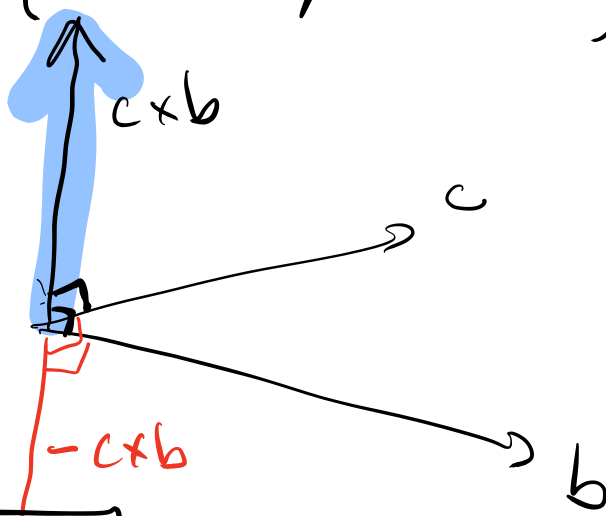
§ 2.4

Geometric/intuitive definition of cross product (vector product)



The $\boxed{\text{cross product}}$ $c \times b$ is the unique vector ① orthogonal to both c & b, ② chosen according to the right hand rule (by convention)

(p167)

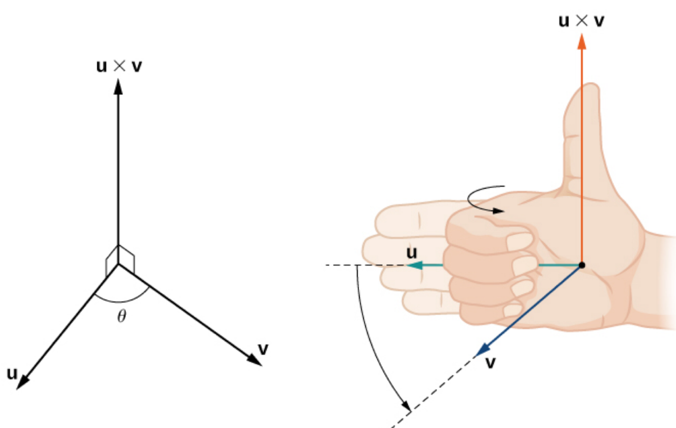
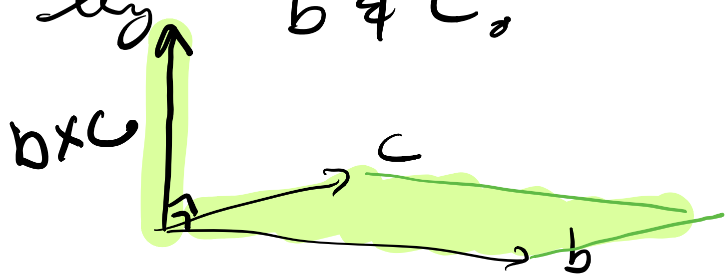


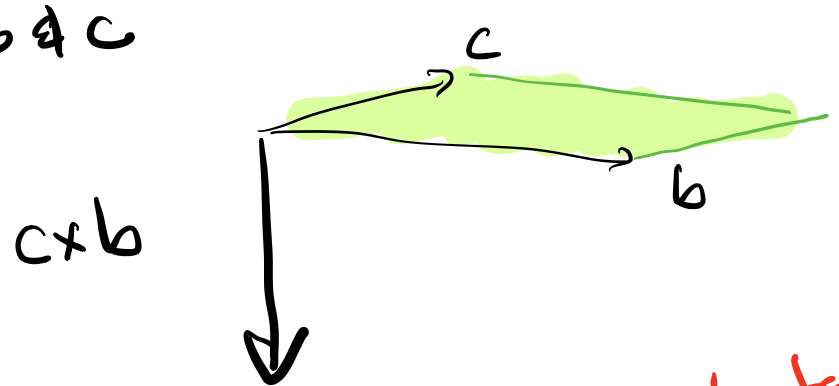
Figure 2.54 The direction of $\mathbf{u} \times \mathbf{v}$ is determined by the right-hand rule.

and ③ has length (magnitude) equal to the area of the parallelogram formed by b & c;



↳ because of this definition:

- switch b & c



so $\boxed{c \times b = -b \times c}$

we only talk about cross product in 3D

- b×c only exists & is unique in 3-space

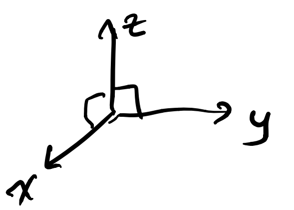
  - in 2-space

cannot find any vector orthogonal to both

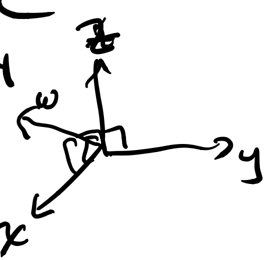
in 2D there's not enough space

  - in 4-space there are too many possible vectors orthogonal to b & c

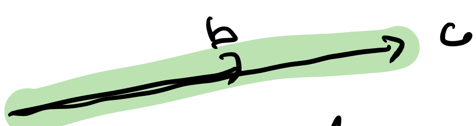
ie $\mathbb{R}^3$



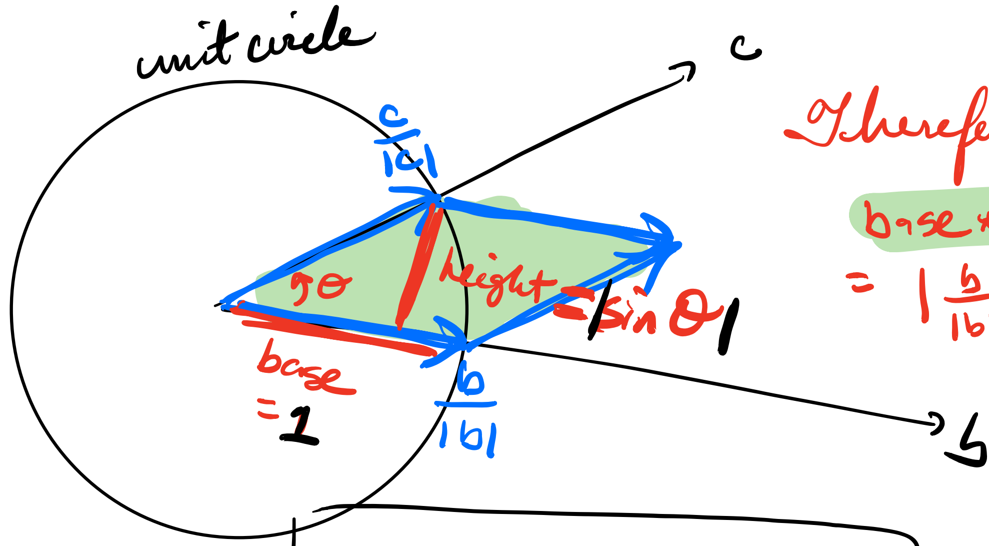
$\mathbb{R}^4$



- if b & c are co-linear, the $b \times c = \langle 0,0,0 \rangle = \mathbf{0}$



the parallelogram is a line, area = 0



Therefore

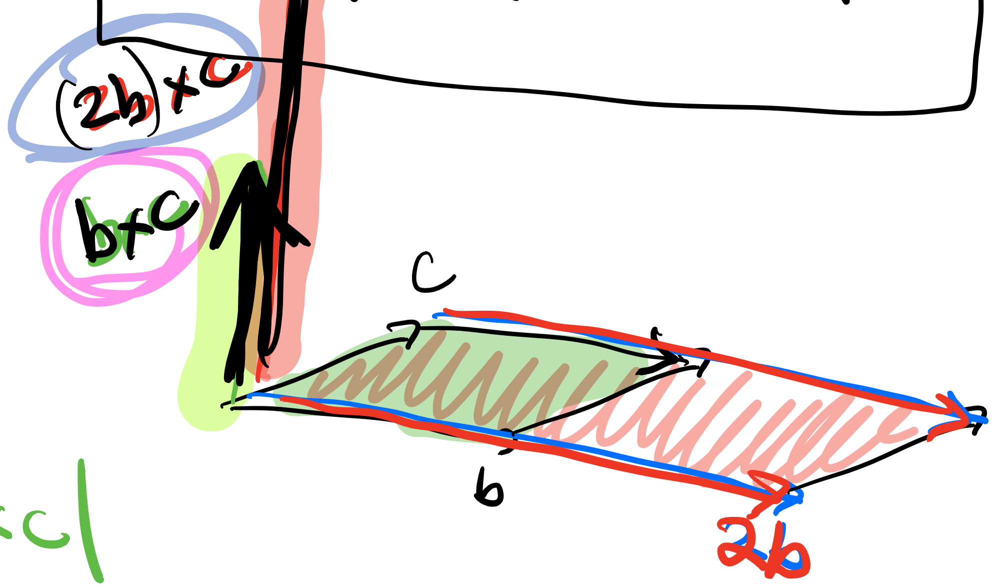
$\text{base} * \text{height} = 1 * |\sin\theta|$

$= \left| \frac{b}{|b|} \times \frac{c}{|c|} \right|$

$$\boxed{1 * |\sin\theta| = \left| \frac{b}{|b|} \times \frac{c}{|c|} \right|}$$

$$\boxed{|\sin\theta| = \frac{1}{|b|}\frac{1}{|c|}\,|b \times c|}$$

$$\boxed{|b|\,|c|\,|\sin\theta| = |b \times c|}$$



$|b \times c|$

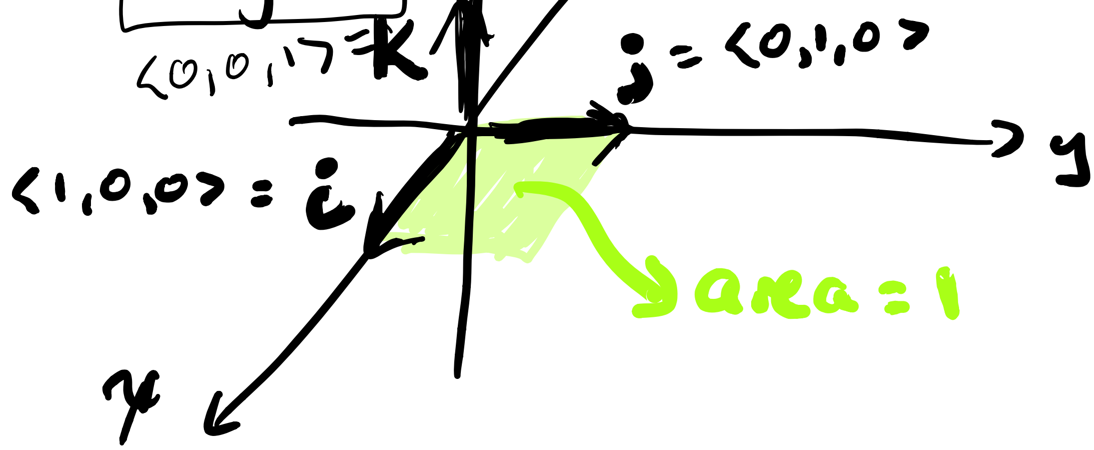
$|2b \times c| = 2\,|b \times c|$

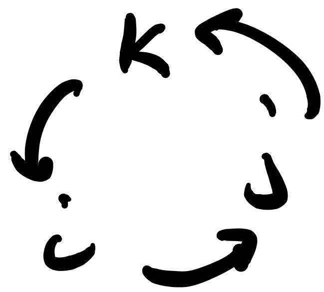
$(2b) \times c = 2\,(b \times c)$

$\Rightarrow$ Multilinearity of cross product

$b \times c = (2b) \times c = b \times (2c)$

– $\boxed{i \times j = k}$

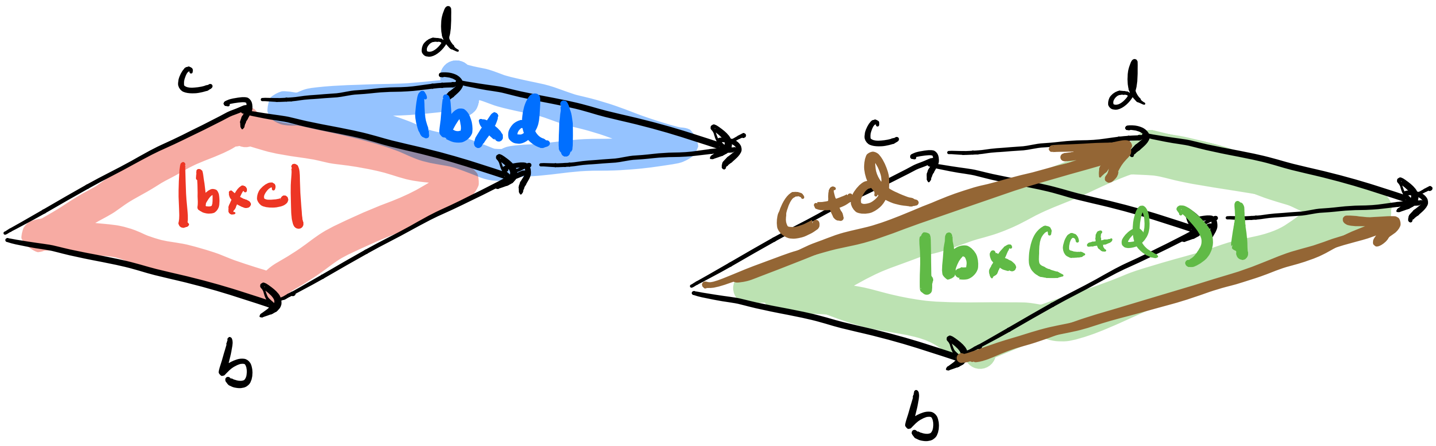




so

$i \times j = k$	$j \times i = -k$
$j \times k = i$	$k \times j = -i$
$k \times i = j$	$i \times k = -j$

–



It turns out $b \times (c+d) = b \times c + b \times d$

– "Triple prodect"

$$a\cdot(b\times c) = (a_1 i + a_2 j + a_3 k)\cdot\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\langle a_1, a_2, a_3\rangle$

$\langle b_1, b_2, b_3\rangle$

$\langle c_1, c_2, c_3\rangle$

$$(a_1 i + a_2 j + a_3 k)\cdot\big((\cdots)i + (\cdots)j + (\cdots)k\big)$$

$$(\cdots)a_1 + (\cdots)a_2 + (\cdots)a_3$$

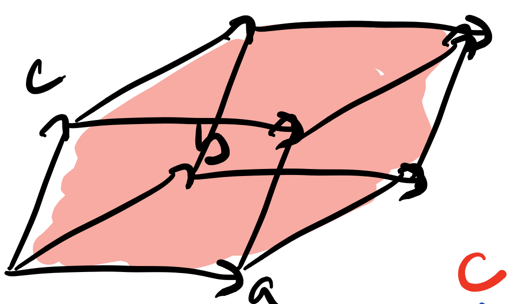
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

triple prodect $a\cdot(b\times c)$

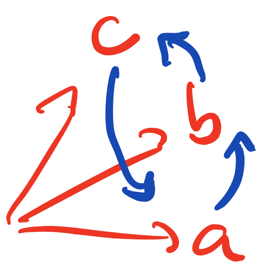
takes 3 vectus $\longrightarrow$ a number

b/c dot product happens last & gives a number

b/c $a\cdot(b\times c) =$ 3×3 determinant, which gives a number



signed area $= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$



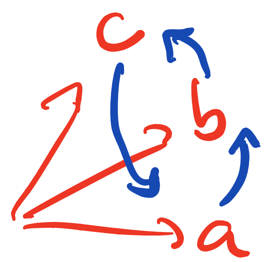
same sig as long as it goes

$= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$

$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

all equal

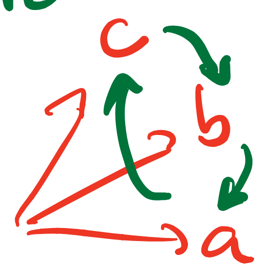
(Facts about triple product)



$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a\cdot(b\times c) = b\cdot(c\times a) = c\cdot(a\times b)$$

reverse the orientation...



$$\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad \text{etc} \qquad \text{etc.}$$

$$a\cdot(c\times b)$$
$$= -\,a\cdot(b\times c)$$

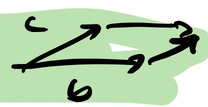
Illustration of why the geometric definition of $b\times c$ is equivalent to $b\times c := \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot (a_1 i + a_2 j + a_3 k)$$

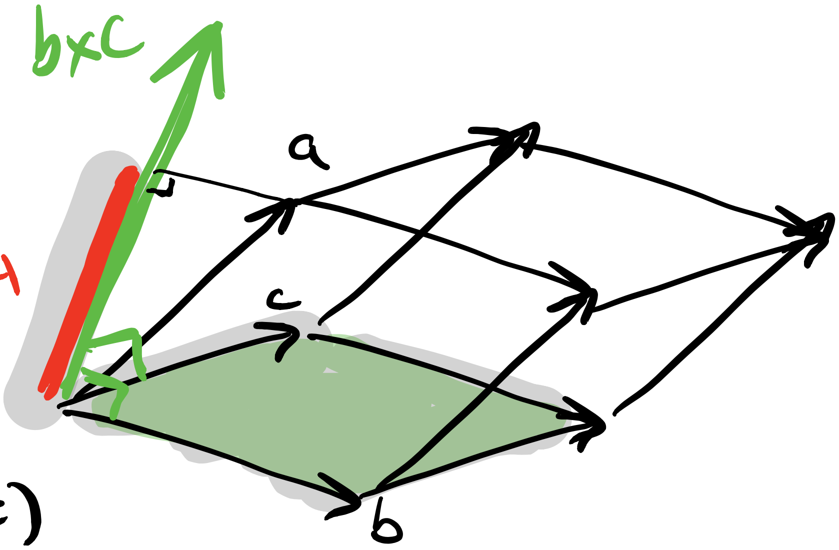
$$= \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot a$$

Goal is to show $\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ must be $b\times c$

using the fact that $(b\times c)\perp b$ and $(b\times c)\perp c$

and $|b\times c|$ is area of 

from last class on dot product



$a \cdot \frac{b\times c}{|b\times c|}$

$\cancel{a \cdot (b\times c)}$

$\cancel{= (b\times c)\cdot a}$

$\cancel{=}$

$$\boxed{a\cdot(b\times c) =} \underbrace{a\cdot\frac{b\times c}{\cancel{|b\times c|}}}_{\text{height of parallelepiped}} * \underbrace{\cancel{|b\times c|}}_{\text{area of base}}$$

$$\boxed{= \text{signed volume of parallel piped}}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Asside

then Vol = 70 cu ft

base area * height

from above (previous page)

from this page

$$a \cdot \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a\cdot(b\times c)$$

therefore $\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ has to equal $(b\times c)$

of $\mathbf{i}$ and $\mathbf{j}$ is parallel to $\mathbf{k}$. Similarly, the vector product of $\mathbf{i}$ and $\mathbf{k}$ is parallel to $\mathbf{j}$, and the vector product of $\mathbf{j}$ and $\mathbf{k}$ is parallel to $\mathbf{i}$. We can use the right-hand rule to determine the direction of each product. Then we have

$$\begin{aligned} \mathbf{i}\times\mathbf{j} &= \mathbf{k} & \mathbf{j}\times\mathbf{i} &= -\mathbf{k} \\ \mathbf{j}\times\mathbf{k} &= \mathbf{i} & \mathbf{k}\times\mathbf{j} &= -\mathbf{i} \\ \mathbf{k}\times\mathbf{i} &= \mathbf{j} & \mathbf{i}\times\mathbf{k} &= -\mathbf{j}. \end{aligned}$$

These formulas come in handy later.

Example 2.33

Cross Product of Standard Unit Vectors

Find $\mathbf{i}\times(\mathbf{j}\times\mathbf{k})$.

Solution

We know that $\mathbf{j}\times\mathbf{k}=\mathbf{i}$. Therefore, $\mathbf{i}\times(\mathbf{j}\times\mathbf{k})=\mathbf{i}\times\mathbf{i}=\mathbf{0}$.

2.32 Find $(\mathbf{i}\times\mathbf{j})\times(\mathbf{k}\times\mathbf{i})$.

As we have seen, the dot product is often called the *scalar product* because it results in a scalar. The cross product results in a vector, so it is sometimes called the **vector product**. These operations are both versions of vector multiplication, but they have very different properties and applications. Let's explore some properties of the cross product. We prove only a few of them. Proofs of the other properties are left as exercises.

Theorem 2.6: Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space, and let c be a scalar.

i.	$\mathbf{u}\times\mathbf{v}$	$=$	$-(\mathbf{v}\times\mathbf{u})$	Anticommutative property
ii.	$\mathbf{u}\times(\mathbf{v}+\mathbf{w})$	$=$	$\mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$	Distributive property
iii.	$c(\mathbf{u}\times\mathbf{v})$	$=$	$(c\mathbf{u})\times\mathbf{v}=\mathbf{u}\times(c\mathbf{v})$	Multiplication by a constant
iv.	$\mathbf{u}\times\mathbf{0}$	$=$	$\mathbf{0}\times\mathbf{u}=\mathbf{0}$	Cross product of the zero vector
v.	$\mathbf{v}\times\mathbf{v}$	$=$	$\mathbf{0}$	Cross product of a vector with itself
vi.	$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$	$=$	$(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$	Scalar triple product

Proof

For property i., we want to show $\mathbf{u}\times\mathbf{v}=-(\mathbf{v}\times\mathbf{u})$. We have

$$\begin{aligned} \mathbf{u}\times\mathbf{v} &= \langle u_1, u_2, u_3\rangle \times \langle v_1, v_2, v_3\rangle \\ &= \langle u_2v_3-u_3v_2, -u_1v_3+u_3v_1, u_1v_2-u_2v_1\rangle \\ &= -\langle u_3v_2-u_2v_3, -u_3v_1+u_1v_3, u_2v_1-u_1v_2\rangle \\ &= -\langle v_1, v_2, v_3\rangle \times \langle u_1, u_2, u_3\rangle \\ &= -(\mathbf{v}\times\mathbf{u}). \end{aligned}$$

Unlike most operations we've seen, the cross product is not commutative. This makes sense if we think about the right-hand rule.

For property iv., this follows directly from the definition of the cross product. We have

$$\begin{aligned}\mathbf{u}\times\mathbf{0} &= \langle\, u_2(0)-u_3(0),\ -(u_2(0)-u_3(0)),\ u_1(0)-u_2(0)\,\rangle \\ &= \langle\, 0,\ 0,\ 0\,\rangle = \mathbf{0}.\end{aligned}$$

Then, by property i., $\mathbf{0}\times\mathbf{u}=\mathbf{0}$ as well. Remember that the dot product of a vector and the zero vector is the *scalar* 0, whereas the cross product of a vector with the zero vector is the *vector* $\mathbf{0}$.

Property vi. looks like the associative property, but note the change in operations:

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \mathbf{u}\cdot\langle\, v_2w_3-v_3w_2,\ -v_1w_3+v_3w_1,\ v_1w_2-v_2w_1\,\rangle \\ &= u_1(v_2w_3-v_3w_2)+u_2(-v_1w_3+v_3w_1)+u_3(v_1w_2-v_2w_1) \\ &= u_1v_2w_3-u_1v_3w_2-u_2v_1w_3+u_2v_3w_1+u_3v_1w_2-u_3v_2w_1 \\ &= (u_2v_3-u_3v_2)w_1+(u_3v_1-u_1v_3)w_2+(u_1v_2-u_2v_1)w_3 \\ &= \langle\, u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1\,\rangle\cdot\langle\, w_1,\ w_2,\ w_3\,\rangle \\ &= (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}.\end{aligned}$$

□

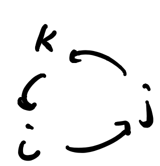
Example 2.34

Using the Properties of the Cross Product

Use the cross product properties to calculate $(2\mathbf{i}\times 3\mathbf{j})\times\mathbf{j}$.

Solution

$$\begin{aligned}&(2i\times 3j)\times j \\ &= 2(i\times 3j)\times j \\ &= 6(i\times j)\times j \\ &= 6\ k\times j \\ &= 6(-i) = \boxed{-6i}\end{aligned}$$



2.33 Use the properties of the cross product to calculate $(\mathbf{i}\times\mathbf{k})\times(\mathbf{k}\times\mathbf{j})$.

So far in this section, we have been concerned with the direction of the vector $\mathbf{u}\times\mathbf{v}$, but we have not discussed its magnitude. It turns out there is a simple expression for the magnitude of $\mathbf{u}\times\mathbf{v}$ involving the magnitudes of $\mathbf{u}$ and $\mathbf{v}$, and the sine of the angle between them.

Theorem 2.7: Magnitude of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors, and let θ be the angle between them. Then, $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\cdot\|\mathbf{v}\|\cdot\sin\theta.$

Proof

Let $\mathbf{u}=\langle\, u_1, u_2, u_3\,\rangle$ and $\mathbf{v}=\langle\, v_1, v_2, v_3\,\rangle$ be vectors, and let θ denote the angle between them. Then

$$\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 3(1) - 5(-2) = 3 + 10 = 13.$$

A 3×3 determinant is defined in terms of 2×2 determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \quad (2.10)$$

Equation 2.10 is referred to as the *expansion of the determinant along the first row*. Notice that the multipliers of each of the 2×2 determinants on the right side of this expression are the entries in the first row of the 3×3 determinant. Furthermore, each of the 2×2 determinants contains the entries from the 3×3 determinant that would remain if you crossed out the row and column containing the multiplier. Thus, for the first term on the right, a_1 is the multiplier, and the 2×2 determinant contains the entries that remain if you cross out the first row and first column of the 3×3 determinant. Similarly, for the second term, the multiplier is a_2, and the 2×2 determinant contains the entries that remain if you cross out the first row and second column of the 3×3 determinant. Notice, however, that the coefficient of the second term is negative. The third term can be calculated in similar fashion.

Example 2.36

Using Expansion Along the First Row to Compute a 3×3 Determinant

Evaluate the determinant $\begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix}$.

Deteminant
3 vectors → number

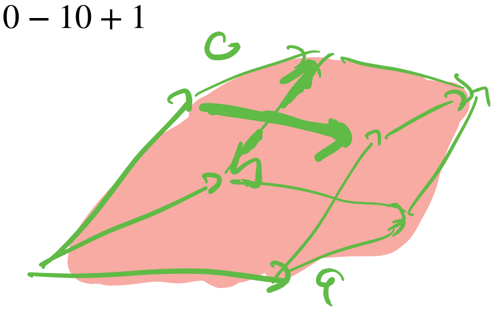
the determinant gives you a number from 3 vectors.

Solution

We have

a = <2, 5, -1>
b = <-1, 1, 3>
c = <-2, 3, 4>

$$\begin{aligned} \begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix} &= 2\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} - 5\begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix} \\ &= 2(4-9) - 5(-4+6) - 1(-3+2) \\ &= 2(-5) - 5(2) - 1(-1) = -10 - 10 + 1 \\ &= -19. \end{aligned}$$



2.35 Evaluate the determinant $\begin{vmatrix} 1 & -2 & -1 \\ 3 & 2 & -3 \\ 1 & 5 & 4 \end{vmatrix}$.

Technically, determinants are defined only in terms of arrays of real numbers. However, the determinant notation provides a useful mnemonic device for the cross product formula.

Formula for cross product.

Rule: Cross Product Calculated by a Determinant

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors. Then the cross product $\mathbf{u} \times \mathbf{v}$ is given by

$$\Rightarrow \quad \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}.$$

Example 2.37

Cross product 2 vectors → another vector

Using Determinant Notation to find $\mathbf{p} \times \mathbf{q}$

Let $\mathbf{p} = \langle -1, 2, 5 \rangle$ and $\mathbf{q} = \langle 4, 0, -3 \rangle$. Find $\mathbf{p} \times \mathbf{q}$.

Solution

we covered determinant last class

We set up our determinant by putting the standard unit vectors across the first row, the components of $\mathbf{u}$ in the second row, and the components of $\mathbf{v}$ in the third row. Then, we have

$$p \times q := \begin{vmatrix} i & j & k \\ -1 & 2 & 5 \\ 4 & 0 & -3 \end{vmatrix} = (-6-0)i - (\overset{-17}{3-20})j + (0-8)k$$

$$= \boxed{-6i + 17j - 8k}$$

Notice that this answer confirms the calculation of the cross product in **Example 2.31**.

2.36 Use determinant notation to find $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = \langle 8, 2, 3 \rangle$ and $\mathbf{b} = \langle -1, 0, 4 \rangle$.

Using the Cross Product

The cross product is very useful for several types of calculations, including finding a vector orthogonal to two given vectors, computing areas of triangles and parallelograms, and even determining the volume of the three-dimensional geometric shape made of parallelograms known as a *parallelepiped*. The following examples illustrate these calculations.

Example 2.38

Finding a Unit Vector Orthogonal to Two Given Vectors

Let $\mathbf{a} = \langle 5, 2, -1 \rangle$ and $\mathbf{b} = \langle 0, -1, 4 \rangle$. Find a unit vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Solution

The cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to both vectors $\mathbf{a}$ and $\mathbf{b}$. We can calculate it with a determinant:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & -1 \\ 0 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 5 & -1 \\ 0 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix}\mathbf{k} \\ &= (8-1)\mathbf{i} - (20-0)\mathbf{j} + (-5-0)\mathbf{k} \\ &= 7\mathbf{i} - 20\mathbf{j} - 5\mathbf{k}. \end{aligned}$$

Normalize this vector to find a unit vector in the same direction:

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{(7)^2 + (-20)^2 + (-5)^2} = \sqrt{474}.$$

Thus, $\left\langle \frac{7}{\sqrt{474}}, \frac{-20}{\sqrt{474}}, \frac{-5}{\sqrt{474}} \right\rangle$ is a unit vector orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

p173

Example 2.38

Finding a Unit Vector Orthogonal to Two Given Vectors

Let **a** = $\langle 5, 2, -1 \rangle$ and **b** = $\langle 0, -1, 4 \rangle$. Find a unit vector orthogonal to both **a** and **b**.

perpendicular

$a \times b$ will be $\perp$ to both a & b

(might not be unit length though)

$$a \times b = \begin{vmatrix} i & j & k \\ 5 & 2 & -1 \\ 0 & -1 & 4 \end{vmatrix} = (8-1)i - (20+0)j + (-5+0)k$$

$$= 7i - 20j - 5k$$

$$|a \times b| = \sqrt{7^2 + (-20)^2 + (-5)^2}$$

$$= \sqrt{474}$$

$$\frac{a \times b}{|a \times b|} = \frac{\langle 7, -20, -5 \rangle}{\sqrt{474}} = \boxed{\left\langle \frac{7}{\sqrt{474}}, -\frac{20}{\sqrt{474}}, -\frac{5}{\sqrt{474}} \right\rangle}$$

Theorem 2.8: Area of a Parallelogram

If we locate vectors $\mathbf{u}$ and $\mathbf{v}$ such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\mathbf{u}\times\mathbf{v}\|$ (Figure 2.57).

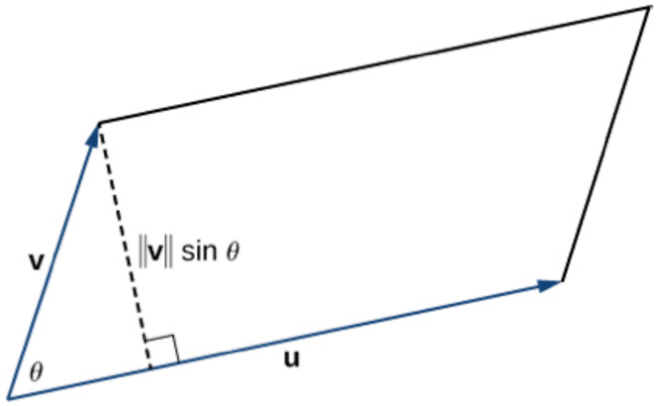


Figure 2.57 The parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$ has base $\|\mathbf{u}\|$ and height $\|\mathbf{v}\|\sin\theta$.

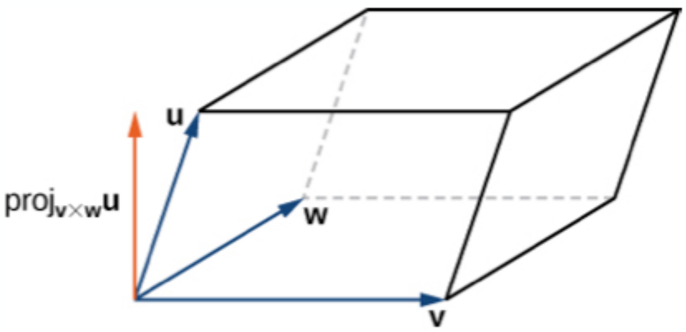


Figure 2.59 The height of the parallelepiped is given by $\|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\|$.

Theorem 2.10: Volume of a Parallelepiped

The volume of a parallelepiped with adjacent edges given by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is the absolute value of the triple scalar product:

$$V = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|.$$

See Figure 2.59.

Theorem 2.9: Calculating a Triple Scalar Product

The triple scalar product of vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ is the determinant of the 3×3 matrix formed by the components of the vectors:

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$