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The Stochastic Preconditioned Conjugate Gradient method

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This paper presents the Stochastic Preconditioned Conjugate Gradient method (SPCG), an iterative equation solver that can greatly reduce the computational effort associated with the repeated calculations required in probabilistic finite element analysis. The method is implemented in a Monte Carlo simulation code and demonstrated for two example problems — a simple cantilever beam and a 3-D space truss. Excellent convergence properties are demonstrated in both cases. For the 3-D space truss problem (99 members, 72 DOF (degrees of freedom)), the method converged in only 3–4 iterations, on average. The method is also compared with the well-known Neumann expansion technique and shown to possess several advantageous properties with regard to speed of convergence. It is also shown that the computational effort required at each iteration of the SPCG method is effectively the same as the effort for each iteration (or expansion term) in Neumann expansion. Finally, it is pointed out that the method is also well-suited for parallel processing implementation. For parallel implementation, solution strategies are necessary that require minimal storage, can be vectorized, and can take advantage of concurrent processing. The SPCG method meets these criteria. Thus, the method promises to contribute toward making probabilistic finite element analysis practical for large, complex structures.

INTRODUCTION

Increases in raw computing power, the advent of large-scale parallel processing, and new numerical techniques are rapidly making probabilistic analysis practical for reliability evaluation of complex structures via the finite element method. While a number of approaches are possible for solution of structural reliability problems, almost all require repeated evaluations of structural response (or a performance function) for different realizations of the problem random variables. For many classes of problems this requires repeated solution of a system of linear equations. To reduce the computational effort associated with these repeated evaluations, one approach that has been suggested, is to use iterative or expansion-based equation solvers.[1–3] These equation solvers differ from more commonly used direct solvers (such as Cholesky decomposition or Gaussian elimination) in that the solution is obtained by beginning with an initial solution estimate and then successively updating the solution until convergence is achieved.

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Iterative approaches can be advantageous (i.e. compared with direct solvers) for several reasons. First, iterative approaches can take advantage of known approximate solutions to rapidly obtain an accurate solution. As mentioned above, almost all probabilistic analysis methods require repeated response calculations when values of the input variables (structure properties and loading) are perturbed about a central point. Typically the solution at the central point is computed before the perturbations begin. Iterative solvers can use knowledge of the solution at the central point to first condition the system of equations and then select a good initial guess to achieve fast convergence. Second, iterative solvers can be more efficient than direct solvers (i.e. even for deterministic analysis) for certain classes of problems and have gained in popularity in recent years.[4,5] Third, the convergence criteria can be adjusted, so that the solution need only be as precise and, therefore, the computational effort as great, as the particular problem requires.

An iterative method is presented here, the Stochastic Preconditioned Conjugate Gradient method (SPCG), that differs in approach from the iterative solvers mentioned earlier. Background on the method's deriva-

tion is given and the method's effectiveness is illlustrated by the solution of example Monte-Carlo simulation problems where excellent convergence properties were obtained. It is believed that the method promises to contribute toward making probabilistic finite element analysis practical for large and complex structures.

PRECONDITIONED CONJUGATE GRADIENT METHOD (PCG)

The preconditioned conjugate gradient method (PCG) is an iterative method for solving systems of linear equations. The method is obtained by combining a preconditioning matrix with the basic conjugate gradient method originally proposed by Hestenes and Stiefel.[6] The PCG method has recently gained popularity in deterministic analysis for two reasons: (1) It requires minimal storage since 'fills' do not occur during solution and sparse storage methods are computationally effective for large problems; and (2) the method is more easily implemented in parallel than direct solvers since all computations are essentially matrix and vector multiplications. Studies by Poole[4] and Nour-Omid[5] have demonstrated that the method is a very powerful approach for solving large systems of equations.

In the basic conjugate gradient method the standard linear structural analysis problem:

$$\boldsymbol{K}\mathbf{x} = \mathbf{f} \tag{1}$$

where $\boldsymbol{K}$ is the stiffness matrix, $\mathbf{f}$ is the loading vector, and $\mathbf{x}$ the structure unknown displacements, is solved by minimizing the function $\phi(\mathbf{x})$; that is:

$$\text{MIN } \phi(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^T\boldsymbol{K}\mathbf{x} - \mathbf{x}^T\mathbf{f} \tag{2}$$

Note that since the minimum value of ϕ is $-\frac{1}{2}\mathbf{f}^T\boldsymbol{K}^{-1}\mathbf{f}$ ($\boldsymbol{K}$ is taken to be positive definite and symmetric), which is achieved by setting $\mathbf{x} = \boldsymbol{K}^{-1}\mathbf{f}$, solution of the minimization problem is equivalent to solving $\boldsymbol{K}\mathbf{x} = \mathbf{f}$. The conjugate gradient method begins with an initial guess for the solution and then proceeds to iterate to an acceptable solution. On each iteration, ϕ is reduced (and an updated solution obtained) by searching the solution space in a direction that is '$\boldsymbol{K}$-conjugate' to all previous search directions. That is, if the current search direction is the vector $\mathbf{p}_k$, then it is said to be $\boldsymbol{K}$-conjugate to all previous search directions if

$$\mathbf{p}_k^T\boldsymbol{K}\mathbf{p}_i = 0 \qquad i = 1, \ldots, k-1 \tag{3}$$

The actual search direction selected is the $\boldsymbol{K}$-conjugate direction that is closest to the steepest descent (negative gradient) direction (where the negative gradient direction is given by the current residual, i.e. the difference of the right and left-hand side of eqn (1) using the current solution estimate). The updated solution, on a given iteration, is obtained by moving an optimal distance, α, in the search direction.

This approach has several advantages over a pure steepest descent, i.e. gradient, approach.[7] Notably, the $\mathbf{p}_i$ are all linearly independent and an exact converged solution is, therefore, guaranteed (if there is no numerical error), in n steps, where n is the number of unknowns (order of the $\boldsymbol{K}$ matrix). Also, this approach avoids a problem that can arise in steepest descent — that the gradient directions become far too similar as the iteration progresses, resulting in very slow convergence.

Due to the large number of steps required to achieve an exact solution (n) and due to the presence of roundoff error (which actually makes achievement of an exact solution impossible), the iteration process is terminated when the solution error becomes acceptably small. Also, in order to reduce the number of steps required to obtain an acceptable solution, that is, speed convergence, particularly for problems with a poorly conditioned $\boldsymbol{K}$ matrix, the $\boldsymbol{K}$ matrix can be preconditioned. This leads to the preconditioned conjugate gradient (PCG) method.

In the PCG method, eqn (1) is modified by premultiplying both sides of the equation by a preconditioning matrix, $\boldsymbol{M}$, to obtain:

$$\boldsymbol{M}^{-1}\boldsymbol{K}\mathbf{x} = \boldsymbol{M}^{-1}\mathbf{f} \tag{4}$$

The preconditioning matrix, $\boldsymbol{M}$, is a symmetric positive definite matrix and is chosen to approximate $\boldsymbol{K}$, thereby improving the conditioning of the system. The PCG method can thus be expressed as:

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \alpha_i\mathbf{p}_i \tag{5}$$

where $\mathbf{p}_i$ is the conjugate search direction, and α_i is a scalar that controls the magnitude of the 'move' in the $\mathbf{p}_i$ direction. The search directions that satisfy the conjugate gradient criteria are obtained as:

$$\mathbf{p}_{i+1} = \mathbf{h}_{i+1} + \beta_i\mathbf{p}_i \tag{6}$$

where

$$\mathbf{h}_{i+1} = \boldsymbol{M}^{-1}\mathbf{r}_{i+1} \tag{7}$$

$$\mathbf{r}_{i+1} = \mathbf{f} - \boldsymbol{K}\mathbf{x}_{i+1} = \mathbf{r}_i - \alpha_i\boldsymbol{K}\mathbf{p}_i \tag{8}$$

and

$$\beta_i = \frac{\mathbf{r}_{i+1} \cdot \mathbf{h}_{i+1}}{\mathbf{r}_i \cdot \mathbf{h}_i} \tag{9}$$

The scalar α is selected so as to minimize the value of ϕ (eqn (2)) for the current search direction, and is obtained as:

$$\alpha_i = \frac{\mathbf{r}_i \cdot \mathbf{h}_i}{\boldsymbol{K}\mathbf{p}_i \cdot \mathbf{p}_i} \tag{10}$$

The above equations can be used recursively to obtain the solution to eqn (4) after beginning with an initial guess $\mathbf{x}_o$ and selecting the initial search direction, $\mathbf{p}_o$, to

be the steepest descent direction, that is,

$$\mathbf{p}_o = \mathbf{h}_o = \boldsymbol{M}^{-1}\mathbf{r}_o = \boldsymbol{M}^{-1}(\mathbf{f} - \boldsymbol{K}\mathbf{x}_o) \tag{11}$$

Typically, iterations are continued until

$$\frac{\mathbf{r}_{i+1} \cdot \mathbf{h}_{i+1}}{\mathbf{r}_o \cdot \mathbf{h}_o} < \eta \tag{12}$$

where η is the convergence tolerance.

Incorporation of the preconditioning matrix, $\boldsymbol{M}$, serves to improve the rate of convergence, but at the cost of additional computations (eqn (7)). The closer the preconditioning matrix approximates the $\boldsymbol{K}$ matrix (improving the conditioning), the faster the rate of convergence, but the greater the additional computational expense (obviously for $\boldsymbol{M} = \boldsymbol{K}$, convergence is in one step). More details on the conjugate gradient method and preconditioning strategies for deterministic analysis can be found in Golub and Van Loan[7] and Axelsson and Barker[8].

STOCHASTIC PRECONDITIONED CONJUGATE GRADIENT METHOD (SPCG)

For probabilistic structural mechanics problems, different pre-conditioning matrices can be appropriate depending on whether the solution is obtained via direct Monte Carlo simulation or via a design point-based approach, such as FORM/SORM, response surface, importance sampling, etc. For Monte Carlo simulation, a candidate preconditioning matrix is the stiffness matrix obtained using mean properties of the problem random variables. We denote this matrix $\boldsymbol{K}_o$. $\boldsymbol{K}_o$ will approximate the stiffness matrix generated on individual histories (samples) of the Monte Carlo simulation, and it need only be factored once prior to the commencement of the simulation loop. In addition, the initial guess for each simulation history can be the solution obtained using $\boldsymbol{K}_o$ and either the mean load vector or the load vector sample that is generated on the particular simulation history. For the design point-based approaches, a similar strategy can be used. Here, however, it may be appropriate to begin with $\boldsymbol{K}_o$ as the preconditioning matrix but to update the preconditioning matrix as the design point estimate is updated. That is, define the preconditioning matrix as the stiffness matrix obtained using the current design point estimate values.

To summarize, the Stochastic Preconditioned Conjugate Gradient method (SPCG) is defined as follows:

(1) Select the preconditioning matrix, **M** (eqn (4)), to be the stiffness matrix obtained using mean properties of the problem random variables, that is $\boldsymbol{K}_o$, or the stiffness matrix obtained using the current design point estimate values and

(2) Select the initial guess to be:

$$\mathbf{x}_o = \boldsymbol{M}^{-1}\mathbf{f}_o \tag{13}$$

where $\boldsymbol{M}$ is the preconditioning matrix selected in Step 1 and $\mathbf{f}_o$ is either the mean load vector, the current simulation history realization of the load, or the loads corresponding to the current design point estimate.

COMPARISON WITH NEUMANN EXPANSION

Neumann expansion can be a very powerful approach as proposed for application in stochastic finite element analysis.[1] Hence, it is interesting to compare the SPCG approach with Neumann expansion. In Neumann expansion the inverse of the stiffness matrix is expanded about the inverse of a 'central matrix', $\boldsymbol{K}_o$, as:

$$\boldsymbol{K}^{-1} = (\boldsymbol{I} - \boldsymbol{P} + \boldsymbol{P}^2 - \boldsymbol{P}^3 + \ldots)\boldsymbol{K}_o^{-1} \tag{14}$$

where

$$\boldsymbol{K} = \boldsymbol{K}_o + \Delta\boldsymbol{K} \tag{15}$$

and

$$\boldsymbol{P} = \boldsymbol{K}_o^{-1}\Delta\boldsymbol{K} \tag{16}$$

To facilitate comparison with the SPCG method, we recast the Neumann expansion in a recursive form as:

$$\mathbf{x}_{i+1} = \mathbf{x}_i + (-\boldsymbol{P})^{i+1}\boldsymbol{K}_o^{-1}\mathbf{f} \qquad i \geq 0$$
$$\mathbf{x}_o = \boldsymbol{K}_o^{-1}\mathbf{f} \tag{17}$$

In this form it is seen that Neumann expansion is equivalent to beginning at some estimate of the solution $\mathbf{x}_i$ and moving in the direction $(-\boldsymbol{P})^{i+1}\boldsymbol{K}_o^{-1}\mathbf{f}$ to obtain an updated solution. Iterations can be continued, just as in the SPCG method, until the specified convergence criteria is met. Note that convergence is guaranteed if the absolute values of all the eigenvalues of $\boldsymbol{P}$ are less than 1. As shown in Yamazaki, Shinozuka and Dasgupta,[1] this convergence criterion can be easily met via scalar adjustment of the central matrix if necessary. The scalar adjustment introduces almost no additional computational effort.

From eqn (11), it can now be shown that when $\boldsymbol{K}_o$ and $\mathbf{x}_o$ are selected to be the same for the SPCG method and Neumann expansion, (i.e. the preconditioning matrix is the same as the central expansion matrix) which is typically the case, then the first search direction in the SPCG method, $\mathbf{p}_o$, is the same as the first search direction in the Neumann expansion, $-\boldsymbol{P}\boldsymbol{K}_o^{-1}\mathbf{f}$. This search direction was shown earlier to be the direction of steepest descent. That is, the Neumann expansion

Table 1. Operation count comparison for SPCG and direct solver

Step	SPCG			Direct		
	Full	Banded	No. of times	Full	Banded	No. of times
Factor matrix	$n^3/6+n^2/2$	$nn_b^2/2$	1	$n^3/6+n^2/2$	$nn_b^2/2$	n_c
Forward-back substitution	n^2	$2nn_b$	1	n^2	$2nn_b$	n_c
Iteration	$n(2n+5)$	$n(4n_b+5)$	n_cn_i	—	—	—
Total	$n^3/6+3n^2/2$ $+n_cn_in(2n+5)$	$nn_b(n_b/2+2)$ $+n_cn_in(4n_b+5)$		$n_c(n^3/6+3n/2)$	$n_cnn_b(n_b/2+2)$	

1. One operation is defined as one multiplication (or division) followed by an addition.
2. n = degrees-of-freedom; n_b = average half-band width; n_i = average number of iterations for SPCG to converge; n_c = number of repeated equation solutions (e.g. number of histories for Monte Carlo Simulation).
3. Operation count for SPCGB assumes some initial guess for each of the n_c calculations. Recomputing initial guess is roughly equivalent to adding 1/2 iteration; thus recomputing the initial guess would lower total operation count if average number of iterations (over the n_c calculations) is reduced by 1/2).

also begins with a move in the steepest descent direction. The difference between the two methods, at this first step, however, is that the SPCG method incorporates the scalar, α. That is, the SPCG method moves the distance that minimizes the error functional, ϕ (eqn (2)). Beyond the first step, there is no longer any similarity in the methods since the methods search for the solution using different directions and different step sizes.

We emphasize here that the SPCG method search directions have certain desirable properties, cited earlier, which are linear independence and 'closeness' to steepest descent. It will be shown in the example problems to follow, that due to these properties, and due to the use of the optimizing scalar, α, very fast convergence can be achieved.

Next, it is appropriate to compare the number of numerical operations required in each step of these two methods. As shown by Yamazaki, Shinozuka and Dasgupta,[1] when the stiffness matrix, $\boldsymbol{K}$, is banded, the most efficient method to obtain the Neumann expansion terms is by using the recursive equation:

$$\boldsymbol{K}_o\mathbf{U}_i = \Delta\boldsymbol{K}\mathbf{U}_{i-1} \tag{18}$$

where $\mathbf{U}_i$ represents the expansion terms that are alternately added and subtracted to obtain the solution $\mathbf{U}$. Solution of eqn (18) requires: 1) evaluation of the right hand side, that is, a banded matrix-vector multiplication; and 2) the evaluation of $\mathbf{U}_i$, by forward and backward substitution using the Cholesky decomposition of $\boldsymbol{K}_o$ (obtained prior to commencement of the recursive procedure).

Examining the SPCG method, it is seen that the SPCG method requires a similar banded matrix-vector multiplication, that is, $\boldsymbol{K}\mathbf{p}_i$ (used in eqns (8) and (10) and a similar forward-backward substitution using the Cholesky decomposition of $\boldsymbol{K}_o$ ($=\boldsymbol{M}$ in eqn (7)). The SPCG method also requires additional scalar-vector products, vector dot products and vector additions. Hence, the number of operations required in each step of the SPCG method is greater than for Neumann expansion; however, these additional operations will represent a small percentage of the total numerical operation count.

COMPARISON WITH DIRECT SOLVER

We can gain some insight to the efficiency of the SPCG method by comparing the number of floating point operations required by the SPCG method and the number required by a direct solver. Table 1 shows an operations count comparison for the two methods. As mentioned earlier, the vast majority of the effort in the SPCG method is associated with the forward-back substitution plus the matrix-vector multiply that is performed once for each iteration (requires $2n^2$ operations for a full matrix and $4nn_b$ operations for a banded matrix). The operation totals in the table can be used to estimate the efficiency of the SPCG method if we can estimate the average number of iterations required for solving the system of equations. Such estimates will be possible as we gain experience with the method.

Generally, the SPCG method will be most effective for problems wherein the bandwidth is large with respect to the number of equations and/or there is a large number of zeros within the band (e.g. large 3-D problems). We note that these are the same types of problems that iterative solvers are generally most effective for in deterministic problems. We caution that Table 1 cannot be used exclusively to evaluate the effectiveness of the SPCG method since operation count alone is not always the decisive factor. Other factors such as matrix sparsity and ability to parallelize the computations must be taken into account for modern computational platforms.

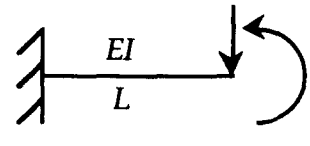

Fig. 1. Cantilever Beam Example.

Table 2. SPCG results for 2-D cantilever beam

	Step		
	0	1	2
$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$	$\begin{Bmatrix} 3 \\ 5 \end{Bmatrix}$	$\begin{Bmatrix} 2{\cdot}0848 \\ 3{\cdot}9270 \end{Bmatrix}$	$\begin{Bmatrix} 2{\cdot}0618 \\ 3{\cdot}8455 \end{Bmatrix}$
$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}^a$	$\begin{Bmatrix} 1{\cdot}0 \\ 1{\cdot}1725 \end{Bmatrix}$	$\begin{Bmatrix} 1{\cdot}0 \\ 3{\cdot}5398 \end{Bmatrix}$	

[a]Directions shown are normalized for comparison to directions in Table 3.

EXAMPLE PROBLEMS

Example 1-Cantilever beam

To demonstrate the mechanics of the SPCG method a simple example is selected, the cantilever beam shown in Fig. 1.

The stiffness matrix for the deflection and rotation at the free end is given by:

$$\boldsymbol{K} = \begin{bmatrix} \dfrac{12EI}{L^3} & \dfrac{-6EI}{L^2} \\ \dfrac{-6EI}{L^2} & \dfrac{4EI}{L} \end{bmatrix} \tag{19}$$

Letting EI and L be random variables (that is, treating EI as a single variable) with mean values of 0·5 and 1·0 respectively,[†] and selecting the preconditioning matrix to be $\boldsymbol{K}_o$, the stiffness matrix obtained using mean properties of the random variables gives:

$$\boldsymbol{K}_o = \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix} \tag{20}$$

Assume now that a solution is required for the case where EI is perturbed by 10% above its mean value (i.e. $EI = 0{\cdot}55$) and L is perturbed by 10% below its mean value (i.e. $L = 0{\cdot}9$), and the load is given by:

$$\mathbf{f} = \begin{Bmatrix} 3 \\ 1 \end{Bmatrix} \tag{21}$$

Thus, we require solution of the system:

$$\begin{bmatrix} 9{\cdot}0535 & -4{\cdot}07407 \\ -4{\cdot}07407 & 2{\cdot}44444 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 1 \end{Bmatrix} \tag{22}$$

The exact solution is easily found to be:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2{\cdot}0618 \\ 3{\cdot}8455 \end{Bmatrix} \tag{23}$$

Table 2 illustrates the results obtained using the SPCG method. The table shows the solution, $\mathbf{x}_i$, and the normalized search direction, $\mathbf{p}_i^*$, for each step, i, of the iterative process. The initial solution guess is computed as $\boldsymbol{K}_o^{-1}\mathbf{f}$. Note the rapid convergence in the first iteration despite the fairly large perturbation ($\approx 50\%$ for k_{11}, $\approx 33\%$ for $k_{12} = k_{21}$, $\approx 25\%$ for k_{22}). As should be expected, the SPCG method produces the exact solution in two iterations for this two degree of freedom example problem.

Next we use Neumann expansion for the same problem. These results are summarized in Table 3 for the first five iterations (i.e. up to expansion order 5). The results show that the method is converging. This is expected since the eigenvalues of the $\boldsymbol{P}$ matrix are all < 1 (or could have been made so, as discussed earlier).[1] The convergence, however, is much slower than the SPCG method. Note that the first search direction is the same for both methods (the direction of steepest descent) although the SPCG method is much closer to the exact solution after 1 iteration due to the use of the optimizing scalar α. Also note that the second search direction in the SPCG method differs significantly from the first search direction, as desired.

Example 2—3-D space truss

In order to investigate the SPCG approach on a more practical scale, it was implemented in a direct Monte Carlo simulation code, and applied to the example 3-D space truss problem shown in Fig. 2. Eight different material type random variables are defined (Table 4), and each material type is taken to be independent. Three random loads were applied to the structure as shown in Fig. 2 and are defined in Table 5. For the Monte Carlo

Table 3. Neumann expansion results for 2-D cantilever beam

	Step, expansion order					
	0	1	2	3	4	5
$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$	$\begin{Bmatrix} 3 \\ 5 \end{Bmatrix}$	$\begin{Bmatrix} 1{\cdot}4733 \\ 3{\cdot}2099 \end{Bmatrix}$	$\begin{Bmatrix} 2{\cdot}4552 \\ 4{\cdot}2606 \end{Bmatrix}$	$\begin{Bmatrix} 1{\cdot}7964 \\ 3{\cdot}5662 \end{Bmatrix}$	$\begin{Bmatrix} 2{\cdot}2413 \\ 4{\cdot}0341 \end{Bmatrix}$	$\begin{Bmatrix} 1{\cdot}9505 \\ 3{\cdot}7179 \end{Bmatrix}$
$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix}^a$	$\begin{Bmatrix} 1{\cdot}0 \\ 1{\cdot}1725 \end{Bmatrix}$	$\begin{Bmatrix} 1{\cdot}0 \\ 1{\cdot}0701 \end{Bmatrix}$	$\begin{Bmatrix} 1{\cdot}0 \\ 1{\cdot}0552 \end{Bmatrix}$	$\begin{Bmatrix} 1{\cdot}0 \\ 1{\cdot}10516 \end{Bmatrix}$	$\begin{Bmatrix} 1{\cdot}0 \\ 1{\cdot}0512 \end{Bmatrix}$	

[a]Directions shown are normalized for comparison to directions in Table 2.

[†]Simple numeric values are chosen here for ease of presentation.

Table 4. Material property random variables for 3-D truss

Material type	E ($\delta = 0{\cdot}10$) (ksi)	A ($\delta = 0{\cdot}10$) (in^2)	ϵ_{int} ($\sigma = 10^{-4}$)
Mv	29000·0	1·590	0·0
Mh_I	29000·0	1·590	0·0
Mh_{II}	29000·0	1·590	0·0
Mh_{III}	29000·0	1·590	0·0
Mb_I	29000·0	0·938	0·0
Mb_{II}	29000·0	0·938	0·0
Mb_{III}	29000·0	0·938	0·0
Mt	29000·0	1·590	0·0

E = mean modulus of elasticity (lognormal r.v.)
A = mean bar cross-section area (lognormal r.v.)
ϵ_{int} = mean initial strain in bar element (normal r.v.)
δ = coefficient of variation
S = standard deviation

Table 5. Loading random variables for 3-D truss

Load	Type	Mean (kips)	δ
S_1	Lognormal	10.0	0·25
S_2	Lognormal	10.0	0·25
S_3	Lognormal	500·0	0·25

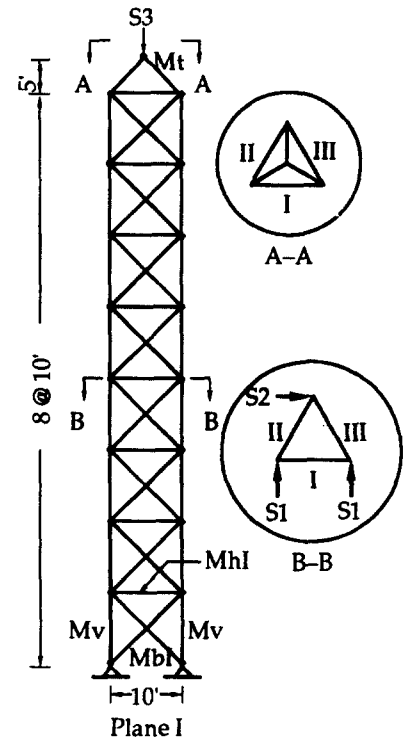

Fig. 2. Front panel of 3-D space truss example.

analysis, 1000 simulation histories were performed and the cumulative distribution function (CDF) for the stress in element 24 (the rear vertical element at the base of the truss) for reliability levels from approximately 0·01 to 0·99 was evaluated. The calculations were carried out on an Alliant FX/80 shared-memory, parallel-multiprocessor computer. Although the code used for these calculations is parallelized to take advantage of the parallelism inherent in Monte Carlo simulation, only a single processor is used for this demonstration.

Table 6 shows performance results of the analysis, and Table 7 shows the computed CDF values. Note that three cases were analyzed, as described below. For each case, the same number of Monte Carlo histories, i.e. 1000, was used. For each individual Monte Carlo history, the number of iterations required to achieve a converged solution was retained. Table 6 contains the summary statistics for the required number of iterations over the 1000 histories, along with the total solution time.

The results in the table show that the number of iterations required for convergence with the SPCG method is again quite small. In the first example, a simple and common preconditioning strategy, often referred to as Jacobi preconditioning was used. Here the preconditioning matrix is selected to be the main diagonal of the stiffness matrix (that is, of the stiffness matrix formed within the particular simulation trial). As shown in the table this method required, on average, 42 iterations to achieve a converged solution. Next, the SPCG method was used. The preconditioning matrix was selected to be the stiffness matrix obtained using mean properties, $\boldsymbol{K}_o$, and the initial guess obtained using $\boldsymbol{K}_o$ and the mean load vector as shown in the table. The mean number of iterations required to achieve a converged solution was reduced by an order of magnitude to 4·2, and the bounds and standard deviations on the number of required iterations were similarly reduced. The solution time was, however, reduced by a smaller factor of 6·3 due to the additional

Table 6. Performance of stochastic preconditioned conjugate gradient method (SPCG) applied to 3-D space truss (1 000 Monte Carlo simulations)

Preconditioning matrix	Initial solution guess	Tolerance η	Number of iterations for convergence				Solution time[a] (sec)
			Min	Mean	Max	Standard deviation	
$\boldsymbol{K}_D$ (Jacobi)	$\boldsymbol{K}_o^{-1}\bar{\mathbf{f}}$	0·01	22	42	61	6·0	638
$\boldsymbol{K}_o$ (SPCG)	$\boldsymbol{K}_o^{-1}\bar{\mathbf{f}}$	0·01	3	4.2	6	0·48	101
$\boldsymbol{K}_o$ (SPCG)	$\boldsymbol{K}_o^{-1}\bar{\mathbf{f}}$	0·10	2	3.2	4	0·41	86

[a]Single processor of an Alliant FX/80.

Table 7.Cumulative distribution function for the stress in element 24 of the 3-D space truss[a]

Member stress (ksi)	CDF		
	Direct solution (Gaussian elimination)	SPCG $\eta = 0{\cdot}01$	SPCG $\eta = 0{\cdot}10$
60	0·027	0·027	0·027
70	0·110	0·110	0·110
80	0·277	0·276	0·278
90	0·451	0·451	0·451
100	0·625	0·625	0·623
110	0·781	0·782	0·781
120	0·871	0·871	0·871
130	0·929	0·929	0·929
140	0·966	0·966	0·966
150	0·986	0·986	0·986

[a]Rear vertical element at the base of the truss.

computational effort required on each iteration of the SPCG method (see eqn (7)). Note that the reduction in number of iterations can be attributed to the preconditioning since the initial guess is identical in both cases.

The effect of reducing the convergence criteria was next investigated. The purpose here was to see if the number of iterations could be further reduced without significantly affecting the accuracy of the Monte-Carlo results. The tolerance was reduced by an order of magnitude from 0·01 to 0·1 and, as shown in Table 6, the mean number of iterations was reduced by another 25%. This reduction in tolerance had no significant affect on the Monte-Carlo results for the range of CDF values evaluated as shown in Table 7.

The significance of the above result is that it may be possible to reduce the tolerance required in Monte-Carlo simulation from that which is normally required in a deterministic evaluation. Further work will be needed to determine what convergence criteria should be used in Monte-Carlo simulation. In fact, for Monte-Carlo a 'smart' convergence criteria could be developed that reflects the basic problem uncertainties. Clearly, a high degree of precision should not be required when the problem uncertainties are very large, if the numerical error can be shown to be unbiased.

SUMMARY AND CONCLUSIONS

This paper has presented the Stochastic Preconditioned Conjugate Gradient method (SPCG), an iterative equation solver that can greatly reduce the computational effort associated with the repeated calculations required in probabilistic finite element analysis. The method was implemented in a Monte Carlo simulation code and demonstrated for two example problems: a simple cantilever beam and a 3-D space truss. Excellent convergence properties were demonstrated in both cases. For the 3-D space truss problem, the method converged in only 3–4 iterations, on average. The method was also compared with Neumann expansion (another powerful approach for stochastic finite element analysis), by interpreting Neumann expansion as an iterative-search type solver. It was shown that the SPCG method has several advantageous properties with regard to speed of convergence. It was also shown that the computational effort required at each iteration of the SPCG method is only slightly greater than the effort for each iteration (or expansion term) in Neumann expansion.

Although the example problem demonstration was for Monte Carlo simulation, the method is equally applicable for design point based approaches. For design point based approaches the only difference would be to use the current design point estimate for evaluation of the preconditioning matrix (rather than mean values as in Monte Carlo simulation).

The method is also well suited for parallel processing implementation. For parallel implementation, solution strategies are necessary that require minimal storage, can be vectorized, and can take advantage of concurrent processing. The SPCG method meets these criteria as was described earlier. For more details on this, the reader is referred to Sues, *et al.*,[9] wherein optimal hardware and software strategies for parallel processing in probabilistic structural analysis are investigated.

The advantages of the SPCG method can be summarized as follows:

1. Convergence of the method is guaranteed, even for large perturbations about the mean, due to use of the conjugate gradient search strategy.
2. Convergence is rapid due to the use of preconditioning.
3. The method is applicable in either Monte Carlo simulation or design point based approaches.
4. The method is well suited to parallel implementation since storage requirements are reduced from direct solvers and the solution method can take advantage of vector-concurrent architectures.

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