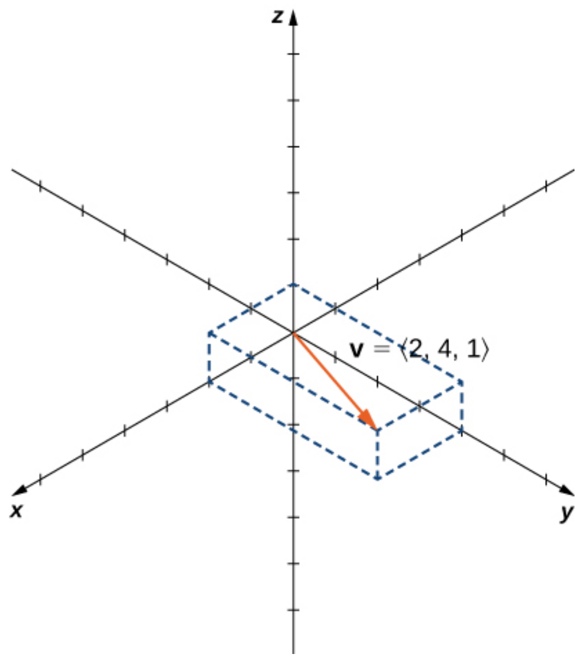


Figure 2.39 Vector $\mathbf{v} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$.

Vector addition and scalar multiplication are defined analogously to the two-dimensional case. If $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2 \rangle$ are vectors, and k is a scalar, then

$$\mathbf{v} + \mathbf{w} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle \text{ and } k\mathbf{v} = \langle kx_1, ky_1, kz_1 \rangle .$$

If $k = -1,$ then $k\mathbf{v} = (-1)\mathbf{v}$ is written as $-\mathbf{v},$ and vector subtraction is defined by $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w}) = \mathbf{v} + (-1)\mathbf{w}.$

The standard unit vectors extend easily into three dimensions as well—$\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ —and we use them in the same way we used the standard unit vectors in two dimensions. Thus, we can represent a vector in $\mathbb{R}^3$ in the following ways:

$$\mathbf{v} = \langle x, y, z \rangle = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Example 2.18

Vector Representations

Let $\overrightarrow{PQ}$ be the vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$ as shown in **Figure 2.40**. Express $\overrightarrow{PQ}$ in both component form and using standard unit vectors.

$\overrightarrow{PQ} = Q - P = \langle -4, -3, 2 \rangle - \langle 3, 12, 6 \rangle = \langle -7, -15, -4 \rangle$

$= -7i - 15j - 4k$

reference.

Rule: Properties of Vectors in Space

Let $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

Scalar multiplication: $k\mathbf{v} = \langle kx_1, ky_1, kz_1 \rangle$

Vector addition: $\mathbf{v} + \mathbf{w} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$

Vector subtraction: $\mathbf{v} - \mathbf{w} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$

Vector magnitude: $\|\mathbf{v}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$

Unit vector in the direction of v: $\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\|\mathbf{v}\|}\langle x_1, y_1, z_1 \rangle = \left\langle \frac{x_1}{\|\mathbf{v}\|}, \frac{y_1}{\|\mathbf{v}\|}, \frac{z_1}{\|\mathbf{v}\|} \right\rangle$, if $\mathbf{v} \neq \mathbf{0}$

We have seen that vector addition in two dimensions satisfies the commutative, associative, and additive inverse properties. These properties of vector operations are valid for three-dimensional vectors as well. Scalar multiplication of vectors satisfies the distributive property, and the zero vector acts as an additive identity. The proofs to verify these properties in three dimensions are straightforward extensions of the proofs in two dimensions.

Example 2.19

Vector Operations in Three Dimensions

Let $\mathbf{v} = \langle -2, 9, 5 \rangle$ and $\mathbf{w} = \langle 1, -1, 0 \rangle$ (**Figure 2.42**). Find the following vectors.

a. $3\mathbf{v} - 2\mathbf{w}$ $= 3\langle -2, 9, 5 \rangle - 2\langle 1, -1, 0 \rangle$

b. $5\|\mathbf{w}\|$ $= \langle -6, 27, 15 \rangle + \langle -2, 2, 0 \rangle = \boxed{\langle -8, 29, 15 \rangle}$

c. $\|5\mathbf{w}\|$

d. A unit vector in the direction of $\mathbf{v}$

$$5\|w\| = 5\|\langle 1, -1, 0 \rangle\| = 5\sqrt{1 + 1 + 0} = 5\sqrt{2}$$

$$\|5w\| = \|5\langle 1, -1, 0 \rangle\| = \|\langle 5, -5, 0 \rangle\| = \sqrt{25 + 25 + 0} = \sqrt{50} = 5\sqrt{2}$$

Notice that $5\|w\| = \|5w\|$

- take a vector
- measure its length
- multiply by 5

- take a vector
- blow it up to 5x the length
- measure the ~~length of the new vector~~ length of the vector you got

"you can exchange (positive) scalar multiplication & magnitude"

2.3 | The Dot Product

Learning Objectives

2.3.1 Calculate the dot product of two given vectors.
2.3.2 Determine whether two given vectors are perpendicular.
2.3.3 Find the direction cosines of a given vector.
2.3.4 Explain what is meant by the vector projection of one vector onto another vector, and describe how to compute it.
2.3.5 Calculate the work done by a given force.

If we apply a force to an object so that the object moves, we say that *work* is done by the force. In **Introduction to Applications of Integration (http://cnx.org/content/m53638/latest/)** on integration applications, we looked at a constant force and we assumed the force was applied in the direction of motion of the object. Under those conditions, work can be expressed as the product of the force acting on an object and the distance the object moves. In this chapter, however, we have seen that both force and the motion of an object can be represented by vectors.

In this section, we develop an operation called the *dot product*, which allows us to calculate work in the case when the force vector and the motion vector have different directions. The dot product essentially tells us how much of the force vector is applied in the direction of the motion vector. The dot product can also help us measure the angle formed by a pair of vectors and the position of a vector relative to the coordinate axes. It even provides a simple test to determine whether two vectors meet at a right angle.

The Dot Product and Its Properties

We have already learned how to add and subtract vectors. In this chapter, we investigate two types of vector multiplication. The first type of vector multiplication is called the dot product, based on the notation we use for it, and it is defined as follows:

Definition

The **dot product** of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by the sum of the products of the components

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3. \tag{2.3}$$

Note that if $\mathbf{u}$ and $\mathbf{v}$ are two-dimensional vectors, we calculate the dot product in a similar fashion. Thus, if $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

When two vectors are combined under addition or subtraction, the result is a vector. When two vectors are combined using the dot product, the result is a scalar. For this reason, the dot product is often called the *scalar product*. It may also be called the *inner product*.

Example 2.21

Calculating Dot Products

$\langle 3,5,2 \rangle \cdot \langle -1,3,0 \rangle = -3+15+0$
$= 12$

a. Find the dot product of $\mathbf{u} = \langle 3, 5, 2 \rangle$ and $\mathbf{v} = \langle -1, 3, 0 \rangle$.

b. Find the scalar product of $\mathbf{p} = 10\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$ and $\mathbf{q} = -2\mathbf{i} + \mathbf{j} + 6\mathbf{k}$.

Solution

$p \cdot q = (10)(-2) + (-4)(1) + (7)(6)$
$= -20 - 4 + 42$
$= \boxed{18}$

a. Substitute the vector components into the formula for the dot product:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= u_1v_1 + u_2v_2 + u_3v_3\\ &= 3(-1) + 5(3) + 2(0) = -3 + 15 + 0 = 12.\end{aligned}$$

b. The calculation is the same if the vectors are written using standard unit vectors. We still have three components for each vector to substitute into the formula for the dot product:

$$\begin{aligned}\mathbf{p}\cdot\mathbf{q} &= u_1v_1 + u_2v_2 + u_3v_3\\ &= 10(-2) + (-4)(1) + (7)(6) = -20 - 4 + 42 = 18.\end{aligned}$$

2.21 Find $\mathbf{u}\cdot\mathbf{v}$, where $\mathbf{u} = \langle 2, 9, -1 \rangle$ and $\mathbf{v} = \langle -3, 1, -4 \rangle$.

Like vector addition and subtraction, the dot product has several algebraic properties. We prove three of these properties and leave the rest as exercises.

can. distribute product into a sum

Theorem 2.3: Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors, and let c be a scalar.

i.	$\mathbf{u}\cdot\mathbf{v} =$	$\mathbf{v}\cdot\mathbf{u}$	Commutative property
ii.	$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) =$	$\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$	Distributive property
iii.	$c(\mathbf{u}\cdot\mathbf{v}) =$	$(c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v})$	Associative property
iv.	$\mathbf{v}\cdot\mathbf{v} =$	$\|\mathbf{v}\|^2$	Property of magnitude

cu•v
its like
2*3*10 = 60
(2*3)*10 = 60
2*(3*10) = 60

please don't accidentally distribute product into product

Proof

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle\\ &= u_1v_1 + u_2v_2 + u_3v_3\\ &= v_1u_1 + v_2u_2 + v_3u_3\\ &= \langle v_1, v_2, v_3 \rangle \cdot \langle u_1, u_2, u_3 \rangle\\ &= \mathbf{v}\cdot\mathbf{u}.\end{aligned}$$

The associative property looks like the associative property for real-number multiplication, but pay close attention to the difference between scalar and vector objects:

$$\begin{aligned}c(\mathbf{u}\cdot\mathbf{v}) &= c(u_1v_1 + u_2v_2 + u_3v_3)\\ &= c(u_1v_1) + c(u_2v_2) + c(u_3v_3)\\ &= (cu_1)v_1 + (cu_2)v_2 + (cu_3)v_3\\ &= \langle cu_1, cu_2, cu_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle\\ &= c\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle\\ &= (c\mathbf{u})\cdot\mathbf{v}.\end{aligned}$$

The proof that $c(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\cdot(c\mathbf{v})$ is similar.

The fourth property shows the relationship between the magnitude of a vector and its dot product with itself:

$$\begin{aligned}\mathbf{v}\cdot\mathbf{v} &= \langle v_1, v_2, v_3\rangle \cdot \langle v_1, v_2, v_3\rangle \\ &= (v_1)^2 + (v_2)^2 + (v_3)^2 \\ &= \left[\sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}\right]^2 \\ &= \|\mathbf{v}\|^2.\end{aligned}$$

□

Note that the definition of the dot product yields $0 \cdot \mathbf{v} = \mathbf{0}$. By property iv., if $\mathbf{v} \cdot \mathbf{v} = 0$, then $\mathbf{v} = 0$.

Example 2.22

Using Properties of the Dot Product

Let $\mathbf{a} = \langle 1, 2, -3 \rangle$, $\mathbf{b} = \langle 0, 2, 4 \rangle$, and $\mathbf{c} = \langle 5, -1, 3 \rangle$. Find each of the following products.

a. $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

b. $\mathbf{a}\cdot(2\mathbf{c})$

c. $\|\mathbf{b}\|^2$

Solution

a. Note that this expression asks for the scalar multiple of **c** by $\mathbf{a}\cdot\mathbf{b}$:

$$\begin{aligned}(\mathbf{a}\cdot\mathbf{b})\mathbf{c} &= (\langle 1, 2, -3\rangle \cdot \langle 0, 2, 4\rangle)\langle 5, -1, 3\rangle \\ &= (1(0) + 2(2) + (-3)(4))\langle 5, -1, 3\rangle \\ &= -8\langle 5, -1, 3\rangle \\ &= \langle -40, 8, -24\rangle.\end{aligned}$$

b. This expression is a dot product of vector **a** and scalar multiple 2**c**:

$$\begin{aligned}\mathbf{a}\cdot(2\mathbf{c}) &= 2(\mathbf{a}\cdot\mathbf{c}) \\ &= 2(\langle 1, 2, -3\rangle \cdot \langle 5, -1, 3\rangle) \\ &= 2(1(5) + 2(-1) + (-3)(3)) \\ &= 2(-6) = -12.\end{aligned}$$

c. Simplifying this expression is a straightforward application of the dot product:

$$\|\mathbf{b}\|^2 = \mathbf{b}\cdot\mathbf{b} = \langle 0, 2, 4\rangle \cdot \langle 0, 2, 4\rangle = 0^2 + 2^2 + 4^2 = 0 + 4 + 16 = 20.$$

2.22 Find the following products for $\mathbf{p} = \langle 7, 0, 2 \rangle$, $\mathbf{q} = \langle -2, 2, -2 \rangle$, and $\mathbf{r} = \langle 0, 2, -3 \rangle$.

a. $(\mathbf{r}\cdot\mathbf{p})\mathbf{q}$

b. $\|\mathbf{p}\|^2$

Using the Dot Product to Find the Angle between Two Vectors

When two nonzero vectors are placed in standard position, whether in two dimensions or three dimensions, they form an angle between them (**Figure 2.44**). The dot product provides a way to find the measure of this angle. This property is a result of the fact that we can express the dot product in terms of the cosine of the angle formed by two vectors.

Proof of property 4 (magnitude property of dot product)

Want to show: $v \cdot v = \|v\|^2$

Let's show $\langle a,b,c\rangle \cdot \langle a,b,c\rangle = \|\langle a,b,c\rangle\|^2$

By definition of magnitude of a vector $\sqrt{a^2+b^2+c^2} = \|\langle a,b,c\rangle\|$

Square both sides $a^2+b^2+c^2 = \|\langle a,b,c\rangle\|^2$

By definition of dot product $\langle a,b,c\rangle \cdot \langle a,b,c\rangle = a^2+b^2+c^2$

Therefore $\langle a,b,c\rangle \cdot \langle a,b,c\rangle = \|\langle a,b,c\rangle\|^2$ ∎

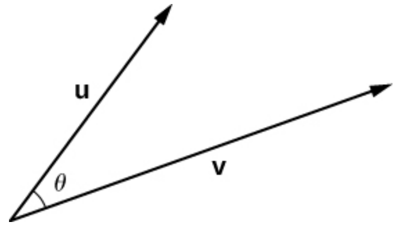


Figure 2.44 Let θ be the angle between two nonzero vectors **u** and **v** such that $0 \le \theta \le \pi$.

Theorem 2.4: Evaluating a Dot Product

The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta. \tag{2.4}$$

Proof

Place vectors **u** and **v** in standard position and consider the vector $\mathbf{v} - \mathbf{u}$ (**Figure 2.45**). These three vectors form a triangle with side lengths $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{v} - \mathbf{u}\|$.

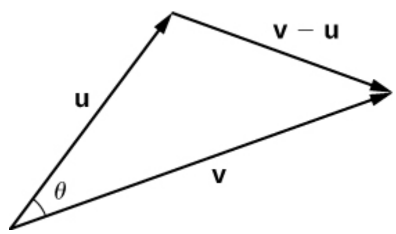


Figure 2.45 The lengths of the sides of the triangle are given by the magnitudes of the vectors that form the triangle.

Recall from trigonometry that the law of cosines describes the relationship among the side lengths of the triangle and the angle θ. Applying the law of cosines here gives

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta.$$

The dot product provides a way to rewrite the left side of this equation:

$$\begin{aligned}
\|\mathbf{v} - \mathbf{u}\|^2 &= (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) \\
&= (\mathbf{v} - \mathbf{u}) \cdot \mathbf{v} - (\mathbf{v} - \mathbf{u}) \cdot \mathbf{u} \\
&= \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} \\
&= \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{u} \\
&= \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2.
\end{aligned}$$

Substituting into the law of cosines yields

$$\begin{aligned}
\|\mathbf{v} - \mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta \\
\|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta \\
-2\mathbf{u} \cdot \mathbf{v} &= -2\|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta \\
\mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta.
\end{aligned}$$

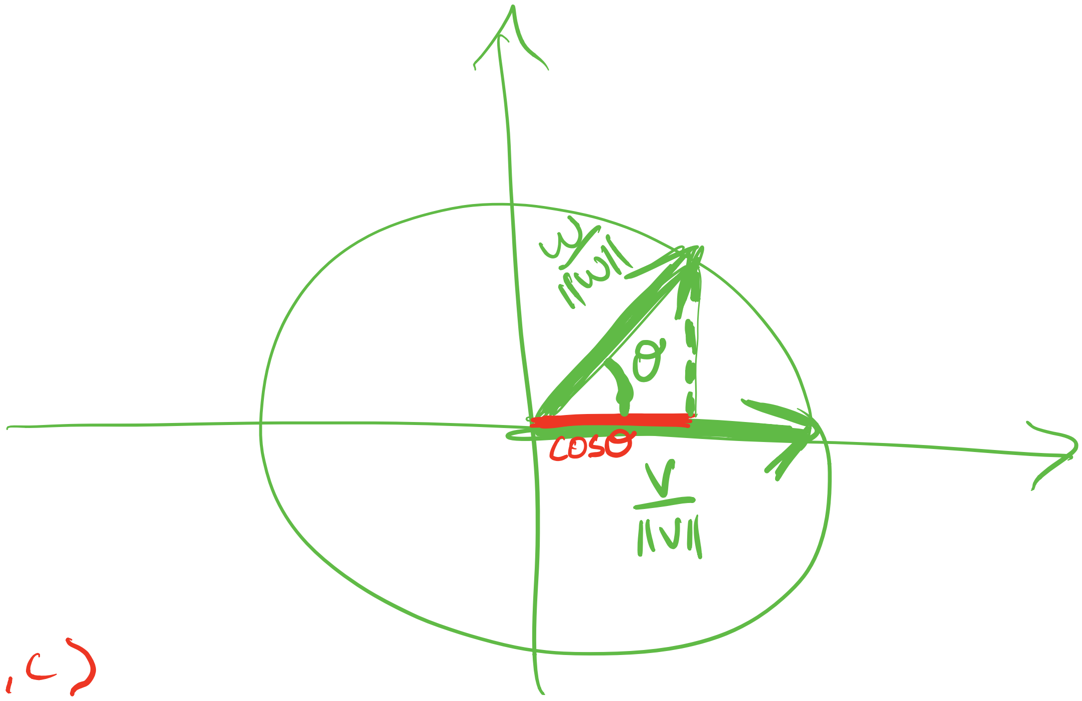
□

We can use this form of the dot product to find the measure of the angle between two nonzero vectors. The following equation rearranges **Equation 2.3** to solve for the cosine of the angle:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}. \tag{2.5}$$

we're going to show this directly.

Show $\frac{V}{\|V\|} \cdot \frac{W}{\|W\|} = \cos\theta$



$V = \langle a, b, c \rangle$

$V = ai + bj + ck$

$i \cdot \langle a, b, c \rangle = a$

$j \cdot \langle a, b, c \rangle = b$

$k \cdot \langle a, b, c \rangle = c$