1.3 | Polar Coordinates

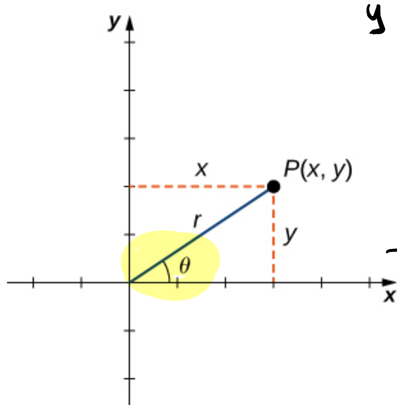
Learning Objectives

- **1.3.1** Locate points in a plane by using polar coordinates.
- **1.3.2** Convert points between rectangular and polar coordinates.
- **1.3.3** Sketch polar curves from given equations.
- **1.3.4** Convert equations between rectangular and polar coordinates.
- **1.3.5** Identify symmetry in polar curves and equations.

The rectangular coordinate system (or Cartesian plane) provides a means of mapping points to ordered pairs and ordered pairs to points. This is called a *one-to-one mapping* from points in the plane to ordered pairs. The polar coordinate system provides an alternative method of mapping points to ordered pairs. In this section we see that in some circumstances, polar coordinates can be more useful than rectangular coordinates.

Defining Polar Coordinates

To find the coordinates of a point in the polar coordinate system, consider **Figure 1.27**. The point P has Cartesian coordinates (x, y). The line segment connecting the origin to the point P measures the distance from the origin to P and has length r. The angle between the positive x-axis and the line segment has measure θ. This observation suggests a natural correspondence between the coordinate pair (x, y) and the values r and θ. This correspondence is the basis of the **polar coordinate system**. Note that every point in the Cartesian plane has two values (hence the term *ordered pair*) associated with it. In the polar coordinate system, each point also two values associated with it: r and θ.



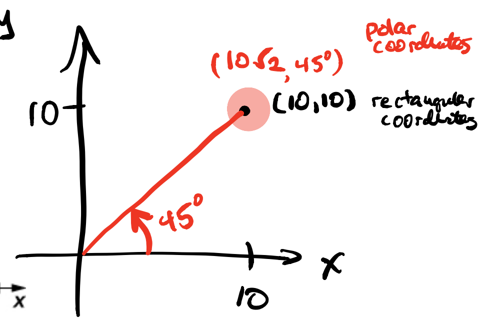


Figure 1.27 An arbitrary point in the Cartesian plane.

Using right-triangle trigonometry, the following equations are true for the point P:

$$\cos\theta = \frac{x}{r} \text{ so } x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \text{ so } y = r\sin\theta.$$

Furthermore, secretly Pythagorean ! ; secretly the distance formula.

$r = \sqrt{x^2 + y^2}$

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}.$$

Each point (x, y) in the Cartesian coordinate system can therefore be represented as an ordered pair (r, θ) in the polar coordinate system. The first coordinate is called the **radial coordinate** and the second coordinate is called the **angular coordinate**. Every point in the plane can be represented in this form.

Note that the equation $\tan\theta = y/x$ has an infinite number of solutions for any ordered pair (x, y). However, if we restrict the solutions to values between 0 and 2π then we can assign a unique solution to the quadrant in which the original point (x, y) is located. Then the corresponding value of r is positive, so $r^2 = x^2 + y^2$.

Theorem 1.4: Converting Points between Coordinate Systems

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$$x = r\cos\theta \text{ and } y = r\sin\theta, \tag{1.7}$$

$$r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}. \tag{1.8}$$

These formulas can be used to convert from rectangular to polar or from polar to rectangular coordinates.

Example 1.10

Converting between Rectangular and Polar Coordinates

Converting polar to rectangular

$(r, \theta) \longleftrightarrow (r\cos\theta, r\sin\theta)$

corresponds to

Convert each of the following points into polar coordinates.

a. $(1, 1)$

b. $(-3, 4)$

c. $(0, 3)$

d. $(5\sqrt{3}, -5)$

θ	0	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Convert each of the following points into rectangular coordinates.

e. $(3, \pi/3)$ $\longleftrightarrow \left(3\cos\frac{\pi}{3}, 3\sin\frac{\pi}{3}\right)$ simplify... $= \left(3 * \frac{1}{2}, 3 * \frac{\sqrt{3}}{2}\right) = \left(3/2, 3\sqrt{3}/2\right)$

f. $(2, 3\pi/2)$

g. $(6, -5\pi/6)$

h.) Convert $(7, 13^\circ) \longleftrightarrow \boxed{(7\cos 13^\circ, 7\sin 13^\circ)}$

Solution

a. Use $x = 1$ and $y = 1$ in **Equation 1.8**:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= 1^2 + 1^2 \\ r &= \sqrt{2} \end{aligned} \quad \text{and} \quad \begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{1}{1} = 1 \\ \theta &= \frac{\pi}{4}. \end{aligned}$$

Therefore this point can be represented as $\left(\sqrt{2}, \frac{\pi}{4}\right)$ in polar coordinates.

b. Use $x = -3$ and $y = 4$ in **Equation 1.8**:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (-3)^2 + (4)^2 \\ r &= 5 \end{aligned} \quad \text{and} \quad \begin{aligned} \tan\theta &= \frac{y}{x} \\ &= -\frac{4}{3} \\ \theta &= -\arctan\left(\frac{4}{3}\right) \\ &\approx 2.21. \end{aligned}$$

Therefore this point can be represented as $(5, 2.21)$ in polar coordinates.

in multidimensional space. Many of the computations are similar to those in the study of single-variable functions, but there are also a lot of differences. In this first chapter, we examine coordinate systems for working in three-dimensional space, along with vectors, which are a key mathematical tool for dealing with quantities in more than one dimension. Let's start here with the basic ideas and work our way up to the more general and powerful tools of mathematics in later chapters.

2.1 | Vectors in the Plane

Learning Objectives

- **2.1.1** Describe a plane vector, using correct notation.
- **2.1.2** Perform basic vector operations (scalar multiplication, addition, subtraction).
- **2.1.3** Express a vector in component form.
- **2.1.4** Explain the formula for the magnitude of a vector.
- **2.1.5** Express a vector in terms of unit vectors.
- **2.1.6** Give two examples of vector quantities.

When describing the movement of an airplane in flight, it is important to communicate two pieces of information: the direction in which the plane is traveling and the plane's speed. When measuring a force, such as the thrust of the plane's engines, it is important to describe not only the strength of that force, but also the direction in which it is applied. Some quantities, such as or force, are defined in terms of both size (also called *magnitude*) and direction. A quantity that has magnitude and direction is called a **vector**. In this text, we denote vectors by boldface letters, such as **v**.

Definition

A vector is a quantity that has both magnitude and direction.

they represent displacement, the difference between 2 points

Vector Representation

A vector in a plane is represented by a directed line segment (an arrow). The endpoints of the segment are called the **initial point** and the **terminal point** of the vector. An arrow from the initial point to the terminal point indicates the direction of the vector. The length of the line segment represents its **magnitude**. We use the notation $\|\mathbf{v}\|$ to denote the magnitude of the vector $\mathbf{v}$. A vector with an initial point and terminal point that are the same is called the **zero vector**, denoted $\mathbf{0}$. The zero vector is the only vector without a direction, and by convention can be considered to have any direction convenient to the problem at hand.

length of vector

zero vector

Vectors with the same magnitude and direction are called equivalent vectors. We treat equivalent vectors as equal, even if they have different initial points. Thus, if $\mathbf{v}$ and $\mathbf{w}$ are equivalent, we write

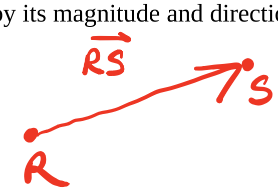
v

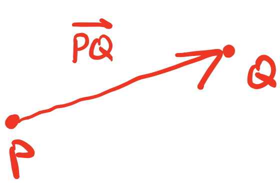
$$\mathbf{v} = \mathbf{w}.$$

Definition

Vectors are said to be **equivalent vectors** if they have the same magnitude and direction.

The arrows in Figure 2.2(b) are equivalent. Each arrow has the same length and direction. A closely related concept is the idea of parallel vectors. Two vectors are said to be parallel if they have the same or opposite directions. We explore this idea in more detail later in the chapter. A vector is defined by its magnitude and direction, regardless of where its initial point is located.





we see that $\overrightarrow{PQ} = \overrightarrow{RS}$

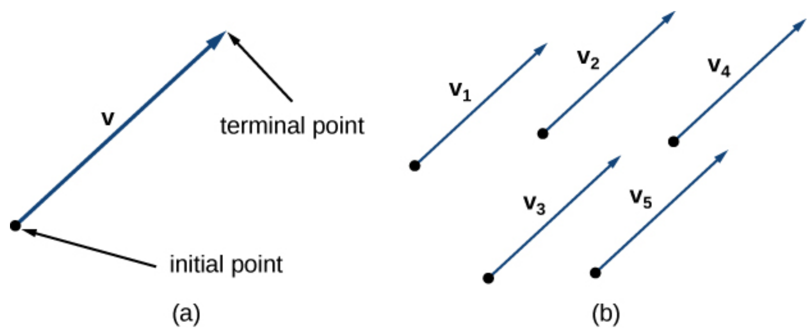


Figure 2.2 (a) A vector is represented by a directed line segment from its initial point to its terminal point. (b) Vectors $\mathbf{v}_1$ through $\mathbf{v}_5$ are equivalent.

The use of boldface, lowercase letters to name vectors is a common representation in print, but there are alternative notations. When writing the name of a vector by hand, for example, it is easier to sketch an arrow over the variable than to simulate boldface type: $\overrightarrow{v}$. When a vector has initial point P and terminal point Q, the notation $\overrightarrow{PQ}$ is useful because it indicates the direction and location of the vector.

Example 2.1

Sketching Vectors

Sketch a vector in the plane from initial point $P(1, 1)$ to terminal point $Q(8, 5)$.

Solution

See **Figure 2.3**. Because the vector goes from point P to point Q, we name it $\overrightarrow{PQ}$.

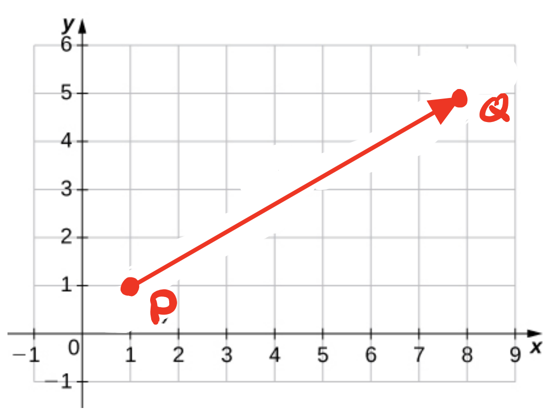


Figure 2.3 The vector with initial point $(1, 1)$ and terminal point $(8, 5)$ is named $\overrightarrow{PQ}$.

2.1 Sketch the vector $\overrightarrow{ST}$ where S is point $(3, -1)$ and T is point $(-2, 3)$.

Combining Vectors

Vectors have many real-life applications, including situations involving force or velocity. For example, consider the forces acting on a boat crossing a river. The boat's motor generates a force in one direction, and the current of the river generates a force in another direction. Both forces are vectors. We must take both the magnitude and direction of each force into account if we want to know where the boat will go.

A second example that involves vectors is a quarterback throwing a football. The quarterback does not throw the ball parallel to the ground; instead, he aims up into the air. The velocity of his throw can be represented by a vector. If we know how hard he throws the ball (magnitude—in this case, speed), and the angle (direction), we can tell how far the ball will travel down the field.

A real number is often called a **scalar** in mathematics and physics. Unlike vectors, scalars are generally considered to have a magnitude only, but no direction. Multiplying a vector by a scalar changes the vector's magnitude. This is called scalar multiplication. Note that changing the magnitude of a vector does not indicate a change in its direction. For example, wind blowing from north to south might increase or decrease in speed while maintaining its direction from north to south.

Definition

The product $k\mathbf{v}$ of a vector $\mathbf{v}$ and a scalar k is a vector with a magnitude that is $|k|$ times the magnitude of $\mathbf{v}$, and with a direction that is the same as the direction of $\mathbf{v}$ if $k > 0$, and opposite the direction of $\mathbf{v}$ if $k < 0$. This is called **scalar multiplication**. If $k = 0$ or $\mathbf{v} = \mathbf{0}$, then $k\mathbf{v} = \mathbf{0}$.

As you might expect, if $k = -1$, we denote the product $k\mathbf{v}$ as

$$k\mathbf{v} = (-1)\mathbf{v} = -\mathbf{v}.$$

Note that $-\mathbf{v}$ has the same magnitude as $\mathbf{v}$, but has the opposite direction (Figure 2.4).

scale a vector by a negative number, the vector's direction flips

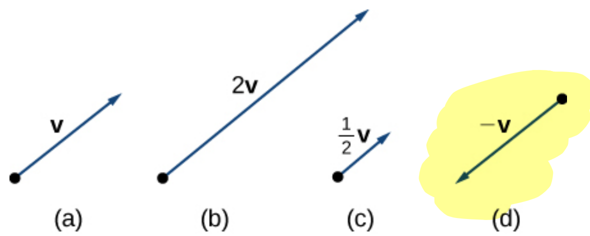


Figure 2.4 (a) The original vector **v** has length n units. (b) The length of $2\mathbf{v}$ equals $2n$ units. (c) The length of $\mathbf{v}/2$ is $n/2$ units. (d) The vectors $\mathbf{v}$ and $-\mathbf{v}$ have the same length but opposite directions.

Another operation we can perform on vectors is to add them together in vector addition, but because each vector may have its own direction, the process is different from adding two numbers. The most common graphical method for adding two vectors is to place the initial point of the second vector at the terminal point of the first, as in Figure 2.5(a). To see why this makes sense, suppose, for example, that both vectors represent displacement. If an object moves first from the initial point to the terminal point of vector $\mathbf{v}$, then from the initial point to the terminal point of vector $\mathbf{w}$, the overall displacement is the same as if the object had made just one movement from the initial point to the terminal point of the vector $\mathbf{v} + \mathbf{w}$. For obvious reasons, this approach is called the **triangle method**. Notice that if we had switched the order, so that $\mathbf{w}$ was our first vector and **v** was our second vector, we would have ended up in the same place. (Again, see Figure 2.5(a).) Thus, $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

A second method for adding vectors is called the **parallelogram method**. With this method, we place the two vectors so they have the same initial point, and then we draw a parallelogram with the vectors as two adjacent sides, as in Figure 2.5(b). The length of the diagonal of the parallelogram is the sum. Comparing Figure 2.5(b) and Figure 2.5(a), we can see that we get the same answer using either method. The vector $\mathbf{v} + \mathbf{w}$ is called the **vector sum**.

Definition

The sum of two vectors $\mathbf{v}$ and $\mathbf{w}$ can be constructed graphically by placing the initial point of $\mathbf{w}$ at the terminal point of $\mathbf{v}$. Then, the vector sum, $\mathbf{v} + \mathbf{w}$, is the vector with an initial point that coincides with the initial point of $\mathbf{v}$ and has a terminal point that coincides with the terminal point of $\mathbf{w}$. This operation is known as **vector addition**.

adding vectors

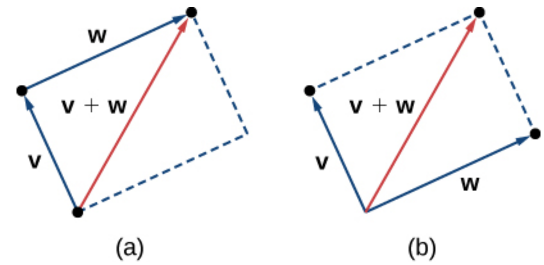


Figure 2.5 (a) When adding vectors by the triangle method, the initial point of $\mathbf{w}$ is the terminal point of $\mathbf{v}$. (b) When adding vectors by the parallelogram method, the vectors $\mathbf{v}$ and $\mathbf{w}$ have the same initial point.

It is also appropriate here to discuss vector subtraction. We define $\mathbf{v} - \mathbf{w}$ as $\mathbf{v} + (-\mathbf{w}) = \mathbf{v} + (-1)\mathbf{w}$. The vector $\mathbf{v} - \mathbf{w}$ is called the **vector difference**. Graphically, the vector $\mathbf{v} - \mathbf{w}$ is depicted by drawing a vector from the terminal point of $\mathbf{w}$ to the terminal point of $\mathbf{v}$ (Figure 2.6).

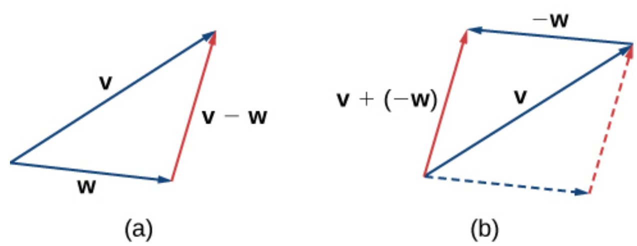
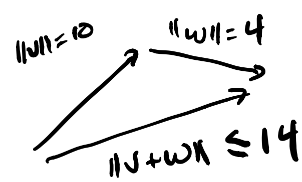


Figure 2.6 (a) The vector difference $\mathbf{v} - \mathbf{w}$ is depicted by drawing a vector from the terminal point of $\mathbf{w}$ to the terminal point of $\mathbf{v}$. (b) The vector $\mathbf{v} - \mathbf{w}$ is equivalent to the vector $\mathbf{v} + (-\mathbf{w})$.



The only "weird" thing with vectors

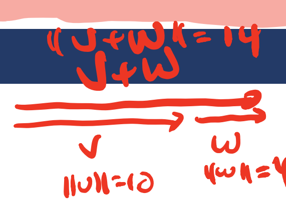
In Figure 2.5(a), the initial point of $\mathbf{v} + \mathbf{w}$ is the initial point of $\mathbf{v}$. The terminal point of $\mathbf{v} + \mathbf{w}$ is the terminal point of $\mathbf{w}$. These three vectors form the sides of a triangle. It follows that the length of any one side is less than the sum of the lengths of the remaining sides. So we have

$$\| \mathbf{v} + \mathbf{w} \| \leq \| \mathbf{v} \| + \| \mathbf{w} \|.$$

magnitudes don't necessarily add up, but there is an inequality

This is known more generally as the **triangle inequality**. There is one case, however, when the resultant vector $\mathbf{u} + \mathbf{v}$ has the same magnitude as the sum of the magnitudes of $\mathbf{u}$ and $\mathbf{v}$. This happens only when $\mathbf{u}$ and $\mathbf{v}$ have the same direction.

Example 2.2

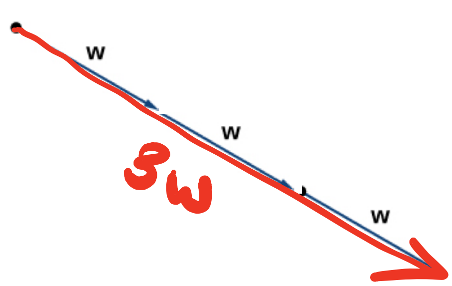


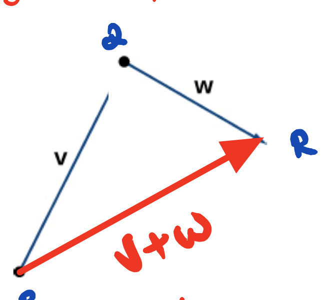
Combining Vectors

Given the vectors $\mathbf{v}$ and $\mathbf{w}$ shown in Figure 2.7, sketch the vectors

a. $3\mathbf{w}$

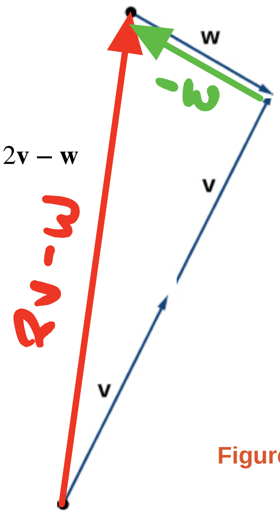
b. $\mathbf{v} + \mathbf{w}$





$(2v-w)+w = 2v$

c. $2\mathbf{v} - \mathbf{w}$



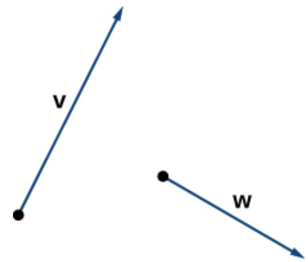
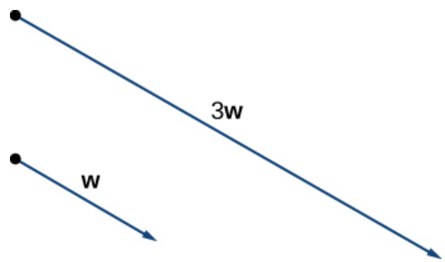


Figure 2.7 Vectors $\mathbf{v}$ and $\mathbf{w}$ lie in the same plane.

Solution

a. The vector $3\mathbf{w}$ has the same direction as $\mathbf{w}$; it is three times as long as $\mathbf{w}$.



Vector $3\mathbf{w}$ has the same direction as $\mathbf{w}$ and is three times as long.

b. Use either addition method to find $\mathbf{v} + \mathbf{w}$.

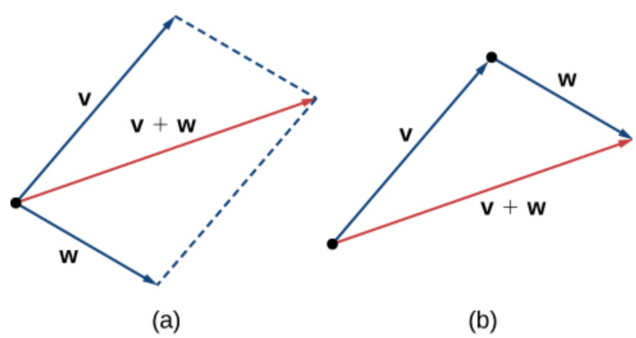


Figure 2.8 To find $\mathbf{v} + \mathbf{w}$, align the vectors at their initial points or place the initial point of one vector at the terminal point of the other. (a) The vector $\mathbf{v} + \mathbf{w}$ is the diagonal of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$ (b) The vector $\mathbf{v} + \mathbf{w}$ is the third side of a triangle formed with $\mathbf{w}$ placed at the terminal point of $\mathbf{v}$.

c. To find $2\mathbf{v} - \mathbf{w}$, we can first rewrite the expression as $2\mathbf{v} + (-\mathbf{w})$. Then we can draw the vector $-\mathbf{w}$, then add it to the vector $2\mathbf{v}$.

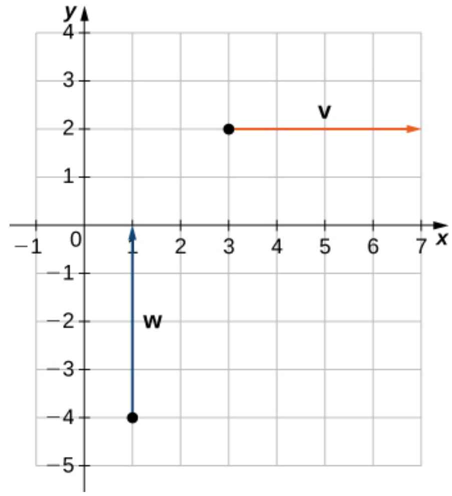


Figure 2.10 These vectors are not equivalent.

b. Based on **Figure 2.11**, and using a bit of geometry, it is clear these vectors have the same length and the same direction, so **v** and **w** are equivalent.

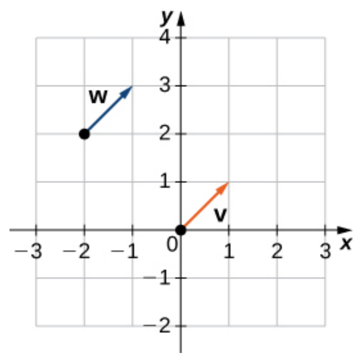
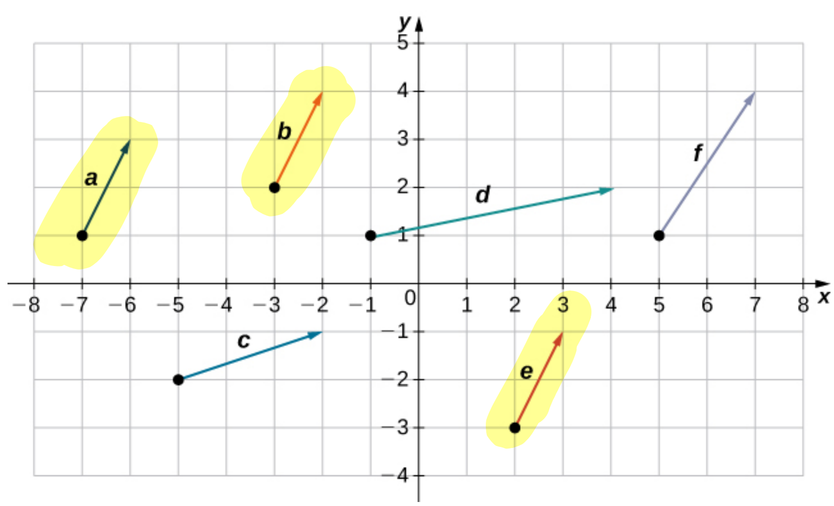


Figure 2.11 These vectors are equivalent.

2.3 Which of the following vectors are equivalent?



We have seen how to plot a vector when we are given an initial point and a terminal point. However, because a vector can be placed anywhere in a plane, it may be easier to perform calculations with a vector when its initial point coincides with the origin. We call a vector with its initial point at the origin a **standard-position vector**. Because the initial point of any vector in standard position is known to be $(0,\ 0)$, we can describe the vector by looking at the coordinates of its terminal point. Thus, if vector **v** has its initial point at the origin and its terminal point at $(x,\ y)$, we write the vector in component form as

$$\mathbf{v} = \langle x,\ y \rangle.$$

When a vector is written in component form like this, the scalars x and y are called the **components** of **v**.

Definition

The vector with initial point $(0,\ 0)$ and terminal point $(x,\ y)$ can be written in component form as

$$\mathbf{v} = \langle x,\ y \rangle.$$

The scalars x and y are called the components of $\mathbf{v}$.

In component form, the zero vector $\mathbf{0} = \langle 0, 0 \rangle$ (see page 102)

Recall that vectors are named with lowercase letters in bold type or by drawing an arrow over their name. We have also learned that we can name a vector by its component form, with the coordinates of its terminal point in angle brackets. However, when writing the component form of a vector, it is important to distinguish between $\langle x,\ y \rangle$ and $(x,\ y)$. The first ordered pair uses angle brackets to describe a vector, whereas the second uses parentheses to describe a point in a plane. The initial point of $\langle x,\ y \rangle$ is $(0,\ 0)$; the terminal point of $\langle x,\ y \rangle$ is $(x,\ y)$.

When we have a vector not already in standard position, we can determine its component form in one of two ways. We can use a geometric approach, in which we sketch the vector in the coordinate plane, and then sketch an equivalent standard-position vector. Alternatively, we can find it algebraically, using the coordinates of the initial point and the terminal point. To find it algebraically, we subtract the x-coordinate of the initial point from the x-coordinate of the terminal point to get the x component, and we subtract the y-coordinate of the initial point from the y-coordinate of the terminal point to get the y component.

Rule: Component Form of a Vector

Let **v** be a vector with initial point $(x_i,\ y_i)$ and terminal point $(x_t,\ y_t)$. Then we can express **v** in component form as $\mathbf{v} = \langle x_t - x_i,\ y_t - y_i \rangle$.

$7-3=4$

3 7

Example 2.4

Expressing Vectors in Component Form

$(-3,4) \longrightarrow (1,2)$

Express vector **v** with initial point $(-3,\ 4)$ and terminal point $(1,\ 2)$ in component form.

$$\begin{array}{r} (1,2) \\ -\ (-3,4) \\ \hline \langle 4,\ -2 \rangle \end{array}$$

Solution

a. Geometric

1. ~~Sketch the vector in the coordinate plane (**Figure 2.12**).~~
2. ~~The terminal point is 4 units to the right and 2 units down from the initial point.~~
3. ~~Find the point that is 4 units to the right and 2 units down from the origin.~~
4. In standard position, this vector has initial point $(0,\ 0)$ and terminal point $(4,\ -2)$:

$$\mathbf{v} = \langle 4, -2 \rangle .$$

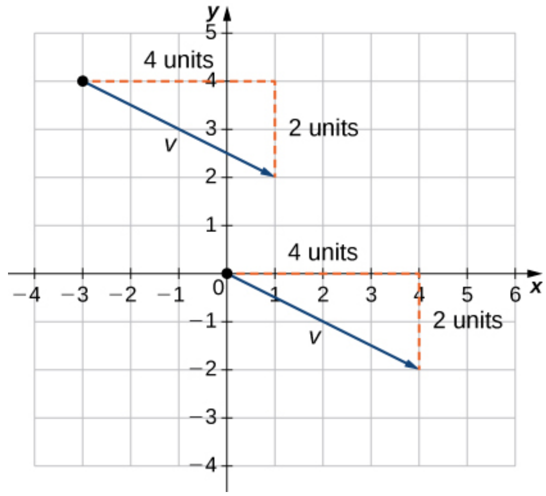


Figure 2.12 These vectors are equivalent.

b. Algebraic
In the first solution, we used a sketch of the vector to see that the terminal point lies 4 units to the right. We can accomplish this algebraically by finding the difference of the x-coordinates:

$$x_t - x_i = 1 - (-3) = 4.$$

Similarly, the difference of the y-coordinates shows the vertical length of the vector.

$$y_t - y_i = 2 - 4 = -2.$$

So, in component form,

$$\begin{aligned} \mathbf{v} &= \langle x_t - x_i, y_t - y_i \rangle \\ &= \langle 1 - (-3), 2 - 4 \rangle \\ &= \langle 4, -2 \rangle . \end{aligned}$$

$$\begin{array}{r} (-1, 2) \\ -\ (-4, -5) \\ \hline \langle 3, 7 \rangle \end{array}$$

2.4 Vector $\mathbf{w}$ has initial point $(-4, -5)$ and terminal point $(-1, 2)$. Express $\mathbf{w}$ in component form.

To find the magnitude of a vector, we calculate the distance between its initial point and its terminal point. The magnitude of vector $\mathbf{v} = \langle x, y \rangle$ is denoted $\|\mathbf{v}\|$, or $|\mathbf{v}|$, and can be computed using the formula

$$\|\mathbf{v}\| = \sqrt{x^2 + y^2}.$$

Note that because this vector is written in component form, it is equivalent to a vector in standard position, with its initial point at the origin and terminal point (x, y). Thus, it suffices to calculate the magnitude of the vector in standard position. Using the distance formula to calculate the distance between initial point $(0, 0)$ and terminal point (x, y), we have

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{(x-0)^2 + (y-0)^2} \\ &= \sqrt{x^2 + y^2}. \end{aligned}$$

Side point.

sqrt outputs "positive or zero" (nonnegative)

Based on this formula, it is clear that for any vector $\mathbf{v}$, $\|\mathbf{v}\| \geq 0$, and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

sqrt outputs zero only here: $\sqrt{0} = 0$

2.5 Let $\mathbf{a} = \langle 7, 1 \rangle$.

a. Find $\|\mathbf{a}\|$.

$\|\mathbf{a}\| = \sqrt{7^2 + 1^2} = \sqrt{50} = 5\sqrt{2}$

"The only vector with magnitude zero is the zero vector."

"$\|v\|=0$ iff $v=0$"

The magnitude of a vector can also be derived using the Pythagorean theorem, as in the following figure.

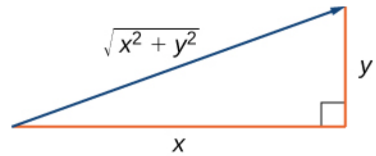


Figure 2.13 If you use the components of a vector to define a right triangle, the magnitude of the vector is the length of the triangle's hypotenuse.

We have defined scalar multiplication and vector addition geometrically. Expressing vectors in component form allows us to perform these same operations algebraically.

In component form

Definition

Let $\mathbf{v} = \langle x_1, y_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2 \rangle$ be vectors, and let k be a scalar.

Scalar multiplication: $k\mathbf{v} = \langle kx_1, ky_1 \rangle$

Vector addition: $\mathbf{v} + \mathbf{w} = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle$

Example 2.5

Performing Operations in Component Form

Let $\mathbf{v}$ be the vector with initial point $(2, 5)$ and terminal point $(8, 13)$, and let $\mathbf{w} = \langle -2, 4 \rangle$.

a. Express $\mathbf{v}$ in component form and find $\|\mathbf{v}\|$. Then, using algebra, find

$v=(8,13)-(2,5)=\boxed{\langle 6,8\rangle}$

$\|v\|=\sqrt{6^2+8^2}=\sqrt{100}=\boxed{10}$

b. $\mathbf{v} + \mathbf{w}$, $\langle 6,8\rangle+\langle -2,4\rangle=\boxed{\langle 4,12\rangle}$

c. $3\mathbf{v}$, and $3\langle 6,8\rangle=\boxed{\langle 18,24\rangle}$

d. $\mathbf{v} - 2\mathbf{w}$.

$=\langle 6,8\rangle-2\langle -2,4\rangle=\boxed{\langle 10,0\rangle}$

$=\langle 6,8\rangle+\langle 4,-8\rangle=$

Solution

a. To place the initial point of $\mathbf{v}$ at the origin, we must translate the vector 2 units to the left and 5 units down (**Figure 2.15**). Using the algebraic method, we can express $\mathbf{v}$ as $\mathbf{v} = \langle 8 - 2, 13 - 5 \rangle = \langle 6, 8 \rangle$:

$$\|\mathbf{v}\| = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10.$$

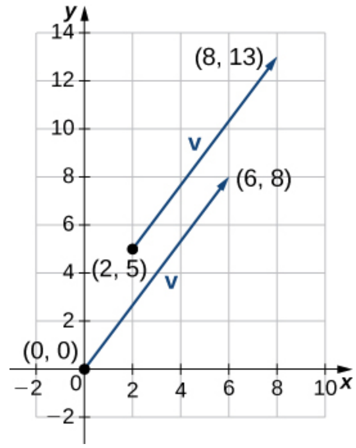


Figure 2.14 In component form, $\mathbf{v} = \langle 6, 8 \rangle$.

b. To find $\mathbf{v} + \mathbf{w}$, add the x-components and the y-components separately:

$$\mathbf{v} + \mathbf{w} = \langle 6, 8 \rangle + \langle -2, 4 \rangle = \langle 4, 12 \rangle.$$

c. To find $3\mathbf{v}$, multiply $\mathbf{v}$ by the scalar $k = 3$:

$$3\mathbf{v} = 3 \cdot \langle 6, 8 \rangle = \langle 3 \cdot 6, 3 \cdot 8 \rangle = \langle 18, 24 \rangle.$$

d. To find $\mathbf{v} - 2\mathbf{w}$, find $-2\mathbf{w}$ and add it to $\mathbf{v}$:

$$\mathbf{v} - 2\mathbf{w} = \langle 6, 8 \rangle - 2 \cdot \langle -2, 4 \rangle = \langle 6, 8 \rangle + \langle 4, -8 \rangle = \langle 10, 0 \rangle.$$

2.5 Let $\mathbf{a} = \langle 7, 1 \rangle$ and let $\mathbf{b}$ be the vector with initial point $(3, 2)$ and terminal point $(-1, -1)$.

a. Find $\|\mathbf{a}\|$.

b. Express $\mathbf{b}$ in component form.

c. Find $3\mathbf{a} - 4\mathbf{b}$.

Now that we have established the basic rules of vector arithmetic, we can state the properties of vector operations. We will prove two of these properties. The others can be proved in a similar manner.

Theorem 2.1: Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a plane. Let r and s be scalars.

i.	$\mathbf{u} + \mathbf{v}$	$=$	$\mathbf{v} + \mathbf{u}$	Commutative property
ii.	$(\mathbf{u} + \mathbf{v}) + \mathbf{w}$	$=$	$\mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property
iii.	$\mathbf{u} + \mathbf{0}$	$=$	$\mathbf{u}$	Additive identity property
iv.	$\mathbf{u} + (-\mathbf{u})$	$=$	$\mathbf{0}$	Additive inverse property
v.	$r(s\mathbf{u})$	$=$	$(rs)\mathbf{u}$	Associativity of scalar multiplication
vi.	$(r + s)\mathbf{u}$	$=$	$r\mathbf{u} + s\mathbf{u}$	Distributive property
vii.	$r(\mathbf{u} + \mathbf{v})$	$=$	$r\mathbf{u} + r\mathbf{v}$	Distributive property
viii.	$1\mathbf{u}$	$=$	$\mathbf{u}$, $0\mathbf{u} = \mathbf{0}$	Identity and zero properties

tl;dr Nothing weird happens with vectors with 1 exception: the triangle rule $\boxed{\text{p105}}$

Proof of Commutative Property

Let $\mathbf{u} = \langle x_1, y_1 \rangle$ and $\mathbf{v} = \langle x_2, y_2 \rangle$. Apply the commutative property for real numbers:

$$\mathbf{u} + \mathbf{v} = \langle x_1 + x_2, y_1 + y_2 \rangle = \langle x_2 + x_1, y_2 + y_1 \rangle = \mathbf{v} + \mathbf{u}.$$

□

Proof of Distributive Property

Apply the distributive property for real numbers:

$$\begin{aligned} r(\mathbf{u} + \mathbf{v}) &= r \cdot \langle x_1 + x_2, y_1 + y_2 \rangle \\ &= \langle r(x_1 + x_2), r(y_1 + y_2) \rangle \\ &= \langle rx_1 + rx_2, ry_1 + ry_2 \rangle \\ &= \langle rx_1, ry_1 \rangle + \langle rx_2, ry_2 \rangle \\ &= r\mathbf{u} + r\mathbf{v}. \end{aligned}$$

□

2.6 Prove the additive inverse property.

We have found the components of a vector given its initial and terminal points. In some cases, we may only have the magnitude and direction of a vector, not the points. For these vectors, we can identify the horizontal and vertical components using trigonometry (**Figure 2.15**).

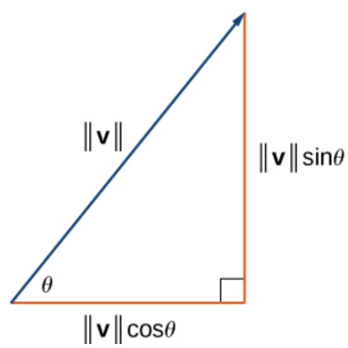


Figure 2.15 The components of a vector form the legs of a right triangle, with the vector as the hypotenuse.

Consider the angle θ formed by the vector $\mathbf{v}$ and the positive x-axis. We can see from the triangle that the components of vector $\mathbf{v}$ are $\langle \|\mathbf{v}\| \cos\theta, \|\mathbf{v}\| \sin\theta \rangle$. Therefore, given an angle and the magnitude of a vector, we can use the cosine and sine of the angle to find the components of the vector.

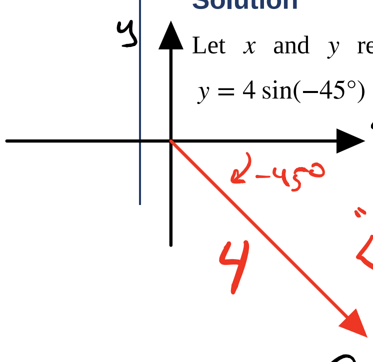
Example 2.6

Finding the Component Form of a Vector Using Trigonometry

Find the component form of a vector with magnitude 4 that forms an angle of $-45°$ with the x-axis.

Solution

Let x and y represent the components of the vector (**Figure 2.16**). Then $x = 4\cos(-45°) = 2\sqrt{2}$ and $y = 4\sin(-45°) = -2\sqrt{2}$. The component form of the vector is $\boxed{\langle 2\sqrt{2}, -2\sqrt{2} \rangle}$.



our textbook does NOT use polar coordinate notation for vectors

simplify to get

" $\langle 4, -45° \rangle$ "

$\boxed{\langle 4\cos(-45)°, 4\sin(-45°) \rangle}$

trig trick: using even/odd symmetry

$\langle 4\cos -45°, 4\sin -45° \rangle$
$= \langle 4\cos 45°, -4\sin 45° \rangle$
$= \langle 4\frac{\sqrt{2}}{2}, -4\frac{\sqrt{2}}{2} \rangle$
$= \boxed{\langle 2\sqrt{2}, -2\sqrt{2} \rangle}$

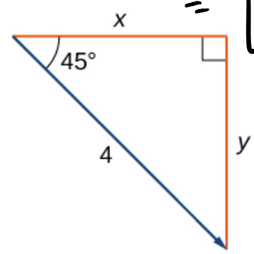


Figure 2.16 Use trigonometric ratios, $x = \| \mathbf{v} \| \cos\theta$ and $y = \| \mathbf{v} \| \sin\theta$, to identify the components of the vector.

17°

2.7 Find the component form of vector $\mathbf{v}$ with magnitude 10 that forms an angle of ~~[illegible]~~ 17° with the positive x-axis.

$\langle 10, 17° \rangle \leftrightarrow \boxed{\langle 10\cos 17°, 10\sin 17° \rangle}$

Unit Vectors

A **unit vector** is a vector with magnitude 1. For any nonzero vector $\mathbf{v}$, we can use scalar multiplication to find a unit vector $\mathbf{u}$ that has the same direction as $\mathbf{v}$. To do this, we multiply the vector by the reciprocal of its magnitude:

$$\mathbf{u} = \frac{1}{\| \mathbf{v} \|}\mathbf{v}.$$

how to find the

Recall that when we defined scalar multiplication, we noted that $\| k\mathbf{v} \| = |k| \cdot \| \mathbf{v} \|$. For $\mathbf{u} = \frac{1}{\| \mathbf{v} \|}\mathbf{v}$, it follows that $\| \mathbf{u} \| = \frac{1}{\| \mathbf{v} \|}(\| \mathbf{v} \|) = 1$. We say that $\mathbf{u}$ is the *unit vector in the direction of* $\mathbf{v}$ (**Figure 2.17**). The process of using scalar multiplication to find a unit vector with a given direction is called **normalization**.

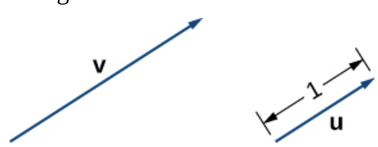


Figure 2.17 The vector $\mathbf{v}$ and associated unit vector $\mathbf{u} = \frac{1}{\| \mathbf{v} \|}\mathbf{v}$. In this case, $\| \mathbf{v} \| > 1$.

check: is $\| \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle \| = 1$?

$\| \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle \| = \sqrt{\frac{1}{5} + \frac{4}{5}} = \sqrt{1} = 1$ ✓✓

Example 2.7

Finding a Unit Vector

Let $\mathbf{v} = \langle 1, 2 \rangle$.

a. Find a unit vector with the same direction as $\mathbf{v}$.

b. Find a vector $\mathbf{w}$ with the same direction as $\mathbf{v}$ such that $\| \mathbf{w} \| = 7$.

$\| v \| = \sqrt{1^2 + 2^2} = \sqrt{5}$

$\frac{1}{\| v \|} v = \frac{1}{\sqrt{5}} \langle 1, 2 \rangle = \boxed{\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle}$

Solution

what about $7\langle 1,2 \rangle = \langle 7, 14 \rangle = \sqrt{245} = 7\sqrt{5}$

vector = (magnitude)(unit vector with the same direction)

$w = 7 \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle = \langle \frac{7}{\sqrt{5}}, \frac{14}{\sqrt{5}} \rangle$

check magnitude

$\| \langle \frac{7}{\sqrt{5}}, \frac{14}{\sqrt{5}} \rangle \| = \sqrt{\frac{49}{5} + \frac{196}{5}} = \sqrt{\frac{245}{5}} = \sqrt{49} = \boxed{7}$

a. First, find the magnitude of $\mathbf{v}$, then divide the components of $\mathbf{v}$ by the magnitude:

$$\| \mathbf{v} \| = \sqrt{1^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5}$$

$$\mathbf{u} = \frac{1}{\| \mathbf{v} \|}\mathbf{v} = \frac{1}{\sqrt{5}} \langle 1, 2 \rangle = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle .$$

b. The vector $\mathbf{u}$ is in the same direction as $\mathbf{v}$ and $\| \mathbf{u} \| = 1$. Use scalar multiplication to increase the length of $\mathbf{u}$ without changing direction:

$$\mathbf{w} = 7\mathbf{u} = 7 \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle = \langle \frac{7}{\sqrt{5}}, \frac{14}{\sqrt{5}} \rangle .$$

2.8 Let $\mathbf{v} = \langle 9, 2 \rangle$. Find a vector with magnitude 5 in the opposite direction as $\mathbf{v}$.

We have seen how convenient it can be to write a vector in component form. Sometimes, though, it is more convenient to write a vector as a sum of a horizontal vector and a vertical vector. To make this easier, let's look at standard unit vectors. The **standard unit vectors** are the vectors $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ (**Figure 2.18**).

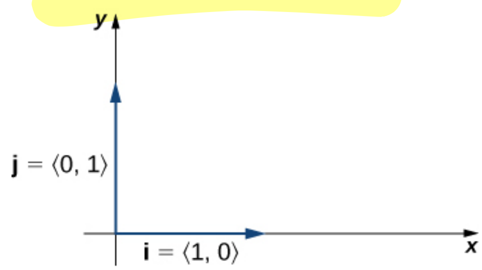


Figure 2.18 The standard unit vectors $\mathbf{i}$ and $\mathbf{j}$.

yes!!

By applying the properties of vectors, it is possible to express any vector in terms of $\mathbf{i}$ and $\mathbf{j}$ in what we call a *linear combination*:

for example $\langle 3,-4 \rangle \stackrel{?}{=} \langle 3,0\rangle+\langle 0,-4\rangle = 3\langle 1,0\rangle-4\langle 0,1\rangle = 3i-4j$

$$\mathbf{v} = \langle x, y \rangle = \langle x, 0 \rangle + \langle 0, y \rangle = x \langle 1, 0 \rangle + y \langle 0, 1 \rangle = x\mathbf{i} + y\mathbf{j}.$$

Thus, $\mathbf{v}$ is the sum of a horizontal vector with magnitude x, and a vertical vector with magnitude y, as in the following figure.

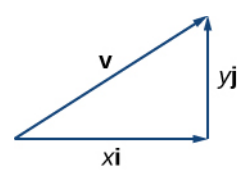


Figure 2.19 The vector $\mathbf{v}$ is the sum of $x\mathbf{i}$ and $y\mathbf{j}$.

Example 2.8

Using Standard Unit Vectors

a. Express the vector $\mathbf{w} = \langle 3, -4 \rangle$ in terms of standard unit vectors.

b. Vector $\mathbf{u}$ is a unit vector that forms an angle of $60°$ with the positive x-axis. Use standard unit vectors to describe $\mathbf{u}$.

means $\|u\| = 1$

$\langle 1, 60° \rangle \longleftrightarrow \langle 1 \cos 60°, 1 \sin 60° \rangle$

Solution

a. Resolve vector $\mathbf{w}$ into a vector with a zero y-component and a vector with a zero x-component:

$$\mathbf{w} = \langle 3, -4 \rangle = 3\mathbf{i} - 4\mathbf{j}.$$

b. Because $\mathbf{u}$ is a unit vector, the terminal point lies on the unit circle when the vector is placed in standard position (**Figure 2.20**).

$$\begin{aligned} u &= \langle \cos 60°, \sin 60° \rangle \\ &= \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle \\ &= \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}. \end{aligned}$$

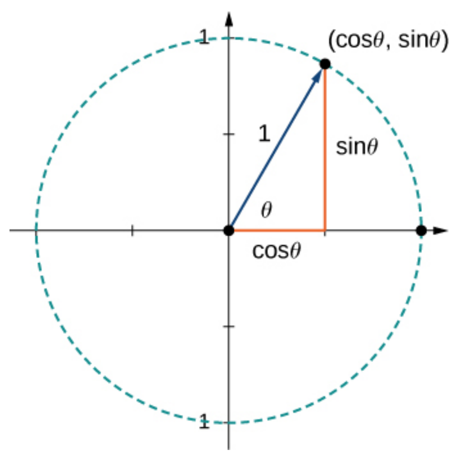


Figure 2.20 The terminal point of $\mathbf{u}$ lies on the unit circle $(\cos\theta, \sin\theta)$.

2.9 Let $\mathbf{a} = \langle 16, -11 \rangle$ and let $\mathbf{b}$ be a unit vector that forms an angle of $225°$ with the positive x-axis. Express $\mathbf{a}$ and $\mathbf{b}$ in terms of the standard unit vectors.

$a = \boxed{16i - 11j}$

Applications of Vectors

Because vectors have both direction and magnitude, they are valuable tools for solving problems involving such applications as motion and force. Recall the boat example and the quarterback example we described earlier. Here we look at two other examples in detail.

Example 2.9