Numerical Methods for Partial Differential Equations
A.Y. 2023/2024

Laboratory 06
Finite Element method for non linear equations and vectorial problems

Exercise 1.

Let $\Omega = (0,1)^3$ be the unit cube and let us consider the following non linear problem:

$$\begin{cases} -\nabla \cdot \left(\left(\mu_0 + \mu_1 u^2 \right) \nabla u \right) = f & \text{in } \Omega, \quad (1a) \\ u = 0 & \text{on } \partial\Omega, \quad (1b) \end{cases}$$

where $\mathbf{x} = (x,y,z)^T$, $\mu_0 = 1$, $\mu_1 = 10$ and $f(\mathbf{x}) = 1$.

1.1. Write the weak formulation of problem (1), expressing it in the residual form $R(u)(v) = 0$.

1.2. Compute the Fréchet derivative $a(u)(\delta, v)$ of the residual $R(u)(v)$, then write Newton's method for the solution of problem (1).

1.3. Using Newton's method, implement a solver for problem (1). Then, solve the problem on the mesh `mesh/mesh-cube-20.msh`, with polynomial degree $r = 1$, and using a tolerance of 10^{-6} on the norm of the residual for the Newton's method.

Exercise 2.

Let $\Omega = (0,1)^3$ be the unit cube and let us consider the following linear elasticity problem: find a displacement field $\mathbf{u} : \Omega \to \mathbb{R}^3$ such that

$$\begin{cases} -\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} & \text{in } \Omega \,, \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma_0 \cup \Gamma_1 \,, \\ \sigma(\mathbf{u})\mathbf{n} = \mathbf{0} & \text{on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \,, \end{cases} \quad (2)$$

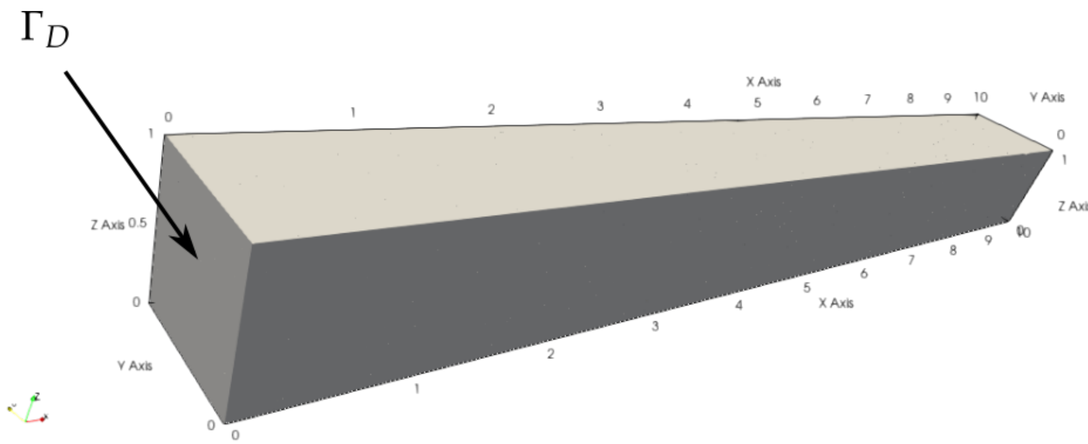

Figure 1: Computational domain for Exercise 2.3. The boundary Γ_D has tag 0 in the file `mesh/mesh-beam-10.msh`.

where

$$
\begin{aligned}
\sigma(\mathbf{u}) &= \mu \nabla \mathbf{u} + \lambda (\nabla \cdot \mathbf{u}) I \ , \\
\Gamma_0 &= \{x = 0, y \in (0,1), z \in (0,1)\} \ , \\
\Gamma_1 &= \{x = 1, y \in (0,1), z \in (0,1)\} \ , \\
\Gamma_2 &= \{x \in (0,1), y = 0, z \in (0,1)\} \ , \\
\Gamma_3 &= \{x \in (0,1), y = 1, z \in (0,1)\} \ , \\
\Gamma_4 &= \{x \in (0,1), y \in (0,1), z = 0\} \ , \\
\Gamma_5 &= \{x \in (0,1), y \in (0,1), z = 1\} \ ,
\end{aligned}
$$

$\mu = 1$, $\lambda = 10$, $\mathbf{g}(\mathbf{x}) = (0.25x, 0.25x, 0)^T$ and $\mathbf{f}(\mathbf{x}) = (0, 0, -1)^T$.

2.1. Write the weak formulation of problem (2).

2.2. Implement in `deal.II` a finite element solver for problem (2).

2.3. Consider now the domain Ω displayed in Figure 1. Solve the following problem:

$$
\begin{cases}
-\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} & \text{in } \Omega \ , \\
\mathbf{u} = \mathbf{0} & \text{on } \Gamma_\mathrm{D} \\
\sigma(\mathbf{u})\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \backslash \Gamma_\mathrm{D} \ ,
\end{cases}
$$

with $\sigma(\mathbf{u}) = \mu \nabla \mathbf{u} + \lambda (\nabla \cdot \mathbf{u}) I$, $\mu = 10$, $\lambda = 1$ and $\mathbf{f}(\mathbf{x}) = (0, 0, -0.1)^T$. The domain is provided in the file `mesh/mesh-beam-10.msh`, and the boundary Γ_D has tag 0.