Use the class data as estimates of the following probabilities. P(change) means the probability that a randomly chosen person in your class has change in his/her pocket or purse. P(bus) means the probability that a randomly chosen person in your class rode a bus within the last month and so on. Discuss your answers.

- Find P(change).
- Find P(bus).
- Find P(change and bus) Find the probability that a randomly chosen student in your class has change in his/her pocket or purse and rode a bus within the last month.
- Find P(change | bus) Find the probability that a randomly chosen student has change given that he/she rode a bus within the last month. Count all the students that rode a bus. From the group of students who rode a bus, count those who have change. The probability is equal to those who have change and rode a bus divided by those who rode a bus.

3.2 Terminology[2]

Probability is a measure that is associated with how certain we are of outcomes of a particular experiment or activity. An **experiment** is a planned operation carried out under controlled conditions. If the result is not predetermined, then the experiment is said to be a **chance** experiment. Flipping one fair coin twice is an example of an experiment.

(or random experiment)

list all the outcomes

The result of an experiment is called an **outcome**. A **sample space** is a set of all possible outcomes. Three ways to represent a sample space are to list the possible outcomes, to create a tree diagram, or to create a Venn diagram. The uppercase letter S is used to denote the sample space. For example, if you flip one fair coin, $S = \{H, T\}$ where H = heads and T = tails are the outcomes.

list some of the outcomes

An **event** is any combination of outcomes. Upper case letters like A and B represent events. For example, if the experiment is to flip one fair coin, event A might be getting at most one head. The probability of an event A is written $P(A)$. probability of A

we look at what actually happens when we simulate the random experiment

we compute using theory

The **probability** of any outcome is the **long-term relative frequency** of that outcome. **Probabilities are between 0 and 1, inclusive** (includes 0 and 1 and all numbers between these values). $P(A) = 0$ means the event A can never happen. $P(A) = 1$ means the event A always happens. $P(A) = 0.5$ means the event A is equally likely to occur or not to occur. For example, if you flip one fair coin repeatedly (from 20 to 2,000 to 20,000 times) the relative fequency of heads approaches 0.5 (the probability of heads).

repeat an experiment you get a data set

"Probability of A is 1" means The event is certain to happen (100% likelihood of happening)

mean event cannot happen (0% chance of happening)

probability means "percent likelihood" so cannot be below 0 (0%) nor above 1 (100%)

Equally likely means that each outcome of an experiment occurs with equal probability. For example, if you toss a **fair**, six-sided die, each face (1, 2, 3, 4, 5, or 6) is as likely to occur as any other face. If you toss a fair coin, a Head(H) and a Tail(T) are equally likely to occur. If you randomly guess the answer to a true/false question on an exam, you are equally likely to select a correct answer or an incorrect answer.

To calculate the probability of an event A **when all outcomes in the sample space are equally likely**, count the number of outcomes for event A and divide by the total number of outcomes in the sample space. For example, if you toss a fair dime and a fair nickel, the sample space is $\{HH, TH, HT, TT\}$ where T = tails and H = heads. The sample space has four outcomes. A = getting one head. There are two outcomes $\{HT, TH\}$. $P(A) = \frac{2}{4}$.

Example

experiment — set of all outcomes = sample space — the event they tell us to consider

Suppose you roll one fair six-sided die, with the numbers {1,2,3,4,5,6} on its faces. Let event E = rolling a number that is at least 5. There are two outcomes $\{5, 6\}$. $P(E) = \frac{2}{6}$. If you were to roll the die only a few times, you would not be surprised if your observed results did not match the probability. If you were to roll the die a very large number of times, you would expect that, overall, 2/6 of the rolls would result in an

[2]This content is available online at <http://cnx.org/content/m16845/1.13/>.

Sample Space $= S = \{1,2,3,4,5,6\}$

$E = \{5,6\}$

$$P(E) = \frac{\text{size}(E)}{\text{size}(S)} = \boxed{\frac{2}{6}} \approx 0.3333$$

- **With replacement**: If each member of a population is replaced after it is picked, then that member has the possibility of being chosen more than once. When sampling is done with replacement, then events are considered to be independent, meaning the result of the first pick will not change the probabilities for the second pick.
- **Without replacement:**: When sampling is done without replacement, then each member of a population may be chosen only once. In this case, the probabilities for the second pick are affected by the result of the first pick. The events are considered to be dependent or not independent.

If it is not known whether A and B are independent or dependent, **assume they are dependent until you can show otherwise**.

3.3.2 Mutually Exclusive Events

A and B are **mutually exclusive** events if they cannot occur at the same time. This means that A and B do not share any outcomes and P(A AND B) = 0.

For example, suppose the sample space $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, and $C = \{7, 9\}$. A AND B $= \{4,5\}$. P(A AND B) = $\frac{2}{10}$and is not equal to zero. Therefore, A and B are not mutually exclusive. A and C do not have any numbers in common so P(A AND C) = 0. Therefore, A and C are mutually exclusive.

If it is not known whether A and B are mutually exclusive, **assume they are not until you can show otherwise**.

The following examples illustrate these definitions and terms.

Example 3.1
Flip two fair coins. (This is an experiment.)

The sample space is $\{HH, HT, TH, TT\}$ where T = tails and H = heads. The outcomes are HH, HT, TH, and TT. The outcomes HT and TH are different. The HT means that the first coin showed heads and the second coin showed tails. The TH means that the first coin showed tails and the second coin showed heads.

- Let A = the event of getting **at most one tail**. (At most one tail means 0 or 1 tail.) Then A can be written as $\{HH, HT, TH\}$. The outcome HH shows 0 tails. HT and TH each show 1 tail.
- Let B = the event of getting all tails. B can be written as $\{TT\}$. B is the **complement** of A. So, $B = A'$. Also, $P(A) + P(B) = P(A) + P(A') = 1$.
- The probabilities for A and for B are $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{4}$.
- Let C = the event of getting all heads. $C = \{HH\}$. Since $B = \{TT\}$, $P(B\ AND\ C) = 0$. B and C are mutually exclusive. (B and C have no members in common because you cannot have all tails and all heads at the same time.)
- Let D = event of getting **more than one** tail. $D = \{TT\}$. $P(D) = \frac{1}{4}$.
- Let E = event of getting a head on the first roll. (This implies you can get either a head or tail on the second roll.) $E = \{HT, HH\}$. $P(E) = \frac{2}{4}$.
- Find the probability of getting **at least one** (1 or 2) tail in two flips. Let F = event of getting at least one tail in two flips. $F = \{HT, TH, TT\}$. P(F) $= \frac{3}{4}$

Example 3.2
Roll one fair 6-sided die. The sample space is $\{1, 2, 3, 4, 5, 6\}$. Let event A = a face is odd. Then $A = \{1, 3, 5\}$. Let event B = a face is even. Then $B = \{2, 4, 6\}$.

- Find the complement of A, A'. The complement of A, A', is B because A and B together make up the sample space. P(A) + P(B) = P(A) + P(A') = 1. Also, P(A) $= \frac{3}{6}$ and P(B) $= \frac{3}{6}$

- Let event C = odd faces larger than 2. Then $C = \{3,5\}$. Let event D = all even faces smaller than 5. Then $D = \{2,4\}$. P(C and D) $= 0$ because you cannot have an odd and even face at the same time. Therefore, C and D are mutually exclusive events.
- Let event E = all faces less than 5. $E = \{1,2,3,4\}$.

  Problem *(Solution on p. 160.)*

  Are C and E mutually exclusive events? (Answer yes or no.) Why or why not?
- Find P(C | A). This is a conditional. Recall that the event C is $\{3,\ 5\}$ and event A is $\{1,\ 3,\ 5\}$. To find P(C | A), find the probability of C using the sample space A. You have reduced the sample space from the original sample space $\{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$ to $\{1,\ 3,\ 5\}$. So, P(C | A) $= \frac{2}{3}$

Example 3.3

Let event G = taking a math class. Let event H = taking a science class. Then, G *AND* H = taking a math class and a science class. Suppose P(G) $= 0.6$, P(H) $= 0.5$, and P(G AND H) $= 0.3$. Are G and H independent?

If G and H are independent, then you must show **ONE** of the following:

- P(G | H) = P(G)
- P(H | G) = P(H)
- P(G AND H) = P(G) · P(H)

NOTE: **The choice you make depends on the information you have.** You could choose any of the methods here because you have the necessary information.

Problem 1

Show that P(G | H) = P(G).

Solution

$$\mathrm{P(G \mid H)} = \frac{\mathrm{P(G\ AND\ H)}}{\mathrm{P(H)}} = \frac{0.3}{0.5} = 0.6 = \mathrm{P(G)}$$

Problem 2

Show P(G AND H) = P(G) · P(H).

Solution

$$P\left(G\right) \cdot P\left(H\right) = 0.6 \cdot 0.5 = 0.3 = \mathrm{P(G\ AND\ H)}$$

Since G and H are independent, then, knowing that a person is taking a science class does not change the chance that he/she is taking math. If the two events had not been independent (that is, they are dependent) then knowing that a person is taking a science class would change the chance he/she is taking math. For practice, show that P(H | G) = P(H) to show that G and H are independent events.

Example 3.4

In a box there are 3 red cards and 5 blue cards. The red cards are marked with the numbers 1, 2, and 3, and the blue cards are marked with the numbers 1, 2, 3, 4, and 5. The cards are well-shuffled. You reach into the box (you cannot see into it) and draw one card.

Let R = red card is drawn, B = blue card is drawn, E = even-numbered card is drawn.

The sample space S = R1, R2, R3, B1, B2, B3, B4, B5. S has 8 outcomes.

- P(R) $= \frac{3}{8}$. P(B) $= \frac{5}{8}$. P(R AND B) $= 0$. (You cannot draw one card that is both red and blue.)

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Example 3.2

Roll one fair 6-sided die. The sample space is $\{1, 2, 3, 4, 5, 6\}$. Let event A = a face is odd. Then $A = \{1, 3, 5\}$. Let event B = a face is even. Then $B = \{2, 4, 6\}$.

- Find the complement of A, A'. The complement of A, A', is B because A and B together make up the sample space. P(A) + P(B) = P(A) + P(A′) = 1. Also, P(A) $= \frac{3}{6}$ and P(B) $= \frac{3}{6}$

- Let event C = odd faces larger than 2. Then $C = \{3, 5\}$. Let event D = all even faces smaller than 5. Then $D = \{2, 4\}$. P(C and D) $= 0$ because you cannot have an odd and even face at the same time. Therefore, C and D are mutually exclusive events.
- Let event E = all faces less than 5. $E = \{1, 2, 3, 4\}$.

Problem *(Solution on p. 160.)*

Are C and E mutually exclusive events? (Answer yes or no.) Why or why not?

- Find P(C | A). This is a conditional. Recall that the event C is $\{3, 5\}$ and event A is $\{1, 3, 5\}$. To find P(C | A), find the probability of C using the sample space A. You have reduced the sample space from the original sample space $\{1, 2, 3, 4, 5, 6\}$ to $\{1, 3, 5\}$. So, P(C | A) $= \frac{2}{3}$

The sample space $S = \{1,2,3,4,5,6\}$

A = "odd" $= \{1, 3, 5\}$

E = "less than 5" $= \{1, 2, 3, 4\}$

$P(A) = P(\text{"odd"}) = \boxed{\frac{3}{6}} = 0.5$

$P(E) = \boxed{\frac{4}{6}} \approx 0.6667$

$\boxed{\text{And}}$

$\boxed{\text{or}}$

$P(A \text{ and } E)$

$= P(\text{odd and less than 5})$

$= \boxed{\frac{2}{6}} \approx 0.3333$

$S = \{1, \cancel{2}, 3, \cancel{4}, \cancel{5}, \cancel{6}\}$

A and $E = \{1, 3\}$

for each outcome ask "is this odd or less than 5"?

$S = \{1, 2, 3, 4, 5, 6\}$

is 6 "odd" or "less than 5"?

no, neither.

A or E = odd or less than 5 $= \{1, 2, 3, 4, 5\}$

$P(A \text{ or } E) = \frac{5}{6} \approx 0.8333$

to summarize this example

sample space $S = \{1, 2, 3, 4, 5, 6\}$

"odd" $A = \{1, 3, 5\}$ $P(A) = 3/6$

"<5" $E = \{1, 2, 3, 4\}$ $P(E) = 4/6$

$A \text{ and } E = \{1, 3\}$ $P(A \text{ and } E) = 2/6$

$A \text{ or } E = \{1, 2, 3, 4, 5\}$ $P(A \text{ or } E) = 5/6$

complement

$\text{not } A = \{2, 4, 6\}$ $P(\text{not } A) = 3/6$

$\text{not } E = \{5, 6\}$ $P(\text{not } E) = 2/6$

$$P(\text{not } E) + P(E) = 6/6 = 1$$

$$P(\text{not } A) + P(A) = 6/6 = 1$$

The complement rule

$$\boxed{P(\text{not } A) = 1 - P(A)}$$

$$P(\text{not } A) + P(A) = 1$$
$$P(\text{not } A) = 1 - P(A)$$

these equations ~~are~~ are equivalent; we subtracted $P(A)$ from both sides

exercise: Suppose $P(G) = 0.2$. What is $P(\text{not } G)$?

$$P(\text{not } G) = 1 - 0.2 = \boxed{0.8}$$

The OR rule

(an example using same events as above)

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 3, 5\}$ $\qquad P(A) = 3/6$

$E = \{1, 2, 3, 4\}$ $\qquad P(E) = 4/6$

"intersection" $A \text{ and } E = \{1, 3\}$ $\qquad P(A \text{ and } E) = 2/6$

$A \text{ or } E$ "the union" $= \{1, 2, 3, 4, 5\}$ $\qquad P(A \text{ or } E) = 5/6$

$\overset{3/6}{P(A)} + \overset{4/6}{P(E)} = 7/6$

$3 + 4 = 7$ instead of 5

why?

Because some outcomes (1 & 3) are in both sets and we don't need to count them twice

So to compute the union,

$$\boxed{P(A) + P(E) - P(A \text{ and } E) = P(A \text{ or } E)}$$

The OR rule

let's check on the example above

$$\frac{3}{6} + \frac{4}{6} - \frac{2}{6} = \frac{5}{6}$$

it works!

- pick a student at random, what's the chance they're female?
- what proportion is female?
(Probability also means proportion of sample space)

A = are female $P(A) = 0.60$
B = have long hair $P(B) = 0.50$

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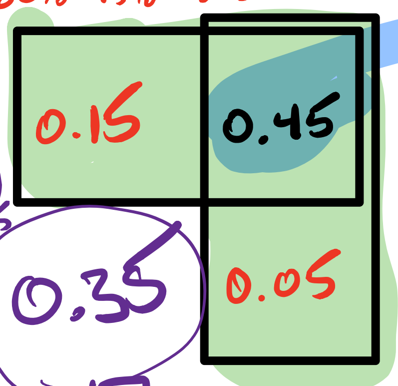
Example 3.5
In a particular college class, 60% of the students are female. 50 % of all students in the class have long hair. 45% of the students are female and have long hair. Of the female students, 75% have

$P(A \text{ and } B) = 0.45$
intersection

$P(B) = 0.50$

60% − 45% = 15%

A



$P(A) = 0.60$

B

$P(\text{not}(A \text{ or } B))$
=
$P(\text{not} A \text{ and not} B)$
$0.15 + 0.45 + 0.05 = 0.65$
$1 - 0.65 = 0.35$
0.35
since all 4 squares add to $\boxed{1}$

$0.50 - 0.45 = 0.05$
$P(B) - P(A \text{ and } B) = P(\text{not} A \text{ and } B)$
$= 0.05$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
$$= 0.60 + 0.50 - 0.45 = 0.65$$

also could have used diagram we made
$0.15 + 0.45 + 0.05 = 0.65$

"De Morgan's Law"
<u>Notice</u> (not A) and (not B) = not (A or B)

to help remember: no eating and no swimming = no eating or swimming.

- P(E) $= \frac{3}{8}$. (There are 3 even-numbered cards, $R2$, $B2$, and $B4$.)
- P(E | B) $= \frac{2}{5}$. (There are 5 blue cards: $B1$, $B2$, $B3$, $B4$, and $B5$. Out of the blue cards, there are 2 even cards: $B2$ and $B4$.)
- P(B | E) $= \frac{2}{3}$. (There are 3 even-numbered cards: $R2$, $B2$, and $B4$. Out of the even-numbered cards, 2 are blue: $B2$ and $B4$.)
- The events R and B are mutually exclusive because P(R AND B) $= 0$.
- Let G = card with a number greater than 3. $G = \{B4, B5\}$. P(G) $= \frac{2}{8}$. Let H = blue card numbered between 1 and 4, inclusive. $H = \{B1, B2, B3, B4\}$. P(G | H) $= \frac{1}{4}$. (The only card in H that has a number greater than 3 is $B4$.) Since $\frac{2}{8} = \frac{1}{4}$, P(G) $=$ P(G | H) which means that G and H are independent.

Example 3.5
In a particular college class, 60% of the students are female. 50 % of all students in the class have long hair. 45% of the students are female and have long hair. Of the female students, 75% have long hair. Let F be the event that the student is female. Let L be the event that the student has long hair. One student is picked randomly. Are the events of being female and having long hair independent?

- The following probabilities are given in this example:
- P(F) $= 0.60$; P(L) $= 0.50$
- P(F AND L) $= 0.45$
- P(L | F) $= 0.75$

NOTE: **The choice you make depends on the information you have.** You could use the first or last condition on the list for this example. You do not know P(F | L) yet, so you can not use the second condition.

Solution 1
Check whether P(F and L) = P(F)P(L): We are given that P(F and L) = 0.45 ; but P(F)P(L) = (0.60)(0.50)= 0.30 The events of being female and having long hair are not independent because P(F and L) does not equal P(F)P(L).

Solution 2
check whether P(L | F) equals P(L): We are given that P(L | F) = 0.75 but P(L) = 0.50; they are not equal. The events of being female and having long hair are not independent.

Interpretation of Results
The events of being female and having long hair are not independent; knowing that a student is female changes the probability that a student has long hair.

**Example 5 contributed by Roberta Bloom

3.4 Two Basic Rules of Probability[4]

3.4.1 The Multiplication Rule

If A and B are two events defined on a **sample space**, then: P(A AND B) $=$ P(B) $\cdot$ P(A | B).

This rule may also be written as : $P(A|B) = \frac{P(A\ AND\ B)}{P(B)}$

[4]This content is available online at <http://cnx.org/content/m16847/1.11/>.

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3.3.2 Mutually Exclusive Events

A and *B* are **mutually exclusive** events if they cannot occur at the same time. This means that *A* and *B* do not share any outcomes and P(A AND B) = 0.

event usually are not mutually exclusive, and we have to check whether they are if asked

example   Roll a fair die.

$S = \{1, 2, 3, 4, 5, 6\}$

even = $E = \{2, 4, 6\}$

odd = $O = \{1, 3, 5\}$

prime (exactly 2 factors) = $R = \{2, 3, 5\}$

square (an odd number of factors) = $Q = \{1, 4\}$

Factors

1:	1			
2:	1	2		
3:	1	3		
4:	1	2	4	
5:	1	5		
6:	1	2	3	6
~~7:~~	~~1~~	~~7~~		

prime means exactly 2 factors

a factor is a number that goes in evenly (all the "nice" denominators for that number)

$R \text{ and } Q = \{\}$

$P(R \text{ and } Q) = 0/6 = 0$

So R, Q are ~~aren't~~ mutually exclusive also $P(E \text{ and } \emptyset) = 0$,

so $E, \emptyset$ are mutually exclusive.

Notice $\text{not } E = \emptyset$.

An ~~Every~~ event E and its complement $\text{not } E$ are always ~~sometimes/never~~ mutually exclusive?

in other words

FACT

$$\boxed{P(E \text{ and not } E) = 0}$$

an event cannot happens and not happen at the same time.

FACT 2

$$\boxed{P(E \text{ or not } E) = 1}$$

an event always happens or doesn't

Because no matter what

Let's use the OR rule on E, not E

$$\underbrace{P(E) + P(\text{not } E)}_{\text{by the complement rule}} - \underbrace{P(E \text{ and not } E)}_{\text{since an event \& its complement are mutually exclusive}} = P(E \text{ or not } E)$$

$$1 - 0 = P(E \text{ or not } E)$$

$$\boxed{1 = P(E \text{ or not } E)}$$

proof of FACT 2

FACT 3

$P(A) + P(B) - P(A \text{ and } B) = P(A \text{ or } B)$

simplifies to $\boxed{P(A) + P(B) = P(A \text{ or } B)}$

if and only if $\boxed{P(A \text{ and } B) = 0}$ $\Updownarrow$ if and only if

See p128

3.4.2 The Addition Rule

If *A* and *B* are defined on a sample space, then: P(A OR B) = P(A) + P(B) − P(A AND B).

If *A* and *B* are **mutually exclusive**, then P(A AND B) = 0. Then P(A OR B) = P(A) + P(B) − P(A AND B) becomes P(A OR B) = P(A) + P(B).

3.8 Summary of Formulas[8]

Formula 3.1: Complement
If A and A' are complements then $P(A) + \mathrm{P(A')} = 1$

Formula 3.2: Addition Rule
$\mathrm{P(A\ OR\ B)} = \mathrm{P(A)} + \mathrm{P(B)} - \mathrm{P(A\ AND\ B)}$

Formula 3.3: Mutually Exclusive
If A and B are mutually exclusive then $\mathrm{P(A\ AND\ B)} = 0$; so $\mathrm{P(A\ OR\ B)} = \mathrm{P(A)} + \mathrm{P(B)}$.

Formula 3.4: Multiplication Rule

- $\mathrm{P(A\ AND\ B)} = \mathrm{P(B)P(A|B)}$
- $\mathrm{P(A\ AND\ B)} = \mathrm{P(A)P(B|A)}$

Formula 3.5: Independence
If A and B are independent then:

- $\mathrm{P(A|B)} = \mathrm{P(A)}$
- $\mathrm{P(B|A)} = \mathrm{P(B)}$
- $\mathrm{P(A\ AND\ B)} = \mathrm{P(A)P(B)}$

[8]This content is available online at <http://cnx.org/content/m16843/1.5/>.