CS 5200 F 19 Take-Home Final Version 1

George Markowsky

December 6, 2019

Due Friday, December 13, 2019 By 11:59 pm
Rules for this Take-home Exam

1. You are not permitted to discuss the take-home exam with anyone other than me. If you have a question, email me and I will try to answer it. You may consult your textbook, class notes, and class handouts for this exam. You may also consult reference documents for Python. You may look up general facts and standard definitions, but you should not search for solutions to these problems on the Internet. You should certainly not copy solutions from the Internet. You may also use tools such as spreadsheets. If you use ideas from sources or tools that are not standard for this course, please cite them in a proper manner.

2. When you write proofs, clearly state what you are proving and how you are proving it. If you have any questions or are confused, please ask for clarification. You may also ask questions about material presented during the semester by email.

3. When you finish your exam, you should submit a PDF file that contains your answers to the problems and a ZIP file that will contain other files that you will be instructed to create. Please following the directions and naming conventions that you will be given. Note that your answers must be typed and no handwritten answers will be accepted with the following exception. **If you need to draw graphs for your answers you may draw them by hand and include photos or scanned versions of the graphs in your document.** The PDF file should include any code that you are asked to write and any reasonably short output that you are asked to supply. No screenshots will be accepted. There is no need for screen shots since Python code is written in ASCII and all output can be produced in ASCII. You just select the text you want and copy it into your answer file. All very long output should be put as a separate file into the ZIP file.

4. All programs must be written in Python and you cannot use packages that code up the routines I am asking you to write. Thus, you cannot use a Binary Search Tree package if I am asking you to write a Binary Search Tree Package.

5. Your PDF should contain the following statement at the beginning:

   I, <your name>, certify that all the material in the submitted files is my original work, that I did not discuss these questions with anyone other than my instructor, and that I did not copy work from anyone for this examination.

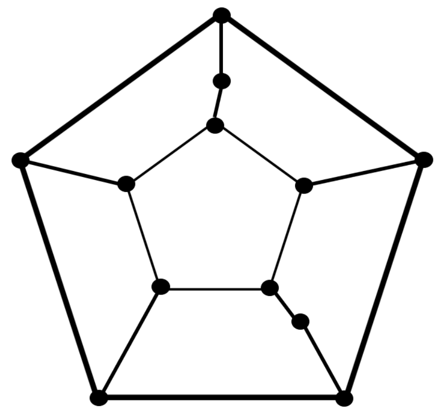
6. Please write clearly and try to keep grammatical and spelling errors to a minimum.

7. **You must not miss the 11:59 pm deadline on 12/13/19. For this reason I suggest that you upload a reasonably finished version of your exam paper some reasonable time before the deadline. If you are even a microsecond late in submitting, submission will be locked and if you don't have something there you will get a 0 for your final exam grade.**

1. (15 points – Hamiltonian Paths and Cycles)

   (a) (2 points) Does the graph below have a Hamiltonian Path? Justify your answer.

   (b) (3 points) Does the graph below have a Hamiltonian Cycle? Justify your answer.



   (c) (10 points) Prove by induction that an n-dimensional cube has a Hamiltonian Cycle. An n-dimensional cube is defined as follows. It consists of the 2^n vertices of the form $(a_0, a_1, ..., a_n)$ where each a_i is either 0 or 1. Two vertices have an edge connecting them if the vertices differ in exactly one component. The following are some examples of n-dimensional cubes.

      i. A 1-dimensional cube consists of the vertices 0 and 1 and the edge (0,1),

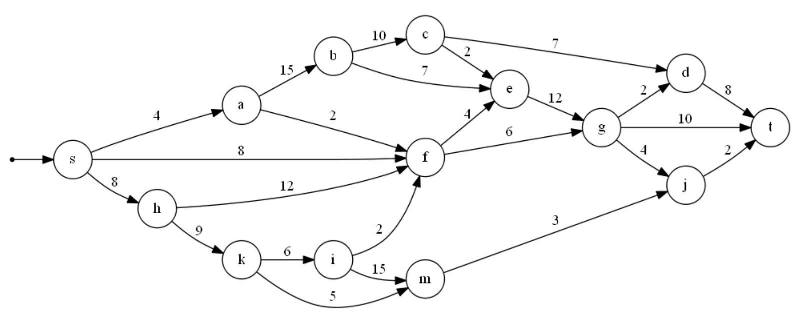
      ii. A 2-dimensional cube consists of the vertices (0,0), (0,1), (1,0), and (1,1). The edges would be ((0,0),(0,1)), ((0,0),(1,0)), ((0,1),(1,1)), and ((1,0)(1,1))

      iii. A 3-dimensional cube consists of the 8 vertices and 12 edges that you would draw when drawing a regular cube.

   Include your solution in your PDF file..

2. (30 points – General Graph Theory)

   (a) (5 points – Max Flow/Min Cut) In the transport network below find the maximal flow and a matching minimal cut. I will be updating the lecture notes for Max Flow by Friday afternoon on 12/6 to complete the discussion of finding the max flow. Include your solution in your PDF file.



   (b) (5 points – Handshaking) There were 10 people at a party. At the end of the party, one of the people (let's call him John) asked each of the other 9 people present to tell him with how many other people at the party did they shake hands. John was surprised to learn the following.

      i. Every person shook hands with at least one other person (no shaking of one's own hands allowed).

      ii. No two of the nine people reported the same number of handshakes.

How many people did John shake hands with? Justify your answer and show all work. Include your solution in your PDF file.

(c) (5 points – Vertex Degrees) Prove that every graph with two or more nodes must have at least two vertices having the same degree. Determine all graphs that contain just a single pair of vertices that have exactly the same degree. Justify your answers. Include your solution in your PDF file.

(d) (5 points – Connectedness) Let G be a graph. The complement of G, G', is the graph having the same nodes, but in which (a,b) in an edge iff (a,b) is not an edge in G. Prove that if G is disconnected, then G' is connected. Include your solution in your PDF file.

(e) (5 points – Chess) Is it possible to move a knight on an 8×8 chessboard so that it completes every move exactly once? A move between two squares of the chessboard is completed when it it made in either direction. Justify your answer. Include your solution in your PDF file.

(f) (5 points – Two Coloring) Is there a planar graph with 17 edges and 10 vertices that can be 2-colored? Justify your answer. Include your solution in your PDF file.

3. (15 points – NP Complete Problems) For each of the 5 following problems determine whether it is NP-Complete or whether it can be solved in polynomial time. Give a reference supporting your claim. Include your solution in your PDF file.

   (a) Suppose you have a logic circuit consisting of AND gates, OR gates, inverters (NOT gates) that has a single output and multiple inputs. Suppose that exactly one of the gates fails so that it always gives the same output value regardless of the input values. Determine a set of input values that will detect that fault. In other words, find a set of input values will produce a different output value if that particular gate gets stuck compared to when it operates normally.

   (b) Determine where a given graph has a spanning tree for which each node has degree 2 or less?

   (c) Let G be a graph with weights on each edge. Determine if G has a spanning tree such that the weights of the edges in G sum to less than some number K.

   (d) Suppose you have a some tasks, T, to be scheduled on m processors. Each task takes some positive amount of time to complete, and once started on a processor, it must remain on that processor until completion. Suppose that there is a deadline by which all tasks must be completed. Find a schedule that allows you to schedule the tasks on the m processors so that you meet the deadline or determine that no such schedule exists.

   (e) Suppose you are given a graph so that each vertex and each edge has a positive weight assigned to it. Suppose you are given two positive integers K and J. Is it possible to partition the vertices V, into subsets $V_1, V_2, ..., V_m$ such that the sum of the weights of the vertices in each $V_i \leq K$ and for each i and j, the sum of the weights of the edges between V_i and V_j is $\leq J$.

4. (15 points – Red-Black Trees, B-Trees, and Binary Search Trees)

   (a) (10 points) Write a program to **implement only the insertion function** of red-black tree. If you are unable to implement this algorithm correctly, describe your difficulties and produce the best code that can. Include your code in both the PDF and ZIP files.

   (b) (5 points) Create 10 sets of 10,000 positive integers. Label the sets Prob4bData0.txt,..., Prob4bData9.txt and include them in the ZIP file. Use the BST and B-Tree programs that you created in Problem 8 of Prelim2 (if you did not code those programs, please do so now), along with the Red-Black Tree Insertion program that you created in Part (a) and your 10 sets of random integers to create BSTs, B-Trees (t = 3), and Red-Black Trees using the appropriate insertion routines. Produce a statistical analysis (mean, median, and standard deviation) of the heights of the trees generated and the actual time that it takes to create the 10,000 element trees. Include the comparison of these algorithms in the PDF file. Include your statistical analysis in both the PDF and ZIP files.

5. (15 points – More Graph Algorithms) All the following questions relate to the file GraphData.txt that was used in the previous two prelims. Refer to the previous prelims for information about how this file is structured. Turn the graph stored in GraphData.txt into a weighted graph by assigning the weight |node1 - node2| to the edge (node1,node2). For example, the first line of GraphData.txt is the edge (277, 325) which gets a weight of |277 - 325| = |-48| = 48.

   (a) (5 points – Minimal Weight Spanning Tree) With the weights assigned find the weight of a minimum weight spanning forest for the graph in GraphData.txt. Put the code you used to find it and the value you found into the PDF file and the ZIP file. In the ZIP file only, include a list of the edges in the minimal weight spanning tree that you found.

   (b) (5 points – Maximal Shortest Paths) Compute the "shortest" distances (costs) between all pairs of nodes in the weighted graph we produced. In the PDF and ZIP files include the code that you wrote to find these values along with the maximum of all the "shortest" paths and the number of pairs of vertices that have this maximum shortest path.

   (c) (5 points – Depth-first Search) Find the length of a longest paths of nodes $v_0, v_1, ..., v_k$ where v_i is an ancestor of v_j if $i < j$. Include your code, this maximum length, and the number of such maximal chains in both your PDF and ZIP files.

6. (10 points – GCD) All integers are positive in this problem. We know that GCD(p,p) = p, for all positive p.

   (a) (2 points) Prove that if p and q are even integers $GCD(p, q) = 2 \times GCD(p/2, q/2)$. Include your solution in your PDF file.

   (b) (2 points) Prove that if p is odd and q is even $GCD(p, q) = GCD(p, q/2)$. There is a similar result if p is even and q is odd.Include your solution in your PDF file.

   (c) (2 points) Prove that if p and q are both odd and $p > q, GCD(p, q) = GCD((p - q)/2, q)$. Include your solution in your PDF file.

   (d) (4 points) Write a program to compute the GCD using the above observations and binary operations where division by 2 of an even number in binary is done by removing the trailing 0 and shifting the string to the right. Similarly, multiplying by 2 means adding a zero to the right end. Write programs that convert numbers between decimal and binary strings consisting of ASCII '0's and '1's. For example, 42 should map to the string '101010' and vice versa. Write the routines to work with binary strings. Generate 100 random pairs of integers, including many large integers, convert them to binary strings and run your binary string GCD algorithm on these numbers. Check your work using a decimal GCD program. Include your program and output in both the PDF and ZIP files.