Homework #3:
Differential Equations

Oscillations (in MATLAB)
Due Oct. 12

1. **Energy Conservation**: Starting with the MATLAB program *pendulum.m*:
   (a) Modify this program to simulate simple harmonic motion. Use the program to calculate the relative change in the total energy during one cycle $\Delta_n = (E_n - E_0)/E_0$. Is the function Δ uniformly small during the cycle? Choose $\theta_0 = 0.25$, $\dot{\theta}_0 = 0$, and $\omega_0^2 = 9$.
   (b) Plot position and velocity as a function of the time t.
   (c) Compute the average value of the kinetic energy and the potential energy during a complete cycle. Plot them on the same graph Is there a relation between the two averages?
   (d) Plot the path of the oscillator in phase space $(\theta, \dot{\theta})$. Set $\omega_0^2 = 9$ and use different initial conditions. Do you find different paths for each of them? What physical quantity distinguishes or characterizes each path? What is the shape of the phase paths? Is the motion of a representative point $(\theta, \dot{\theta})$ always in the clockwise or counterclockwise direction?

2. **Large Oscillations**: Again starting with the MATLAB file *pendulum.m*
   (a) Modify your program to simulate large amplitude oscillations in a pendulum. Set $g/L = 9$. Check the stability by calculating the total energy and ensuring that it does not drift from its initial value.
   (b) Set $d\theta/dt|_{t=0} = 0$ and make plots of $\theta(t)$ and $d\theta/dt(t)$ for the initial conditions $\theta(t = 0) = 0.1$, 0.2, 0.4, 0.8, 1.0. Describe the qualitative behavior of θ and $d\theta/dt$. What is the period T and the maximum amplitude θ_{max} in each case? Plot T versus θ_{max} and discuss the qualitative dependence of the period on the amplitude. How do the results compare in the linear and non-linear cases, that is, which period is larger? Explain the relative values of T in physical terms.

   Use the exact pendulum equation for the rest of the assignment

3. **Damped Oscillations**
   (a) Incorporate the effects of damping in your program and plot the time dependence of position and velocity. Make runs for $\omega_0^2 = 9$, $\theta_0 = 1$, $\dot{\theta}_0 = 0$, and $\gamma = 0.5$.
   (b) Compare the period and angular frequency to the undamped case. Is the period longer or shorter? Make additional runs for $\gamma = 1, 2, 3$. Does the frequency increase or decrease with greater damping?
   (c) Compute the average value of the kinetic energy, potential energy, and total energy over a complete cycle. Plot these averages as a function of the number of cycles. Due to the presence of damping, these averages decrease with time. Is the decrease uniform?
   (d) Compute the time-dependence of $\theta(t)$ and $\dot{\theta}(t)$ for $\gamma = 4, 5, 6, 7, 8$. Is the motion oscillatory for all γ? Consider a condition for equilibrium $\theta < 0.0001$; how quickly does $\theta(t)$ decay to equilibrium? Include an $events$ function in your program that automatically stops when the pendulum reaches ”equilibrium”.
   (e) Construct the phase space diagram for cases $\omega_0^2 = 9$ and $\gamma = 0.5, 2, 4, 6, 8$. Are the qualitative features of the paths independent of γ? If not, discuss the qualitative differences.

4. **Linear Response to Forced Oscillations**
   (a) Modify your program so that an external force of the form $\frac{1}{m}F(t) = A_0 \cos \omega t$ is included. Set $\omega_0^2 = 9$, $\gamma = 0.5$, $A_0 = 1$ and $\omega = 2$ (well use these values for the rest of the exercise). These values correspond

to a lightly damped oscillator. Plot $\theta(t)$ versus t for the initial conditions $(\theta_0 = 1, \dot{\theta}_0 = 0)$. How does the qualitative behavior differ from the unperturbed case? What is the period and angular frequency of $\theta(t)$ after several oscillations? Obtain a similar plot with $(\theta_0 = 0, \dot{\theta}_0 = 1)$. What is the period and angular frequency after several oscillations? Does $\theta(t)$ approach a limiting behavior that is independent of the initial conditions? Identify a transient part of $\theta(t)$ which depends on the initial conditions and decays in time, and a steady state part which dominates at longer times and which is independent of the initial conditions.

(b) Compute $\theta(t)$ for $\omega = 1$ and $\omega = 4$. What is the period and angular frequency of the steady state in each case? On the basis of these results, explain which parameters determine the frequency of the steady state behavior.

(c) Verify that the steady state behavior is given by

$$\theta(t) = A(\omega)\cos\omega t + \delta,$$

where δ is the phase difference between the applied force and the steady state motion. Compute δ for $\omega_0^2 = 9, \gamma = 0.5$,$\omega = 0$, 1.0, 2.0, 2.2, 2.4, 2.6, 2.8, 3.0, 3.2, 3.4. Repeat the computation for $\gamma = 1.5$ and plot δ versus ω for the two values of γ. Discuss the qualitative dependence of $\delta(\omega)$ in the two cases.

5. **Resonance**:

(a) Adopt the initial condition $(\theta_0 = 0, \dot{\theta}_0 = 0)$. Compute $A(\omega)$ for $\omega = 0$, 1.0, 2.0, 2.2, 2.4, 2.6, 2.8, 3.0, 3.2, 3.4 with $\omega_0 = 3$ and $\gamma = 0.5$. Plot $A(\omega)$ versus ω and describe its qualitative behavior. If $A(\omega)$ has a maximum, determine the resonance angular frequency ω_{max}, which is the frequency at the maximum of A. Is the value of ω_{max} close to the natural angular frequency ω_0?

(b) Compute A_{max}, the value of the amplitude at ω_{max}, and the ratio $\Delta\omega/\omega_{max}$, where $\Delta\omega$ is the width of the resonance. Define $\Delta\omega$ as the frequency interval between points on the amplitude curve which are $1/\sqrt{2}A_{max}$. Set $\omega_0 = 3$ and consider $\gamma = 0.1, 0.5, 1.0, 2.0$. Describe the qualitative dependence of A_{max} and $\Delta\omega/\omega_{max}$ on γ. The quantity $\Delta\omega/\omega_{max}$ is proportional to $1/Q$, where Q is the quality factor of the oscillator.

Heat Equation (in Python)
Due Oct. 19

6. Complete Exercise 1 in the jupyter notebook *HeatFlow.ipynb*. Start from the given code.

7. **Two bars in contact**: Two identical bars, 25cm long each, are in contact. One bar is initially at $50°C$, and the other at $100°C$. The free ends are kept at $0°C$. Calculate and plot the temperature distribution as a function of time. Use the *jsanimation* package as shown in *HeatFlow.ipynb*.